Multi-state perceptrons: learning rule and perceptron of maximal stability

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Abstract

A new perceptron learning rule which works with multilayer neural networks made of multi-state units is obtained, and the corresponding convergence theorem is proved. The definition of perceptron of maximal stability is enlarged in order to include these new multi-state perceptrons, and a proof of existence and uniqueness of such optimal solutions is outlined.

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1 Introduction

Perceptrons constitute the simplest architecture for a layered feed-forward neural network [Rosenblatt 58]. An input layer feeds the only unit of the second layer, where the output is read. Thus, there are as many weights ω_k as input units —say N— and just one threshold U. The activation function which decides the final state of the output unit is usually taken to be

$$g(h) \equiv \begin{cases} 0 \text{ if } h < 0, \\ 1 \text{ if } h \geq 0, \end{cases}$$

where the field h is calculated, as a function of the input pattern $\boldsymbol{\xi}$, through the formula

$$h \equiv \boldsymbol{\omega} \cdot \boldsymbol{\xi} - U.$$

Learning amounts to finding the weights $\boldsymbol{\omega}$ and the threshold U which map a set of input patterns $\{\boldsymbol{\xi}^\mu\}_{\mu=1,\ldots,p}$ into their corresponding outputs $\{\zeta^\mu\}_{\mu=1,\ldots,p}$. Among all the possible input-output associations, only the so-called *linearily separable* problems have perceptron solutions. The well-known *perceptron convergence theorem* gives a *perceptron learning rule* to obtain these solutions whenever they exist [Minsky and Papert 69]. Moreover, recent works have developed fast converging algorithms to find the perceptron *solution of maximal stability* (e.g. [Krauth and Mézard 87, Ruján 91]).

The binary perceptron divides the input space in two half-spaces, one for each possible value of the output. The problem of classifying in more than two classes with the aid of a collection of perceptrons is well-known in the literature (see e.g. [Duda and Hart 73]). Likewise, if the mapping to be learned has a continuous output, it can be related to the previous classification scheme in two steps: partition of the interval of variation of the continuous parameter in a finite number of pieces —to arbitrary precision— and assignment of each one to a certain base 2 vector (see [Gallant 90]).

For instance, a 'thermometer' representation for the interval [0,1] could be

$$\boldsymbol{\zeta} = \begin{cases} (0,0,0,0) & \text{for} \quad y \in [0, 0.2), \\ (1,0,0,0) & \text{for} \quad y \in [0.2, 0.4), \\ (1,1,0,0) & \text{for} \quad y \in [0.4, 0.6), \\ (1,1,1,0) & \text{for} \quad y \in [0.6, 0.8), \\ (1,1,1,1) & \text{for} \quad y \in [0.8, 1], \end{cases} \tag{1.1}$$

which reduces the learning problem to a five classes classification one. However, even if this four perceptrons network has learned the thermometer-like $\boldsymbol{\xi}^\mu \longmapsto \boldsymbol{\zeta}^\mu$, $\mu = 1, \ldots, p$ correspondence, new inputs supplied to the net may produce ouputs such as $(0,0,1,1)$ or $(1,0,1,0)$, which cannot be interpreted within this representation; in fact, most of the available codifying schemes suffer from the same inconsistency.

One natural way of avoiding these problematic and rather artificial conversions from continuous to binary data is the use of *multi-state units* perceptrons (see e.g. [Rieger 90, Nadal and Rau 91]). With them, only the first of the two steps mentioned above is necessary, i.e. the discretization of the continuous interval. Geometrically, multi-state units define a vector in the input space which points to the direction of increase of the output parameter, the boundaries being parallel hyperplanes. That is why this method gets rid of meaningless patterns, since this partition clearly incorporates the underlying relation of order.

At first sight it may seem that the structure derived from a set of binary perceptrons is richer than that arising from a single multi-state unit. Nevertheless, it must be taken into account that combinations of multi-state perceptrons will be needed whenever the learning problem is not *multi-state-separable*, giving rise to multilayer neural networks made of multi-state units.

In this article we shall first introduce a new *multi-state perceptron learning rule*, and shall prove the corresponding convergence theorem. Then, the concept of solution with *maximal stability* will be extended to multi-state perceptrons, and their most remarkable properties will be stated. Finally, possible applications of the model of multi-state neural networks and some open problems will be discussed.

