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CONTRIBUTION TO THE STUDY OF ALLIANCES IN GRAPHS

DOCTORAL THESIS

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I STATE that the present study, entitled "Contribution to the study of alliances in graphs", presented by Ismael González Yero for the award of the degree of Doctor, has been carried out under my supervision at the Department of Computer Engineering and Mathematics of this university, and that it fulfils all the requirements to be eligible for the European Doctorate Award.

Tarragona, October 20th, 2010

Doctoral Thesis Supervisor
Dr. Juan Alberto Rodríguez Velázquez

To my sister, my mother and my father...

The mere formulation of a problem is far more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old problems from a new angle requires creative imagination and marks real advances in science.

Albert Einstein

We often hear that mathematics consists mainly of "proving theorems".
Is a writer's job mainly that of "writing sentences"?

Gian-Carlo Rota

A mathematician is a blind man in a dark room
looking for a black cat which isn't there.

Charles Darwin

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Contents

Introduction

Alliances are present in several ways in real world. General speaking, an alliance can be understood as a collection of elements sharing similar objectives or having similar properties among all elements of the collection. In this sense, there exist alliances like the following ones:

- Group of people united by a common friendship.
- Group of plants belonging to the same botanical family.
- Group of companies sharing the same economic interest.
- Group of Twitter users following or being followed among themselves.
- Group of Facebook users sharing a common activity.

For instance, Facebook can be seen as an enormous network (or graph) in which each user is a vertex and two vertices are connected if they are "friends", in the sense of the system. With this idea, an alliance in Facebook can be realized as a collection of users (or vertices) having more "friends" inside the collection than outside. Analogously, Twitter can be understood as a graph in which each user is a vertex and two vertices are adjacent if at least one of them is "following" the other one. Hence, an alliance in Twitter can be realized as a collection of users following (or being followed) more users (by more users) inside the collection than outside.

Similar ideas were used by Kristiansen, Hedetniemi and Hedetniemi in [52] to define the concepts of alliances in graphs. In this work the authors described different kind of alliances named defensive, offensive and powerful[1]. For instance, a defensive alliance in a graph G is a set S of vertices of G with the property that every vertex in S has at most one more neighbor outside of S than it has in S. Similarly, an offensive alliance in a graph G is a set S of vertices of G with the property that every vertex in the neighborhood of S has at least one more neighbor in S than it has outside of S. The combination of these two kind of alliances is called powerful alliance, i.e., a powerful alliance is a set S of vertices of G, which is both, defensive and offensive. Also, in [52], it was stated the problem of finding the minimum cardinality of any alliance in a graph. This problem was proved to be NP-complete in [11, 30, 31, 32, 46, 47, 48] for all the cases of alliances. The Ph. D. Thesis [45] contains a complete study of complexity of computing minimum cardinality of any alliance in a graph, even in the case of weighted graphs.

A generalization of alliances was presented by Shafique and Dutton in [64], where they defined a defensive k-alliance as a set S of vertices of G with the property that every vertex in S has at least k more neighbors in S than it has outside of S. Analogously, an offensive k-alliance is a set S of vertices of G with the property that every vertex in the neighborhood of S has at least k more neighbors in S than it has outside of S. Notice that, a defensive alliance is a defensive (-1)-alliance and an offensive alliance is an offensive 1-alliance. Thus, a powerful k-alliance is a set S of vertices of G which is a defensive k-alliance and an offensive $(k+2)$-alliance. The authors of [64] also defined the concepts of alliance free sets and alliance cover sets as those set of vertices, such that they do not contain any alliance and they contain at least one vertex from each alliance in a graph, respectively.

[1] Also called dual alliances.

Applications of alliances can be found in the Ph. D. Thesis [67] where the authors studied problems of partitioning graphs into alliances and its application to data clustering. On the other hand, defensive alliances represent the mathematical model of web communities, by adopting the definition of Web Community proposed by Flake, Lawrence and Giles in [35], "a Web Community is a set of web pages having more hyperlinks (in either direction) to members of the set than to non-members". Other applications of alliances were presented in [19, 41, 50, 73].

Diverse investigations have been developed about alliances. For instance, defensive alliances have been studied in [1, 5, 10, 12, 14, 24, 25, 36, 38, 42, 43, 59, 61, 62, 63, 70, 71], offensive alliances in [3, 4, 13, 15, 16, 17, 28, 32, 58, 59, 60, 61, 72] and powerful alliances in [6, 8, 9, 31, 59, 61]. Moreover, the Ph. D. Thesis [23, 45, 67, 69] are important compilations of the principal results obtained about this topic.

The principal motivation of this work is based mainly on the NP-completeness of computing minimum cardinality of (defensive, offensive, powerful) k-alliances in graphs. Another motivation is the relative increasing interest of investigations about alliances, which can be seen throughout more than 50 published papers and four Doctoral Thesis presented in the last five years.

An interesting problem in graph theory is related to the study of graph products, and particularly, there are many investigations about obtaining relationships between invariants of Cartesian product graphs and the corresponding invariants of its factors. For instance, is well known the Vizing's conjecture[2] [74] related to the domination number of Cartesian product graphs and the domination number of the factors. In this sense, we emphasize into obtaining relationships between alliances in Cartesian product graphs and alliances in its factors.

[2] The domination number of Cartesian product graph is greater or equal than the product of domination numbers of its factors.

On the other hand, other important problem in graph theory is related to obtaining partitions of the vertex set of a graph satisfying an specific property. Thus, in this work we are interested into obtaining partitions of a graph formed by alliances of diverse types.

The work is structured in the following way: the first three chapters are centered into offensive, defensive and powerful k-alliances, respectively. There we obtain some mathematical properties of the respective alliances, we study the alliances in Cartesian product graphs and the partitions of a graph into alliances of the respective type. The last chapter is about alliance free sets and alliance cover sets. There, we obtain some bounds for the maximum cardinality of alliance free sets and the minimum cardinality of alliance cover sets. Moreover, we study the (defensive, offensive, powerful) alliance free sets of Cartesian product graphs.

We begin by establishing the principal terminology and notation which we will use throughout the thesis. We refer to Glossary to complete all the used notation. Through the thesis, $G = (V, E)$ represents a undirected finite graph without loops and multiple edges with set of vertices V and set of edges E. The order of G is $|V| = n(G)$ and the size $|E| = m(G)$ (If there is no ambiguity we will use only n and m). We denote two adjacent vertices $u, v \in V$ by $u \sim v$ and in this case we say that uv is an edge of G or $uv \in E$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v) := \{u \in X : u \sim v\}$ and the degree of v in X is denoted by $\delta_X(v) = |N_X(v)|$. In the case $X = V$ we will use only $N(v)$, which is also called the open neighborhood of a vertex $v \in V$, and $\delta(v)$ to denote the degree of v in G. The close neighborhood of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. The minimum and maximum degree of G are denoted by δ and Δ, respectively.

The subgraph induced by $S \subset V$ is denoted by $\langle S \rangle$ and the complement of the set S in V is denoted by $\overline{S}$. Moreover, $\partial(S)$ denotes the neighborhood of the set S in V, i.e., $\partial(S) = \bigcup_{v\in S} N_{\overline{S}}(v)$. The complement of a graph $G=(V,E)$ is the graph $\overline{G}=(V,\overline{E})$ in which the edge $uv\in\overline{E}$ if and only $uv\notin E$. The line graph of a graph G is the graph $\mathcal{L}(G)$, obtained from G, by associating a vertex of $\mathcal{L}(G)$ with each edge of the graph G and connecting two vertices by an edge if and only if the corresponding edges of G meet at one endpoint. The domination number of a graph is denoted by $\gamma(G)$ and the k-domination number[3] by $\gamma_k(G)$. We recall that the Cartesian product of two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is the graph $G_1\times G_2=(V,E)$, such that $V=\{(a,b)\ :\ a\in V_1,\ b\in V_2\}$ and two vertices $(a,b)\in V$ and $(c,d)\in V$ are adjacent in $G_1\times G_2$ if and only if either ($a=c$ and $bd\in E_2$) or ($b=d$ and $ac\in E_1$).

[3] A set S is a k-dominating set of G if for every vertex $v\in\overline{S}$, it follows $\delta_S(V)\geq k$. If $k=1$, then S is a standard dominating set.

Chapter 1

Offensive Alliances

Abstract

We investigate the relationship between global offensive k-alliances and some characteristic sets of a graph including r-dependent sets, τ-dominating sets and standard dominating sets. In addition, we discuss the close relationship that exists between the offensive alliances in Cartesian product graph and the offensive alliances in its factors. Also, we study the problem of estimating the maximum number of sets belonging to a partition of the vertex set of a graph into offensive k-alliances.

1.1 Introduction

A nonempty set $S \subseteq V$ is an *offensive k-alliance* in $G = (V, E)$, $k \in \{2 - \Delta, ..., \Delta\}$, if for every $v \in \partial(S)$

$$\delta_S(v) \geq \delta_{\overline{S}}(v) + k. \tag{1.1}$$

An offensive k-alliance S is called *global* if it is a dominating set. Figure 1.1 shows examples of (global) offensive k-alliances. Notice that equation (1.1) is equivalent to

$$\delta(v) \geq 2\delta_{\overline{S}}(v) + k. \tag{1.2}$$

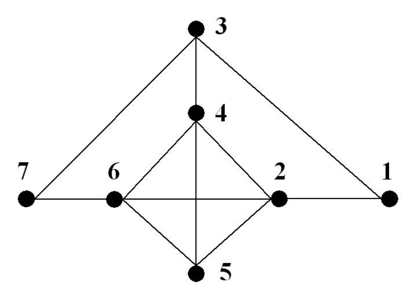

Figure 1.1: $\{2, 6\}$ is an offensive 0-alliance and $\{2, 4, 6\}$ is a global offensive (-1)-alliance.

It is clear that if $k > \Delta$, no set S satisfies (1.1) and, if $k < 2 - \Delta$, all the subsets of V satisfy it. The *offensive k-alliance number* of G, denoted by $a_k^o(G)$, is defined as the minimum cardinality of an offensive k-alliance in G and the *global offensive k-alliance number* of G, denoted by $\gamma_k^o(G)$, is defined as the minimum cardinality of a global offensive k-alliance in G. Notice that $\gamma_k^o(G) \geq a_k^o(G)$ and $\gamma_{k+1}^o(G) \geq \gamma_k^o(G) \geq \gamma(G)$.

Offensive alliances have been studied in several ways. The first results about offensive alliances were presented in [28] and after that some works have been appearing in the literature, like those in [3, 4, 13, 14, 15, 16, 17, 32, 58, 59, 60, 61, 69, 72]. The complexity of computing minimum cardinality of (global) offensive k-alliances in graphs was studied in [32, 45, 46, 48], where it was proved that this is an NP-complete problem. In [3, 4, 14, 15], [13], [58] and [61] were studied the global offensive alliances in trees, bipartite graphs, cubic graphs and planar graphs, respectively. Here we present some of the principal known results about offensive alliances.

Due to the the NP-completeness of computing minimum cardinality of (global) offensive k-alliances, several researches are centered into obtaining lower and upper bounds for the (global) offensive k-alliance number of a graph. In this sense, in [28] was obtained that for all graphs G of minimum degree δ, $a_1^o(G) \geq \frac{\delta+1}{2}$ and $a_2^o(G) > \frac{\delta+1}{2}$. A generalization of that was presented in [32].

Theorem 1. [32] *For any graph G of order n and minimum degree δ, and for every $k \in \{2-\delta, ..., \delta\}$,*

$$\left\lceil \frac{\delta+k}{2} \right\rceil \leq a_k^o(G) \leq \gamma_k^o(G) \leq n - \left\lceil \frac{\delta-k+2}{2} \right\rceil.$$

Among other interesting results, in [28] and [69] appeared some upper bounds, like the following ones, for the offensive alliance number of a graph in terms of its order.

Theorem 2. [28] *For every graph G of order $n \geq 2$, $a_1^o(G) \leq \frac{2n}{3}$.*

Theorem 3. [69] *For every graph G of order n, $a_0^o(G) \leq \frac{n}{2}$.*

We recall that the above result was first presented in [28] restricted to those graphs having every vertex with odd degree.

Theorem 4. [28] *If G has n vertices and domination number $\gamma(G)$, then*

$$a_1^o(G) \leq \frac{n+\gamma(G)}{2}.$$

As a kind of generalization of the above result, it was proved in [32] the following result for the case of global offensive k-alliances.

Theorem 5. [32] *For any graph G of order n and k-domination number $\gamma_k(G)$,*

$$\gamma_k^o(G) \le \frac{n + \gamma_k(G)}{2}.$$

There are also some other results about the general case of global offensive k-alliances. As an examples we have the following lower bound and upper bound, obtained in [69] and [32], respectively.

Theorem 6. [69] *For every graph G of order n, size m and maximum degree Δ,*

$$\gamma_k^o(G) \ge \left\lceil \frac{2m + kn}{3\Delta + k} \right\rceil.$$

Theorem 7. [32] *For any simple graph G of order n, minimum degree δ, and for every $k \in \{1, ..., \delta\}$,*

$$\gamma_k^o(G) \le \left\lfloor \frac{n(2k+1)}{2k+2} \right\rfloor.$$

Moreover, in [16] was obtained the following result which improves the above bound for the cases $k \in \{2, ..., \delta - 2\}$.

Theorem 8. [16] *Let G be a graph of order n and minimum degree δ. Then, for every $k \in \{2, ..., \delta - 2\}$*

$$\gamma_k^o(G) \le \frac{nk}{k+1}.$$

Also, in [16] was obtained the following relationship between the global offensive $(k+1)$-alliance number and the global offensive k-alliance number of a graph and there were characterized the extremal graphs satisfying such a relation.

Theorem 9. [16] *Let G be a graph of order n and minimum degree δ. Then, for every $k \in \{1, ..., \delta - 1\}$*

$$\gamma_{k+1}^o(G) \leq \frac{\gamma_k^o(G) + n}{2}.$$

On the other hand, offensive alliances have been also related with other parameters of the graph. For instance, in [59] were obtained the following bounds in terms of the order, minimum degree and Laplacian spectral radius[1].

Theorem 10. [59] *Let G be a simple graph of order n and minimum degree δ. Let μ_* be the Laplacian spectral radius of G. The global offensive 1-alliance number of G is bounded by*

$$\gamma_1^o(G) \geq \left\lceil \frac{n}{\mu_*} \left\lceil \frac{\delta + 1}{2} \right\rceil \right\rceil$$

and the global offensive 2-alliance number of G is bounded by

$$\gamma_2^o(G) \geq \left\lceil \frac{n}{\mu_*} \left(\left\lceil \frac{\delta}{2} \right\rceil + 1 \right) \right\rceil .$$

Some other investigations have been centered into studying global offensive alliances in particular classes of graph. In this sense, in [58], [3, 4, 14, 15], [13, 17] and [61] were studied the (global) offensive alliances in cubic graphs, trees, bipartite graphs and planar graphs, respectively, where the authors obtained several tight bounds for the (global) offensive alliance number and some relationships between offensive alliance numbers and invariants of the graph like domination number, $\gamma(G)$, independence number, $\beta_0(T)$, or independence domination number, $i(G)$.

Theorem 11. [58] *Let G be a connected cubic graph of order n.*

(i) $\frac{n}{2} \leq \gamma_2^o(G) \leq \frac{3n}{4}$.

[1]The largest eigenvalue of the Laplacian matrix is called the Laplacian spectral radius of G.

(ii) $\gamma_2^o(G) = \frac{n}{2}$ *if and only if* G *is a bipartite graph.*

(iii) $\gamma_2^o(G) = \frac{3n}{4}$ *if and only if* G *is isomorphic to the complete graph* K_4.

Also, in [58] was obtained the following interesting chain of inequalities for cubic graphs, where $\gamma_1^{io}(G)$ represents the minimum cardinality of a global offensive 1-alliance which is an independent set.

$$\frac{2n}{5} \leq \gamma_1^{io}(G) \leq \frac{n}{2} \leq \gamma_2^o(G) \leq \frac{3n}{4} \leq \gamma_2^o(\mathcal{L}(G)) = \gamma_1^o(\mathcal{L}(G)) \leq n.$$

For the case of trees we emphasize the following results.

Theorem 12. [3] *For every tree* T *of order* $n \geq 3$, s *support vertices and domination number* $\gamma(T)$,

$$\gamma_1^o(T) \leq \frac{3}{2}\gamma(T) + \frac{s-2}{2}.$$

Let $\mathcal{F}$ be the family of trees of order at least three which is obtained from r disjoint stars by adding $r-1$ edges between the centers of the stars in such a way that the resulting graph is connected, and then by subdividing the new added edges exactly once.

Theorem 13. [4] *Let* T *be a tree of order* $n \geq 3$ *with* l *leaves and* s *support vertices. Then* $\gamma_1^o(T) \geq \frac{n-l+s+1}{3}$ *with equality if and only if* $T \in \mathcal{F}$.

Theorem 14. [14] *For any tree* T, $\gamma_1^o(T) \leq \beta_0(T) \leq \gamma_2^o(T)$, *and these bounds are sharp.*

Theorem 15. [15] *For every nontrivial tree* T,

(i) $\gamma_2^o(G) \geq \gamma(T) + 1$, *with equality if and only if* T *is a subdivided star, a corona of a star, or a subdivided double star.*

(ii) $\gamma_2^o(G) \geq i(T) + 1$, *with equality if and only if* $\gamma_2^o(G) \geq \gamma(T) + 1$.

(iii) *If G has order $n \geq 3$, s support vertices and l leaves, then* $\gamma_2^o(G) \geq \gamma_1^o(T) + l - s$

Some results on bipartite graphs are the following ones.

Theorem 16. [13] *For every bipartite graph G without isolated vertices, l vertices of degree one and s support vertices,*

$$\gamma_1^o(G) \leq \frac{n - l + s}{2}.$$

Theorem 17. [13] *For every bipartite graph G without isolated vertices and l vertices of degree one,*

$$\gamma_2^o(G) \leq \frac{n + l}{2}.$$

Theorem 18. [17] *If G is a connected bipartite graph of order at least three, then*

$$\gamma_2^o(G) \leq \frac{3\beta_0(G)}{2}.$$

Moreover, equality holds if and only if G is a corona graph of a connected bipartite graph H with a bipartition (X, Y) such that $|X| = |Y|$ and $\gamma_1^o(H) = |H|/2$.

In the case of planar graphs we emphasize the following results.

Theorem 19. [61] *Let $G = (V, E)$ be a planar graph of order $n > 2$. If S is a global offensive 1-alliance in G such that the subgraph $\langle V \setminus S \rangle$ has c connected components, then*

(i) $|S| \geq \left\lceil \frac{n+4-2c}{3} \right\rceil$.

(ii) *If S is a global offensive 2-alliance, then* $|S| \geq \left\lceil \frac{n-c+2}{2} \right\rceil$.

We refer to the Ph. D. Thesis [67] and [69] to have a more complete idea about the principal results related to offensive alliances.

1.2 On global offensive k-alliances

As it was mentioned in the above section, the problem of finding the global offensive k-alliance number is NP-complete [32, 48]. Even so, for some graphs it is possible to obtain this number. For instance, it is satisfied that for the family of complete graphs, K_n, of order n, $\gamma_k^o(K_n) = \left\lceil \frac{n+k-1}{2} \right\rceil$, for any cycle, C_n, of order n

$$\gamma_k^o(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{for } k = 0, \\ \left\lceil \frac{n}{2} \right\rceil, & \text{for } k = 1, 2, \end{cases}$$

and for any path, P_n, of order n

$$\gamma_k^o(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{for } k = 0, \\ \left\lfloor \frac{n}{2} \right\rfloor + k - 1, & \text{for } k = 1, 2. \end{cases}$$

Now, for bipartite graphs we obtain the following result.

Remark 1.1. *Let $G = K_{r,t}$ be a complete bipartite graph with $t \le r$. For every $k \in \{2 - r, ..., r\}$,*

(i) *if $k \ge t + 1$, then $\gamma_k^o(G) = r$,*

(ii) *if $k \le t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil \ge t$, then $\gamma_k^o(G) = t$,*

(iii) *if $-t < k \le t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil < t$, then $\gamma_k^o(G) = \left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil$,*

(iv) *if $k \le -t$ and $\left\lceil \frac{r+k}{2} \right\rceil + \left\lceil \frac{t+k}{2} \right\rceil < t$, then $\gamma_k^o(G) = \min\{t, 1 + \left\lceil \frac{r+k}{2} \right\rceil\}$.*

Proof. (i) Let $\{V_t, V_r\}$ be the bi-partition of the vertex set of G. Since V_r is a global offensive k-alliance, we only need to show that for every global offensive k-alliance S, $V_r \subseteq S$. If $v \in \overline{S}$, then we have $\delta_S(v) \ge \delta_{\overline{S}}(v) + k > t$, in consequence $v \in V_t$. Thus, $\overline{S} \subseteq V_t$ or, equivalently, $V_r \subseteq S$. Therefore, we conclude that $\gamma_k^o(G) = r$.

(ii) If $k \leq t$, it is clear that V_t is a global offensive k-alliance, then $\gamma_k^o(G) \leq t$. We suppose that $\lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil \geq t$ and there exists a global offensive k-alliance $S = A \cup B$ such that $A \subseteq V_r$, $B \subseteq V_t$ and $|S| < t$. In such a case, as S is a dominating set, $B \neq \emptyset$. Since S is a global offensive k-alliance, $2|B| \geq t + k$ and $2|A| \geq r + k$. Then we have, $t > |S| \geq \lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil \geq t$, a contradiction. Therefore, $\gamma_k^o(G) = t$.

