Centrality Rankings in Multiplex Networks

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ABSTRACT

The vertiginous increase of e-platforms for social communication has boosted the ways people use to interact each other. Micro-blogging and decentralized posts are used indistinctly for social interaction, usually by the same individuals acting simultaneously in the different platforms. Multiplex networks are the natural abstraction representation of such "layered" relationships and others, like co-authorship. Here, we re-define the betweenness centrality measure to account for the inherent structure of multiplex networks and propose an algorithm to compute it in an efficient way. To show the necessity and the advantage of the proposed definition, we analyze the obtained centralities for two real multiplex networks, a social multiplex of two layers obtained from Twitter and Instagram and a co-authorship network of four layers obtained from arXiv. Results show that the proposed definition provides more accurate results than the current approach of evaluating the classical betweenness centrality on the aggregated network, in particular for the middle ranked nodes. We also analyze the computational cost of the presented algorithm.

Categories and Subject Descriptors

H.4 [**Information Systems Applications**]: Miscellaneous; J.4 [**Social and Behavioral Sciences**]: Web Science

Keywords

Betweenness centrality; Multiplex networks; Multilayer networks

WebSci'14, June 23–26, 2014, Bloomington, IN, USA.

http://dx.doi.org/10.1145/2615569.2615687.

1. INTRODUCTION

Complex networks have become a natural abstraction of the interactions between elements in complex systems [20]. When the type of interaction is essentially identical between any two elements, the theory of complex networks provides with a wide set of tools and diagnostics that turn out to be very useful to gain insight of the system under study. However, there exist particular cases where this classical approach may lead to misleading results. Specifically, when the entities under study are related with each other using different types of relations. Representative examples are provided by temporal networks [19, 11], where edge connectivity may vary on time, transportation networks [6], where two geographic places may be connected by different transport modes, or social networks [18] where users are connected using several platforms or different categorical layers (for example, in co-authorship networks the categories of the field of study).

Here, we focus our study on a particular type of interconnected multilayer network [7] called "multiplex", where each object, if it exists, is univocally represented in each independent layer and so the interconnectivity pattern among layers becomes one-to-one. Figure 1b, 1c shows some representative examples of multiplex networks where it is possible to see the characteristics of the related topology. The examples show a multiplex with two layers where entities s, b, t exist in both layers. The interconnectivity between them is different in both layers.

Note that the existence of several topological structures similar to the one used in this work requires special care in their differentiation. Several works [16, 23, 3, 2, 25] have used a structure similar to multiplex networks. The main difference being that the topology adopted here relies on the connectivity between nodes representing the same entity in the different layers. In some of these studies, each layer is treated independently and metrics are evaluated on each of them to aggregate later the results in different ways. In the others, the different layers are projected into a single layer network by aggregating the edges of the individual layers and the metrics are computed over this aggregated network. Another approach that has been considered is the aggregation of all layers into a single layer with some differentiation between the connectivity of the different layers, like edge-colored graphs. Other structures similar to multiplex networks are interdependent networks [10]. The main difference between those type of networks and the one used here is more conceptual than structural. In the case of interconnected networks elements of layers have no counterparts on the other layers and consequently the one-to-one structural relation between elements of the different layers does not exist. For further details about the classification of such multilayer networks we refer to [12] and references therein. In the proposed approach, we do not perform any type of prior aggregation and keep the inherent structure of the interconnected layers in the multiplex to define the desired diagnostic, in our case the computation of shortest-path betweenness centrality.

The paper is structured as follows. Section 2 reviews centrality measures on multiplex networks and particularly focus on the problem of defining the shortest-path betweenness centrality for these type of networks. Section 2.1 describes our new proposed definition. Section 2.2 presents a computationally efficient algorithm to compute the proposed betweenness. Next, Section 3 presents results on real data, compares this results to the ones obtained by the aggregated network and experimentally evaluates the computational complexity of the algorithm. Eventually, Section 4 summarizes the main findings.

2. CENTRALITY MEASURES ON MULTIPLEX NETWORKS

In network theory, centrality diagnostics are aimed to measure the relative importance of a node, an edge, or some other subgraph [26, 20]. This diagnostic measure is specially interesting in social sciences since it is a proxy to determine influential nodes. However, centrality measures are also extremely important to address the problem of blind search and efficient navigation, e.g. PageRank [24] or HITS [13]. The generalization of these particular measures to multiplex is proposed in [22, 27, 15, 14, 17, 8].

