

Research



Cite this article: Gómez-Gardeñes J, de Domenico M, Gutiérrez G, Arenas A, Gómez S. 2015 Layer–layer competition in multiplex complex networks. *Phil. Trans. R. Soc. A* **373**: 20150117.
<http://dx.doi.org/10.1098/rsta.2015.0117>

Accepted: 1 September 2015

One contribution of 13 to a theme issue 'Topics on non-equilibrium statistical mechanics and nonlinear physics (II)'.

Subject Areas:

statistical physics, complexity

Keywords:

complex networks, multilayer and interconnected networks, multiplex network, competition

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Layer–layer competition in multiplex complex networks

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The coexistence of multiple types of interactions within social, technological and biological networks has moved the focus of the physics of complex systems towards a multiplex description of the interactions between their constituents. This novel approach has unveiled that the multiplex nature of complex systems has strong influence in the emergence of collective states and their critical properties. Here we address an important issue that is intrinsic to the coexistence of multiple means of interactions within a network: their competition. To this aim, we study a two-layer multiplex in which the activity of users can be localized in each of the layers or shared between them, favouring that neighbouring nodes within a layer focus their activity on the same layer. This framework mimics the coexistence and competition of multiple communication channels, in a way that the prevalence of a particular communication platform emerges as a result of the localization of user activity in one single interaction layer. Our results indicate that there is a transition from localization (use of a preferred layer) to delocalization (combined usage of both layers) and that the prevalence of a particular layer (in the localized state) depends on the structural properties.

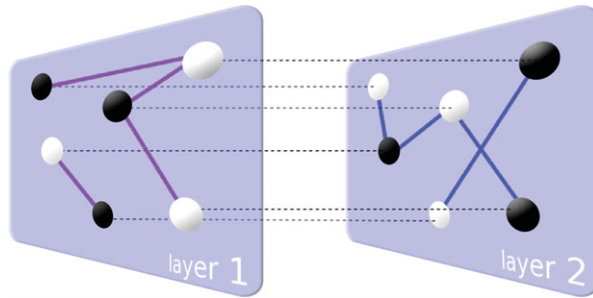


Figure 1. Schematic of a two-layer multiplex network. The multiplex is composed of six nodes so that each node appears in each layer and it is connected (dashed lines) with its representation in the other layer. When layers compete, each user must choose the layer in which it is active (black), thus remaining inactive (white) in the other one. (Online version in colour.)

1. Introduction

During the last 15 years, many statistical physics methods and nonlinear models have undergone a reformulation in order to take into account non-regular interaction patterns [1–3]. This reformulation is rooted in the availability of datasets capturing the relationships among the constituents of macroscopic systems of diverse nature, such as technological, biological and social ones, and their successful description in terms of complex networks [4,5]. As a result, many important collective phenomena taking place in these systems, such as synchronization [6], epidemics [7], cooperation and consensus [8] among others, have been revisited under the paradigm of complex networks [9,10].

As data gathering techniques increase their resolution, new properties of the interaction patterns in complex systems are captured. Main features include the spatial, temporal and multiplex nature of interaction networks. This latter ingredient has greatly focused the attention of network science in the last few years, leading to a number of works about the structure and dynamics of multilayer and multiplex networks [11,12].

Multiplex networks [13] are often described as the framework for capturing the interactions among a set of elements (nodes) when these interactions can take different forms or be established through different means, each of them defining a network often referred to as an interaction layer. Thus, a multiplex network can be seen as a collection of layers so that each node is represented in all of them (figure 1). Typical examples of multiplex networks are transportation systems [14–17], in which different transportation modes can be used to travel between cities or urban areas, or social systems [18], in which individuals can choose among different means and communication platforms for interacting with each other.

Most of the time the different interaction layers forming the multiplex coexist in a competitive way [19–23]. For instance, think of two of the most important applications for mobile communication, such as *WhatsApp* and *Line*; here the competition relies on the usage of each platform. The more users decide to use one platform the more value has the platform. From the point of view of users the final choice between these two platforms relies on two main issues. Obviously, the intrinsic quality of the platform plays a key role in the final decision of individuals. However, there is social added value that comes from the degree of usage of the platform among the acquaintances of an individual. It is thus interesting how these local and context-driven decisions affect the onset of a collective state, here represented as the localization of the multiplex activity in one of its layers.