(iii) In the proof of (ii) we have shown that if there exists a global offensive k-alliance S of cardinality $|S| < t$, then $|S| \geq \lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil$. Taking $A \subset V_r$ of cardinality $\lceil \frac{r+k}{2} \rceil$ and $B \subset V_t$ of cardinality $\lceil \frac{t+k}{2} \rceil$ we obtain a global offensive k-alliance $S = A \cup B$ of cardinality $|S| = \lceil \frac{r+k}{2} \rceil + \lceil \frac{t+k}{2} \rceil$.

(iv) Finally, if $S = A \cup B$ where $A \subseteq V_r$, $B \subseteq V_t$, $|A| = \lceil \frac{r+k}{2} \rceil$ and $|B| = 1$, then S is a global offensive k-alliance. Moreover, S is a minimum global offensive k-alliance if and only if $|S| = 1 + \lceil \frac{r+k}{2} \rceil \leq t$. □

1.2.1 Global offensive k-alliances and r-dependent sets

A set $S \subseteq V$ is an *r-dependent set* in G if the maximum degree of a vertex in the subgraph $\langle S \rangle$ is at most r, i.e., $\delta_S(v) \leq r, \quad \forall v \in S$ [29]. We denote by $\alpha_r(G)$ the maximum cardinality of an r-dependent set in G.

Theorem 1.2. *Let G be a graph of order n, minimum degree δ and maximum degree Δ.*

(i) *If S is an r-dependent set in G, $r \in \left\{0, ..., \lfloor \frac{\delta-1}{2} \rfloor \right\}$, then $\overline{S}$ is a global offensive $(\delta - 2r)$-alliance.*

(ii) *If S is a global offensive k-alliance in G, $k \in \{2 - \Delta, ..., \Delta\}$, then $\overline{S}$ is a $\left\lfloor \frac{\Delta-k}{2} \right\rfloor$-dependent set.*

(iii) *Let G be a δ-regular graph ($\delta > 0$). S is an r-dependent set in G, $r \in \left\{0, ..., \lfloor \frac{\delta-1}{2} \rfloor \right\}$, if and only if $\overline{S}$ is a global offensive $(\delta - 2r)$-alliance.*

Proof. (i) Let S be an r-dependent set in G, then $\delta_S(v) \le r$ for every $v \in S$. Therefore, $\delta_S(v) + \delta \le 2\delta_S(v) + \delta_{\overline{S}}(v) \le 2r + \delta_{\overline{S}}(v)$. As a consequence, $\delta_{\overline{S}}(v) \ge \delta_S(v) + \delta - 2r$, for every $v \in S$. That is, $\overline{S}$ is a global offensive $(\delta - 2r)$-alliance in G.

(ii) If S is a global offensive k-alliance in G, then $\delta(v) \ge 2\delta_{\overline{S}}(v) + k$ for every $v \in \overline{S}$. As a consequence, $\delta_{\overline{S}}(v) \le \frac{\delta(v)-k}{2} \le \frac{\Delta-k}{2}$ for every $v \in \overline{S}$, that is, $\overline{S}$ is a $\lfloor \frac{\Delta-k}{2} \rfloor$-dependent set in G.

(iii) The result follows immediately from (i) and (ii). □

Corollary 1.3. *Let G be a graph of order n, minimum degree δ and maximum degree Δ.*

(i) *For every $k \in \{2-\Delta, ..., \Delta\}$,*

$$n - \alpha_{\lfloor \frac{\Delta-k}{2} \rfloor}(G) \le \gamma_k^o(G).$$

(ii) *For every $k \in \{1, ..., \delta\}$,*

$$\gamma_k^o(G) \le n - \alpha_{\lfloor \frac{\delta-k}{2} \rfloor}(G).$$

(iii) *If G is a δ-regular graph ($\delta > 0$), for every $k \in \{1, ..., \delta\}$,*

$$\gamma_k^o(G) = n - \alpha_{\lfloor \frac{\delta-k}{2} \rfloor}(G).$$

1.2.2 Global offensive k-alliances and τ-dominating sets

Let G be a graph without isolated vertices. For a given $\tau \in (0, 1]$, a set $S \subseteq V$ is called *τ-dominating set* in G if $\delta_S(v) \ge \tau\delta(v)$ for every $v \in \overline{S}$ [20]. We denote by $\gamma_\tau(G)$ the minimum cardinality of a τ-dominating set in G.

Theorem 1.4. *Let G be a graph of minimum degree $\delta > 0$ and maximum degree Δ.*

(i) *If* $0 < \tau \leq \min\{\frac{k+\delta}{2\delta}, \frac{k+\Delta}{2\Delta}\}$*, then every global offensive* k*-alliance in* G *is a* τ*-dominating set.*

(ii) *If* $\max\{\frac{k+\delta}{2\delta}, \frac{k+\Delta}{2\Delta}\} \leq \tau \leq 1$*, then every* τ*-dominating set in* G *is a global offensive* k*-alliance.*

Proof. (i) If S is a global offensive k-alliance in G, then $2\delta_S(v) \geq \delta(v) + k$ for every $v \in \overline{S}$. Therefore, if $0 < \tau \leq \min\{\frac{1}{2}, \frac{k+\delta}{2\delta}\}$, then

$$\delta_S(v) \geq \frac{\delta(v)+k}{2} \geq \frac{\delta(v)+\delta(2\tau-1)}{2} \geq \tau\delta(v).$$

Moreover, if $\frac{1}{2} \leq \tau \leq \frac{k+\Delta}{2\Delta}$, then

$$\delta_S(v) \geq \frac{\delta(v)+k}{2} \geq \frac{\delta(v)+\Delta(2\tau-1)}{2} \geq \tau\delta(v).$$

(ii) Since $\delta > 0$, it is clear that every τ-dominating set is a dominating set. If $\tau \geq \frac{1}{2}$, then $\delta(2\tau-1) \leq \delta(v)(2\tau-1)$, for every vertex v in G. Hence, if S is a τ-dominating set and $\frac{k+\delta}{2\delta} \leq \tau$, we have $k \leq (2\tau-1)\delta(v) \leq 2\delta_S(v) - \delta(v)$, for every $v \in \overline{S}$. Thus, S is a global offensive k-alliance in G.

On the other hand, if $\tau \leq \frac{1}{2}$, then $\Delta(2\tau-1) \leq \delta(v)(2\tau-1)$, for every vertex v in G. Hence, if S is a τ-dominating set and $\frac{k+\Delta}{2\Delta} \leq \tau$, we have $k \leq (2\tau-1)\delta(v) \leq 2\delta_S(v) - \delta(v)$, for every $v \in \overline{S}$. Thus, S is a global offensive k-alliance in G. □

Corollary 1.5. *S is a global offensive* 0*-alliance in* G *if, and only if,* S *is a* $(\frac{1}{2})$*-dominating set.*

Corollary 1.6. *S is a global offensive* k*-alliance in a* δ*-regular graph* G *if, and only if,* S *is a* $(\frac{k+\delta}{2\delta})$*-dominating set in* G*.*

Theorem 1.7. *Let* G *be a graph of order* n*, minimum degree* $\delta > 0$ *and maximum degree* $\Delta \geq 2$*. For every* $j \in \{2-\Delta, ..., 0\}$ *and* $k \leq -\frac{j\delta}{\Delta}$ *it is satisfied*

$$\gamma_k^o(G) + \gamma_j^o(G) \leq n.$$

Proof. If $j \in \{2-\Delta, ..., 0\}$, then there exists $\tau \in \left[\frac{1}{\Delta}, \frac{1}{2}\right]$ such that $j = \Delta(2\tau - 1)$. Therefore, if S is a τ-dominating set, then (by Theorem 1.4 (ii)) S is a global offensive j-alliance. In consequence, $\gamma_j^o(G) \leq \gamma_\tau(G)$. Moreover, if $k \leq -\frac{j\delta}{\Delta} = \delta(1-2\tau)$, then $1-\tau \geq \max\{\frac{1}{2}, \frac{k+\delta}{2\delta}\}$. Hence, by Theorem 1.4 (ii) we have that every $(1-\tau)$-dominating set is a global offensive k-alliance. Thus, $\gamma_k^o(G) \leq \gamma_{1-\tau}(G)$. Using that $\gamma_\tau(G) + \gamma_{1-\tau}(G) \leq n$ for $0 < \tau < 1$ (see Theorem 9, [20]), we obtain the required result. □

Notice that from Theorem 1.7 we have the following result.

Corollary 1.8. *If G is a graph of order n and minimum degree $\delta > 0$, then $\gamma_0^o(G) \leq \frac{n}{2}$.*

1.2.3 Global offensive k-alliances and dominating sets

We say that a global offensive k-alliance S is *minimal* if no proper subset $S' \subset S$ is a global offensive k-alliance.

Theorem 1.9. *Let G be a graph without isolated vertices and let $k \leq 1$. If S is a minimal global offensive k-alliance in G, then $\overline{S}$ is a dominating set in G.*

Proof. We suppose there exists $u \in S$ such that $\delta_{\overline{S}}(u) = 0$ and let $S' = S \setminus \{u\}$. Since S is a minimal global offensive k-alliance, and G has no isolated vertices, there exists $v \in \overline{S'}$ such that $\delta_{S'}(v) < \delta_{\overline{S'}}(v) + k$. If $v = u$, we have $\delta_S(u) = \delta_{S'}(u) < \delta_{\overline{S'}}(u) + k = k$, a contradiction. If $v \neq u$, we have $\delta_S(v) = \delta_{S'}(v) < \delta_{\overline{S'}}(v) + k = \delta_{\overline{S}}(v) + k$, which is a contradiction too. Thus, $\delta_{\overline{S}}(u) > 0$ for every $u \in S$. □

Lemma 1.10. *Let G be a graph of order n. A dominating set S in $\overline{G}$ is a global offensive k-alliance in $\overline{G}$ if and only if $\delta_S(v) - \delta_{\overline{S}}(v) + n + k - 1 \leq 2|S|$ for every $v \in \overline{S}$ in G.*

Proof. We know that a dominating set S in $\overline{G}$ is a global offensive k-alliance in $\overline{G}$ if and only if $\overline{\delta}_S(v) \geq \overline{\delta}_{\overline{S}}(v) + k$ for every $v \in \overline{S}$, where $\overline{\delta}_S(v)$ and $\overline{\delta}_{\overline{S}}(v)$ denote the number of vertices that v has in S and $\overline{S}$, respectively, in $\overline{G}$. Now, using that $\overline{\delta}_S(v) = |S| - \delta_S(v)$ and $\overline{\delta}_{\overline{S}}(v) = |\overline{S}| - 1 - \delta_{\overline{S}}(v) = n - |S| - 1 - \delta_{\overline{S}}(v)$, we get that S is a global offensive k-alliance in $\overline{G}$ if and only if $|S| - \delta_S(v) \geq n - |S| - 1 + k - \delta_{\overline{S}}(v)$ or, equivalently, if $\delta_S(v) - \delta_{\overline{S}}(v) + n + k - 1 \leq 2|S|$ for every $v \in \overline{S}$. □

Theorem 1.11. *Let G be a graph of order n, minimum degree δ and maximum degree Δ.*

(i) *Every dominating set in $\overline{G} = (V, \overline{E})$, $S \subseteq V$, of cardinality $|S| \geq \left\lceil \frac{n+k+\Delta-1}{2} \right\rceil$ is a global offensive k-alliance in $\overline{G}$.*

(ii) *Every dominating set in $G = (V, E)$, $S \subseteq V$, of cardinality $|S| \geq \left\lceil \frac{2n+k-\delta-2}{2} \right\rceil$ is a global offensive k-alliance in G.*

Proof. If S is a dominating set in $\overline{G}$ and it satisfies $|S| \geq \left\lceil \frac{n+k+\Delta-1}{2} \right\rceil$, then

$$|S| \geq \frac{n+k+\Delta-1}{2} \geq \frac{\delta_S(v) - \delta_{\overline{S}}(v) + n + k - 1}{2}$$

for every vertex v. Therefore, by Lemma 1.10 we have that S is a global offensive k-alliance in $\overline{G}$. Thus, (i) follows.

Analogously, by replacing G by $\overline{G}$ and by taking into account that the maximum degree in $\overline{G}$ is $n - 1 - \delta$, we obtain (ii). □

1.3 Cartesian product of offensive k-alliances

In this section we discuss the relationship that exist between the (global) offensive k_i-alliance number of G_i, $i \in \{1, 2\}$ and the (global) offensive k-alliance number of $G_1 \times G_2$, for some specific values of k.

Theorem 1.12. *Let $G_i = (V_i, E_i)$ be a graph of minimum degree δ_i and maximum degree Δ_i, $i \in \{1, 2\}$.*

(i) *If S_i is an offensive k_i-alliance in G_i, $i \in \{1, 2\}$, then, for $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$, $S_1 \times S_2$ is an offensive k-alliance in $G_1 \times G_2$.*

(ii) *Let $S_i \subset V_i$, $i \in \{1, 2\}$. If $S_1 \times S_2$ is an offensive k-alliance in $G_1 \times G_2$, then S_1 is an offensive $(k + \delta_2)$-alliance in G_1 and S_2 is an offensive $(k + \delta_1)$-alliance in G_2, moreover, $k \leq \min\{\Delta_1 - \delta_2, \Delta_2 - \delta_1\}$.*

Proof. If $X = S_1 \times S_2$, then $(u, v) \in \partial(X)$ if and only if, either $u \in \partial(S_1)$ and $v \in S_2$ or $u \in S_1$ and $v \in \partial(S_2)$. We differentiate two cases:

Case 1: If $u \in \partial(S_1)$ and $v \in S_2$, then $\delta_X(u, v) = \delta_{S_1}(u)$ and $\delta_{\overline{X}}(u, v) = \delta_{\overline{S_1}}(u) + \delta(v)$.

Case 2: If $u \in S_1$ and $v \in \partial(S_2)$, then $\delta_X(u, v) = \delta_{S_2}(v)$ and $\delta_{\overline{X}}(u, v) = \delta(u) + \delta_{\overline{S_2}}(v)$.

(i) In Case 1 we have $\delta_X(u, v) = \delta_{S_1}(u) \geq \delta_{\overline{S_1}}(u) + k_1 = \delta_{\overline{X}}(u, v) - \delta(v) + k_1 \geq \delta_{\overline{X}}(u, v) - \Delta_2 + k_1$ and in Case 2 we obtain $\delta_X(u, v) = \delta_{S_2}(v) \geq \delta_{\overline{S_2}}(v) + k_2 = \delta_{\overline{X}}(u, v) - \delta(u) + k_2 \geq \delta_{\overline{X}}(u, v) - \Delta_1 + k_2$. Hence, for every $(u, v) \in \partial(X)$, $\delta_X(u, v) \geq \delta_{\overline{X}}(u, v) + k$, with $k \leq \min\{\Delta_1 - \delta_2, \Delta_2 - \delta_1\}$. So, the result follows.

(ii) In Case 1 we have $\delta_{S_1}(u) = \delta_X(u, v) \geq \delta_{\overline{X}}(u, v) + k = \delta_{\overline{S_1}}(u) + \delta(v) + k \geq \delta_{\overline{S_1}}(u) + \delta_2 + k$ and in Case 2 we deduce $\delta_{S_2}(v) = \delta_X(u, v) \geq \delta_{\overline{X}}(u, v) + k = \delta_{\overline{S_2}}(v) + \delta(u) + k \geq \delta_{\overline{S_2}}(v) + \delta_1 + k$. Hence, for every $u \in \partial(S_1)$, $\delta_{S_1}(u) \geq \delta_{\overline{S_1}}(u) + \delta_2 + k$ and for every $v \in \partial(S_2)$, $\delta_{S_2}(v) \geq \delta_{\overline{S_2}}(v) + \delta_1 + k$. So, the result follows. □

Corollary 1.13. *Let G_i be a graph of maximum degree Δ_i, $i \in \{1, 2\}$. Then for every $k \leq \min\{k_1 - \Delta_2, k_2 - \Delta_1\}$,*

$$a_k^o(G_1 \times G_2) \leq a_{k_1}^o(G_1) a_{k_2}^o(G_2).$$

For the particular case of the graph $C_4 \times K_4$, we have $a^o_{-3}(C_4 \times K_4) = 2 = a^o_0(C_4)a^o_1(K_4)$.

Theorem 1.14. *Let $G_2 = (V_2, E_2)$ be a graph of maximum degree Δ_2 and minimum degree δ_2.*

(i) *If S is a global offensive k-alliance in G_1, then $S \times V_2$ is a global offensive $(k - \Delta_2)$-alliance in $G_1 \times G_2$.*

(ii) *If $S \times V_2$ is a global offensive k-alliance in $G_1 \times G_2$, then S is a global offensive $(k + \delta_2)$-alliance in G_1, moreover, $k \leq \Delta_1 - \delta_2$, where Δ_1 denotes the maximum degree of G_1.*

Proof. (i) We first note that, as S is a dominating set in G_1, $X = S \times V_2$ is a dominating set in $G_1 \times G_2$. In addition, for every $x_{ij} = (u_i, v_j) \in \overline{X}$ we have $\delta_X(x_{ij}) = \delta_S(u_i)$ and $\delta_{\overline{S}}(u_i) + \Delta_2 \geq \delta_{\overline{S}}(u_i) + \delta(v_j) = \delta_{\overline{X}}(x_{ij})$, so $\delta_X(x_{ij}) = \delta_S(u_i) \geq \delta_{\overline{S}}(u_i) + k \geq \delta_{\overline{X}}(x_{ij}) - \Delta_2 + k$. Thus, X is a global offensive $(k - \Delta_2)$-alliance in $G_1 \times G_2$.

(ii) From Theorem 1.12 (ii) we obtain that S is an offensive $(k + \delta_2)$-alliance in G_1 and $k \leq \Delta_1 - \delta_2$. We only need to show that S is a dominating set. As $S \times V_2$ is a dominating set in $G_1 \times G_2$, we have that for every $u \in \overline{S}$ and $v \in V_2$ there exists $(a, b) \in S \times V_2$ such that (a, b) is adjacent to (u, v), hence, $b = v$ and a is adjacent to u, so the result follows. □

It is easy to see the following result on domination,

$$\gamma(G_1 \times G_2) \leq n_2\gamma(G_1),$$

where n_2 is the order of G_2. An “analogous” result on global offensive k-alliances can be deduced from Theorem 1.14 (i).

Corollary 1.15. *For any graph G_1 and any graph G_2 of order n_2 and maximum degree Δ_2,*

$$\gamma^o_{k-\Delta_2}(G_1 \times G_2) \leq n_2\gamma^o_k(G_1).$$

We emphasize the following particular cases of Corollary 1.15.

Remark 1.16. *For any graph G,*

(i) $\gamma^o_{k-2}(G \times C_t) \le t\gamma^o_k(G)$,

(ii) $\gamma^o_{k-2}(G \times P_t) \le t\gamma^o_k(G)$.

(iii) $\gamma^o_{k-t+1}(G \times K_t) \le t\gamma^o_k(G)$.

Notice also that if G_2 is a regular graph, Theorem 1.14 (i) can be simplified as follow.

Corollary 1.17. *Let $G_2 = (V_2, E_2)$ be a δ-regular graph. A set S is a global offensive k-alliance in G_1 if and only if $S \times V_2$ is a global offensive $(k - \delta)$-alliance in $G_1 \times G_2$.*

1.4 Partitions into offensive k-alliances

For any graph $G = (V, E)$, the (*global*) *offensive k-alliance partition number* of G, $(\psi^{go}_k(G))$ $\psi^o_k(G)$, $k \in \{2 - \Delta, ..., \Delta\}$, is defined to be the maximum number of sets in a partition of V such that each set is (a global offensive) an offensive k-alliance.

If V can be partitioned into global offensive k-alliances, then there exist a global offensive k-alliance S and a vertex of minimum degree v such that $v \notin S$ and $\delta = \delta(v) \ge 2\delta_{\overline{S}}(v) + k$. Therefore, if $k > \delta$, then V can not be partitioned into global offensive k-alliances. Hereafter we will say that $(\Pi^{go}_r(G))$ $\Pi^o_r(G)$ is a partition of G into r (global) offensive k-alliances. Now on we will say that a graph G is partitionable into (global) offensive k-alliances if $(\psi^{go}_k(G) \ge 2)$ $\psi^o_k(G) \ge 2$.

Notice that if every vertex of G has even degree and k is odd, or every vertex of G has odd degree and k is even, then every (global) offensive k-

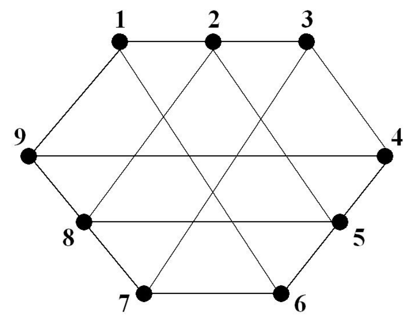

Figure 1.2: $\{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$ is a partition of the graph into three offensive (-2)-alliances.

alliance in G is an offensive (a global offensive) $(k+1)$-alliance and vice versa. Hence, in such a case, $\psi_k^o(G) = \psi_{k+1}^o(G)$ and $\psi_k^{go}(G) = \psi_{k+1}^{go}(G)$.

1.4.1 Partitioning CR(n,2)

We now introduce an important class of graphs that will provide useful examples in the later sections. Let $\mathbb{Z}_n$ be the additive group of integers modulo n and let $M \subset \mathbb{Z}_n$, such that, $i \in M$ if and only if $-i \in M$. We can construct a graph $G = (V, E)$ as follows, the vertices of V are the elements of $\mathbb{Z}_n$ and (i, j) is an edge in E if and only if $j-i \in M$. This graph is called a *circulant of order* n and we will denote it by $CR(n, M)$. The set M is called the *set of generators* of the circulant graph. With this notation, a cycle graph is $CR(n, \{-1, 1\})$ and the complete graph is $CR(n, \mathbb{Z}_n)$. To simplify the notation we will use $CR(n, t)$, $0 < t \le \frac{n}{2}$, instead of $CR(n, \{-t, -t+1, ..., -1, 1, 2, ..., t\})$. We emphasize that $CR(n, t)$ is a $(2t)$-regular graph.