Here, we focus on the definition of shortest path (or geodesic) betweenness centrality [9] for multiplex networks. As motivated in the previous section, social networks are usually composed by several types of relations, or ties, between individuals. Consider, for example, the ego network of an individual. Family, work, and hobbies are likely to be among his relationships as well as some others are likely to be maintained with on-line social platforms. In this scenario, the classical procedure of studying such networks is to project all this information into a single network by collapsing all the relations. However, using this aggregation procedure, as we will show, the resulting network may not accurately reflect the real topology.

Consider the scenario described in Figure 1. Figure 1a shows a network of three individuals $\{s, b, t\}$, edges between individuals indicate a relation between them. However, since the aggregation procedure is not injective several scenarios can lead to the same aggregated network, as shown in Figure 1b and 1c. In Figure 1b, individuals s and t can communicate through individual b in layer 1. This reduces to the classical approach where all the relations are within a single layer. Consider now the scenario of Figure 1c, this time individuals s and t are disconnected in both layers and individual b acts as a bridge allowing information to flow from layer 1 to layer 2. Note that without individual b connecting the two layers, individuals s and t will be disconnected. Undoubtedly, individual b in the scenario described by Figure 1c has more importance than individual b in the scenario of Figure 1b and centrality measures should reflect this fact. Thus, to obtain reliable centrality measures that accurately reflect the real structure it is mandatory to analyze the relations between individuals considering the full multiplex architecture when performing the respective analysis.

2.1 Shortest path betweenness centrality on multiplex networks

First of all, we define a path, $p_{s_\alpha \to t_\beta} \in \mathcal{P}_{s_\alpha \to t_\beta}$, on a multiplex network consisting of L layers and N nodes per layer, as an ordered sequence of nodes which starts from node s in layer α and finish in node t in layer β, with the restriction that an edge exists between every pair of consecutive nodes in p. $\mathcal{P}_{s_\alpha \to t_\beta}$ indicates the set of all possible paths between node s in layer α and node t in layer β. For every path $p_{s_\alpha \to t_\beta}$ it is possible to define a distance function $d\left(p_{s_\alpha \to t_\beta}\right)$, usually depending on the weight of the edges the path traverses to account for the "length" of the path. Without loss of generality, we define this distance function as the number of traversed edges in the path. Hence, the set of shortest-paths $P^*_{s\to t}$, from node s to node t, in the multiplex is defined as the set of paths which minimize the distance function between the two nodes,

$$P^*_{s\to t} = \underset{\substack{p_{s_\alpha \to t_\beta} \in \mathcal{P}_{[s\alpha \to t\beta]} \\ \alpha,\beta \in \{1,\dots,L\}}}{\arg\min} d(p_{s_\alpha \to t_\beta}) \tag{1}$$

That is, a shortest-path between two individuals, in a multiplex network, is a minimum path that starts from the source node in any layer, and reaches the destination node in any layer. See that this definition is coherent with the definition of a multiplex network since the same node in the different layers represent the same physical entity.

Considering (1), the shortest-path betweenness of node v on layer l, $g(v_l)$ is defined as the sum, for every possible origin-destination pair (s,t), of the fraction of times that node v on layer l, belongs to a path in $P^*_{[s\to t]}$. Specifically, the shortest-path betweenness centrality on a multiplex network is obtained by:

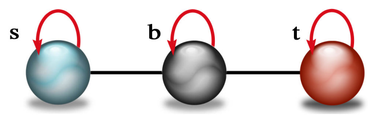
$$g(v_l) = \sum_{\substack{s,t=1 \\ s\neq t\neq v}}^{N} \frac{\sigma_{s,t}(v_l)}{\sigma_{s,t}}, \tag{2}$$

where $\sigma_{s,t} = |P^*_{[s\to t]}|$ is the number of shortest-paths from s to t and $\sigma_{s,t}(v_l)$ is the number of times node v_l is in a shortest-path from s to t.