The article is organized as follows. In §2, we present the mathematical formulation of the multiplex network model and derive the relevant equations to be solved. In §3, we show the numerical results corresponding to the competition between different interaction layers. Finally, in §4, we round off the article with the conclusions of our work and the future perspectives that it may open.

2. Mathematical formulation of multiplex layer competition

The physical framework used to study the layer–layer competition relies on a two-state model (reminiscent of an Ising-like model but much simpler). In this case, nodes are two-state (up and down) systems and the interaction neighbourhood of a node depends on its (up or down) state. Thus, at the macroscopic scale, this translates into the existence of two interaction layers, one associated with the up state and another with the down one. In the following, we first describe in §2a the general Hamiltonian capturing the interactions between the nodes of a multiplex of L layers to particularize in §2b to the case of competitive interactions in a two-layer multiplex.

(a) General formulation: multiplexes of L layers

In general, a multiplex network is composed of L network layers of N nodes each. As each individual i is represented in each of the L networks, each pair of networks α and β are interconnected by N links connecting the pair of nodes that represent the same individual i . This set-up can be seen as a collection of independent layers or platforms available for the communication between individuals (such as WhatsApp, Line, Tango, etc.) or for data-sharing (such as Dropbox, iCloud, Box, etc.), the nodes of the layers, the users and the links within a layer being the connections established by the users via a particular platform.

The availability of different platforms oriented towards the same goal poses a natural competition for the choice of the users. The essential ingredients of this competition can be cast in a mathematical formulation in which the state of a node i in layer α can be explained as the probability that node i is active in layer α (for communication platforms) or the fraction of resources that node i shares with its neighbours in this layer (for data-allocation systems). In this way, the state vector of a node i in the interconnected multilayer network is denoted by $\mathbf{p}_i \equiv (p_i^{(1)}, \dots, p_i^{(L)})$, together with the constraints that the sum of the probabilities of finding a node active in each layer is equal to 1:

$$\sum_{\alpha=1}^L p_i^{(\alpha)} = 1, \quad i = 1, 2, \dots, N. \quad (2.1)$$

The state of the multiplex can be represented by a $L \times N$ matrix $\mathbf{P} \equiv [(\mathbf{p}_1)^\dagger, \dots, (\mathbf{p}_N)^\dagger]$. The *state matrix*, accounting for the intensity of interaction between the nodes is given by all possible products between the states, i.e. by the $LN \times LN$ matrix $\Sigma = \mathbf{P} \otimes \mathbf{P}^\dagger$ with elements $\Sigma_{ij}^{(\alpha\beta)} = p_i^{(\alpha)} p_j^{(\beta)}$. In addition, we define \mathbf{J} as a $LN \times LN$ *interaction matrix* capturing both intra- and inter-layer links:

$$\mathbf{J} = \bigoplus \mathbf{W}^{(\alpha)} + \mathbf{D} \otimes \mathbf{I}, \quad (2.2)$$

where \mathbf{I} indicates the $N \times N$ identity matrix, \mathbf{D} is the $L \times L$ matrix accounting for the network of layers [24] and $W_{ij}^{(\alpha)}$ is the weight of the interaction between nodes i and j in layer α . More specifically, this topology describes nodes that are present in multiple layers simultaneously and inter-layer connections are allowed only between a node and its counterparts in the other layers. If α and β are indices indicating two given layers, the block matrix structure of \mathbf{J} can be indexed by four indices, two for nodes and two for layers, i.e. by $J_{ij}^{(\alpha\beta)}$.

Therefore, given the ensemble of all possible states, i.e. the set $\{\mathbf{P}\}$ of all matrices satisfying constraints equation (2.1), we can define the Hamiltonian of a specific configuration \mathbf{P} as

$$\mathcal{H}(\mathbf{P}) = - \sum_{\alpha, \beta=1}^L \sum_{i, j=1}^N J_{ij}^{(\alpha\beta)} p_i^{(\alpha)} p_j^{(\beta)}. \quad (2.3)$$

(b) Competition in two-layer multiplexes

For the sake of simplicity, hereafter we consider the case of the competition in a multiplex composed of two layers, i.e. $L = 2$. In this specific case, following equation (2.1), the state of a

node is completely determined by its probability of being active in one of the two layers, e.g. the first one. Thus, the state of the whole multiplex can be described by the vector $\mathbf{p} \equiv (p_1, \dots, p_N)$, where for simplicity we have omitted the layer index explicitly. Moreover, we also consider a uniform and undirected connection between the two layers, so that the interaction strength of a node i in layer 1 and its counterpart in layer 2 is captured by the parameter J_x .