Note that, if n is even, $\Pi_2^{go}(CR(n,2)) = \{\{1,3,5,...,n-1\},\{2,4,6,...,n\}\}$ is a partition of $CR(n,2)$ into global offensive 0-alliances, moreover, if $n = 4j$, $\Pi_4^{go}(CR(n,2)) = \{\{1,5,...,n-3\},\{2,6,...,n-2\},\{3,7,...,n-1\},\{4,8,...,n\}\}$ is a partition of $CR(n,2)$ into global offensive (-2)-alliances. At next we compute the global offensive k-alliance partition number of $CR(n,2)$, for

$k \in \{-2, .., 4\}$.

Claim 1.18. *For the circulant graph $G = CR(n, 2)$, $\gamma(G) = \lceil \frac{n}{5} \rceil$.*

Claim 1.19. *Any dominating set in $G = CR(n, 2)$ is a global offensive (-2)-alliance.*

Proof. As S is a dominating set in G, then for every $v \in \overline{S}$, we have $\delta_S(v) \geq 1 = 3 - 2 \geq \delta_{\overline{S}}(v) - 2$. So, S is a global offensive (-2)-alliance in G. □

Remark 1.20. *In the case of the circulant graph $G = CR(n, 2)$ we have the following:*

(i) *G is not partitionable into global offensive 3-alliances or global offensive 4-alliances.*

(ii) *$\psi_1^{go}(G) = \psi_2^{go}(G) = 2$ if and only if $n = 4j$.*

(iii) *$\psi_{-1}^{go}(G) = \psi_0^{go}(G) = 3$ if and only if $n = 3j$.*

(iv) *$\psi_{-2}^{go}(G) = \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$.*

Proof. We first emphasize that, since G is a 4-regular graph, $\psi_{-1}^{go}(G) = \psi_0^{go}(G)$, $\psi_1^{go}(G) = \psi_2^{go}(G)$ and $\psi_3^{go}(G) = \psi_4^{go}(G)$. So, we will only consider the study of $\psi_k^{go}(G)$ for $k = 4, 2, 0, -2$. Let us denote the vertices of G by $\{v_1, v_2, \ldots, v_n\}$ such that v_i is adjacent to $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$.

(i) By Corollary 1.24 we know that if G is partitionable into global offensive k-alliances, for $k \geq 1$, then $\psi_k^{go}(G) = 2$. So, suppose $\{S_1, S_2\}$ is a partition of the graph into two global offensive 4-alliances. If $v_i \notin S_j$, then $\delta_{S_j}(v_i) = 4 = \delta(v_i)$, so $CR(n, 2)$ is a bipartite graph, a contradiction.

(ii) As above, let us suppose $\{S_1, S_2\}$ is a partition of the graph into two global offensive 2-alliances. If $v_i \notin S_1$, then $\delta_{S_1}(v_i) \geq 3$. If $\delta_{S_1}(v_i) = 4$, then $\delta_{S_1}(v_{i+1}) \geq 2$, so $\delta_{S_2}(v_{i+1}) \leq 2 < \delta_{\overline{S_2}}(v_{i+1}) + 2$, a contradiction. Thus $\delta_{S_1}(v_i) = 3$. Analogously for S_2, if $v_i \notin S_2$, then $\delta_{S_2}(v_i) = 3$.

Now, let $v_i \in S_2$, if $v_{i-2}, v_{i-1}, v_{i+1} \in S_1$ (or $v_{i-1}, v_{i+1}, v_{i+2} \in S_1$), we obtain that $\delta_{S_2}(v_{i-1}) \leq 2$ (or $\delta_{S_2}(v_{i+1}) \leq 2$), which is a contradiction. Therefore, if $v_l, v_{l+1} \in S_1$, then necessarily, $v_{l+2}, v_{l+3} \in S_2$ and this is possible if and only if $n = 4j$.

(iii) Let us suppose $n = 3j$. So, the sets $\{v_1, v_4, ..., v_{n-2}\}$, $\{v_2, v_5, ..., v_{n-1}\}$ and $\{v_3, v_6, ..., v_n\}$ form a partition of G into three global offensive 0-alliances, therefore $\psi_0^{go}(G) \geq 3$. From Corollary 1.25 we have that $\psi_0^{go}(G) \leq 3$, so $\psi_0^{go}(G) = 3$. On the contrary, let us suppose $\psi_0^{go}(G) = 3$, then by Theorem 1.27 and Remark 1.29 each alliance in the partition is a maximal independent set and the chromatic number of G is 3, so there exist three color classes among the vertices of G, $\dot{v}_1$, $\dot{v}_2$ and $\dot{v}_3$, which contain those vertices with subindexes congruent to 1, 2 and 3, respectively, hence v_n belongs to the class $\dot{v}_3$.

(iv) We have that $\psi_{-2}^{go}(G)\gamma_{-2}^{go}(G) \leq n$, now, by using Claims 1.18 and 1.19 we obtain $\psi_{-2}^{go}(G) \leq \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$. By taking $q = \left\lfloor \frac{n}{\lceil \frac{n}{5} \rceil} \right\rfloor$, let us form a partition of the graph into q dominating sets. Note that $2 < q \leq 5$. Hence, we have the following cases:

Case 1: $q = 5$ if and only if $n = 5j$, $j \in \mathbb{Z}_+$. The sets $\{v_1, v_6, ..., v_{n-4}\}$, $\{v_2, v_7, ..., v_{n-3}\}$, $\{v_3, v_8, ..., v_{n-2}\}$, $\{v_4, v_9, ..., v_{n-1}\}$ and $\{v_5, v_{10}, ..., v_n\}$ form a partition of G into five dominating sets.

Case 2: $q = 4$ if and only if $n \neq 6, 7, 11, 5j$, $j \in Z_+$. So, if $n = 4j + r$, $r \in \{0, 1, 2, 3\}$, then P_r is a partition of G into dominating sets:

$$P_0 = \{\{v_1, v_5, ..., v_{n-3}\}, \{v_2, v_6, ..., v_{n-2}\},$$
$$\{v_3, v_7, ..., v_{n-1}\}, \{v_4, v_8, ..., v_n\}\},$$
$$P_1 = \{\{v_1, v_6, v_{10}, v_{14} ..., v_{n-3}\}, \{v_2, v_7, v_{11}, v_{15}, ..., v_{n-2}\},$$
$$\{v_3, v_8, v_{12}, v_{16}, ..., v_{n-1}\}, \{v_4, v_5, v_9, v_{13}, ..., v_n\}\},$$

$$P_2 = \{\{v_1, v_6, v_{11}, v_{15}, v_{19}, ..., v_{n-3}\}, \{v_2, v_7, v_{12}, v_{16}, v_{20}, ..., v_{n-2}\},$$
$$\{v_3, v_8, v_{13}, v_{17}, v_{21}, ..., v_{n-1}\}, \{v_4, v_5, v_9, v_{10}, v_{14}, v_{18}, ..., v_n\}\},$$
$$P_3 = \{\{v_1, v_6, v_{11}, v_{16}, v_{20}, v_{24}, ..., v_{n-3}\}, \{v_2, v_7, v_{12}, v_{17}, v_{21}, v_{25}, ..., v_{n-2}\},$$
$$\{v_3, v_8, v_{13}, v_{18}, v_{22}, v_{26}, ..., v_{n-1}\}, \{v_4, v_5, v_9, v_{10}, v_{14}, v_{15}, v_{19}, v_{23}, ..., v_n\}\}.$$

Case 3: $q = 3$ if and only if $n = 6, 7, 11$. In these cases we have $P_6 = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}\}$, $P_7 = \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3, v_6, v_7\}\}$ and $P_{11} = \{\{v_1, v_4, v_7, v_{11}\}, \{v_2, v_5, v_8, v_{10}\}, \{v_3, v_6, v_9\}\}$ form partitions into three dominating sets. Therefore, by using Claim 1.19 we conclude the proof. □

1.4.2 Relations between $\psi_k^{go}(G)$ and k

Proposition 1.21. *For any graph G without isolated vertices, there exists $k \in \{0, ..., \delta\}$ such that G is partitionable into global offensive k-alliances.*

Proof. If $\delta \geq 1$ and $\{X, Y\}$ is a partition of V such that the edge cut-set between X and Y has maximum cardinality, then X and Y are dominating sets. Moreover, for every $x_i \in X$ there exists $t_i \in \mathbb{Z}$, $t_i \geq 0$, such that, $\delta_Y(x_i) = \delta_X(x_i) + t_i$. Taking $t = \min_{x_i \in X}\{t_i\}$, then we have that Y is a global offensive t-alliance in G. Analogously we obtain that there exists $r \in \mathbb{Z}$, $r \geq 0$, such that X is a global offensive r-alliance in G. Therefore, taking $k = \min\{t, r\}$ we conclude that $\{X, Y\}$ is a partition of V into two global offensive k-alliances in G. □

Corollary 1.22. *Any graph without isolated vertices is partitionable into global offensive 0-alliances.*

Theorem 1.23. *If a graph is partitionable into $r \geq 3$ global offensive k-alliances, then $k \leq 3 - r$.*

Proof. We suppose that $\Pi_r^{go}(G) = \{S_1, ...S_r\}$ is a partition of G into $r \geq 3$

global offensive k-alliances. For every $v \in S_r$ we have

$$\begin{aligned}
\delta_{S_1}(v) &\geq \delta_{\overline{S_1}}(v) + k \geq \sum_{j=2}^{r-1} \delta_{S_j}(v) + k \\
&\geq \sum_{j=2}^{r-1} (\delta_{\overline{S_j}}(v) + k) + k \\
&\geq \sum_{j=2}^{r-1} \sum_{i=1;i\neq j}^{r-1} \delta_{S_i}(v) + \sum_{j=2}^{r-1} k + k \\
&= \sum_{j=2}^{r-1} \delta_{S_1}(v) + \sum_{j=2}^{r-1} \sum_{i=2;i\neq j}^{r-1} \delta_{S_i}(v) + k(r-1) \\
&\geq (r-2)\delta_{S_1}(v) + \sum_{j=2}^{r-1} \sum_{i=2;i\neq j}^{r-1} 1 + k(r-1) \\
&= (r-2)\delta_{S_1}(v) + (r-2)(r-3) + k(r-1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &\geq (r-3)\delta_{S_1}(v) + (r-2)(r-3) + k(r-1) \\
&\geq (r-3) + (r-2)(r-3) + k(r-1) \\
&= (r-1)(r-3+k),
\end{aligned}$$

that is, $k \leq 3 - r$. □

From Theorem 1.23 we have that if a graph is partitionable into $r \geq 3$ global offensive k-alliances, then $k \leq 0$, so we obtain the following interesting consequence.

Corollary 1.24. *If G is partitionable into global offensive k-alliances for $k \geq 1$, then $\psi_k^{go}(G) = 2$.*

From Corollary 1.22 we have that any graph without isolated vertices is partitionable into global offensive 0-alliances. In consequence, from Theorem 1.23 we obtain the following result.

Corollary 1.25. *Let G be a graph without isolated vertices. If $k \in \{2-\Delta, ..., 0\}$, then $2 \leq \psi_k^{go}(G) \leq 3-k$.*

An example of graph where $\psi_0^{go}(G) = 2$ is the complete graph $G = K_n$ and an example of graph where $\psi_0^{go}(G) = 3$ is the cycle graph C_{3t}, $t \geq 1$.

Theorem 1.26. *Let G be a graph of order n such that $\psi_k^{go}(G) > 2$. Then, for every $l \in \{1, ..., \psi_k^{go}(G) - 2\}$, there exists a subgraph, G_l, of G of order $n(G_l) \leq n - l\gamma_k^o(G)$ such that $\psi_{l+k}^{go}(G_l) + l \geq \psi_k^{go}(G)$.*

Proof. Let $\Pi_r^{go}(G) = \{S_1, S_2, ..., S_r\}$ be a partition of V into $r > 2$ global offensive k-alliances and let $t \in \{2, ..., r-1\}$. We take $l = r - t$ and $G_l = \langle \bigcup_1^t S_i \rangle$. For every $i \leq t$, S_i is a dominating set in G_l, in addition, for every $v \in \overline{S_i} \cap (\bigcup_1^t S_i)$, we have

$$\begin{aligned}\delta_{S_i}(v) \geq \delta_{\overline{S_i}}(v) + k &= \sum_{j=1, j\neq i}^{t} \delta_{S_j}(v) + \sum_{j=t+1}^{r} \delta_{S_j}(v) + k \\ &\geq \sum_{j=1, j\neq i}^{t} \delta_{S_j}(v) + r - t + k,\end{aligned}$$

that is, S_i is a global offensive $(l+k)$-alliance in G_l. Moreover, the order of G and G_l are related as follow,

$$n = \sum_{i=1}^{r} |S_i| = n(G_l) + \sum_{i=t+1}^{r} |S_i| \geq n(G_l) + l\gamma_k^o(G).$$

□

1.4.3 Partition number and chromatic number

In this section, motivated by Corollary 1.25, we will study the cases $\psi_0^{go}(G) = 2$ and $\psi_0^{go}(G) = 3$. As a consequence of the study, we will show the relationship that exists between the chromatic number of G, $\chi(G)$, and $\psi_0^{go}(G)$.

We recall that, given a positive integer t, a t-dependent set in G is a set of vertices of G such that no vertex in the set is adjacent to more than t vertices of the set. A 0-dependent set in G is simply an independent set of vertices in G.

Theorem 1.27. *Any set belonging to a partition of a graph into $r \geq 3$ global offensive k-alliances, is a $(-k)$-dependent*[2] *set.*

Proof. Let $\Pi_r^{go}(G) = \{S_1, S_2, ..., S_r\}$ be a partition of G into $r \geq 3$ global offensive k-alliances. For every $v \in S_r$,

$$\begin{aligned}\delta_{S_1}(v) &\geq \delta_{\overline{S_1}}(v) + k \geq \delta_{S_2}(v) + \delta_{S_r}(v) + k \\ &\geq \delta_{\overline{S_2}}(v) + \delta_{S_r}(v) + 2k \geq \delta_{S_1}(v) + 2\delta_{S_r}(v) + 2k.\end{aligned}$$

Therefore, $\delta_{S_r}(v) \leq -k$ and, as a consequence, S_r is a $(-k)$-dependent set. Analogously we obtain that S_i, $1 \leq i \leq r-1$ is a $(-k)$-dependent set too. □

Notice that, if $k = 0$ in the above result, then $r = 3$ and as a consequence, every set in a partition into three global offensive 0-alliances is an independent set, so it leads to the following result.

Corollary 1.28. *If $\psi_0^{go}(G) = 3$, then $\chi(G) \leq 3$.*

A trivial example of graph where $\psi_0^{go}(G) = 3$ and $\chi(G) = 3$ is the cycle graph C_3, and a graph where $\psi_0^{go}(G) = 3$ and $\chi(G) = 2$ is the cycle graph $G = C_6$.

Remark 1.29. *If G is a non bipartite graph and $\psi_0^{go}(G) = 3$, then $\chi(G) = 3$.*

An example of graph where $\chi(G) > 3$ and $\psi_0^{go}(G) = 2$ is the complete graph $G = K_n$ with $n \geq 4$.

[2]We recall that, by Theorem 1.23, if $r \geq 3$, then $k \leq 0$.

Corollary 1.30. *For any graph G without isolated vertices and chromatic number greater than 3, $\psi_0^{go}(G) = 2$.*

Let us see another sufficient condition for the global offensive 0-alliance number to be 2.

Theorem 1.31. *For any graph G without isolated vertices containing a vertex of odd degree, it is satisfied $\psi_0^{go}(G) = 2$.*

Proof. By Corollary 1.22 and Corollary 1.25 we have that $2 \leq \psi_0^{go}(G) \leq 3$. Let us suppose $\{S_1, S_2, S_3\}$ is a partition of G into global offensive 0-alliances. Without loss of generality, let us suppose S_1 contains a vertex v of odd degree. From Theorem 1.27 we have $\delta_{S_1}(v) = 0$. As S_2 and S_3 are global offensive 0-alliances, we obtain $\delta_{S_2}(v) \geq \delta_{\overline{S_2}}(v) = \delta_{S_3}(v) \geq \delta_{\overline{S_3}}(v) = \delta_{S_2}(v)$, in consequence, $\delta(v) = \delta_{S_2}(v) + \delta_{S_3}(v) = 2\delta_{S_2}(v)$, a contradiction. □

Note that Theorem 1.31 is equivalent to saying that if $\psi_0^{go}(G) = 3$, then every vertex in G has even degree. As a consequence, for k odd, every partition of G into (global) offensive k-alliances is a partition of G into (global) offensive $(k+1)$-alliances and vice versa.

Corollary 1.32. *If $\psi_0^{go}(G) = 3$ and k is odd, then $a_k^o(G) = a_{k+1}^o(G)$, $\gamma_k^o(G) = \gamma_{k+1}^o(G)$, $\psi_k^o(G) = \psi_{k+1}^o(G)$ and $\psi_k^{go}(G) = \psi_{k+1}^{go}(G)$.*

1.4.4 Bounds on $\psi_k^o(G)$ and $\psi_k^{go}(G)$

From the following relation between the offensive k-alliance number and the offensive k-alliance partition number, we obtain that lower bounds on $a_k^o(G)$ lead to upper bounds on $\psi_k^o(G)$:

$$a_k^o(G)\psi_k^o(G) \leq n.$$

From Theorem 1 we have that $a_k^o(G) \geq \left\lceil \frac{\delta+k}{2} \right\rceil$, hence

$$\psi_k^o(G) \leq \begin{cases} \left\lfloor \frac{2n}{\delta+k} \right\rfloor, & \delta+k \quad \text{even} \\ \\ \left\lfloor \frac{2n}{\delta+k+1} \right\rfloor, & \delta+k \quad \text{odd.} \end{cases}$$

This bound is attained, for instance, for every δ-regular graph, $\delta \geq 1$, by taking $k = 2-\delta$. In such a case, each vertex is an offensive $(2-\delta)$-alliance and $\psi_k^o(G) = n$. Another example is $G = CR(8,2)$ where $\{1,2,5,6\}$ and $\{3,4,7,8\}$ are (global) offensive 2-alliances and the above bound leads to $\psi_2^o(G) \leq 2$.

Analogously, lower bounds on $\gamma_k^o(G)$ lead to upper bounds on $\psi_k^{go}(G)$:

$$\gamma_k^o(G)\psi_k^{go}(G) \leq n.$$

Now, from Theorem 6 we have that $\gamma_k^o(G) \geq \left\lceil \frac{2m+kn}{3\Delta+k} \right\rceil$, hence

$$\psi_k^{go}(G) \leq \left\lfloor \frac{n}{\left\lceil \frac{2m+kn}{3\Delta+k} \right\rceil} \right\rfloor.$$

This bound is attained, for instance, for the circulant graph $CR(n,2)$ for $k=-2$ and, if $n=3j$, it is also attained for $k \in \{-1,0\}$.

Theorem 1.33. *If a graph G is partitionable into global offensive k-alliances, then*

(i) $\psi_k^{go}(G) \leq \left\lfloor \frac{2m-n(k-4)}{2n} \right\rfloor$,

(ii) $\psi_k^{go}(G) \leq \left\lfloor \frac{\delta-k+4}{2} \right\rfloor$,

(iii) $\psi_k^{go}(G) \leq \left\lfloor \frac{4-k+\sqrt{k^2+2(\delta-k)}}{2} \right\rfloor$.

Proof. Let $\Pi_r^{go}(G) = \{S_1, S_2, ..., S_r\}$ be a partition of G into global offensive k-alliances. Since S_i is a dominating set for every $i \in \{1, ..., r\}$, we have that for every $v \in \overline{S_i}$, $\delta_{\overline{S_i}}(v) \geq r-2$. Thus, the bounds are obtained as follow.

(i) $\delta(v)-(r-2) \geq \delta_{S_i}(v) \geq \delta_{\overline{S_i}}(v)+k \geq r-2+k$, so $2m = \sum_{v\in V}\delta(v) \geq n(2r-4+k)$. Hence, the bound follows.

(ii) If v is a vertex of minimum degree δ, there exists $S_i \in \Pi_r^{go}(G)$ such that $v \notin S_i$, thus, $\delta = \delta(v) \geq 2\delta_{\overline{S_i}}(v)+k \geq 2(r-2)+k$.

(iii) As above, if v is a vertex of minimum degree δ, there exists $S_i \in \Pi_r^{go}(G)$ such that $v \in S_i$, thus, for every $j \neq i$, $\delta = \delta(v) \geq 2\delta_{\overline{S_j}}(v)+k \geq 2\sum_{l\neq i,j}\delta_{S_l}(v)+k$. Also, as each offensive k-alliance belonging to $\Pi_r^{go}(G)$ is a dominating set, $\delta_{S_l}(v) \geq \delta_{\overline{S_l}}(v)+k \geq r-2+k$. So, we obtain $\delta \geq 2(r-2)(r-2+k)+k = 2r^2+2(k-4)r-3k+8$ and, as a consequence, $2r^2+2(k-4)r-3k+8-\delta \leq 0$. Therefore $r \leq \frac{4-k+\sqrt{k^2+2(\delta-k)}}{2}$. □

In order to compare (ii) and (iii) for $\delta \geq 1$, we note that

$$\frac{\delta-k+4}{2} < \frac{4-k+\sqrt{k^2+2(\delta-k)}}{2}$$

if and only if, $k < 2-\delta$. Examples of equality in above theorem are the following ones. Bound (i) is attained for the cycle graph C_{3t}, where $\psi_0^{go}(C_{3t}) = 3$ and (ii) is attained in the case of the circulant graph $G = CR(5n,2)$ and $k=-2$, where $\psi_{-2}^{go}(G) = 5$. For the case of the cube graph Q_3 bound (ii) is attained for $k = 2,3$ where $\psi_2^{go}(Q_3) = \psi_3^{go}(Q_3) = 2$ and bound (iii) is attained for $k \in \{-2,-1\}$, where $\psi_{-2}^{go}(Q_3) = \psi_{-1}^{go}(Q_3) = 4$.