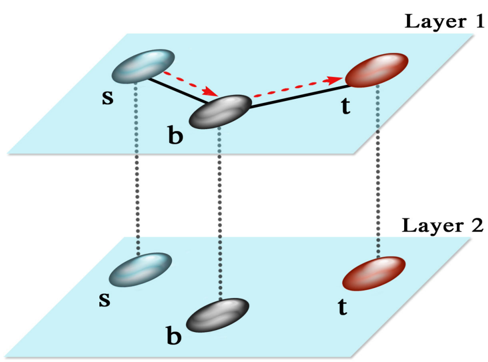
Note that with the given definition the shortest path degeneration increases. $P^*_{[s\to t]}$ may contain several shortest paths between s and t in same layer (classical shortest path degeneration) together with shortest paths that start and end in the same node but in different layers (multiplex shortest path degeneration).

Eventually, the shortest-path betweenness of a node, in a multiplex network, can be obtained by:

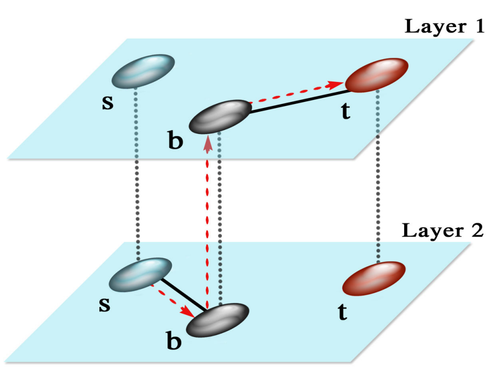
$$g(v) = \sum_{l=1}^{L} g(v_l). \tag{3}$$



(a) Aggregated Network.

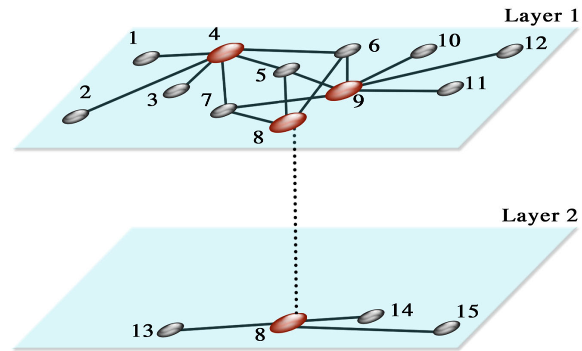


(b) Individuals are only connected in one layer.

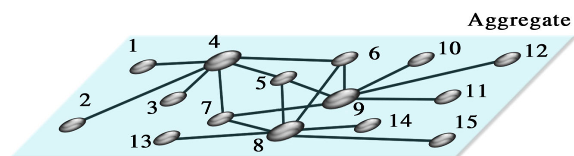


(c) Individual b acts as a bridge between the layers.

Figure 1: Example of possible multiplex configurations for the same aggregated network



(a) Example of a multiplex with three hubs, one acting as a bridge between layers.



(b) Aggregated network of Figure 2a. Still contains three hubs, but all have the same centrality.

Figure 2: Example of how the multiplex obtains different centrality rankings than the aggregated network.

Usually, not only the numerical value of the betweenness centrality is of interest [1] about a particular network but also the ranking of the nodes it provides [4]. In the scenarios described in Figure 1, although the betweenness given by the aggregated network is different from the betweenness given by the multiplex network, the rankings are equivalent. In both cases, node b becomes the most central node in the network while s and t are ranked as second. However, this situation is not common. Usually, rankings are substantially different. To illustrate the situation, consider the scenario given in Figure 2. Figure 2a represents a multiplex network with two hubs in layer 1 and a third hub linked to some nodes on layer 1 and some nodes on layer 2. Figure 2b shows the aggregated version of the multiplex in Figure 2a. It is easy to see that the shortest-path betweenness on the aggregate network of hubs numbered 4, 8 and 9 is the highest and the same. These nodes are ranked the first on the aggregated network. However, in the multiplex network node 8 is ranked the first and nodes 4 and 9 are ranked second exhibiting equal betweenness. It is worth noting how the multiplex representation disambiguates the betweenness of the aggregated network providing more centrality to the hub which acts as a bridge between nodes connected in the different layers. This change in the centrality allows that a node, which is not central in any single layer, to be ranked the first in the multiplex network.