2 Multi-state perceptron convergence theorem

A Q-state neuron may be in anyone of Q different output values or 'colors' $\sigma_1 < \cdots < \sigma_Q$. They constitute the result of the processing of an incoming stimulus through an activation function of the form

$$g_U(h) \equiv \begin{cases} \sigma_1 & \text{if} \quad h < U_1, \\ \sigma_v & \text{if} \quad U_{v-1} \leq h < U_v, \ v = 2, \ldots, Q-1, \\ \sigma_Q & \text{if} \quad U_{Q-1} \leq h. \end{cases} \tag{2.1}$$

Therefore, $Q-1$ thresholds $U_1 < \cdots < U_{Q-1}$ have to be defined for each updating unit, which in the case of the perceptron is reduced to just the output unit. The field now simply reads

$$h \equiv \boldsymbol{\omega} \cdot \boldsymbol{\xi}. \tag{2.2}$$

Let us distribute the input patterns in the following subsets:

$$\mathcal{F}_v \equiv \{\boldsymbol{\xi}^\mu \mid \zeta^\mu = \sigma_v\}, \ v = 1, \ldots, Q. \tag{2.3}$$

From a geometrical point of view [Ruján 90] the output processor corresponds to the set of $Q-1$ *parallel hyperplanes*

$$\boldsymbol{\omega} \cdot \boldsymbol{\xi} = U_v, \ v = 1, \ldots, Q-1, \tag{2.4}$$

which divide the input space into Q ordered regions, one for each of the colors $\sigma_1, \ldots, \sigma_Q$. Thus, the map $\boldsymbol{\xi}^\mu \longmapsto \zeta^\mu$, $\mu = 1, \ldots, p$, is said to be *learnable* or *separable* if it is possible to choose hyperplanes such that each $\mathcal{F}_v$ be in the zone of color σ_v.

This picture make us realize that the fundamental parameters to be searched for while learning are the components of the unit vector $\hat{\boldsymbol{\omega}}$ and *not* the thresholds, since these can be assigned a value as follows. If the input-output map is learnable then

$$\zeta^\mu = g_U(\boldsymbol{\omega} \cdot \boldsymbol{\xi}^\mu), \ \mu = 1, \ldots, p \tag{2.5}$$

yields

$$\left.\begin{array}{c} \forall \boldsymbol{\xi}_v^\rho \in \mathcal{F}_v \\ \forall \boldsymbol{\xi}_{v+1}^\gamma \in \mathcal{F}_{v+1} \end{array}\right\} \implies \boldsymbol{\omega} \cdot \boldsymbol{\xi}_v^\rho < \boldsymbol{\omega} \cdot \boldsymbol{\xi}_{v+1}^\gamma \tag{2.6}$$

which means that, defining

$$\begin{array}{rcl} \boldsymbol{\xi}_v^\alpha \in \mathcal{F}_v \;\; | \;\; \boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\alpha & \geq & \boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\rho \quad \forall \boldsymbol{\xi}_v^\rho \in \mathcal{F}_v \\ \boldsymbol{\xi}_v^\beta \in \mathcal{F}_v \;\; | \;\; \boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\beta & \leq & \boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\gamma \quad \forall \boldsymbol{\xi}_v^\gamma \in \mathcal{F}_v \end{array} \tag{2.7}$$

we get

$$U_v \in \left]\boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\alpha \;,\; \boldsymbol{\omega}\cdot\boldsymbol{\xi}_{v+1}^\beta\right[, \; v = 1, \ldots, Q-1. \tag{2.8}$$

Hence, during the learning process it is possible to choose

$$U_v = \frac{\boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\alpha + \boldsymbol{\omega}\cdot\boldsymbol{\xi}_{v+1}^\beta}{2}, \; v = 1, \ldots, Q-1, \tag{2.9}$$

which is the best choice for the thresholds with the given $\boldsymbol{\omega}$. Here lies the difference between our approach and that of recent papers such as [Mertens et al 91], where the thresholds are compelled to be inside certain intervals given beforehand. Consequently, we have somehow enlarged their notion of learnability.