1.4.5 On the cardinality of sets belonging to a partition

In this subsection we obtain bounds for the cardinality of the sets belonging to a partition of a graph into global offensive k-alliances.

Theorem 1.34. *If S belongs to a partition of G into global offensive k-alliances, then*

$$\left\lceil \frac{n(2\delta-\Delta+k)}{\Delta+2\delta+k} \right\rceil \leq |S| \leq \left\lfloor \frac{2n\Delta}{\Delta+2\delta+k} \right\rfloor.$$

Proof. If X is a t-dependent set in G, then for every $v \in X$ we have $\delta_X(v) \leq t$. So, $\delta(v) - \delta_{\overline{X}}(v) \leq t$. Hence,

$$\Delta(n - |X|) \geq \sum_{v \in \overline{X}} \delta_X(v) = \sum_{v \in X} \delta_{\overline{X}}(v) \geq \sum_{v \in X} (\delta - t) = |X|(\delta - t),$$

which leads to,

$$|X| \leq \frac{n\Delta}{\Delta + \delta - t}. \tag{1.3}$$

Now, since the union of offensive k-alliances is an offensive k-alliance too, if S belongs to a partition of V into global offensive k-alliances, then $\{S, \overline{S}\}$ is a partition of V into two global offensive k-alliances and, by Theorem 1.2 (ii) it is also a partition of V into two $\left\lfloor \frac{\Delta-k}{2} \right\rfloor$-dependent sets. Therefore, by taking $t = \left\lfloor \frac{\Delta-k}{2} \right\rfloor \leq \frac{\Delta-k}{2}$ in (1.3) we obtain the upper bound on $|S|$. The lower bound on $|S|$ is deduced from the upper bound on $|\overline{S}| = n - |S|$. □

The circulant graph $CR(n, 2)$ contains a partition into two global offensive 0-alliances S and $\overline{S}$, such that $|S| = \lceil \frac{n}{3} \rceil$ and $|\overline{S}| = \lfloor \frac{2n}{3} \rfloor$, where the bounds of the above theorem are attained.

The Laplacian spectral radius contains important information about the graph. This eigenvalue is related to several important graph invariants and it imposes reasonably good bounds on the values of several parameters of graphs which are very hard to compute.

The Laplacian spectral radius, μ_*, satisfy the following equality shown by Fiedler [33]:

$$\mu_* = 2n \max \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} \right\}, \tag{1.4}$$

where not all the components of the vector $(w_1, w_2, ..., w_n) \in \mathbb{R}^n$ are equal.

The following theorem shows the relationship between the Laplacian spectral radius of a graph and the cardinality of sets belonging to a partition into global offensive k-alliances.

Theorem 1.35. *Let $G = (V, E)$ be a graph with Laplacian spectral radius μ_*. If S belongs to a partition of G into global offensive k-alliances, $-\delta \leq k \leq \mu_* - \delta$, then*

$$\left\lceil \frac{n}{2} - \sqrt{\frac{n^2(\mu_* - k) - 2nm}{4\mu_*}} \right\rceil \leq |S| \leq \left\lfloor \frac{n}{2} + \sqrt{\frac{n^2(\mu_* - k) - 2nm}{4\mu_*}} \right\rfloor .$$

Proof. If S belongs to a partition of G into global offensive k-alliances, we know that S and $\overline{S}$ are global offensive k-alliances in G, then

$$\sum_{v \in \overline{S}} \delta(v) \leq 2 \sum_{v \in \overline{S}} \delta_S(v) - k|\overline{S}|$$

and

$$\sum_{v \in S} \delta(v) \leq 2 \sum_{v \in S} \delta_{\overline{S}}(v) - k|S|.$$

Therefore

$$2m \leq 4 \sum_{v \in S} \delta_{\overline{S}}(v) - kn. \tag{1.5}$$

On the other hand, by equation (1.4), taking $w \in \mathbb{R}^n$ defined as

$$w_i = \begin{cases} 1 & \text{if} \quad v_i \in S; \\ 0 & \text{otherwise,} \end{cases} \tag{1.6}$$

we have

$$\mu_* \geq \frac{n \sum_{v \in S} \delta_{\overline{S}}(v)}{|S||\overline{S}|}. \tag{1.7}$$

Therefore, by using the expression (1.5) in (1.7) we obtain

$$\frac{2m + nk}{4} \leq \frac{|S|(n - |S|)\mu_*}{n}.$$

By solving the above inequality for $|S|$ and by considering that it is an integer we obtain the bounds on $|S|$. □

The above bounds are attained for the complete graph K_n for n even and $k = 1$. In this case K_n is partitioned into two global offensive 1-alliances of cardinality $\frac{n}{2}$.

1.4.6 On the edge cut-set

Theorem 1.36. *Let G be a graph of order n and size m. If $C^{go}_{(r,k)}(G)$ is the minimum number of edges having its endpoints in different sets of a partition of G into $r \geq 2$ global offensive k-alliances, then*

(i) $C^{go}_{(r,k)}(G) \geq \left\lceil \frac{(r-1)(2m+nk)}{4} \right\rceil$,

(ii) *if $r \geq 3$, then* $C^{go}_{(r,k)}(G) \leq \left\lfloor \frac{(r-1)(2m-nk)}{4(r-2)} \right\rfloor$,

(iii) *if $r > 3$, then* $C^{go}_{(r,k)}(G) \leq \left\lfloor -\frac{nk(r-1)}{2r-6} \right\rfloor$.

Proof. Let $\Pi^{go}_r = \{S_1, S_2, ..., S_r\}$ be a partition of G into r global offensive k-alliances. The number of edges in G with one endpoint in S_i and the other endpoint in S_j is $C(S_i, S_j) = \sum_{v \in S_i} \delta_{S_j}(v) = \sum_{v \in S_j} \delta_{S_i}(v)$. Hence, taking into account that for every $v \in \overline{S_i}$, $\delta(v) \leq 2\delta_{S_i}(v) - k$, we have that

$$\begin{aligned} 2(r-1)m &= \sum_{i=1}^{r} \sum_{v \in \overline{S_i}} \delta(v) \leq 2\sum_{i=1}^{r} \sum_{v \in \overline{S_i}} \delta_{S_i}(v) - k\sum_{i=1}^{r}(n - |S_i|) \\ &= 2\sum_{i=1}^{r} \sum_{j=1; j\neq i}^{r} \sum_{v \in S_j} \delta_{S_i}(v) - nk(r-1) \\ &= 2\sum_{i=1}^{r} \sum_{j=1; j\neq i}^{r} C(S_i, S_j) - nk(r-1) \\ &= 4C^{go}_{(r,k)}(G) - nk(r-1). \end{aligned}$$

So, (i) follows. On the other hand if $r \geq 3$, then for every $v \in \overline{S_i}$, $\delta(v) \geq$

$2\delta_{\overline{S_i}}(v)+k$, we have

$$\begin{aligned}
2(r-1)m &= \sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta(v) \ge 2\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{\overline{S_i}}(v)+k\sum_{i=1}^{r}(n-|S_i|)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\neq i}^{r}\sum_{v\in S_j}\delta_{\overline{S_i}}(v)+nk(r-1)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\neq i}^{r}\sum_{v\in S_j}\sum_{l=1;l\neq i}^{r}\delta_{S_l}(v)+nk(r-1)\\
&\ge 2\sum_{i=1}^{r}\sum_{j=1;j\neq i}^{r}\sum_{l=1;l\neq i,j}^{r}\sum_{v\in S_j}\delta_{S_l}(v)+nk(r-1)\\
&= 2\sum_{i=1}^{r}\sum_{j=1;j\neq i}^{r}\sum_{l=1;l\neq i,j}^{r}C(S_l,S_j)+nk(r-1)\\
&= 4(r-2)C^{go}_{(r,k)}(G)+nk(r-1).
\end{aligned}$$

Therefore, (ii) follows. Finally, as each S_i is a global offensive k-alliance, we have

$$\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{S_i}(v) \ge \sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{\overline{S_i}}(v)+kn(r-1).$$

Hence, from the proof of (i) we have $2C^{go}_{(r,k)}(G)=\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{S_i}(v)$ and, from the proof of (ii), we have $\sum_{i=1}^{r}\sum_{v\in\overline{S_i}}\delta_{\overline{S_i}}(v)\ge 2(r-2)C^{go}_{(r,k)}(G)$. Therefore, we obtain $2C^{go}_{(r,k)}(G)\ge 2(r-2)C^{go}_{(r,k)}(G)+nk(r-1)$ and (iii) follows. □

From the above result we have that if $\psi^{go}_k(G)\ge 3$ then $\psi^{go}_k(G)\le \left\lfloor\frac{6m+nk}{2m+nk}\right\rfloor$. Also, notice that, for $k\le\delta$, $2\le\left\lfloor\frac{6m+nk}{2m+nk}\right\rfloor$, so we obtain the following bound on $\psi^{go}_k(G)$.

Corollary 1.37. *For any graph G of order n and size m,*

$$\psi^{go}_k(G)\le\left\lfloor\frac{6m+nk}{2m+nk}\right\rfloor.$$

The above bound is attained, for instance, for the circulant graph $CR(5n, 2)$, where $\psi^{go}_{-2}(G) = 5$.

1.4.7 Partitioning $G_1 \times G_2$ into offensive k-alliances

From Theorem 1.12, if $G_i = (V_i, E_i)$ is a graph of minimum degree δ_i and maximum degree Δ_i, $i \in \{1, 2\}$ and S_i is an offensive k_i-alliance in G_i, $i \in \{1, 2\}$, then, for $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$, $S_1 \times S_2$ is an offensive k-alliance in $G_1 \times G_2$. Thus, we deduce that, a partition

$$\Pi^o_{r_i}(G_i) = \{S^{(i)}_1, S^{(i)}_2, ..., S^{(i)}_{r_i}\}$$

of G_i into r_i offensive k_i-alliances, $i \in \{1, 2\}$, induces a partition of $G_1 \times G_2$ into $r_1 r_2$ offensive k-alliances, with $k = \min\{k_2 - \Delta_1, k_1 - \Delta_2\}$:

$$\Pi^o_{r_1 r_2}(G_1 \times G_2) = \left\{ \begin{array}{ccc} S^{(1)}_1 \times S^{(2)}_1 & \cdots & S^{(1)}_1 \times S^{(2)}_{r_2} \\ S^{(1)}_2 \times S^{(2)}_1 & \cdots & S^{(1)}_2 \times S^{(2)}_{r_2} \\ \vdots & \vdots & \vdots \\ S^{(1)}_{r_1} \times S^{(2)}_1 & \cdots & S^{(1)}_{r_1} \times S^{(2)}_{r_2} \end{array} \right\}.$$

So, we obtain the following result.

Corollary 1.38. *For any graph G_i of maximum degree Δ_i, $i \in \{1, 2\}$, and for every $k \le \min\{k_1 - \Delta_2, k_2 - \Delta_1\}$, $\psi^o_k(G_1 \times G_2) \ge \psi^o_{k_1}(G_1)\psi^o_{k_2}(G_2)$.*

For the particular case of the graph $C_4 \times K_4$, we have $\psi^o_{-3}(C_4 \times K_4) = 8 = 4 \cdot 2 = \psi^o_0(C_4)\psi^o_1(K_4)$.

Theorem 1.39. *Let $G_i = (V_i, E_i)$ be a graph of order n_i and let $\Pi^{go}_{r_i}(G_i)$ be a partition of G_i into r_i global offensive k_i-alliances, $i \in \{1, 2\}$. If $x_i = \min_{S \in \Pi^{go}_{r_i}(G_i)} \{|S|\}$ and $k \le \min\{k_1, k_2\}$, then*

(i) $\gamma^o_k(G_1 \times G_2) \le \min\{n_2 x_1, n_1 x_2\}$,

(ii) $\psi_k^{go}(G_1 \times G_2) \geq \max\{\psi_{k_1}^{go}(G_1), \psi_{k_2}^{go}(G_2)\}$.

Proof. If we consider the set $M_j = S_j^{(1)} \times V_2$ where $S_j^{(1)} \in \Pi_{r_1}^{go}(G_1)$, then for every $(u, v) \notin M_j$ it is satisfied that

$$\delta_{S_j^{(1)} \times V_2}(u, v) = \delta_{S_j^{(1)}}(u) \geq \delta_{\overline{S_j^{(1)}}}(u) + k_1 = \delta_{\overline{S_j^{(1)} \times V_2}}(u, v) + k_1.$$

Thus, M_j is a global offensive k_1-alliance in $G_1 \times G_2$. The same argument shows that $N_l = V_1 \times S_l^{(2)}$ is a global offensive k_2-alliance for every $S_l^{(2)} \in \Pi_{r_2}^{go}(G_2)$. Thus, by taking $S_j^{(1)}$ and $S_l^{(2)}$ of cardinality x_1 and x_2, respectively, we obtain $|M_j| = x_1 n_2$ and $|N_l| = x_2 n_1$, so (i) follows. Moreover, as $\{M_1, ..., M_{r_1}\}$ and $\{N_1, ..., N_{r_2}\}$ are partitions of $G_1 \times G_2$ into global offensive k-alliances, (ii) follows. □

Suppose G_j is partitionable into global offensive k_j-alliances, for $k_j \geq 1$ and $j \in \{1, 2\}$. Bound (ii) is attained for $1 \leq k \leq \min\{k_1, k_2\}$, where $\psi_k^{go}(G_1 \times G_2) = 2 = \max\{2, 2\} = \max\{\psi_{k_1}^{go}(G_1), \psi_{k_2}^{go}(G_2)\}$. From (ii) we deduce the following result.

Corollary 1.40. *If a graph G_i of order n_i is partitionable into global offensive k_i-alliances, $i \in \{1, 2\}$, then for $k \leq \min\{k_1, k_2\}$,*

$$\gamma_k^o(G_1 \times G_2) \leq \frac{n_1 n_2}{\max\{\psi_{k_1}^{go}(G_1), \psi_{k_2}^{go}(G_2)\}}.$$

Example of equality is $\gamma_1^o(C_4 \times K_2) = \dfrac{4 \cdot 2}{\max\{\psi_1^{go}(C_4), \psi_1^{go}(K_2)\}} = 4.$

Chapter 2

Defensive Alliances

Abstract

We introduce the concept of boundary defensive k-alliance and we investigate some of its mathematical properties. Also, we discuss the relationships that exist between the defensive k-alliances in Cartesian product graphs and the defensive k-alliances in its factors. We study the problem of estimating the maximum number of sets belonging to a partition of the vertex set of a graph into defensive k-alliances. Moreover, we obtain some relationships between this maximum number of sets and other invariants of a graph like isoperimetric number, bisection and bipartition width.

2.1 Introduction

A nonempty set $S \subseteq V$ is a *defensive k-alliance* in $G = (V, E)$, $k \in \{-\Delta, \dots, \Delta\}$, if for every $v \in S$,

$$\delta_S(v) \geq \delta_{\overline{S}}(v) + k. \tag{2.1}$$

A defensive k-alliance S is called *global* if it forms a dominating set. Figure 2.1 shows examples of (global) defensive k-alliances. Notice that equation (2.1) is equivalent to

$$\delta(v) \geq 2\delta_{\overline{S}}(v) + k. \tag{2.2}$$

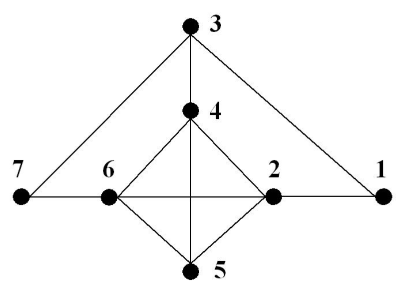

Figure 2.1: $\{2, 5, 6\}$ is a defensive 0-alliance and $\{3, 4, 5\}$ is a global defensive (-1)-alliance.

If $k > 1$, the star graph $K_{1,t}$ has no defensive k-alliances and every set composed by two adjacent vertices in a cubic graph is a defensive (-1)-alliance. For graphs having defensive k-alliances, the *defensive k-alliance number* of G, denoted by $a_k^d(G)$, is defined as the minimum cardinality of a defensive k-alliance in G. For graphs having global defensive k-alliances, the *global defensive k-alliance number* of G, denoted by $\gamma_k^d(G)$, is the minimum cardinality of a global defensive k-alliance in G.
Notice that $a_{k+1}^d(G) \geq a_k^d(G)$, $\gamma_{k+1}^d(G) \geq \gamma_k^d(G) \geq \gamma(G)$ and $\gamma_k^d(G) \geq a_k^d(G)$.

Defensive alliances have been studied in different ways. The first results about defensive alliances were presented in [36, 52] and after that some results have been appearing in the literature, like those in [1, 5, 10, 12, 14, 24, 25, 27, 37, 38, 42, 43, 59, 61, 62, 63, 69, 70, 71]. The complexity of computing minimum cardinality of defensive k-alliances in graphs was studied in [11, 30, 45, 46, 48, 70], where it was proved that this is an NP-complete problem. An spectral study of alliances in graphs was presented in [59, 63], where the authors obtained some bounds for the defensive alliance number in terms of the algebraic connectivity, the Laplacian spectral radius and the spectral radius[1] of the graph. In [5, 37] and [61] were studied the global defensive alliances in trees and planar graphs, respectively. In [1] were studied the defensive alliances in regular graphs and circulant graphs. Moreover, the alliances in complement graphs, line graphs and weighted graphs were studied in [70], [63, 71] and [47], respectively. In [14, 27] were obtained some relations between the independence number and the defensive alliances number of a graph. Also, in [24, 25, 42] were investigated the partitions of a graph into defensive (-1)-alliances. Here we present some of the principal known results about defensive alliances.

The first results about alliances appeared in [36, 52]. For instance, there were obtained the following bounds

$$a_{-1}^d(G) \le \min\left\{ n - \left\lceil \frac{\delta}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil \right\},$$

and also

$$a_0^d(G) \le n - \left\lfloor \frac{\delta}{2} \right\rfloor.$$

After that, in [63] were presented generalizations of the above results for the case of defensive k-alliances.

[1]The second smallest eigenvalue of the Laplacian matrix of a graph G is called the algebraic connectivity of G. The largest eigenvalue of the adjacency matrix of G is the spectral radius of G.

Theorem 20. [63] *For every* $k \in \{-\delta, \ldots, \Delta\}$,

$$\left\lceil \frac{\delta + k + 2}{2} \right\rceil \leq a_k^d(G) \leq n - \left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

Theorem 21. [63] *For every* $k \in \{-\delta, \ldots, 0\}$, $a_k^d(G) \leq \left\lceil \frac{n+k+1}{2} \right\rceil$.

Moreover, the global defensive k-alliances in graphs have been studied in [62] where the authors presented the following interesting results.

Theorem 22. [62] *Let S be a global defensive k-alliance of minimum cardinality in a graph G. If $W \subset S$ is a dominating set in G, then for every $r \in \mathbb{Z}$ such that $0 \leq r \leq \gamma_k^d(G) - |W|$,*

$$\gamma_{k-2r}^d(G) + r \leq \gamma_k^d(G).$$

Theorem 23. [62] *For any graph G of order n and maximum degree Δ and for every $k \in \{-\Delta, ..., \Delta\}$,*

$$\gamma_k^d(G) \geq \left\lceil \frac{n}{\left\lfloor \frac{\Delta - k}{2} \right\rfloor + 1} \right\rceil.$$

It is well-known that the algebraic connectivity of a graph is probably the most important information contained in the Laplacian spectrum. This eigenvalue is related to several important graph invariants and it imposes reasonably good bounds on the values of several parameters of graphs which are very hard to compute. Now we present a result about defensive alliances, obtained in [63].

Theorem 24. [63] *For any connected graph G and for every $k \in \{-\delta, \ldots, \Delta\}$,*

$$a_k^d(G) \geq \left\lceil \frac{n(\mu + k + 1)}{n + \mu} \right\rceil.$$

The cases $k = -1$ and $k = 0$ in the above theorem were studied previously in [59]. Other relations between defensive alliances and the eigenvalues of a graph appeared in [69], in this case related to the spectral radius.

Theorem 25. [69] *For every graph G of order n and spectral radius λ,*

$$\gamma_k^d(G) \geq \left\lceil \frac{n}{\lambda - k + 1} \right\rceil.$$

The particular cases of the above theorem $k = -1$ and $k = 0$ were studied previously in [59]. Now, as a special cases of graphs in which have been investigated their defensive alliances we find the complement graph and the line graph. In [70] and [63, 71], respectively, were proved the following results about defensive alliances in complement graphs and line graphs.