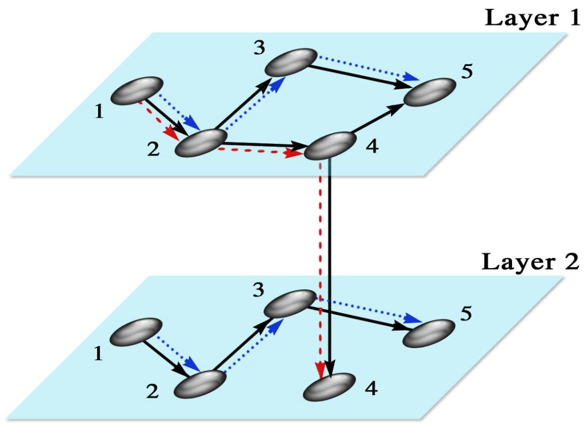


Figure 3: Shortest-paths acyclic graph with multiple paths reaching the same node in different layers.

2.2 Computation of Shortest Path betweenness

It is a common approach to compute the betweenness centrality in two steps. In the first, the shortest-paths, in form of a predecessor list, are computed using well known algorithms such as Breadth First Search (with a computational cost of $\mathcal{O}(NE)$ per source node), Dijkstra (with a computational cost of $\mathcal{O}(NE+N^2\log N)$ per source node, when implemented using a Fibonacci Heap) or Floyd-Warshall (with a computational cost of $\mathcal{O}(N^3)$), where N and E are the number of nodes and edges, respectively. Once the shortest-paths information is obtained, each shortest-path is reconstructed and the corresponding betweenness is accumulated for the traversed nodes. It is easy to see that the betweenness computation yields to $\mathcal{O}(N^3)$ algorithm. Note that, except for the Floyd-Warshall algorithm, the computational complexity of the full algorithm is dominated by the calculus of the betweenness not by the shortest-path. To reduce the complexity, as Brandes [5] and Newman [21] show, it is possible to perform a recursive computation of the betweenness using a backtracking procedure over the shortest-paths acyclic graph.

Our procedure is inspired in that proposed in [5], and is outlined in Algorithm 1 for the simple unweighted case[1].The algorithm is divided in two steps. In the first step (line 11 to 27), for a single source node s all the shortest path to all other nodes are computed. The shortest-path acyclic graph is stored in variable P as well as the the number of shortest path that pass thought each node σ. The initialization of the Breadth First Search differs from the classical one to consider the source node s can be localized in any layer. Thus, the neighbors of source node s are the union of the neighbors of s in all layers (line 17). Besides, note that the shortest paths are computed considering the destination nodes in the different layers correspond to different entities. The equivalence of entities in the different layers is performed in the computation of the betweenness (second step). To correctly account for these equivalences, we keep track of the first accessed node (independently of the layer) though variable $vOrder$ and of shortest path distance in the multiplex though variable d_M. The necessity of variable d_M should not be confused with d which stores the distance to every node in every layer. d_M is used to keep track of the first time a node is accessed independently of the layer as well as to account for multiplex path degeneracy. However, d is still necessary since the shortest path search procedure must travel through the different layers. Once the all shortest path are found, to correctly account for the multiplex path degeneracy the number of shortest paths that pass though each node independently of the layer is computed (lines 28 to 31). In the second step, the contribution of each shortest path in P to the betweenness is accumulated in the betweenness vector C_B. To account for all shortest path contributions in an efficient way the shortest path acyclic graph is traversed starting from the farthest nodes to the source. That is, the shortest-path acyclic graph is traversed in a backtracking fashion. In a single layer graph, where only classical path degeneracy needs to be accounted, at each traversed node w, the paths that go thought w plus the path that starts at w are correct distributed among the predecessors v considering the number of paths that reach w and the number of paths that reach each predecessor v ($\sigma[w]$ and $\sigma[v]$). Each fraction of paths is accumulated in each $\delta[v]$. Eventually, when all nodes farther than w to the source are explored, $\delta[w]$ can be safely accumulated in the betweenness of w. However, in a multiplex network this procedure is substantially more complex since we need to account not only for the first node it is accessed independently of the layer but also for multiplex path degeneracy. To illustrate these particularities, consider Figure 3, which represents a possible acyclic graph (in black solid arrows) that generates all shortest-path from node labelled 1, independently on the layer, to all other nodes. There are two possible shortest-paths (shown in dashed red arrows) from individual 1 to individual 4 in the acyclic graph, $\{1_{L1}, 2_{L1}, 4_{L1}\}$ and $\{1_{L1}, 2_{L1}, 4_{L1}, 4_{L2}\}$. However, the shortest-path reaching node 4 in layer 2 is not a valid one, since it exists a shorter path that reaches node 4 in layer 1. To avoid counting these paths, we only consider a new path starts at w if w is the first accessed node considering its replicas in the different layers (see this check in line 35). The second particularity that we need to account for is multiplex path degeneracy, the case of node 5 (in layers 1 and 2) in Figure 3. There are two shortest-paths (shown in dotted blue) that reach node 5, one reaches node 5 in layer 1 and the other reaches node 5 in layer 2. Thus, for walks that end at node 5, the betweenness contribution to its predecessors, such as node 3 in layer 1, corresponds to the number of times we reach 5 considering the layer where it was reached divided by the times we reach 5 independently of the layer where it was reached. See in the Algorithm (line 36) how the contribution to predecessor v of w of the path that ends at w is given by $\frac{\sigma[w]}{\sigma_M[w]}$.