To incorporate the competition between layers we consider two essential ingredients of the interactions at the local level. On the one hand, the communication between two agents that are connected within one of the layers is more efficient when both of them are always active in this layer or when they allocate all of their shared resources in the same platform. However, as the sets of contacts an individual has in the two layers are, in principle, different, by splitting the activity between the two layers an individual will increase the number of simultaneous contacts. The Hamiltonian capturing these two ingredients can be obtained from the general one in equation (2.3) as

$$H(\mathbf{p}) = - \sum_{i,j=1}^N W_{ij}^{(1)} p_i p_j - \sum_{i,j=1}^N W_{ij}^{(2)} (1 - p_i)(1 - p_j) - 2J_x \sum_{i=1}^N p_i (1 - p_i). \quad (2.4)$$

From this Hamiltonian, it becomes clear that the first two terms on the right are those favouring the localization of the activity of each individual in layers 1 and 2, respectively. In its turn, the third term favours the splitting of the activity of each individual.

The relative importance of this third term with respect to those favouring the localization of the activity within a single layer is controlled by the inter-layer coupling J_x . Note that the limit $J_x \gg 1$ means that nodes are prone to combine their activity in both layers which, for instance, in the case of data-sharing or mobile communication platforms would represent information (pictures, files, tweets, etc.) that can be easily transferred from one platform to the other one. On the other hand, the case $J_x \ll 1$ implies that a simultaneous use of platforms is hard to achieve.

3. Results

Having introduced the mathematical framework, our goal is to study the competition between the two layers as a function of the inter-layer strength J_x and the structural patterns of each of the network layers. To this aim, it is useful to check the behaviour in the two asymptotic limits: $J_x \gg 1$ and $J_x = 0$. First, when $J_x \gg 1$ the first two terms in the Hamiltonian equation (2.4) become negligible, so that the configuration of minimum energy is achieved for $p_i = 1/2 \forall i$, i.e. when the individuals split their activity between the two layers. On the other hand, for $J_x = 0$ the multiplex becomes a set of two independent networks and the configurations localized in the first layer ($\mathbf{p} = \mathbf{1}$, i.e. $p_i = 1 \forall i$) and the second one ($\mathbf{p} = \mathbf{0}$) compete. In this case, the minimum energy configuration is achieved by concentrating all the activity in the layer α with the largest total strength

$$s^{(\alpha)} = \sum_{i,j=1}^N W_{ij}^{(\alpha)}. \quad (3.1)$$

For the particular case of unweighted networks, this means that the layer with the largest average degree (or largest number of links) will concentrate all the activity when $J_x = 0$. From now on, we will consider that $s^{(1)} > s^{(2)}$ so that in the absence of inter-layer interactions the activity focuses on the first layer: $\mathbf{p} = \mathbf{1}$.

Considering these two asymptotic behaviours, we are thus interested in characterizing the transition from the localized activity regime at small values of inter-layer coupling J_x to that of the mixed one for large J_x . A first proxy is to check when the fully localized solution (e.g. $\mathbf{p} = \mathbf{1}$) ceases to be the one with the minimum energy $H = -s^{(\alpha)}$ ($\alpha = 1$ in the case of $\mathbf{p} = \mathbf{1}$). To this aim,

we calculate the gradient of the multivariate Hamiltonian $H(\mathbf{p})$:

$$\frac{\partial H}{\partial p_i} = -2 \sum_{j=1}^N W_{ij}^{(1)} p_j + 2 \sum_{j=1}^N W_{ij}^{(2)} (1 - p_j) - 2J_x(1 - 2p_i). \quad (3.2)$$

The values of these derivatives for the localized solution in the first layer ($\mathbf{p} = \mathbf{1}$) become

$$\left. \frac{\partial H}{\partial p_i} \right|_{\mathbf{p}=\mathbf{1}} = 2(J_x - s_i^{(1)}), \quad (3.3)$$

where $s_i^{(\alpha)} = \sum_{j=1}^N W_{ij}^{(\alpha)}$ is the strength of node i in layer α . For values of J_x smaller than the strength $s_i^{(1)}$ of all nodes in the first layer, the derivatives at $\mathbf{p} = \mathbf{1}$ are all negative and thus the gradient points to the interior of the hypercube $[0, 1]^N$, which contains all the possible feasible states of the system. This means that the energy of the system around $\mathbf{p} = \mathbf{1}$ is always increased for any small change of \mathbf{p} inside the hypercube, showing that $\mathbf{p} = \mathbf{1}$ has minimum energy whenever the inter-layer coupling J_x is below its critical value:

$$J_x^c = \min_{i=1, \dots, N} (s_i^{(1)}) = s_{\min}^{(1)}. \quad (3.4)$$

For unweighted networks, it reduces to the minimum degree of the nodes in the first layer, $k_{\min}^{(1)}$. Above this critical inter-layer coupling J_x^c the minimum energy moves from $\mathbf{p} = \mathbf{1}$ to a new position inside the hypercube $[0, 1]^N$, thus starting to distribute the activity between the two layers.

From a mathematical point of view, the finding of the state with minimal energy is a quadratic (the Hamiltonian) programming problem with linear equality (normalization of the probabilities) and inequality (probabilities in range $[0, 1]$) constraints. In general, for two layers, the candidate minimum of the Hamiltonian is calculated by setting $\partial H / \partial p_i = 0 \forall i$, which can be expressed as the following linear system:

$$[2J_x \mathbf{I} - (\mathbf{W}^{(1)} + \mathbf{W}^{(2)})] \mathbf{p} = J_x \mathbf{1} - \mathbf{s}^{(2)}. \quad (3.5)$$

However, its solution \mathbf{p}^* does not always fulfil the constraints, does not constitute a minimum, or even both conditions fail at the same time. When any of these happens, the minimum is placed in the boundaries of the $[0, 1]^N$ hypercube, with at least one probability equal to 1 or 0. Supposing \mathbf{p}^* is inside the hypercube, it is a minimum if the Hessian matrix, with components

$$\frac{\partial^2 H}{\partial p_i \partial p_j} = 2(2J_x \delta_{ij} - W_{ij}^{(1)} - W_{ij}^{(2)}), \quad (3.6)$$

is positive definite. When the Hessian is not positive definite, the quadratic programming problem becomes NP-hard, and no polynomial time algorithm is known to solve it. As the Hessian is proportional to the matrix of the system in equation (3.5), a positive definite Hessian implies the system has a non-singular matrix and thus a unique solution.

Summarizing, we proceed as follows to find the state with minimum energy. For each value of the coupling J_x , we first solve equation (3.5) and obtain a solution \mathbf{p}^* . Then, we check if $\mathbf{p}^* \in [0, 1]^N$ and if all the eigenvalues of the Hessian equation (3.6) are positive. If both conditions are fulfilled, \mathbf{p}^* is the ground state for this value of the coupling and we have finished. Otherwise, a heuristics is needed to solve the problem. We have chosen particle swarm optimization [25] for its simplicity, ability to cope with continuous and bounded variables, and outstanding performance in many fields [26].

In order to represent the ground state of the Hamiltonian for each value of J_x , we make use of magnetization $M(\mathbf{p})$:

$$M(\mathbf{p}) = \frac{1}{N} \sum_{i=1}^N (2p_i - 1) \quad (3.7)$$

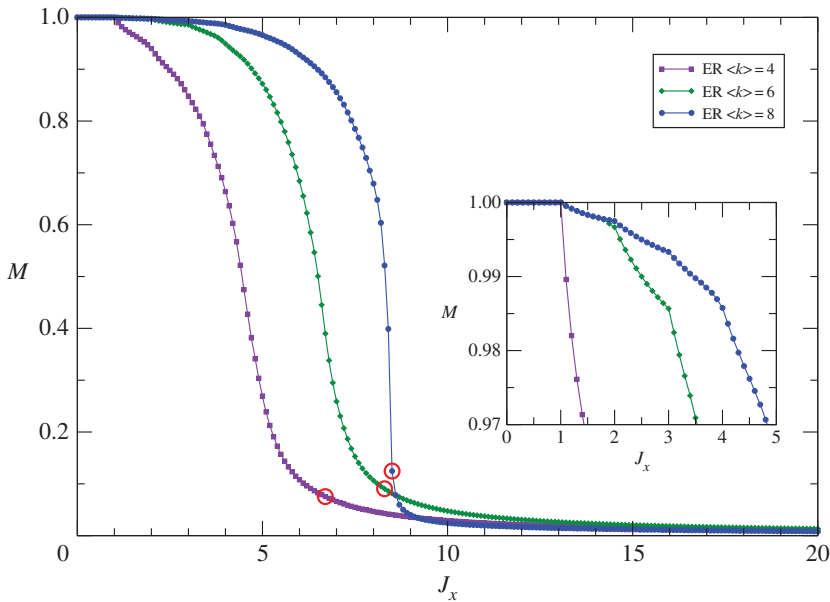


Figure 2. Magnetization for multiplex networks consisting of two Erdős–Rényi (ER) layers and varying mean degree. All ER networks have 200 nodes and minimum degree $k_{\min} = 1$. The red circles indicate the minimum values of J_x so that minimum energy configurations can be calculated from equation (3.5). For values of J_x to the left of the red circle, particle swarm optimization has been used to compute the solutions. Inset: zoom to show when activity starts to be distributed in both layers. (Online version in colour.)

that characterizes the level of activity between layers: $M = 1$ when all the activity is concentrated in the first layer ($p = 1$), and $M = -1$ when it is concentrated in the second layer ($p = 0$). Obviously, when activity is shared between the two layers ($p = 0.5$) the magnetization vanishes, $M = 0$.

In figure 2, we show the magnetization as a function of J_x for three multiplex networks composed of two Erdős–Rényi (ER) layers. The layers of the three multiplex networks have average degree $\langle k \rangle = 4, 6$ and 8 , respectively. The layers were produced by means of the algorithm introduced in [27] that allows one to interpolate between ER and scale-free networks and, as a by-product of the procedure, enables one to control the minimum degree of the resulting graphs. In this way, we have set the minimum degree to $k_{\min} = 1$ in the three cases. However, the method in [27] produces (for a given value of k_{\min}) networks of identical strength. Thus, we take the first network layer and add $0.05 \times N$ links at random to ensure that $s^{(1)} > s^{(2)}$. The three curves in figure 2 display the transition from localized activity for small J_x values to mixed one for large ones. Interestingly, the inset shows that localized activity is lost as soon as $J_x = 1 = k_{\min}$ in agreement with our former estimation. From this point, the decay of M is slower for those multiplexes with larger mean degree $\langle k \rangle$ in the layers.

These results are further corroborated in figure 3. In this case, we show again three two-layer multiplexes (again composed of coupled ER–ER networks) with the same average degrees as in figure 2 ($\langle k \rangle = 4, 6$ and 8) but with different values of the minimum degree, $k_{\min} = 1, 2$ and 4 , respectively. This latter feature is revealed in the inset of the plot where we show that the state of localized activity ($M = 1$) is no longer the ground state of the multiplex as soon as $J_x > k_{\min} = 1, 2$ and 4 . On the other hand, as in figure 2, the final mixed activity state is achieved first for those multiplexes with smaller average degree.

In both figures 2 and 3, the insets show that the evolution of $M(J_x)$ close to $M = 1$ shows different cusps. In particular, these cusps appear at integer values of J_x , which at the same time correspond to the values at which the number of nodes for which equation (3.3) changes sign

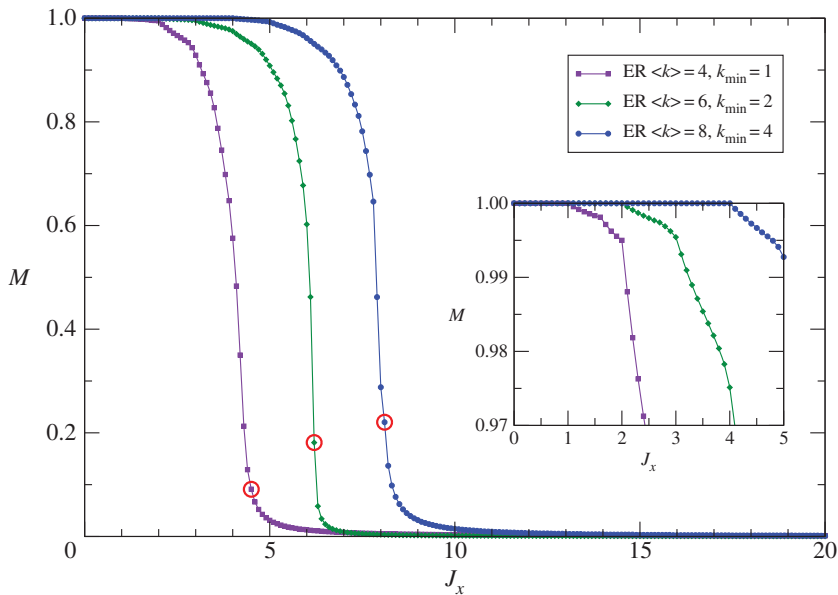


Figure 3. Magnetization for multiplex networks consisting of two Erdős–Rényi (ER) layers and varying mean and minimum degrees. All ER networks have 200 nodes. The red circles indicate the minimum values of J_x so that minimum energy configurations can be calculated from equation (3.5). For values of J_x to the left of the red circle, particle swarm optimization has been used to compute the solutions. Inset: zoom to show when activity starts to be distributed in both layers. (Online version in colour.)