Theorem 26. [70] *If G is a graph of order n with maximum degree Δ, then*

$$\left\lceil \frac{n - \Delta + k + 1}{2} \right\rceil \leq a_k^d(\overline{G}) \leq \left\lceil \frac{n + \Delta + k + 1}{2} \right\rceil.$$

Theorem 27. [70] *Let G be a graph of order n such that $\gamma(G) > 3$ and $k \in \{-\delta, ..., 0\}$. If the minimum defensive k-alliance in G is not global, then*

$$a_k^d(\overline{G}) \leq \begin{cases} \left\lfloor \frac{3n + k + 5}{4} - \frac{\gamma(G) + \gamma(\overline{G})}{2} \right\rfloor, & \text{if } n + k \text{ is odd} \\ \left\lfloor \frac{3n + k + 6}{4} - \frac{\gamma(G) + \gamma(\overline{G})}{2} \right\rfloor, & \text{if } n + k \text{ is even.} \end{cases}$$

Theorem 28. [63] *For any simple graph G of maximum degree Δ, and for every $k \in \{2(1 - \Delta), ..., 0\}$,*

$$a_k^d(\mathcal{L}(G)) \leq \Delta + \left\lceil \frac{k}{2} \right\rceil.$$

Theorem 29. [63] *Let $G = (V, E)$ be a simple graph of maximum degree Δ. Let $v \in V$ such that $\delta(v) = \Delta$, let $\delta_v = \max\{\delta(u) : u \sim v\}$ and let $\delta_* = \min\{\delta_v : \delta(v) = \Delta\}$. For every $k \in \{2 - \delta_* - \Delta, ..., \Delta - \delta_*\}$,*

$$a_k^d(\mathcal{L}(G)) \leq \left\lceil \frac{\Delta + \delta_* + k}{2} \right\rceil.$$

Moreover, if $\delta_1 \geq \delta_2 \geq ... \geq \delta_n$ is the degree sequence of G, then for every $k \in \{2 - \delta_1 - \delta_2, ..., \delta_1 + \delta_2 - 2\}$,

$$a_k^d(\mathcal{L}(G)) \geq \left\lceil \frac{\delta_n + \delta_{n-1} + k}{2} \right\rceil.$$

As a consequence of the above results, in [63] was obtained the following interesting result.

Corollary 30. [63] *For any δ-regular graph G, $\delta > 0$, and for every $k \in \{2(1-\delta), ..., 0\}$,*

$$a_k^d(\mathcal{L}(G)) = \delta + \left\lceil \frac{k}{2} \right\rceil.$$

The cases $k = -1$ and $k = 0$ in the above results were studied previously in [71]. On the other hand, several researches about defensive alliances have been centered into specific classes of graphs. As an example, in [1] were studied the defensive alliances in regular graphs and circulant graphs. In order to present some results from [1] it is necessary to introduce some notation. The (k, δ)-induced alliances set is the set of graphs H of order t, minimum degree $\delta_H \geq \left\lfloor \frac{\delta}{2} \right\rfloor$, and maximum degree $\Delta_H \leq \delta$, with no proper subgraph of minimum degree greater than $\left\lfloor \frac{\delta}{2} \right\rfloor$. This set is denoted by $\mathcal{S}_{(t,\delta)}$.

Theorem 31. [1] *If G is a δ-regular graph, then S is a critical alliance*[2] *of G of cardinality t if and only if $\langle S \rangle \in \mathcal{S}_{(t,\delta)}$.*

Also, in [1] were characterized the (6)-regular graphs G satisfying that $a_{-1}^d(G) \in \{4, 5, 6, 7\}$. For the case of circulant graphs[3] in [1] were obtained the following results.

Theorem 32. [1] *Let $G = CR(n, M)$ be a circulant graph with $|M|$ generators.*

(i) *If $\delta = 2|M|$, then $|M| + 1 \leq a_{-1}^d(G) \leq 2^{|M|}$.*

(ii) *If $\delta = 2|M| - 1$, then $|M| \leq a_{-1}^d(G) \leq 2^{|M|-1}$.*

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[2] A critical alliance is an alliance such that it does not contain other alliance as a proper subset.

[3] See page 23 for the definition of circulant graphs.

As a consequence, in [1] was obtained that for the case of $|M| = 3$, it is satisfied that $4 \le a^d_{-1}(G) \le 8$. Moreover, the authors of that article characterized the circulant graphs G such that $a^d_{-1}(G) \in \{4, 5, 6, 7\}$.

Other class of graphs in which have been studied its defensive alliances is the case of planar graphs. For instance, [61] was dedicated to study defensive alliances in planar graphs, where are some results like the following ones.

Theorem 33. [61] *Let G be a planar graph of order n.*

(i) *If $n > 6$, then $\gamma^d_{-1}(G) \ge \left\lceil \frac{n+12}{8} \right\rceil$.*

(ii) *If $n > 6$ and G is a triangle-free graph, then $\gamma^d_{-1}(G) \ge \left\lceil \frac{n+8}{6} \right\rceil$.*

(iii) *If $n > 4$, then $\gamma^d_0(G) \ge \left\lceil \frac{n+12}{7} \right\rceil$.*

(iv) *If $n > 4$ and G is a triangle-free graph, then $\gamma^d_0(G) \ge \left\lceil \frac{n+8}{5} \right\rceil$.*

Theorem 34. [61] *For any tree T of order n,*

$$\gamma^d_{-1}(G) \ge \left\lceil \frac{n+2}{4} \right\rceil \text{ and } \gamma^d_0(G) \ge \left\lceil \frac{n+2}{3} \right\rceil.$$

Global defensive alliances in trees have been also studied separately, for instance, [37] is an example of that. A t-ary tree is a rooted tree where each node has at most t children. A complete t-ary tree is a t-ary tree in which all the leaves have the same depth and all the nodes except the leaves have t children. We let $T_{t,d}$ be the complete t-ary tree with depth/height d. With the above notation we present the following results obtained in [37].

Theorem 35. [37] *Let n be the order of $T_{2,d}$. Then for any d,*

$$\gamma^d_{-1}(T_{2,d}) = \left\lceil \frac{2n}{5} \right\rceil.$$

Theorem 36. [37] *Let d be an integer greater than three,*

(i) *If d is odd, then $\gamma^d_{-1}(T_{3,d}) = \left\lfloor \frac{19n}{36} \right\rfloor$.*

(ii) *If d is even, then* $\gamma_{-1}^d(T_{3,d}) = \lceil \frac{19n}{36} \rceil$.

Theorem 37. [37] *For $d \geq 2$ and $t \geq 2$,*

$$t^{d-1}\left\lceil \frac{t-1}{2} \right\rceil + t^{d-1} + t^{d-2} \leq \gamma_{-1}^d(T_{t,d}) \leq t^{d-1}\left\lceil \frac{t-1}{2} \right\rceil + t^{d-1} + t^{d-2} + t^{d-3}.$$

Defensive alliances in trees have been also studied in [5] where it was obtained the following bound in terms of the number of leaves and support vertices of a tree. Also, in this paper were characterized the extremal graphs satisfying this bound.

Theorem 38. [5] *Let T be a tree of order $n \geq 2$ with l leaves and s support vertices. Then*

$$\gamma_0^d(T) \geq \frac{3n - l - s + 4}{6}.$$

In [14, 27] were investigated some relationships between the independence number (independent domination number) and the global defensive alliance number of a graph. For instance, there were obtained the following results.

Theorem 39. [14] *For any tree T, $\gamma_{-1}^d(T) \leq \beta_0(T)$, and this bound is sharp.*

Theorem 40. [14] *If T is a tree of order $n \geq 3$ with s support vertices, then*

(i) $\gamma_0^d(G) \leq \frac{3\beta_0(T)-1}{2}$,

(ii) $\gamma_0^d(G) \leq \beta_0(T) + s - 1$.

In order to present some results from [27] we introduce some notation defined in the mentioned article.

$\mathcal{F}_1$ is the family of graphs obtained from a clique S isomorphic to K_t by attaching $t = \delta_S(u) + 1$ leaves at each vertex $u \in S$.

$\mathcal{F}_2$ is the family of bipartite graphs obtained from a balanced complete bipartite graph S isomorphic to $K_{t,t}$ by attaching $t+1$ leaves at each vertex $u \in S$.

$\mathcal{F}_3$ is the family of trees obtained from a tree S by attaching a set L_u of $\delta_S(u)+1$ leaves at each vertex $u \in S$.

Theorem 41. [27]

(i) *Every graph G satisfies $i(G) \le (\gamma^d_{-1}(G))^2 - \gamma^d_{-1}(G) + 1$ with equality if and only if $G \in \mathcal{F}_1$.*

(ii) *Every bipartite graph G satisfies $i(G) \le \frac{(\gamma^d_{-1}(G))^2}{4} + \gamma^d_{-1}(G)$ with equality if and only if $G \in \mathcal{F}_2$.*

(iii) *Every tree G satisfies $i(G) \le 2\gamma^d_{-1}(G) - 1$ with equality if and only if $G \in \mathcal{F}_3$.*

Similarly to the above result, in [27] were obtained some relationships between the independent domination number and the global defensive 0-alliance number of a graph.

On the other hand, defensive alliances in Cartesian product graphs were studied in [52], where the authors obtained the following result.

Theorem 42. [52] *For any Cartesian product graph $G_1 \times G_2$,*

(i) $a^d_{-1}(G_1 \times G_2) \le \min\{a^d_{-1}(G_1)a^d_0(G_2), a^d_0(G_1)a^d_{-1}(G_2)\}$.

(ii) $a^d_0(G_1 \times G_2) \le a^d_0(G_1)a^d_0(G_2)$.

Other topic of interest into investigating defensive alliances is related to graph partitions in which each set is formed by a defensive alliance. In [24, 25] were studied the partitions of a graph into defensive (-1)-alliances. In these articles was defined the concept of (global) defensive alliance partition number, $(\psi^{gd}_{-1}(G))$ $\psi^d_{-1}(G)$, as the maximum number of sets in a partition of a graph such that every set of the partition is a (global) defensive (-1)-alliance.

Theorem 43. [25] *Let G be a connected graph of order $n \geq 3$. Then*

$$1 \leq \psi_{-1}^{d}(G) \leq \left\lfloor n + \frac{3}{2} - \frac{\sqrt{1+4n}}{2} \right\rfloor.$$

Theorem 44. [25] *Let G be a graph with minimum degree δ. Then*

$$\psi_{-1}^{d}(G) \leq \left\lfloor \frac{n}{\lceil \frac{\delta+1}{2} \rceil} \right\rfloor.$$

Moreover, in [24] and [42] were studied the partitions into (global) defensive (-1)-alliances in trees and grid graphs, respectively.

Theorem 45. [24] *Let G be a connected graph with minimum degree δ. Then*

$$\psi_{-1}^{gd}(G) \leq 1 + \left\lfloor \frac{\delta}{2} \right\rfloor.$$

As a consequence of the above result, in [24] was obtained the following interesting result.

Corollary 46. [24] *Let T be a tree of order $n \geq 3$. Then $1 \leq \psi_{-1}^{gd}(T) \leq 2$.*

Moreover, in [24] were characterized some families of trees satisfying that $\psi_{-1}^{gd}(T) = 1$ or $\psi_{-1}^{gd}(T) = 2$. From [42] is known the following results for the class of grid graphs $P_r \times P_c$.

Theorem 47. [42] *For $4 \leq r \leq c$,*

$$\psi_{-1}^{d}(P_r \times P_c) = \left\lfloor \frac{r-2}{2} \right\rceil \left\lfloor \frac{c-2}{2} \right\rceil + r + c - 2.$$

Theorem 48. [42] *For $2 \leq r \leq c$, $\psi_{-1}^{gd}(P_r \times P_c) = 2$.*

We refer to the Ph. D. Thesis [67] and [69] to have a more complete idea about the principal results related to defensive alliances.

2.2 Boundary defensive k-alliances

Defensive k-alliances are formed by vertices of a graphs that satisfy equation (2.1), i.e., each vertex belonging to a defensive k-alliance has at least k-more neighbors inside of the alliance than outside of the alliance. Figure 2.2 shows an example in which every vertex of the set $S = \{1, 2\}$ has exactly one vertex more outside of S than inside of S. In this sense, we are interested into study the limit case of equation (2.1). A set $S \subset V$ is a *boundary defensive k-alliance* in G, $k \in \{-\Delta, \ldots, \Delta\}$, if

$$\delta_S(v) = \delta_{\overline{S}}(v) + k, \quad \forall v \in S. \tag{2.3}$$

A boundary defensive k-alliance in G is called *global* if it forms a dominating set in G. Figure 2.2 shows examples of (global) boundary defensive k-alliances. Notice that equation (2.3) is equivalent to

$$\delta(v) = 2\delta_S(v) - k \quad \forall v \in S. \tag{2.4}$$

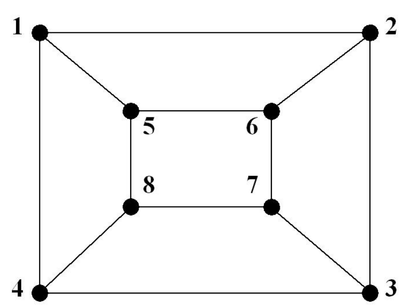

Figure 2.2: $\{1, 2\}$ is a boundary defensive (-1)-alliance and $\{5, 6, 7, 8\}$ is a global boundary defensive 1-alliance.

Note that there are graphs which does not contain any boundary defensive k-alliance for some values of k. For instance, the cube graph of Figure 2.2 has no boundary defensive 0-alliances.

Remark 2.1. *Let G be a simple graph and let $k \in \{-\Delta, \ldots, \Delta\}$. If for every $v \in V$, $\delta(v) - k$ is an odd number, then G does not contain any boundary defensive k-alliance.*

Corollary 2.2. *Let G be a δ-regular graph and let $k \in \{-\delta, \ldots, \delta\}$. If $\delta - k$ is odd, then G does not contain any boundary defensive k-alliance.*

Corollary 2.3. *If every vertex of a graph G has odd degree, then G does not contain any boundary defensive 0-alliance.*

Remark 2.4. *If S is a defensive k-alliance in G and $\bar{S}$ is a global offensive $(-k)$-alliance in G, then S is a boundary defensive k-alliance in G.*

Theorem 2.5. *Let $G = (V, E)$ be a graph and let $S \subset V$. Let $m(\langle S \rangle)$ be the size of $\langle S \rangle$ and let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a boundary defensive k-alliance in G, then*

(i) $m(\langle S \rangle) = \dfrac{c + |S|k}{2}$.

(ii) *If G is a δ-regular graph, then* $m(\langle S \rangle) = \dfrac{|S|(\delta + k)}{4}$ *and* $c = \dfrac{|S|(\delta - k)}{2}$.

Proof. If S is a boundary defensive k-alliance in G, then

$$2m(\langle S \rangle) = \sum_{v \in S} \delta_S(v) = \sum_{v \in S} \delta_{\overline{S}}(v) + |S|k = c + |S|k.$$

Thus, (i) follows. Moreover,

$$\delta(v) = 2\delta_{\overline{S}}(v) + k, \quad \forall v \in S.$$

Hence,

$$\sum_{v \in S} \delta(v) = 2 \sum_{v \in S} \delta_{\overline{S}}(v) + |S|k = 2c + |S|k.$$

Therefore, if G is δ-regular, $\delta|S| = 2c + |S|k$. Thus, (ii) follows. □

Notice that if S is a boundary defensive k-alliance in a graph G, then $a_k^d(G) \leq |S|$. So, lower bounds for defensive k-alliance number are also lower bounds for the cardinality of any boundary defensive k-alliance. Moreover, upper bounds for the cardinality of any boundary defensive k-alliance are upper bounds for the defensive k-alliance number. For instance, the lower bound shown in Theorem 20 leads to a lower bound for the cardinality of any boundary defensive k-alliance. In the next result we obtain an upper bound for the cardinality of any boundary defensive k-alliance, which is the same obtained in Theorem 20 for the defensive k-alliance number. By completeness we add also the lower bound from Theorem 20 and its proof.

Remark 2.6. *If S is a boundary defensive k-alliance in a graph G, then*

$$\left\lceil \frac{\delta + k + 2}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta + k}{2} \right\rfloor .$$

Proof. Since $S \subseteq V$ is a boundary defensive k-alliance in G, by equation (2.4) we have

$$\frac{\delta(v) + k}{2} = \delta_S(v) \leq |S| - 1, \quad \forall v \in S.$$

$$\frac{\delta + k + 2}{2} \leq |S|.$$

Hence, the lower bound follows. On the other hand, if S is a boundary defensive k-alliance in G, then

$$\frac{\delta - k}{2} \leq \frac{\delta(v) - k}{2} = \delta_{\overline{S}}(v) \leq n - |S|, \quad \forall v \in S.$$

Thus, the upper bound follows. □

As the following corollary shows, the above bounds are tight.

Corollary 2.7. *The cardinality of every boundary defensive k-alliance S in the complete graph of order n is $|S| = \frac{n+k+1}{2}$.*

As a consequence of the above corollary we conclude that the complete graph $G = K_n$ has boundary defensive k-alliances if and only if $n + k + 1$ is even. The next equality about the algebraic connectivity of G, μ, shown by Fiedler in [33] is useful to obtain the following results:

$$\mu = 2n \min \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} \right\}, \tag{2.5}$$

where not all the components of the vector $(w_1, w_2, ..., w_n) \in \mathbb{R}^n$ are equal.

The following theorems show the relationship between the algebraic connectivity (and the Laplacian spectral radius) of a graph and the cardinality of its boundary defensive k-alliances.

Theorem 2.8. *Let G be a connected graph. If S is a boundary defensive k-alliance in G, then*

$$\left\lceil \frac{n(\mu - \left\lfloor \frac{\Delta - k}{2} \right\rfloor)}{\mu} \right\rceil \leq |S| \leq \left\lfloor \frac{n(\mu_* - \left\lceil \frac{\delta - k}{2} \right\rceil)}{\mu_*} \right\rfloor.$$

Proof. Since S is a boundary defensive k-alliance in G,

$$\delta_{\overline{S}}(v) = \frac{\delta(v) - k}{2} \geq \left\lceil \frac{\delta - k}{2} \right\rceil, \quad \forall v \in S. \tag{2.6}$$

Now, from the proof of Theorem 1.35, equation (1.7) we have

$$\mu_* \geq \frac{n \sum_{v \in S} \delta_{\overline{S}}(v)}{|S|(n - |S|)}. \tag{2.7}$$

Then, by using (2.6) the above relation leads to

$$\mu_* \geq \frac{n \left\lceil \frac{\delta - k}{2} \right\rceil}{n - |S|}. \tag{2.8}$$

Therefore, by solving (2.8) for $|S|$ and by considering that it is an integer, we obtain the upper bound.

On the other hand,

$$\delta_{\overline{S}}(v) = \frac{\delta(v) - k}{2} \leq \left\lfloor \frac{\Delta - k}{2} \right\rfloor, \quad \forall v \in S. \tag{2.9}$$

Then, the lower bound is obtained as above by using (2.9) and (2.5) instead of (2.6) and (1.4), respectively. □

If $G = K_n$, then $\mu = \mu_* = n$ and $\Delta = \delta = n - 1$. Therefore, the above theorem leads to the same result as Corollary 2.7.

The following result, given by Fiedler in [34], gives another relationship between the algebraic connectivity μ and the minimum and maximum degrees of the graph, which we will use to obtain bounds on the cardinality of boundary defensive k-alliances.

Lemma 2.9. [34] *If G is a graph of order n, then $\mu \leq \frac{n}{n-1}\delta$.*

Theorem 2.10. *Let G be a connected graph. If S is a boundary defensive k-alliance in G, then*

$$\left\lceil \frac{n(\mu + k + 2) - \mu}{2n} \right\rceil \leq |S| \leq n - \left\lceil \frac{n(\mu - k) - \mu}{2n} \right\rceil.$$

Proof. Since S is a boundary defensive k-alliance in G,

$$\delta \leq \delta(v) = 2\delta_S(v) - k \leq 2(|S| - 1) - k, \quad \forall v \in S, \tag{2.10}$$

and

$$\delta \leq \delta(v) = 2\delta_{\overline{S}}(v) + k \leq 2(n - |S|) + k, \quad \forall v \in S. \tag{2.11}$$

By Lemma 2.9, we have

$$\mu \leq \frac{n}{n-1}\delta. \tag{2.12}$$

Therefore, by using (2.10) and (2.11) in (2.12) we obtain both bounds. □

Notice that in the case of the complete graph $G = K_n$, the above theorem leads to Corollary 2.7.

2.2.1 Boundary defensive k-alliances and planar subgraphs

The Euler formula states that for a connected planar graph of order n, size m and f faces, $n - m + f = 2$.

As a direct consequence of Theorem 2.5 and the Euler formula we obtain the following result.

Corollary 2.11. *Let $G = (V, E)$ be a graph and let $S \subset V$. Let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a boundary defensive k-alliance in G such that $\langle S \rangle$ is planar connected with f faces, then*

(i) $|S| = \dfrac{c + 4 - 2f}{2 - k}$, *for* $k \neq 2$.

(ii) *If G is a δ-regular graph, then* $|S| = \dfrac{4f - 8}{\delta + k - 4}$ *and* $c = \dfrac{2(\delta - k)(f - 2)}{\delta + k - 4}$, *for* $k \in \{5 - \delta, ..., \delta\}$.

Theorem 2.12. *Let G be a graph and let S be a boundary defensive k-alliance in G such that $\langle S \rangle$ is planar connected with f faces; then*

$$|S| \leq \left\lfloor \frac{\sqrt{16 - 8f + (n + k - 2)^2} + n + k - 2}{2} \right\rfloor.$$

Proof. If S denotes a boundary defensive k-alliance in G, then

$$\sum_{v \in S} \delta_S(v) = \sum_{v \in S} \delta_{\overline{S}}(v) + k|S| \leq |S|(n - |S|) + k|S|.$$

By the Euler formula on $\langle S \rangle$ we have $\sum_{v \in S} \delta_S(v) = 2(|S| + f - 2)$, so the result follows. □

The above bound is tight. For instance, the bound is attained for the complete graph $G = K_5$ where any set of cardinality four forms a boundary

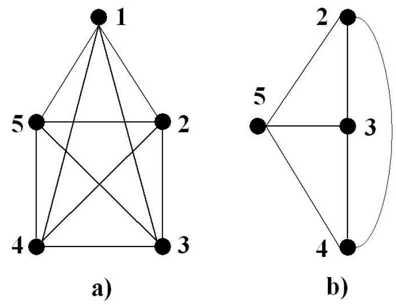

Figure 2.3: (a) The complete graph $G = (V, E) \cong K_5$ is an example of a 4-regular graph where the set $S = \{2, 3, 4, 5\} \subset V$ is a boundary defensive 2-alliance. (b) $\langle S \rangle \cong K_4$ is planar with four faces. In this case $|S| = \frac{4f-8}{\delta+k-4}$.

defensive 2-alliance and $\langle S \rangle \cong K_4$ is planar with $f = 4$ faces (See Figure 2.3).