3. EXPERIMENTAL RESULTS

To analyze the computational cost of the algorithm and provide empirical evidences that the ranking of nodes is af-

[1]The adaptation to the weighted case is straightforward, it is only required to replace the shortest-paths acyclic graph generation for the Dijkstra algorithm instead of the current Breadth First Search procedure.

fected by the mutiplex structure, we performed numerical experiments on two real-multiplex networks. The first multiplex corresponds to a co-authorship network obtained from the arXiv repository (http://arxiv.org/). It is composed by 4 layers and 310 nodes per layer. Each layer corresponds to a category specially selected to have a certain fraction of authors in common in the layers. We selected: Physics and Society (physics.soc-ph), Condensed Matter (cond-mat.soft), Adaptation and Self-Organizing Systems (nlin.ao) and Social and Information Networks (cs.si). Within each layer, we only consider authors with at least six co-authors, and two authors have a tie if they co-authored at least two papers. With the largest connected component of each layer, we created the multiplex network by connecting the same author in the different layers with an edge. We did not consider any disambiguation mechanism since, in the selected subset, we detected less than 1% author names with a normalized Levenshtein distance greater than 0.95. The second multiplex corresponds to a directed ego multiplex of two layers, built gathering data from two on-line social networks: Twitter (https://twitter.com) and Instagram (http://instagram.com). From a list of 13297 users, obtained using an on-line ranking platform, we obtained their Twitter and Instagram Id. Neither disambiguation of names nor matching between them in both platforms was needed since users provide their user Id on both platforms. To construct the network, we selected users which have more than 21 friends on Twitter (obtaining 2000 individuals) and 10 friends in Instagram (obtaining 2756 individuals). The topology of each on-line social network was derived from the user's friend list. Thus, a user has a tie with another one if he/she is in his/her friend list. Equivalently to the co-authorship, to create the multiplex network, we inter-connected the users in both layers when possible.

Figure 4a and 4b show two plots with the difference of the rankings obtained with the aggregated network and with the multiplex network for the co-authorship and social networks, respectively. The values of the ranking have been computed using the Dense Ranking approach, and then normalized to be on the same scale. We see that the amount of entities that obtain different ranking on the two networks is notable in both cases. In the Arxiv co-authorship network 54% of the entities obtain different ranking and in the Twitter+Instagram network 87%. With respect to the relationship of the rankings obtained in the multiplex and in the aggregated network, we observe that there is a correlation of both measures. The amount of entities that obtained the same ranking can be seen in the central plateau of the figures. This plateau is greater in the co-authorship network where the lower differences are observed in the first ranked and last ranked entities. This tendency is justifiable since the centrality of the first ranked nodes is notably higher than middle ranked ones and the changes on the rankings is of few positions. In order to see this, we provide in Table 1 the first twelve ranked authors for the co-authorship network. Last ranked nodes have zero centrality in both rankings. On the Twitter+Instagram network, we also observe the lower differences for first ranked and last ranked entities. However, these differences are smaller, giving a narrower plateau and wider tails. The maximum difference on the rankings provided by the multiplex network and aggregated network is an increase 21% in the ranking for the co-authorship network and a decrease of 25% in the ranking for the on-line

Algorithm 1: Shortest path betweenness for multiplex networks. N corresponds to the number of nodes per layer, and L to the number of layers in the multiplex.