Our proposal for the *multi-state perceptron learning rule* stems from the following **Theorem**. If there exists $\boldsymbol{\omega}^*$ such that $\boldsymbol{\omega}^*\cdot\boldsymbol{\xi}_v^\rho < \boldsymbol{\omega}^*\cdot\boldsymbol{\xi}_{v+1}^\gamma$ for all $\boldsymbol{\xi}_v^\rho \in \mathcal{F}_v$ and $\boldsymbol{\xi}_{v+1}^\gamma \in \mathcal{F}_{v+1}$, $v = 1, \ldots, Q-1$, then the program

Start choose any value for $\boldsymbol{\omega}$ and $\eta > 0$;
Test choose $v \in \{1, \ldots, Q-1\}$, $\boldsymbol{\xi}_v^\rho \in \mathcal{F}_v$ and $\boldsymbol{\xi}_{v+1}^\gamma \in \mathcal{F}_{v+1}$;
if $\boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\rho < \boldsymbol{\omega}\cdot\boldsymbol{\xi}_{v+1}^\gamma$ then go to **Test**
else go to **Add**;
Add replace $\boldsymbol{\omega}$ by $\boldsymbol{\omega} + \eta(\boldsymbol{\xi}_{v+1}^\gamma - \boldsymbol{\xi}_v^\rho)$;
go to **Test**.

will go to Add only a finite number of times.

Corollary. The previous algorithm finds a multi-state perceptron solution to the map $\boldsymbol{\xi}^{\boldsymbol{\mu}} \longmapsto \zeta^\mu$, $\mu = 1, \ldots, p$ whenever it exists, provided the maximum number of passes through Add is reached. This may be achieved by continuously choosing pairs $\{\boldsymbol{\xi}_v^\rho, \; \boldsymbol{\xi}_{v+1}^\gamma\}$ such that $\boldsymbol{\omega}\cdot\boldsymbol{\xi}_v^\rho \geq \boldsymbol{\omega}\cdot\boldsymbol{\xi}_{v+1}^\gamma$.

Proof. Define

$$G(\boldsymbol{\omega}) \;\equiv\; \frac{\boldsymbol{\omega}\cdot\boldsymbol{\omega}^*}{\|\boldsymbol{\omega}\|\,\|\boldsymbol{\omega}^*\|} \leq 1, \tag{2.10}$$

$$\delta \equiv \min_{v,\rho,\gamma} (\boldsymbol{\omega}^* \cdot \boldsymbol{\xi}_{v+1}^{\gamma} - \boldsymbol{\omega}^* \cdot \boldsymbol{\xi}_{v}^{\rho}) > 0, \tag{2.11}$$

$$M^2 \equiv \max_{v,\rho,\gamma} \|\boldsymbol{\xi}_{v+1}^{\gamma} - \boldsymbol{\xi}_{v}^{\rho}\|^2 > 0. \tag{2.12}$$

On successive passes of the program through `Add`,

$$\boldsymbol{\omega}^* \cdot \boldsymbol{\omega}_{t+1} \geq \boldsymbol{\omega}^* \cdot \boldsymbol{\omega}_t + \eta\delta, \tag{2.13}$$

$$\|\boldsymbol{\omega}_{t+1}\|^2 \leq \|\boldsymbol{\omega}_t\|^2 + \eta^2 M^2. \tag{2.14}$$

Therefore, after n applications of `Add`,

$$G(\boldsymbol{\omega}_n) \geq L(\boldsymbol{\omega}_n), \tag{2.15}$$

$$L(\boldsymbol{\omega}_n) \equiv \frac{\boldsymbol{\omega}^* \cdot \boldsymbol{\omega}_0 + n\eta\delta}{\|\boldsymbol{\omega}^*\|\sqrt{\|\boldsymbol{\omega}_0\|^2 + n\eta^2 M^2}}, \tag{2.16}$$

which for large n goes as

$$L(\boldsymbol{\omega}_n) \approx \sqrt{n}\,\frac{\delta}{\|\boldsymbol{\omega}^*\| M}. \tag{2.17}$$

However, n cannot grow at will since $G(\boldsymbol{\omega}) \leq 1$, $\forall \boldsymbol{\omega}$, which implies that the number of passes through `Add` has to be finite.

It is interesting to note that no assumption has been made on the number and nature of the input patterns. Thus, the theorem applies even when an infinite number of pairs of patterns is present and also to inputs not belonging to the 'lattice' $\{\sigma_1, \ldots, \sigma_Q\}^N$.

3 Multi-state perceptron of maximal stability

In the previous section an algorithm for finding a set of parallel hyperplanes which separate the $\mathcal{F}_v$ sets in the correct order has been found, under the assumption that such solutions exist. The problem we are going to address now is that of selecting the 'best' of all such solutions.