Theorem 2.13. *Let G be a graph and let S be a boundary defensive k-alliance in G such that $\langle S \rangle$ is planar connected with $f > 2$ faces.*

(i) *If $k \in \{5 - \Delta, \dots, \Delta\}$, then* $|S| \geq \left\lceil \dfrac{4f-8}{\Delta+k-4} \right\rceil$,

(ii) *If $k \in \{5 - \delta, \dots, \Delta\}$, then* $|S| \leq \left\lfloor \dfrac{4f-8}{\delta+k-4} \right\rfloor$.

Proof. Since S is a boundary defensive k-alliance in G,

$$\sum_{v\in S} \delta_S(v) = \sum_{v\in S} \delta_{\overline{S}}(v) + k|S|.$$

Hence,

$$|S|\frac{\delta-k}{2} + k|S| \leq \sum_{v\in S} \delta_S(v) \leq |S|\frac{\Delta-k}{2} + k|S|. \qquad (2.13)$$

Therefore, by the Euler formula on $\langle S \rangle$ and the above inequalities, the bounds on $|S|$ follow. □

By Corollary 2.11 the above bounds are tight.

2.3 Defensive k-alliances in Cartesian product graphs

Let $S \subset V_1 \times V_2$ be a set of vertices of $G_1 \times G_2$. Let $P_{G_i}(S)$ the projection of the set S over G_i. Then for every $u \in P_{G_1}(S)$, we define $X_u = \{(x, v) \in S \; : \; x = u\}$ and $Y_v = \{(u, y) \in S \; : \; y = v\}$.

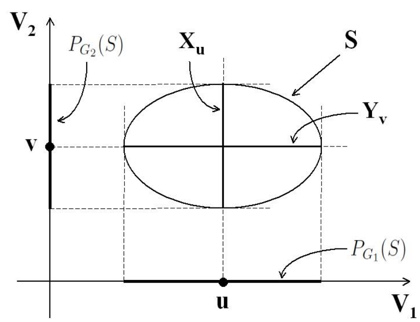

Figure 2.4: X_u, Y_v and the projections of S over G_1 and G_2.

Theorem 2.14. *If $S \subset V_1 \times V_2$ is a defensive k-alliance in $G_1 \times G_2$, then for every $u \in P_{G_1}(S)$ and for every $v \in P_{G_2}(S)$, $P_{G_2}(X_u)$ and $P_{G_1}(Y_v)$ are a defensive $(k - \Delta_1)$-alliance in G_2 and a defensive $(k - \Delta_2)$-alliance in G_1, respectively.*

Proof. Let $S \subset V_1 \times V_2$. Now, for $u \in P_{G_1}(S)$ and $v \in P_{G_2}(S)$ we have

$$\begin{aligned}
\delta_{P_{G_2}(X_u)}(u) + \Delta_1 &\geq \delta_{P_{G_2}(X_u)}(u) + \delta_{P_{G_1}(Y_v)}(v) \\
&= \delta_{X_u}(u, v) + \delta_{Y_v}(u, v) \\
&= \delta_S(u, v) \\
&\geq \delta_{\overline{S}}(u, v) + k \\
&= \delta_{\overline{X_a}}(u, v) + \delta_{\overline{Y_v}}(u, v) + k \\
&= \delta_{\overline{P_{G_2}(X_u)}}(u) + \delta_{\overline{P_{G_2}(Y_v)}}(v) + k \\
&\geq \delta_{\overline{P_{G_2}(X_u)}}(u) + k.
\end{aligned}$$

So, $P_{G_2}(X_u)$ is a defensive $(k-\Delta_1)$-alliance in G_2. To prove that $P_{G_1}(Y_v)$ is a defensive $(k-\Delta_2)$-alliance in G_1 we develop an analogous procedure. □

Notice that

$$P_{G_2}(S) = \bigcup_{u \in P_{G_1}(S)} P_{G_2}(X_u) \text{ and } P_{G_1}(S) = \bigcup_{v \in P_{G_2}(S)} P_{G_1}(Y_v).$$

Also, as the union of defensive k-alliances in a graph is a defensive k-alliance in the graph, we obtain the following consequence of the above result.

Corollary 2.15. *Let $G_i = (V_i, E_i)$ be a graph of maximum degree Δ_i, $i \in \{1,2\}$. If $S \subset V_1 \times V_2$ is a defensive k-alliance in $G_1 \times G_2$, then the projections $P_{G_1}(S)$ and $P_{G_2}(S)$ of S over the graphs G_1 and G_2 are a defensive $(k-\Delta_2)$-alliance and a defensive $(k-\Delta_1)$-alliance in G_1 and G_2, respectively.*

Corollary 2.16. *Let G_i be a graph of maximum degree Δ_i, $i \in \{1,2\}$. If $G_1 \times G_2$ contains defensive k-alliances, then G_i contains defensive $(k-\Delta_j)$-alliances, with $i, j \in \{1,2\}$, $i \neq j$ and, as a consequence,*

$$a_k^d(G_1 \times G_2) \geq \max\{a_{k-\Delta_2}^d(G_1), a_{k-\Delta_1}^d(G_2)\}.$$

Now we continue with the study of relationships between $a_{k_1+k_2}^d(G_1 \times G_2)$ and $a_{k_i}^d(G_i)$, $i \in \{1,2\}$.

Theorem 2.17. *For any graph G_i, if S_i is a defensive k_i-alliance in G_i, $i \in \{1,2\}$, then $S_1 \times S_2$ is a defensive (k_1+k_2)-alliance in $G_1 \times G_2$ and*

$$a_{k_1+k_2}^d(G_1 \times G_2) \leq a_{k_1}^d(G_1) a_{k_2}^d(G_2).$$

Proof. Let $X = S_1 \times S_2$. Then for every $x = (u,v) \in X$,

$$\begin{aligned} \delta_X(x) &= \delta_{S_1}(u) + \delta_{S_2}(v) \\ &\geq \big(\delta_{\overline{S_1}}(u) + k_1\big) + \big(\delta_{\overline{S_2}}(v) + k_2\big) \\ &= \delta_{\overline{X}}(x) + k_1 + k_2. \end{aligned}$$

Thus, X is a defensive (k_1+k_2)-alliance in $G_1 \times G_2$. □

Notice that the bound of the above theorem is a general case of the results obtained in Theorem 42. In the particular case of the Petersen graph, P, and the 3-cube graph, Q_3, we have $a^d_{-2}(P \times Q_3) = 4 = a^d_{-1}(P)a^d_{-1}(Q_3)$. An example where we cannot apply Theorem 2.17 is the graph $K_{1,4} \times K_2$, for $k_1 = 2$ and $k_2 = 0$; the star graph $K_{1,4}$ does not contain defensive 2-alliances, although $K_{1,4} \times K_2$ contains some of them and $a^d_2(K_{1,4} \times K_2) = 8$. We note that from the above theorem we obtain $a^d_{2k}(G_1 \times G_2) \leq a^d_k(G_1)a^d_k(G_2)$. Another interesting consequence of Theorem 2.17 is the following.

Corollary 2.18. *Let G_1 and G_2 be two graphs of order n_1 and n_2 and maximum degree Δ_1 and Δ_2, respectively. Let $s \in \mathbb{Z}$ such that $\max\{\Delta_1, \Delta_2\} \leq s \leq \Delta_1 + \Delta_2 + k$. Then*

$$a^d_{k-s}(G_1 \times G_2) \leq \min\{a^d_k(G_1), a^d_k(G_2)\}.$$

As example of equalities we take $G_1 = P$, $G_2 = Q_3$, $k = 1$ and $s = 3$. In such a case, $4 = a^d_{-2}(P \times Q_3) = \min\{a^d_1(P), a^d_1(Q_3)\} = \min\{5, 4\}$. As a consequence of Theorem 2.17 we obtain the following relationship between global defensive alliances in Cartesian product graphs and global defensive alliances in its factors.

Corollary 2.19. *Let G_i be a graph of minimum degree δ_i and maximum degree Δ_i, $i \in \{1, 2\}$.*

(i) *If G_1 contains a global defensive k_1-alliance, then for every integer $k_2 \in \{-\Delta_2, ..., \delta_2\}$, $G_1 \times G_2$ contains a global defensive (k_1+k_2)-alliance and*

$$\gamma^d_{k_1+k_2}(G_1 \times G_2) \leq \gamma^d_{k_1}(G_1)n_2.$$

(ii) *If G_2 contains a global defensive k_2-alliance, then for every integer $k_1 \in \{-\Delta_1, ..., \delta_1\}$, $G_1 \times G_2$ contains a global defensive (k_1+k_2)-alliance and*

$$\gamma^d_{k_1+k_2}(G_1 \times G_2) \leq \gamma^d_{k_2}(G_2)n_1.$$

Proof. From Theorem 2.17 we obtain that for every defensive k-alliance S_1 of G_1 and every defensive k-alliance S_2 of G_2, the sets $S_1 \times V_2$ and $V_1 \times S_2$ are defensive $(k_1 + k_2)$-alliances in $G_1 \times G_2$. Moreover, $S_1 \times V_2$ and $V_1 \times S_2$ are dominating sets in $G_1 \times G_2$. Thus, the results follow. □

For the graph $C_4 \times Q_3$, by taking $k_1 = 0$ and $k_2 = 1$, we obtain equality in the above theorem.

2.4 Partitions into defensive k-alliances

For any graph $G = (V, E)$, the (*global*) *defensive k-alliance partition number* of G, $(\psi_k^{gd}(G))$ $\psi_k^d(G)$, $k \in \{-\Delta, ..., \delta\}$, is defined to be the maximum number of sets in a partition of V such that each set of the partition is a (global) defensive k-alliance.

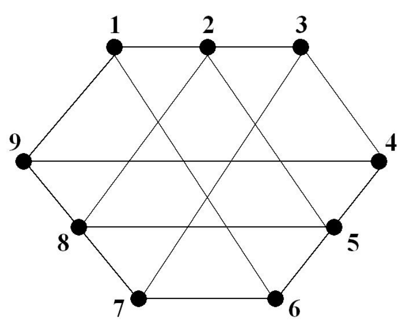

Figure 2.5: $\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$ is a partition of the graph into three defensive (-1)-alliances.

Extreme cases are $\psi_{-\Delta}^d(G) = n$, where each set composed of one vertex is a defensive $(-\Delta)$-alliance, and $\psi_\delta^d(G) = 1$ for the case of a connected δ-regular graph where V is the only defensive δ-alliance. A graph G is *partitionable* into (global) defensive k-alliances if $(\psi_k^{gd}(G) \geq 2)$ $\psi_k^d(G) \geq 2$. Figure 2.5 shows an example of a partition of a graph into three defensive (-1)-alliances.

Hereafter we will say that $(\Pi_r^{gd}(G))$ $\Pi_r^d(G)$ is a partition of G into r (global) defensive k-alliances.

Notice that if every vertex of G has even degree and k is odd, $k = 2l-1$, then every (global) defensive $(2l-1)$-alliance in G is a (global) defensive $(2l)$-alliance and vice versa. Hence, in such a case, $\psi_{2l-1}^d(G) = \psi_{2l}^d(G)$ and $\psi_{2l-1}^{gd}(G) = \psi_{2l}^{gd}(G)$. Analogously, if every vertex of G has odd degree and k is even, $k = 2l$, then every defensive $(2l)$-alliance in G is a defensive $(2l+1)$-alliance and vice versa. Hence, in such a case, $\psi_{2l}^d(G) = \psi_{2l+1}^d(G)$ and $\psi_{2l}^{gd}(G) = \psi_{2l+1}^{gd}(G)$.

2.4.1 Partitions into boundary defensive k-alliances

Let $G = (V, E)$ be a graph and let $\Pi_r^d(G) = \{S_1, S_2, ...S_r\}$ be a partition of V into r boundary defensive k-alliances. Suppose $x = \max_{1\le i\le r}\{|S_i|\}$ and $y = \min_{1\le i\le r}\{|S_i|\}$. Thus, $\frac{n}{x} \le r \le \frac{n}{y}$. Examples of bounds of r are the following two corollaries.

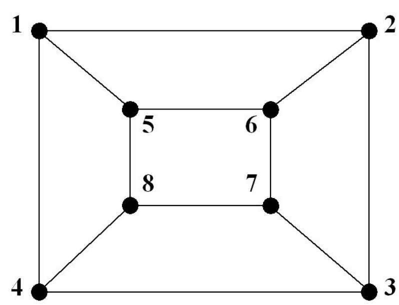

Figure 2.6: $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ is a partition of the graph into four boundary defensive (-1)-alliances.

As a consequence of Remark 2.6 we obtain the following bounds.

Corollary 2.20. *If G can be partitioned into r boundary defensive k-alliances,* *then*

$$\frac{2n}{2n-\delta+k} \le r \le \frac{2n}{\delta+k+2}.$$

The above bounds are tight. For instance, from the above result we obtain that the complete graph $G = K_n$ can be partitioned into $r = \frac{2n}{n+k+1}$ boundary defensive k-alliances. In particular, if n is even, each pair of vertices of K_n forms a boundary defensive $(3-n)$-alliance. Thus, K_n can be partitioned into $\frac{n}{2}$ of these alliances. Moreover, the upper bound is attained, for instance, in the case of $G = K_{t_1} \times C_{t_2}$, where C_{t_2} denotes a cycle of order t_2. In such a case, G is a (t_1+1)-regular graph of order $n = t_1 t_2$. Thus, for $k = t_1 - 3$ we obtain $r = t_2$. Notice that each one of the t_2 copies of K_{t_1} is a boundary defensive (t_1-3)-alliance in G.

Remark 2.21. *The complete graph of order n, $G = K_n$, can be partitioned into r boundary defensive k-alliances if and only if $n \equiv 0(r)$ and $k = \frac{2n}{r} - n - 1$.*

As a consequence of Theorem 2.8 we obtain the following result.

Corollary 2.22. *If G can be partitioned into r boundary defensive k-alliances, then*

$$\frac{2\mu_*}{2\mu_*-\delta+k} \le r \le \frac{2\mu}{2\mu-\Delta+k}.$$

The above bounds are tight. An example where the bounds are attained is the complete graph $G = K_n$. Moreover, by Corollary 2.22 we conclude, for instance, that if the Petersen graph (Figure 2.7) can be partitioned into r boundary defensive k-alliances, then $k = 1$ and $r = 2$ (in this case $\Delta = \delta = 3$, $\mu = 2$ and $\mu_* = 5$).

Theorem 2.23. *Let $G = (V, E)$ be a graph and let $M \subset E$ be a cut set partitioning V into two boundary defensive k-alliances S and $\overline{S}$, where $k \neq \Delta$ and $k \neq \delta$. Then*

$$\left\lceil \frac{2m-kn}{2(\Delta-k)} \right\rceil \le |S| \le \left\lfloor \frac{2m-kn}{2(\delta-k)} \right\rfloor \text{ and } |M| = \frac{2m-kn}{4}.$$

Proof. Since S is a boundary defensive k-alliance in G, for every $v \in S$ we have that $\delta(v) = 2\delta_{\overline{S}}(v) + k$. Hence

$$\sum_{v\in S} \delta(v) = 2\sum_{v\in S} \delta_{\overline{S}}(v) + k|S|.$$

Similarly, as $\overline{S}$ is a boundary defensive k-alliance in G,

$$\sum_{v\in \overline{S}} \delta(v) = 2\sum_{v\in \overline{S}} \delta_{S}(v) + k(n-|S|).$$

Therefore, from these two equalities we obtain

$$2m = 4\sum_{v\in S} \delta_{\overline{S}}(v) + kn. \tag{2.14}$$

So, we have $|M| = \sum_{v\in S} \delta_{\overline{S}}(v) = \frac{2m-nk}{4}$. Moreover, by using (2.6) and (2.9) in (2.14), we obtain the bounds on $|S|$. □

Corollary 2.24. *Let $G = (V, E)$ be a δ-regular graph and let $M \subset E$ be a cut set partitioning V into two boundary defensive k-alliances S and $\overline{S}$. Then $|S| = \frac{n}{2}$ and $|M| = \frac{n(\delta-k)}{4}$.*

Theorem 2.25. *If $\{X, Y\}$ is a partition of V into two boundary defensive k-alliances in $G = (V, E)$, then, without loss of generality,*

$$\left\lceil \sqrt{\frac{n(kn-2m+n\mu)}{4\mu}} + \frac{n}{2} \right\rceil \le |X| \le \left\lfloor \sqrt{\frac{n(kn-2m+n\mu_*)}{4\mu_*}} + \frac{n}{2} \right\rfloor$$

and

$$\left\lceil \frac{n}{2} - \sqrt{\frac{n(kn-2m+n\mu_*)}{4\mu_*}} \right\rceil \le |Y| \le \left\lfloor \frac{n}{2} - \sqrt{\frac{n(kn-2m+n\mu)}{4\mu}} \right\rfloor.$$

Proof. By Theorem 2.23,

$$\sum_{v\in X} \delta_Y(v) = \frac{2m-kn}{4}. \tag{2.15}$$

Moreover, as we have shown in the proof of Theorem 2.8

$$\mu \leq \frac{n \sum_{v \in X} \delta_Y(v)}{|X|(n-|X|)} \leq \mu_*. \tag{2.16}$$

Therefore, by using (2.15) in both sides of (2.16) we obtain the bounds on $|X|$ and $|Y| = n - |X|$. □

The above bounds are tight. For instance, in the case of the complete graph $G = K_n$, the above theorem leads to $|X| = \frac{n}{2} + \sqrt{\frac{n(k+1)}{4}}$ and $|Y| = \frac{n}{2} - \sqrt{\frac{n(k+1)}{4}}$. By using Remark 2.21 we have $k = -1$ and, as a consequence, $|X| = |Y| = \frac{n}{2}$.

By Corollary 2.24 and Theorem 2.25 we obtain the following interesting consequence.

Theorem 2.26. *Let $G = (V, E)$ be a δ-regular graph. If G is partitionable into two boundary defensive k-alliances, then the algebraic connectivity of G is $\mu = \delta - k$ (an even number).*

By the above necessary condition of existence of a partition of V into two boundary defensive k-alliances we obtain, for instance, that the icosahedron cannot be partitioned into two boundary defensive k-alliances, because its algebraic connectivity is $\mu = 5 - \sqrt{5} \notin \mathbb{Z}$. Moreover, the Petersen graph (See Figure 2.7) can only be partitioned into two boundary defensive k-alliances for the case of $k = 1$, because $\delta = 3$ and $\mu = 2$.

2.4.2 Partitions into r defensive k-alliances

Example 2.27. Let k and r be integers such that $r > 1$ and $r + k > 0$ and let $\mathcal{H}$ be a family of graphs whose vertex set is $V = \cup_{i=1}^{r} V_i$ where, for every V_i, $\langle V_i \rangle \cong K_{r+k}$ and $\delta_{V_j}(v) = 1$, for every $v \in V_i$ and $j \neq i$. Notice that $\{V_1, V_2, ..., V_r\}$ is a partition of the graphs belonging to $\mathcal{H}$ into r

global defensive k-alliances. A particular family of graphs included in $\mathcal{H}$ is $K_{r+k} \times K_r$.

Hereafter, $\mathcal{H}$ will denote the family of graphs defined in the above example.

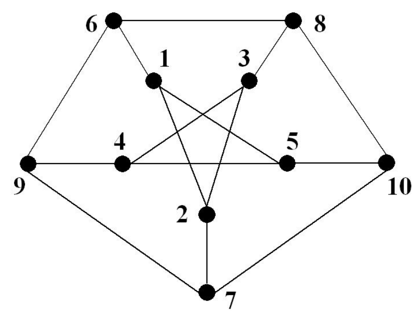

Figure 2.7: The sets $\{1,2,3,4,5\}$ and $\{6,7,8,9,10\}$ form a partition of the Petersen graph into two defensive 1-alliances.

From the following relation between the defensive k-alliance number, $a_k^d(G)$, and $\psi_k^d(G)$ we obtain that lower bounds on $a_k^d(G)$ lead to upper bounds on $\psi_k^d(G)$:

$$a_k^d(G)\psi_k^d(G) \leq n. \tag{2.17}$$

For instance, from Theorem 20 we have that

$$a_k^d(G) \geq \left\lceil \frac{\delta+k+2}{2} \right\rceil. \tag{2.18}$$

An example of equality in the above bound is provided by the graphs belonging to the family $\mathcal{H}$, for which we obtain $a_k^d(G) = r + k$.

By (2.17) and (2.18) we obtain the following bound,

$$\psi_k^d(G) \leq \begin{cases} \left\lfloor \frac{2n}{\delta+k+2} \right\rfloor, & \delta+k \quad \text{even} \\ \\ \left\lfloor \frac{2n}{\delta+k+3} \right\rfloor, & \delta+k \quad \text{odd.} \end{cases}$$

This bound gives the exact value of $\psi_k^d(G)$, for instance, for every $G \in \mathcal{H}$, where $\psi_k^d(G) = r$, and in the following cases: $\psi_{-1}^d(K_4 \times C_4) = 5$, $\psi_0^d(K_3 \times C_4) = \psi_{-1}^d(K_2 \times C_4) = 4$ and $\psi_1^d(K_2 \times C_4) = 2$.