Data: G
Result: C_B

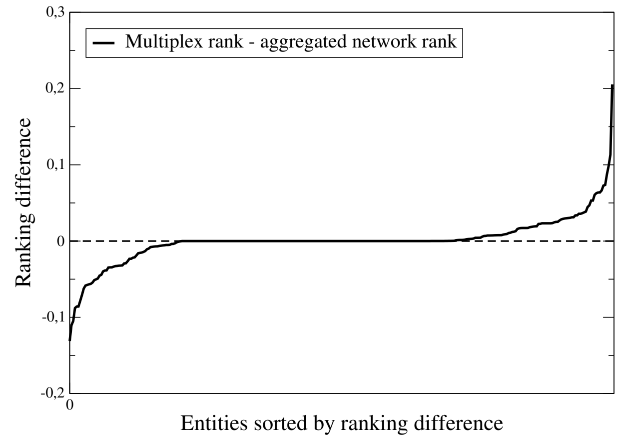
- $C_B[1..N] \leftarrow 0$;
- **for** $s \in 1 \dots N$ **do**
  - $S \leftarrow$ empty stack;
  - $P[1..NL] \leftarrow$ empty list;
  - $\sigma[1..NL] \leftarrow 0, \sigma[w] \leftarrow 1, s \equiv w \mod N$;
  - $d[1..NL] \leftarrow -1, d[w] \leftarrow 0, s \equiv w \mod N$;
  - $d_M[1..NL] \leftarrow -1, d_M[w] \leftarrow 0, s \equiv w \mod N$;
  - $vOrder[1..N] \leftarrow$ empty list;
  - $Q \leftarrow$ empty queue;
  - Q enqueue s;
  - **while** Q *not empty* **do**
    - $v \leftarrow \text{first}(Q)$;
    - S push v;
    - **if** $v \neq s$ **then**
      - $W =$ neighbor of v in G
    - **else**
      - $W = \bigcup_{\substack{v' \in \{1..NL\} \\ v' \equiv s \mod N}}$ neighbor of v' in G
    - **for** $w \in W$ **do**
      - **if** $d[w] < 0$ **then**
        - Q enqueue w;
        - $d[w] = d[v] + 1$;
        - **if** $d_M[w \mod N] < 0 \vee d_M[w \mod N] == d[w]$ **then**
          - $d_M[w \mod N] = d[w]$;
          - $vOrder[w \mod N]$ add w;
      - **if** $d[w] = d[v] + 1$ **then**
        - $\sigma[w] \leftarrow \sigma[w] + \sigma[v]$;
        - $P[w]$ add v
  - **for** $w \in \{1..N\}$ **do**
    - $\sigma_M[w] \leftarrow 0$;
    - **for** $v \in vOrder[w]$ **do**
      - $\sigma_M[w] \leftarrow \sigma_M[w] + \sigma[v]$
  - **while** S *not empty* **do**
    - $w \leftarrow pop(S)$;
    - **for** $v \in P[w]$ **do**
      - **if** $w \in vOrder[w \mod N]$ **then**
        - $\delta[v] \leftarrow \delta[v] + \frac{\sigma[v]}{\sigma[w]} \left(\frac{\sigma[w]}{\sigma_M[w]} + \delta[w] \right)$
      - **else**
        - $\delta[v] \leftarrow \delta[v] + \frac{\sigma[v]}{\sigma[w]} \delta[w]$
      - **if** $w \neq s$ **then**
        - $C_B[w \mod M] \leftarrow C_B[w \mod N] + \delta[w]$

social network. As a result, we can conclude that the centrality computed on the aggregated network is usually different than the centrality on the multiplex network. Thus, to obtain accurate betweenness centrality rankings it is crucial to compute these centralities directly on the multiplex structure.