It is our precise prescription that the *multi-state perceptron of maximal stability* has to be defined as the one whose smallest gap between the pairs $\{\mathcal{F}_v,\ \mathcal{F}_{v+1}\}$, $v = 1, \ldots, Q-1$ is maximal. These gaps are given by the numbers

$$\begin{aligned} R_v(\boldsymbol{\omega}) &\equiv \min_{\rho,\gamma} \left(\frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \cdot (\boldsymbol{\xi}_{v+1}^{\gamma} - \boldsymbol{\xi}_{v}^{\rho}) \right) \\ &= \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \cdot \left(\boldsymbol{\xi}_{v+1}^{\beta} - \boldsymbol{\xi}_{v}^{\alpha} \right), \end{aligned} \tag{3.1}$$

where to obtain the second expression we have made use of the definitions in (2.7). Therefore, calling $\mathcal{D} \subset \boldsymbol{R}^N$ the set of all the solutions to the multi-state perceptron problem, the function to be maximized is

$$R(\boldsymbol{\omega}) \equiv \begin{cases} \min\limits_{v=1,\ldots,Q-1} R_v(\boldsymbol{\omega}) & \text{if} \quad \boldsymbol{\omega} \in \mathcal{D}, \\ 0 & \text{if} \quad \boldsymbol{\omega} \notin \mathcal{D}. \end{cases} \tag{3.2}$$

Since $R(\lambda\boldsymbol{\omega}) = R(\boldsymbol{\omega}),\ \forall\lambda > 0$, it is actually preferable to restrict the domain of R to the hyper-sphere $S^N \subset \boldsymbol{R}^N$, i.e.

$$\begin{array}{rcl} \widetilde{R} : S^N & \longrightarrow & \mathcal{R}^N \\ \hat{\boldsymbol{\omega}} & \longmapsto & \widetilde{R}(\hat{\boldsymbol{\omega}}) \equiv R(\hat{\boldsymbol{\omega}}) \end{array} \tag{3.3}$$

The basic properties of $\widetilde{R}$ are:

1. $\widetilde{R}(\hat{\boldsymbol{\omega}}) > 0 \iff \hat{\boldsymbol{\omega}} \in \mathcal{D} \cap S^N$.
2. The set $\mathcal{D}$ is convex.
3. The restriction of $\widetilde{R}$ to $\mathcal{D} \cap S^N$ is a strictly concave function.
4. The restriction of $\widetilde{R}$ to $\mathcal{D} \cap S^N$ has a unique maximum.

This last property assures the existence and uniqueness of a Perceptron of Maximal Stability, and it is a direct consequence of the preceding propositions. Moreover, it asserts that *no other* relative maxima are present, which is of great practical interest whenever this optimal perceptron has to be explicitly found.

In [Mertens et al 91] the optimization procedure constitutes a forward generalization of the AdaTron algorithm ([Anlauf and Biehl 89]). Here the situation is much more complicated because the function we want to maximize is not simply quadratic with linear constraints, but a piecewise combination of them due to the previous discrete minimization taken over the gaps. Thus, we have not been able to find a suitable optimization method which could take advantage of the particularities of this problem. Of course, the designing of such converging algorithms is an open question which deserves further investigation.

4 Conclusions

A new *perceptron learning rule* which can be used with perceptrons made of *multi-state units* has been derived. Its convergence has been proven to be guaranteed whenever the set of input-output patterns is *multi-state-separable*. This last concept constitutes an extension of the so-called linear separability, which involves a single hyperplane lying in the input space. Moreover, we have generalized the definition of *perceptron of maximal stability* to encompass the case of multi-state simple perceptrons, and their main characteristics and properties have been shown. In particular, the existence and uniqueness of such solutions implies that any standard optimizing method can in principle be used, but the determination of the best procedure is an open question left to future research.

The comparison between the actual performances of binary vs. multi-state units in multilayer neural networks is of great interest. Nevertheless, it cannot be established in practice until a good multilayer learning method is found —i.e. a learning rule which may be used even when non multi-state-separable problems are treated. To be specific, for binary units there do exist some learning algorithms (e.g. the 'tiling algorithm' [Mézard and Nadal 89] and 'sequential learning' [Marchand et al 90]) which add hidden layers and units in such a way that a correct mapping between binary input and output patterns is always assured; and their main tool is the repeated use of the 'binary perceptron learning rule'.

Therefore, our first objective will consist now in designing an always-converging multilayer learning rule with hidden multi-state units. Its existence is assured by the fact that it is actually possible to learn the separation of each of the colors from the rest (with the help of two-color units) and then the resulting hidden representation turns out to be multi-state-separable; but this is certainly not a good solution because most of the hidden units turn out to be binary.

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