increases. The effect is similar to that at k_{\min} : many nodes have $p_i = 1$ until J_x is equal to k_i in the first layer. This rule is exact only for nodes with $k_i = k_{\min}$, and also holds for an important fraction of nodes when $k_i > k_{\min}$. The collective effect is reflected in the form of cusps in the curve $M(J_x)$.

4. Conclusion

In this work, we have introduced a model to analyse how layers compete for the activity of users in a multiplex network inspired in the simultaneous interplay of communication and data-sharing online platforms. To this aim, we have focused on a multiplex composed of two layers and we have relied on a two-state model in which each of the two states of a node is associated to be active in the top layer and the bottom one, respectively. At variance with the usual Ising model in a network [28], here a node interacts only with those neighbours in the layer it is active. We have set two competing mechanisms, one favouring activity localization and another favouring the splitting of the node's activity between the two layers. On the one hand, if a node and all its neighbours in a layer are active in it, this would favour an efficient communication between them. However, when a node splits its activity between the two layers this would favour the passage of information from a neighbour in one layer to a neighbour in the other one, thus maximizing the outreach of information. The competition between these two mechanisms is controlled by the inter-layer coupling J_x which can be seen as the ability that a node has to pass information from one layer to the other.

Our results show that, regardless of the average connectivity and total strength of the leading layer (the one that focuses all the activity for small inter-layer coupling), it is its minimum degree that causes nodes to start to use the other layer, i.e. controls the onset of the transition from localized to mixed activity. On the other hand, it is the average degree of the leading layer that controls when the state of full mixing is reached. These two results point out that the transition

from localized to mixed activity occurs via a cascade from poorly connected nodes in the leading layers (the ones that obtain more benefits from leaving first the leading layer) to those highly connected ones (being the ones that are less prone to leave the layer in which they are well connected). Thus, the larger the average degree of the leading layer, the more inter-layer coupling is needed to persuade all the degree classes to leave the localized state.

We expect that the simple model introduced here will stimulate more research about the coexistence and competition of interaction layers in multiplex networks, making possible the characterization of how and when the coexistence of different layers in real multiplex systems is possible. Future research avenues include the study of other types of layer topologies and the presence of correlations between the degrees of a node in different layers. Moreover, a more challenging problem is the competition in multiplexes composed of more than two layers, characterized by the Hamiltonian in equation (2.3). It is clear that the existence of multiple parameters for the interaction between the L layers poses a mathematical and computational difficulty. On the other hand, this general framework provides an interesting scenario in which many different transitions between the localization in different layers are observed due to the multiple competition between them. It is also interesting to note that we have tackled the analysis of the two-state model by considering continuous variables, $\{p_i\}$, associated to each node. However, another possibility is to consider binary states for the nodes, as in Ising-like models, so that metastable states can show up due to the multi-stable character of the node states [29,30].

Authors' contributions. J.G.-G., M.D., A.A. and S.G. conceived of the study, coordinated the study and helped draft the manuscript; J.G.-G., M.D., A.A. and S.G. designed and analysed the model; G.G. and S.G. carried out the numerical analyses. All authors gave final approval for publication.

Competing interests. We declare we have no competing interests.

Funding. J.G.-G. acknowledges financial support from MINECO through the Ramón y Cajal Program. Financial support came from MINECO (projects FIS2011-25167 and FIS2012-38266-C02-01), Departamento de Industria e Innovación del Gobierno de Aragón y Fondo Social Europeo (FENOL group). A.A., J.G.-G., M.d.D. and S.G. were supported by the European Commission FET-Proactive project PLEXMATH (grant no. 317614), and MULTIPLEX (grant no. 317532). A.A. also acknowledges financial support from the ICREA Academia and the James S. McDonnell Foundation, and S.G. and A.A. were supported by FIS2012-38266 and the Generalitat de Catalunya 2009-SGR-838.

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