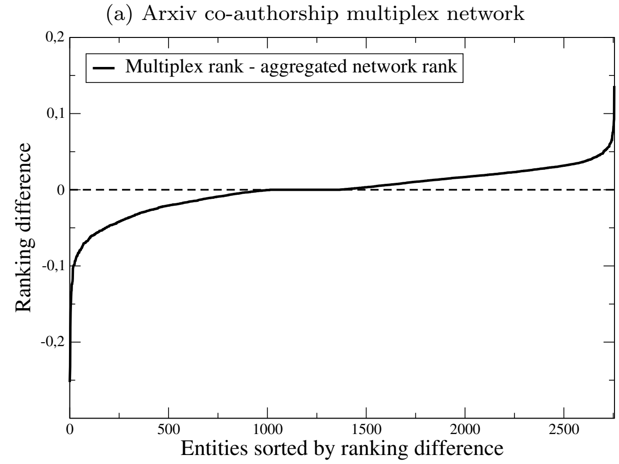
A crucial point to consider on the definition of measures for networks is the computational cost of the algorithm required to compute them. Since most of real networks contain large number of nodes and also large number of edges, only algorithms with low computational cost are able to run over those networks. The theoretical computational complexity of the algorithm developed here is $\mathcal{O}(NLE)$ for unweighted mutiplex networks and $\mathcal{O}(NLE+N^2L^2 \log NL)$ for weighted multiplex networks. N corresponds to the number of nodes per layer, L to the number of layers and E to the number of edges in the mutiplex structure.

Ranking	Multiplex	Aggregated
1	s.havlin	s.havlin
2	m.barthelemy	z.di
3	z.di	m.barthelemy
4	j.wu	h.e.stanley
5	h.e.stanley	j.wu
6	p.holme	h.jeong
7	a.l.barabasi	v.latora
8	r.lambiotte	r.lambiotte
9	m.barahona	s.sreenivasan
10	h.jeong	m.barahona
11	a.vespignani	p.holme
12	v.latora	a.l.barabasi

Table 1: First twelve ranked authors of the arXiv co-authorship dataset.



(a) Arxiv co-authorship multiplex network



(b) Twitter+Instagram multiplex network.

Figure 4: Plot showing the difference of the rankings obtained with the multiplex and with the aggregated network.

4. SUMMARY

Multiplex networks are state-of-the-art structures to represent social interaction, allowing to accurately represent different types of relation between individuals such as family relations, friendship relations or on-line social platforms communication. However, classical measures developed for single layer networks cannot be trivially extended (e.g. degree of the node, clustering coefficient or centrality) to this type of networks. This situation requires a careful re-definition of the classical measures and a brand new set of measures specific for multiplex networks. Among those measures, in this paper, we focus on shortest-path betweenness centrality. In the first part of the paper, we extended the classical definition and provide an appropriate interpretation of its meaning. We showed, my means of representative examples that shortest-path betweenness on multiplex networks tend to favor individuals which act as a bridge between layers allowing to connect individuals which are disconnected inside layers. In the second part of the paper, we provide an algorithm to compute the shortest-path betweenness with a computational cost of $\mathcal{O}(NLE)$ for unweighted multiplex networks and $\mathcal{O}(NLE + N^2L^2 \log NL)$ for weighted multiplex networks.

To validate the convenience and the accuracy of the given centrality measure we conducted experiments on two real data multiplex networks, a co-authorship multiplex of 4 layers and an on-line social multiplex of 2 layers. Results show a clear difference between the rankings computed on the multiplex structure and the ones computed with the classical shortest-path betweenness on the aggregated network. From the results, we can conclude that the rankings computed on the aggregated network are a proxy of the multiplex rankings but to obtain accurate results these need to be computed directly on the multiplex structure.

5. ACKNOWLEDGMENTS

Manlio de Domenico, and Albert Solé-Ribalta are supported by the European Commission FET-Proactive project PLEXMATH (Grant No. 317614), Alex Arenas and Sergio Gómez acknowledge support from the project MULTIPLEX (grant 317532) and the Generalitat de Catalunya 2009-SGR-838. Alex Arenas also acknowledges financial support from the ICREA Academia and the James S. McDonnell Foundation, and Sergio Gómez and Alex Arenas are also supported by FIS2012-38266.

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