Analogously, for global alliances we have

$$\gamma_k^d(G)\psi_k^{gd}(G) \leq n. \tag{2.19}$$

One example of bound on $\gamma_k^d(G)$ is the following from Theorem 23.

$$\gamma_k^d(G) \geq \left\lceil \frac{n}{\lfloor \frac{\Delta - k}{2} \rfloor + 1} \right\rceil . \tag{2.20}$$

For the graphs in $\mathcal{H}$, the above bound gives the exact value $\gamma_k^d(G) = r + k$. Thus, the bound obtained by combining (2.19) and (2.20),

$$\psi_k^{gd}(G) \leq \left\lfloor \frac{\Delta - k}{2} \right\rfloor + 1,$$

leads to the exact value of $\psi_k^{gd}(G) = r$ for every $G \in \mathcal{H}$. Even so, this bound can be improved.

Theorem 2.28. *For every graph G partitionable into global defensive k-alliances,*

(i) $\psi_k^{gd}(G) \leq \lfloor \frac{\sqrt{k^2+4n}-k}{2} \rfloor$,

(ii) $\psi_k^{gd}(G) \leq \lfloor \frac{\delta - k + 2}{2} \rfloor$.

Proof. Let $\Pi_r^{gd}(G) = \{S_1, S_2, ..., S_r\}$ be a partition of G into r global defensive k-alliances. Since, every $S_i \in \Pi_r^{gd}(G)$ is a dominating set, we have that for every $v \in S_i$, $\delta_{\overline{S_i}}(v) \geq r - 1$. Thus, the bounds are obtained as follow.

(i) $|S_i| - 1 \geq \delta_{S_i}(v) \geq \delta_{\overline{S_i}}(v) + k \geq r - 1 + k$, so $n = \sum_{i=1}^r |S_i| \geq r(r+k)$. By solving the inequality $r^2 + kr - n \leq 0$ we obtain the result.

(ii) Taking $v \in S_i$ as a vertex of minimum degree we obtain the result from $\delta = \delta(v) \geq 2\delta_{\overline{S_i}}(v) + k \geq 2(r-1) + k$. □

The above bounds are attained, for instance, in the following cases: $\psi^{gd}_{-1}(K_4 \times C_4) = 4$, $\psi^{gd}_0(K_3 \times C_4) = 3$, $\psi^{gd}_1(K_2 \times C_4) = 2$ and $\psi^{gd}_1(P) = 2$, where P denotes the Petersen graph.

Remark 2.29. *For every $k \in \{1-\delta, ..., \delta\}$, if $\psi^{gd}_k(G) \geq 2$, then*

$$\gamma^d_k(G) + \psi^{gd}_k(G) \leq \frac{n+4}{2}.$$

Proof. By equation (2.19), we have $\gamma^d_k(G) + \psi^{gd}_k(G) \leq \frac{n+\left(\psi^{gd}_k(G)\right)^2}{\psi^{gd}_k(G)}$. On the other hand, if $k \in \{1-\delta, ..., \delta\}$, then $\gamma^d_k(G) \geq 2$. Moreover, if $\psi^{gd}_k(G) \geq 2$, then $\gamma^d_k(G) \leq \frac{n}{2}$. So, $2 \leq \psi^{gd}_k(G) \leq \frac{n}{\gamma^d_k(G)} \leq \frac{n}{2}$. As a consequence, the result is obtained as follow,

$$\max_{2 \leq x \leq \frac{n}{\gamma^d_k(G)}} \left\{ \frac{n+x^2}{x} \right\} = \max \left\{ \frac{n+4}{2}, \frac{n+(\gamma^d_k(G))^2}{\gamma^d_k(G)} \right\} = \frac{n+4}{2}.$$

□

Example of equality in the above relation is $\gamma^d_{-1}(C_4 \times K_2) + \psi^{gd}_{-1}(C_4 \times K_2) = 6$.

Theorem 2.30. *If $\psi^{gd}_k(G) > 2$, then, for every $l \in \{1, ..., \psi^{gd}_k(G) - 2\}$, there exists a subgraph, G_l, of G of order $n(G_l) \leq n(G) - l\gamma^d_k(G)$ such that $\psi^{gd}_{l+k}(G_l) + l \geq \psi^{gd}_k(G)$.*

Proof. Let $\Pi^{gd}_r(G) = \{S_1, S_2, ..., S_r\}$ be a partition of V into $r > 2$ global defensive k-alliances. Let $S_i \in \Pi^{gd}_r(G)$. As S_i is a dominating set in G, it is also a dominating set in $\langle \cup_{j=1}^t S_j \rangle$. In addition, for every $v \in S_i$, we have

$$\begin{aligned}
\delta_{S_i}(v) &\geq \delta_{\overline{S_i}}(v) + k \\
&= \sum_{j=1, j\neq i}^{t} \delta_{S_j}(v) + \sum_{j=t+1}^{r} \delta_{S_j}(v) + k \\
&\geq \sum_{j=1, j\neq i}^{t} \delta_{S_j}(v) + r - t + k.
\end{aligned}$$

Hence, S_i is a global defensive $(r-t+k)$-alliance in $\langle \cup_{j=1}^{t} S_j \rangle$. Thus, for every $t \in \{2, ..., r-1\}$, $\{S_1, S_2, ..., S_t\} \subset \Pi_r^{gd}(G)$ is a partition of $\langle \cup_{j=1}^{t} S_j \rangle$ into t global defensive $(r-t+k)$-alliances.

Therefore, we can take $G_l = \langle \cup_{j=1}^{t} S_j \rangle$, where $l = r - t$. Then the order of G and G_l are related as follow, $n(G) = \sum_{i=1}^{r} |S_i| = n(G_l) + \sum_{i=t+1}^{r} |S_i| \geq n(G_l) + l\gamma_k^d(G)$. □

One example where $\psi_{l+k}^{gd}(G_l) + l = \psi_k^{gd}(G)$ and $n(G_l) = n(G) - l\gamma_k^d(G)$ is the following. Let $G = K_4 \times C_4$, the Cartesian product of the complete graph K_4 by the cycle graph C_4. $\psi_{-1}^{gd}(K_4 \times C_4) = 4$ and we can take each set of $\Pi_4^{gd}(K_4 \times C_4)$ as the vertex set of a copy of C_4, so $G_1 = K_3 \times C_4$ and $G_2 = K_2 \times C_4$ (the 3-cube graph). Hence, $4 = \psi_{-1}^{gd}(K_4 \times C_4) = \psi_0^{gd}(K_3 \times C_4) + 1 = \psi_1^{gd}(K_2 \times C_4) + 2$ and $8 = n(K_2 \times C_4) = n(K_3 \times C_4) - \gamma_{-1}^d(K_3 \times C_4) = [n(K_4 \times C_4) - \gamma_{-1}^d(K_4 \times C_4)] - \gamma_{-1}^d(K_3 \times C_4) = n(K_4 \times C_4) - 2\gamma_{-1}^d(K_4 \times C_4) = 16 - 2 \cdot 4$.

Theorem 2.31. *Let $C_{(r,k)}^{gd}(G)$ be the minimum number of edges having its endpoints in different sets of a partition of G into $r \geq 2$ global defensive k-alliances. Then*

(i) $C_{(r,k)}^{gd}(G) \geq \frac{1}{2} r(r-1)\gamma_k^d(G)$,

(ii) $C_{(r,k)}^{gd}(G) \geq \frac{1}{2} r(r-1)(r+k)$,

(iii) $C_{(r,k)}^{gd}(G) \leq \frac{2m-nk}{4}$.

(iv) $C_{(r,k)}^{gd}(G) = \frac{1}{2} r(r-1)\gamma_k^d(G) = \frac{1}{2} r(r-1)(r+k) = \frac{2m-nk}{4}$ *if and only if* $G \in \mathcal{H}$.

Proof. Let $\Pi_r^{gd}(G) = \{S_1, S_2, ..., S_r\}$ be a partition of V into r global defensive k-alliances and Let $x = \min_{S_i \in \Pi_r^{gd}(G)} |S_i|$. From the fact that every set of $\Pi_r^{gd}(G)$ is a dominating set, we obtain that the number of edges adjacent to $v \in S_i$ with one endpoint in $\cup_{j=i+1}^{r} S_j$ is bounded by $\sum_{j=i+1}^{r} \delta_{S_j}(v) \geq r - i$. Therefore,

$$C_{(r,k)}^{gd}(G) \geq \sum_{i=1}^{r-1}(r-i)|S_i| \geq x\sum_{i=1}^{r-1}(r-i) = \frac{x}{2}r(r-1). \tag{2.21}$$

Since every $S_i \in \Pi_r^{gd}(G)$ is a global defensive k-alliance, we have $x \geq r+k$ and $x \geq \gamma_k^d(G)$, as a consequence, (i) and (ii) follow.

In order to obtain the upper bound (iii) we note that the number of edges in G with one endpoint in S_i and the other endpoint in S_j is $C(S_i, S_j) = \sum_{v \in S_i} \delta_{S_j}(v) = \sum_{v \in S_j} \delta_{S_i}(v)$. Hence,

$$\begin{aligned}
2m = \sum_{i=1}^{r}\sum_{v \in S_i}\delta(v) &\geq 2\sum_{i=1}^{r}\sum_{v\in S_i}\delta_{\overline{S_i}}(v) + k\sum_{i=1}^{r}|S_i| \\
&= 2\sum_{i=1}^{r}\sum_{v\in S_i}\sum_{j=1,j\neq i}^{r}\delta_{S_j}(v) + kn \\
&= 2\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\sum_{v\in S_i}\delta_{S_j}(v) + kn \\
&= 2\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}C(S_i,S_j) + nk \\
&= 4C_{(r,k)}^{gd}(G) + nk.
\end{aligned}$$

Therefore, (iii) follows.

If for some $S_i \in \Pi_r^{gd}(G)$ there exists $v \in S_i$ such that $\delta_{S_i}(v) > \delta_{\overline{S_i}}(v) + k$, then, by analogy to the proof of (iii) we obtain $C_{(r,k)}^{gd}(G) < \frac{2m-nk}{4}$. Therefore, if $C_{(r,k)}^{gd}(G) = \frac{2m-nk}{4}$, then for every $S_i \in \Pi_r^{gd}(G)$, and for every $v \in S_i$, we have

$$\delta_{S_i}(v) = \delta_{\overline{S_i}}(v) + k. \tag{2.22}$$

Moreover, if for some $S_i \in \Pi_r^{gd}(G)$ there exists a vertex $v \in S_i$ such that $\sum_{j \neq i} \delta_{S_i}(v) > r - 1$, then, by analogy to the proof of (i) and (ii) we obtain

$C^{gd}_{(r,k)}(G) > \frac{1}{2}r(r-1)\gamma^d_k(G)$ and $C^{gd}_{(r,k)}(G) > \frac{1}{2}r(r-1)(r+k)$. Therefore, if $C^{gd}_{(r,k)}(G) = \frac{1}{2}r(r-1)\gamma^d_k(G) = \frac{1}{2}r(r-1)(r+k)$, then for every $S_i \in \Pi^{gd}_r(G)$, and for every $v \in S_i$, we have

$$\delta_{\overline{S_i}}(v) = \sum_{j\neq i} \delta_{S_i}(v) = r-1. \tag{2.23}$$

So, by (2.22) and (2.23) we obtain that for every $S_i \in \Pi^{gd}_r(G)$, $\langle S_i \rangle$ is regular of degree $r+k-1$. Thus, G is a regular graph of degree $2(r-1)+k$ and, by $\frac{1}{2}r(r-1)\gamma^d_k(G) = \frac{1}{2}r(r-1)(r+k) = \frac{2m-nk}{4}$ we have $n = r(r+k)$ and $\gamma^d_k(G) = r+k$. Hence, $|S_i| = r+k$, so $\langle S_i \rangle \cong K_{r+k}$. Moreover, as every $S_j \in \Pi^{gd}_r(G)$ is a dominating set, by (2.23) we have $\delta_{S_j}(v) = 1$, for every $v \in S_i$, $i \neq j$. Therefore, $G \in \mathcal{H}$. The opposite implication is immediate. $\square$

By (2.21) and Theorem 2.31 (iii) we obtain the following result.

Corollary 2.32. *For every graph G partitionable into r global defensive k-alliances of equal cardinality, $r \leq \frac{2(m+n)-kn}{2n}$.*

A family of graphs that achieve equality for Corollary 2.32 is the family $\mathcal{H}$ defined in Example 2.27.

By Theorem 2.31 and equation (2.18) we obtain the following two necessary conditions for the existence of a partition of a graph into r global defensive k-alliances.

Corollary 2.33. *If for a graph G, $k > \frac{2m-r(r-1)(\delta+2)}{n+r(r-1)}$ or $k > \frac{2(m-r^2(r-1))}{n+2r(r-1)}$, then G cannot be partitioned into r global defensive k-alliances.*

By the above corollary we conclude, for instance, that the 3-cube graph cannot be partitioned into $r > 2$ global defensive k-alliances.

Remark 2.34. *The size of the subgraph induced by a set belonging to a partition of G into r global defensive k-alliances is bounded below by $\frac{1}{2}\gamma^d_k(G)(r+k-1)$.*

Proof. The result follows from the fact that for every $S \in \Pi_r^{gd}(G)$,

$$\sum_{v\in S} \delta_S(v) \geq ((r-1)+k)|S| \geq (r-1+k)\gamma_k^d(G).$$

□

The above bound is tight as we can check by taking $G \in \mathcal{H}$.

2.4.3 Isoperimetric number, bisection and k-alliances

The *isoperimetric number* of a graph $G = (V, E)$, defined as

$$\mathbf{i}(G) := \min_{S\subset V(G):|S|\leq \frac{n}{2}} \left\{ \frac{\sum_{v\in S} \delta_{\overline{S}}(v)}{|S|} \right\}$$

has been extensively studied. For instance, we cite the papers by Mohar [56], Kahale [49] and Kwak *et. al.* [53]. This graph invariant is very hard to compute, and even obtaining bounds on $\mathbf{i}(G)$ is not straightforward. Here we consider the case of graphs which are partitionable into defensive k-alliances and, for these graphs, we obtain a tight bound on $\mathbf{i}(G)$.

As a consequence of Theorem 2.31 (iii) we obtain the following result.

Corollary 2.35. *If there exists a partition $\Pi_r^{gd}(G)$ into $r \geq 2$ global defensive k-alliances such that, for every $S_i \in \Pi_r^{gd}(G)$, $|S_i| \leq \frac{n}{2}$, then*

$$\mathbf{i}(G) \leq \frac{2m - nk}{2n}.$$

Proof. For every $S_i \in \Pi_r^{gd}(G)$ we have $|S_i|\mathbf{i}(G) \leq \sum_{v\in S_i} \delta_{\overline{S_i}}(v) = \sum_{v\in S_i} \sum_{j=1,j\neq i}^{r} \delta_{S_j}(v)$. Hence,

$$n\mathbf{i}(G) = \mathbf{i}(G)\sum_{i=1}^{r}|S_i| \leq \sum_{i=1}^{r}\sum_{v\in S_i}\sum_{j=1,j\neq i}^{r} \delta_{S_j}(v) = 2C_{(r,k)}^{gd}(G) \leq \frac{2m-nk}{2}.$$

□

Example of equality in above bound is the graph $G = C_3 \times C_3$ for $k = 0$. That is, $C_3 \times C_3$ can be partitioned into $r = 3$ global defensive 0-alliances of cardinality 3, moreover, $\mathbf{i}(C_3 \times C_3) = 2$. Other example is the 3-cube graph $G = C_4 \times K_2$, for $k = 1$. In this case each copy of the cycle C_4 is a global defensive 1-alliance and $\mathbf{i}(C_4 \times K_2) = 1$.

Notice that if $\mathbf{i}(G) > \frac{2m-nk}{2n}$, then G cannot be partitioned into $r \geq 2$ global defensive k-alliances with the condition that the cardinality of every set in the partition is at most $\frac{n}{2}$. One example of this is the graph $G = C_3 \times C_3$ for $k \geq 1$.

Theorem 2.36. *For any graph G,*

(i) *if G is partitionable into global defensive k-alliances, then*

$$\psi_k^{gd}(G) \leq \Delta + 1 - \mathbf{i}(G) - k,$$

(ii) *if G is partitionable into defensive k-alliances, then*

$$a_k^d(G) \geq \mathbf{i}(G) + k + 1.$$

Proof. (i) Let $\Pi_r^{gd}(G) = \{S_1, S_2, ..., S_r\}$ be a partition of G into $r \geq 2$ global defensive k-alliances. Then, there exists $S_i \in \Pi_r^{gd}(G)$ such that $|S_i| \leq \frac{n}{2}$. Hence,

$$\begin{aligned} |S_i|\mathbf{i}(G) &\leq \sum_{v \in S_i} \delta_{\overline{S_i}}(v) \\ &\leq \sum_{v \in S_i} (\delta_{S_i}(v) - k) \\ &\leq \sum_{v \in S_i} (\delta(v) - r + 1 - k) \\ &\leq |S_i|(\Delta - r + 1 - k). \end{aligned}$$

Thus, $r \leq \Delta + 1 - \mathbf{i}(G) - k$.

(ii) If $\psi_k^d(G) \geq 2$, then there exists a defensive k-alliance S such that $|S| \leq \frac{n}{2}$. Therefore,

$$|S|\mathbf{i}(G) \leq \sum_{v \in S} \delta_{\overline{S}}(v) \leq \sum_{v \in S} (\delta_S(v) - k) \leq |S|(|S| - 1) - k|S|.$$

Thus, the result follows. □

The following relation between the algebraic connectivity and the isoperimetric number of a graph was shown by Mohar in [56]: $\mathbf{i}(G) \geq \frac{\mu}{2}$.

Corollary 2.37. *For any graph G,*

(i) *if G is partitionable into global defensive k-alliances, then*

$$\psi_k^{gd}(G) \leq \left\lfloor \Delta + 1 - \frac{\mu}{2} - k \right\rfloor,$$

(ii) *if G is partitionable into defensive k-alliances, then*

$$a_k^d(G) \geq \left\lceil \frac{\mu + 2(k+1)}{2} \right\rceil.$$

Example of equality in the above bounds is the graph $G = C_3 \times C_3$ for $k = 0$, in this case $\mu = 3$.

From the above corollary, we emphasize that if $\mu > 2(\Delta - 1 - k)$, then G cannot be partitioned into global defensive k-alliances. For instance, we conclude that $G = C_3 \times C_3$ cannot be partitioned into global defensive k-alliances for $k > 1$. Moreover, by Corollary 2.37 (ii) we conclude that if $a_k^d(G) < \left\lceil \frac{\mu+2(k+1)}{2} \right\rceil$, then G cannot be partitioned into defensive k-alliances.

A *bisection* of G is a 2-partition $\{X, Y\}$ of the vertex set V in which $|X| = |Y|$ or $|X| = |Y| + 1$. The bisection problem is to find a bisection for which $\sum_{v \in X} \delta_Y(v)$ is as small as possible. The *bipartition width*, $bw(G)$, is defined as

$$bw(G) := \min_{X \subset V(G), |X| = \lfloor \frac{n}{2} \rfloor} \left\{ \sum_{v \in X} \delta_{\overline{X}}(v) \right\}.$$

It was shown by Merris [55] and Mohar [56] that

$$bw(G) \geq \begin{cases} \left\lceil \frac{n\mu}{4} \right\rceil & \textit{if } n \textit{ is even;} \\ \\ \left\lceil \frac{(n^2-1)\mu}{4n} \right\rceil & \textit{if } n \textit{ is odd.} \end{cases}$$

We are interested in the bisection of a graph into global defensive k-alliances, i.e., the bisection $\{X, Y\}$ of V such that X and Y are global defensive k-alliances. An example of bisection into global defensive (t-1)-alliances is obtained for the family of hypercube graphs $Q_{t+1} = Q_t \times K_2$, by taking $\{X, Y\}$ such that $\langle X \rangle \cong Q_t \cong \langle Y \rangle$.

By Theorem 2.31 (iii) and the above bound we obtain the following result.

Corollary 2.38. *If* $\left\lfloor \frac{2m-nk}{4} \right\rfloor < \left\lceil \frac{n\mu}{4} \right\rceil$, *for* n *even, or* $\left\lfloor \frac{2m-nk}{4} \right\rfloor < \left\lceil \frac{(n^2-1)\mu}{4n} \right\rceil$, *for* n *odd, then* G *cannot be bisectioned into global defensive* k*-alliances.*

For example, according to Corollary 2.38 we can conclude that, for $k > 0$, the graph $C_3 \times C_3$ cannot be bisectioned into global defensive k-alliances.

2.4.4 Partitioning $G_1 \times G_2$ into defensive k-alliances

In this subsection we will discuss the close relationships that exists between $\psi^d_{k_1+k_2}(G_1 \times G_2)$ and $\psi^d_{k_i}(G_i)$, $i \in \{1, 2\}$. From Theorem 2.17 we have that if G_i contains a defensive k_i-alliance, $i \in \{1, 2\}$, then $G_1 \times G_2$ contains a defensive $(k_1 + k_2)$-alliance. Therefore, we obtain the following result.

Theorem 2.39. *For any graphs* G_1 *and* G_2*, if there exists a partition of* G_i *into defensive* k_i*-alliances,* $i \in \{1, 2\}$*, then there exists a partition of* $G_1 \times G_2$ *into defensive* $(k_1 + k_2)$*-alliances and*

$$\psi^d_{k_1+k_2}(G_1 \times G_2) \geq \psi^d_{k_1}(G_1)\psi^d_{k_2}(G_2).$$

Proof. Every partition

$$\Pi^d_{r_i}(G_i) = \{S^{(i)}_1, S^{(i)}_2, ..., S^{(i)}_{r_i}\}$$

of G_i into r_i defensive k_i-alliances, $i \in \{1,2\}$, induces a partition of $G_1 \times G_2$ into r_1r_2 defensive (k_1+k_2)-alliances:

$$\Pi^d_{r_1r_2}(G_1 \times G_2) = \left\{ \begin{array}{ccc} S^{(1)}_1 \times S^{(2)}_1 & \cdots & S^{(1)}_1 \times S^{(2)}_{r_2} \\ S^{(1)}_2 \times S^{(2)}_1 & \cdots & S^{(1)}_2 \times S^{(2)}_{r_2} \\ \vdots & \vdots & \vdots \\ S^{(1)}_{r_1} \times S^{(2)}_1 & \cdots & S^{(1)}_{r_1} \times S^{(2)}_{r_2} \end{array} \right\}.$$

Therefore, the result follows. □

In the particular case of the Petersen graph, P, and the 3-cube graph, Q_3, we have $\psi^d_{-2}(P \times Q_3) = 20 = \psi^d_{-1}(P)\psi^d_{-1}(Q_3)$ and $5 = \psi^d_2(P \times Q_3) > \psi^d_1(P)\psi^d_1(Q_3) = 4$. We note that from Theorem 2.39 we obtain that $\psi^d_{2k}(G_1 \times G_2) \geq \psi^d_k(G_1)\psi^d_k(G_2)$. Another interesting consequence of Theorem 2.39 is the following.

Corollary 2.40. *Let G_i be a graph of order n_i maximum degree Δ_i, $i \in \{1,2\}$. Let $s \in \mathbb{Z}$ such that* $\max\{\Delta_1, \Delta_2\} \leq s \leq \Delta_1 + \Delta_2 + k$*. Then*

$$\psi^d_{k-s}(G_1 \times G_2) \geq \max\{n_2\psi^d_k(G_1), n_1\psi^d_k(G_2)\}.$$

As example of equality we take $G_1 = P$, $G_2 = Q_3$, $k = 1$ and $s = 3$. In such a case, $20 = \psi^d_{-2}(P \times Q_3) = \max\{8\psi^d_1(P), 10\psi^d_1(Q_3)\} = \max\{16, 20\}$.

At next we study the case of global defensive k-alliances.

Theorem 2.41. *Let $\Pi^{gd}_{r_i}(G_i)$ be a partition of a graph G_i, of order n_i, into $r_i \geq 1$ global defensive k_i-alliances, $i \in \{1,2\}$, $r_1 \leq r_2$. Let $x_i = \min_{X \in \Pi^{gd}_{r_i}(G_i)} \{|X|\}$. Then,*

(i) $\gamma^{d}_{k_1+k_2}(G_1 \times G_2) \leq \min\{x_1 n_2, x_2 n_1\}$,

(ii) $\psi^{gd}_{k_1+k_2}(G_1 \times G_2) \geq \max\left\{\psi^{gd}_{k_1}(G_1), \psi^{gd}_{k_2}(G_2)\right\}$.

Proof. From the procedure showed in the proof of Theorem 2.17 we obtain that for every $S^{(1)}_j \in \Pi^{gd}_{r_1}(G_1)$ and every $S^{(2)}_l \in \Pi^{gd}_{r_2}(G_2)$, the sets $M_j = S^{(1)}_j \times V_2$ and $N_l = V_1 \times S^{(2)}_l$ are defensive $(k_1 + k_2)$-alliances in $G_1 \times G_2$. Moreover M_j and N_l are dominating sets in $G_1 \times G_2$. Thus, by taking $S^{(1)}_j$ and $S^{(2)}_l$ of cardinality x_1 and x_2, respectively, we obtain $|M_j| = x_1 n_2$ and $|N_l| = x_2 n_1$, so (i) follows. Moreover, as $\{M_1, ..., M_{r_1}\}$ and $\{N_1, ..., N_{r_2}\}$ are partitions of $G_1 \times G_2$ into global defensive $(k_1 + k_2)$-alliances, (ii) follows. □

Corollary 2.42. *If G_i is a graph of order n_i such that $\psi^{gd}_{k_i}(G_i) \geq 1$, $i \in \{1, 2\}$, then*

$$\gamma^{d}_{k_1+k_2}(G_1 \times G_2) \leq \frac{n_1 n_2}{\max_{i\in\{1,2\}}\left\{\psi^{gd}_{k_i}(G_i)\right\}}.$$

For the graph $C_4 \times Q_3$, by taking $k_1 = 0$ and $k_2 = 1$, we obtain equalities in Theorem 2.41 and Corollary 2.42.

Chapter 3

Powerful Alliances

Abstract

We introduce the concept of boundary powerful k-alliances and we investigate some of its mathematical properties. We study the relationships that exist between powerful k-alliances in Cartesian product graphs and powerful k-alliances in its factors. Moreover, we study the partitions of a graph into powerful k-alliances.

3.1 Introduction

A nonempty set of vertices $S \subseteq V$ is a *powerful k-alliance*[1] in a graph $G = (V, E)$, $k \in \{-\Delta, ..., \Delta - 2\}$, if S is a defensive k-alliance and an offensive $(k+2)$-alliance. A powerful k-alliance is called *global* if it is a dominating set. Figure 3.1 shows two examples of powerful k-alliances.

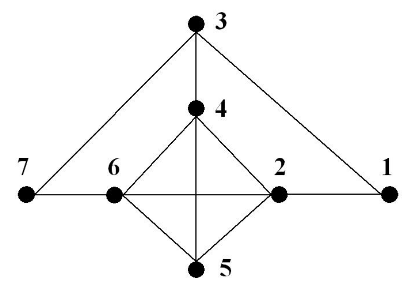

Figure 3.1: $\{2, 6\}$ is a powerful (-2)-alliance and $\{2, 3, 4, 6\}$ is a global powerful (-1)-alliance.

For $k \in \{-\Delta, ..., \Delta - 2\}$, the (*global*) *powerful k-alliance number* of G, denoted by $(\gamma_k^p(G))$ $a_k^p(G)$, is defined as the minimum cardinality of a (global) powerful k-alliance in G.

The concept of powerful alliance was introduced first in [52] and after that just a few works have been developed, like those in [6, 8, 9, 31, 59, 61, 69]. The complexity of computing minimum cardinality of powerful k-alliances in graphs was studied in [31], where it was proved that this is a NP-complete problem. Here we present some of the principal known results about powerful alliances.

Based on the NP-completeness of computing minimum cardinality of powerful k-alliances in graphs, some of the principal results about powerful alliances are centered into obtaining some bounds for the (global) powerful

[1] Also called dual alliance.

alliance number of a graph. For instance, in [9] was proved that for any graph G of order n and minimum degree $\delta > 0$, $\gamma^p_{-1}(G) \le n - \lceil \frac{\delta}{2} \rceil$. This result was generalized in [31] to powerful k-alliances.

Theorem 49. [31] *Let G be a graph of order n, size m and minimum degree δ. If G contains global powerful k-alliances, then*

$$\left\lceil \frac{\sqrt{8m + 4n(k+2) + (k+1)2} + k + 1}{4} \right\rceil \le \gamma^p_k(G) \le n - \left\lfloor \frac{\delta - k}{2} \right\rfloor.$$

Moreover, among other interesting results, in [9] were shown some relationships between the (global) powerful alliance number and the domination number of a graph. For instance, there were characterized those trees having equal domination number and global powerful (-1)-alliance number, and equal powerful (-1)-alliance number and global powerful (-1)-alliance number. Diverse kind of bounds for the powerful (-1)-alliance number of arbitrary graphs or specific families of graphs, like trees for instance, were also obtained in [9].

Theorem 50. [9] *For any graph G of order n, $\gamma^{gp}_{-1}(G) \ge \frac{n(\delta+1)}{\Delta+\delta+2}$.*

Theorem 51. [9] *If T is a tree of order n and $T \neq P_n$, then $a^p_{-1}(T) \le \left\lfloor \frac{n+3}{2} \right\rfloor$ and this bound is sharp.*

Theorem 52. [9] *Let T be any tree and t any integer such that $1 \le t \le a^p_{-1}(T)$. Then T has a subtree T' with $a^p_{-1}(T') = t$.*

Theorem 53. [9] *A graph $G = (V, E)$ has $\gamma(G) = \gamma^d_{-1}(G)$ if and only if $N[v]$ contains at least $\left\lceil \frac{N[v]}{2} \right\rceil$ support vertices for every vertex $v \in V$.*

Now, in order to present other results from [9] we need to introduce some notation. A vertex w in a tree T is said to have a *tail* if there is a leaf v for which all vertices in the $v - w$ path have degree two. The length of a tail is the distance from v to w. Let T be the tree formed from a star by subdividing any number of its edges any number of times, that is, T has at most one vertex of degree three or more. We call such a tree T a *spider*. A path, for example, is a special case of a spider. The subdivided edges are the tails of the central vertex x. Suppose x has r tails of length one, s tails of length two, and $t = \Delta - r - s$ tails of length at least three. Let $\mathcal{T}_1$ be the set of spiders T such that either T is a path or $\Delta \geq 3$ and:

- $r + s = \Delta$, or
- Δ is even, $r \leq \frac{\Delta}{2}$, $r + s = \Delta - 1$, and there is one tail of length three, or
- Δ is even, $r \leq \frac{\Delta-2}{2}$, $r + s = \Delta - 1$, and there is one tail of length four, or
- $\Delta = 4$, $s = 2$, and there are two tails of length four.

On the other hand, let $\mathcal{T}_2$ be the two trees shown in Figure 3.2.

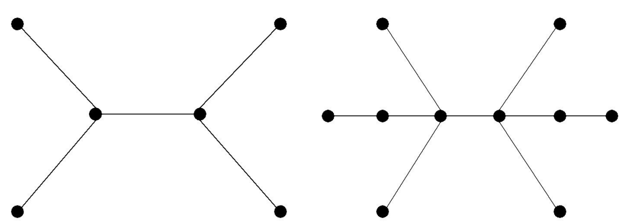

Figure 3.2: The two trees of $\mathcal{T}_2$.

Theorem 54. [9] *A tree T has $a^d_{-1}(G) = \gamma^d_{-1}(G)$ if and only if $T \in \mathcal{T}_1 \cup \mathcal{T}_2$.*

There are also some results, like the following one, about the relationship between powerful k-alliance number and spectral radius of the graph [31].

Theorem 55. [31] *Let G be a graph of order n, size m and spectral radius λ. If G contains global powerful k-alliances, then*

$$\gamma_k^p(G) \geq \left\lceil \frac{2m + n(k-2)}{4\lambda - 2k + 2} \right\rceil.$$

The particular cases $k = -1$ and $k = 0$ in Theorems 49 and 55 were studied previously in [59]. Also, in [31] were investigated some relationships between the total r-domination number[2], $\gamma_{rt}(G)$, and the global powerful k-alliance number of a graph. For instance, there were proved the following results.

Theorem 56. [31]

(i) *Each global powerful k-alliance, $k \geq 1$, is a total k-dominating set.*

(ii) *Each total r-dominating set is a global powerful k-alliance, where $-\Delta < k \leq 2(r-1) - \Delta$.*

(iii) *For $-\Delta < k \leq 2(r-1) - \Delta$, $\gamma_{rt}(G) \geq \gamma_k^p(G)$.*

(iv) *For $k \geq 1$, $\gamma_k^p(G) \geq \gamma_{kt}(G)$.*

Also, in [31] have been investigated the global powerful k-alliances in planar graphs.

Theorem 57. [31] *Let G be a graph of order n and size m. Let S be a global powerful k-alliance in G such that $\langle S \rangle$ is a planar graph.*

(i) *If $n > 2(2-k)$, then $|S| \geq \left\lceil \frac{2(m+24)+n(k+2)}{2(13-k)} \right\rceil$.*

(ii) *If $n > 2(2-k)$ and $\langle S \rangle$ is a triangle free graph, then $|S| \geq \left\lceil \frac{2(m+16)+n(k+2)}{2(9-k)} \right\rceil$.*

[2] A set of vertices S is a total r-dominating set in a graph G if for every vertex v of G, $\delta_S(v) \geq r$.

Theorem 58. [31] *Let G be a graph of order n. Let S be a global powerful k-alliance in G such that $\langle S\rangle$ is planar connected with f faces. Then,*

$$|S| \geq \left\lceil \frac{2(m-4f+8)+n(k+2)}{2(5-k)} \right\rceil .$$

The case $k=-1$ in the above two theorems was studied previously in [61]. On the other hand, in [6] were studied the global powerful k-alliances in graphs, but in this article was used other name for the same structure. The authors of [6] defined the concept of *excess-t global powerful alliance* as a set of vertices S of a graph $G=(V,E)$ such that for every vertex $v\in V$, $|N[v]\cap S| \geq |N[v]\cap (V-S)|+t$. Notice that this expression is equivalent to that S is a global defensive t-alliance and a global offensive $(t+2)$-alliance. In this work were obtained several results about global powerful k-alliances in graphs.

Theorem 59. [6] *For any graph G of maximum degree Δ,*

$$\gamma_k^p(G) \geq \left\lceil \frac{\Delta+k+1}{2} \right\rceil .$$

Theorem 60. [6] *Let $G=(V,E)$ be a graph.*

(i) *If $e\in E$, then $\gamma_{-1}^p(G)-1 \leq \gamma_{-1}^p(G-e) \leq \gamma_{-1}^p(G)+2$.*

(ii) *If $f\notin E$, then $\gamma_{-1}^p(G)-2 \leq \gamma_{-1}^p(G+f) \leq \gamma_{-1}^p(G)+1$.*

Theorem 61. [6] *For any graph G on n vertices,*

$$\gamma_{-1}^p(G) \geq \begin{cases} \left\lceil \sqrt{n+0.25}-0.5 \right\rceil, & \text{if } n \text{ is even,} \\ \left\lceil \sqrt{n} \right\rceil, & \text{if } n \text{ is odd,} \end{cases}$$

and these bounds are sharp.

Theorem 62. [6] *If $\gamma_{-1}^p(G) \leq \left\lceil \frac{n}{2} \right\rceil - 1$, then $\gamma_{-1}^p(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.*

Corollary 63. [6] *For any graph G either* $\max\{\gamma^p_{-1}(G), \gamma^p_{-1}(\overline{G})\} \le \left\lceil \frac{n+1}{2} \right\rceil$ *or* $\left\lfloor \frac{n+1}{2} \right\rfloor \le \min\{\gamma^p_{-1}(G), \gamma^p_{-1}(\overline{G})\}$.

Theorem 64. [6] *Let G be a graph of minimum degree δ and maximum degree Δ. Then*

$$\gamma^p_{-1}(G) + \gamma^p_{-1}(\overline{G}) \ge \left\lceil \frac{n+\Delta-\delta+1}{2} \right\rceil.$$

Theorem 65. [6] *Let G be a graph of minimum degree δ and maximum degree Δ. Then*

$$\gamma^p_{-1}(G) + \gamma^p_{-1}(\overline{G}) \le \left\lceil \frac{3n+\Delta-\delta+1}{2} \right\rceil.$$

A particular case of Cartesian product graph and its powerful alliances was studied in [8] where was obtained that for any cycles C_s and C_t, $a^p_{-1}(C_s \times C_t) \ge \frac{7st}{12}$. We refer to the Ph. D. Thesis [69] to have a more complete idea about the principal known results related to powerful alliances.

3.2 Boundary powerful k-alliances

In Chapter 2 we studied the boundary defensive k-alliances in graphs, i.e., defensive k-alliances having exactly k more neighbors inside of the alliance than outside. Similarly, a boundary offensive k-alliance is a set of vertices of a graph, such that every vertex of its neighborhood has exactly k-more neighbors inside of the set than it has outside. The study of boundary offensive k-alliances is completely analogous to the study of boundary defensive k-alliances, based on the following fact.

Remark 3.1. *Let $G=(V,E)$ be a graph. $S \subset V$ is a boundary defensive k-alliance in a graph G if and only if $\partial(S)$ is a boundary offensive $(-k)$-alliance in G.*

A set $S \subseteq V$ is a *boundary powerful k-alliance* in $G = (V, E)$, $k \in \{-\Delta, \dots, \Delta - 2\}$, if S is a boundary defensive k-alliance and a boundary offensive $(k+2)$-alliance. A boundary powerful k-alliance in G is called *global* if it forms a dominating set in G. Figure 3.3 shows examples of (global) boundary powerful k-alliances.

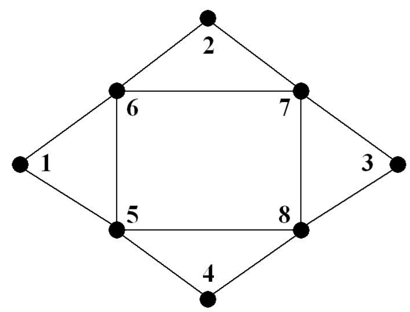

Figure 3.3: $\{1, 2, 3, 4\}$ is a boundary powerful (-2)-alliance and $\{5, 6, 7, 8\}$ is a boundary powerful 0-alliance.

3.2.1 Cardinality of boundary powerful k-alliances

It was shown in Theorem 2.6 that the cardinality of a boundary defensive k-alliance S is bounded by

$$\left\lceil \frac{\delta + k + 2}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta + k}{2} \right\rfloor.$$

Analogously, in the case of a boundary offensive k-alliance S we have

$$\left\lceil \frac{\delta + k}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta + k - 2}{2} \right\rfloor.$$

Thus, by replacing k by $k+2$ in the second equation we obtain the following result.

Remark 3.2. *If S is a boundary powerful k-alliance in a graph, then*

$$\left\lceil \frac{\delta + k + 2}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta + k}{2} \right\rfloor.$$

Notice that the above result gives a closed formula, for instance, in the case of complete graphs.

Corollary 3.3. *If S is a boundary powerful k-alliance in a complete graph $G = K_n$, then $|S| = \left\lceil \frac{n+k+1}{2} \right\rceil$.*

Theorem 3.4. *If S is a global boundary powerful k-alliance in a graph, then*

$$\left\lceil \frac{2m + n(k+2)}{2\Delta + 2} \right\rceil \leq |S| \leq \left\lfloor \frac{2m + n(k+2)}{2\delta + 2} \right\rfloor.$$

Proof. Since S is a global boundary powerful k-alliance in G, then, for every $v \in S$, $\delta(v) = 2\delta_{\overline{S}}(v) + k$, and for every $v \in \overline{S}$, $\delta(v) = 2\delta_S(v) - (k+2)$. Hence,

$$\sum_{v \in S} \delta(v) = 2 \sum_{v \in S} \delta_{\overline{S}}(v) + k|S|,$$

and

$$\sum_{v \in \overline{S}} \delta(v) = 2 \sum_{v \in \overline{S}} \delta_S(v) - (k+2)(n - |S|).$$

Now, as $\sum_{v \in S} \delta_{\overline{S}}(v) = \sum_{v \in \overline{S}} \delta_S(v)$, we have

$$2m = 4 \sum_{v \in S} \delta_{\overline{S}}(v) + |S|(2k+2) - n(k+2). \tag{3.1}$$

On the other hand, for every $v \in S$,

$$\frac{\delta - k}{2} \leq \delta_{\overline{S}}(v) \leq \frac{\Delta - k}{2}. \tag{3.2}$$

Therefore, by using the above inequalities in equation (3.1) we obtain the bounds on $|S|$. □

Since for any δ-regular graph, $m = \frac{\delta n}{2}$, the above theorem gives a closed formula for the cardinality of any global boundary powerful k-alliance.

Corollary 3.5. *If S is a global boundary powerful k-alliance in a δ-regular graph, then*

$$|S| = \left\lceil \frac{n(\delta+k+2)}{2(\delta+1)} \right\rceil .$$

As we mention in Subsection 2.2.1 for every planar graph of order n, size m and f faces, the Euler formula states that $m = n + f - 2$. Hence, we obtain the following corollary of Theorem 3.4.

Corollary 3.6. *Let G be a planar connected graph with f faces. If S is a global boundary powerful k-alliance in G, then*

$$\left\lceil \frac{n(k+4)+2f-4}{2\Delta+2} \right\rceil \le |S| \le \left\lfloor \frac{n(k+4)+2f-4}{2\delta+2} \right\rfloor$$

and, if G is δ-regular,

$$|S| = \frac{n(k+4)+2f-4}{2(\delta+1)}.$$

Theorem 3.7. *If S is a global boundary powerful k-alliance in a graph, then*

$$\left\lceil \frac{n(2\delta+k+2)-2m}{2\delta+2} \right\rceil \le |S| \le \left\lfloor \frac{n(2\Delta+k+2)-2m}{2\Delta+2} \right\rfloor .$$

Proof. Since S is a global boundary offensive $(k+2)$-alliance in G, then for every $v \in \overline{S}$,

$$\frac{\delta+k+2}{2} \le \delta_S(v) \le \frac{\Delta+k+2}{2}. \tag{3.3}$$

Now, as $\sum_{v\in S} \delta_{\overline{S}}(v) = \sum_{v\in\overline{S}} \delta_S(v)$, by using (3.3), in equation (3.1), we obtain both bounds on $|S|$. □

Notice that the above theorem leads to Corollary 3.5 for the case of regular graphs.

Theorem 3.8. *Let $G = (V, E)$ be a graph and let $S \subset V$. Let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a global boundary powerful k-alliance in G, with $k \neq -1$, then*

$$|S| = \frac{2(m + n - 2c) + nk}{2(k + 1)}.$$

Proof. Let $m(\langle S\rangle)$ be the size of $\langle S\rangle$. Since S is a boundary defensive k-alliance in G,

$$2m(\langle S\rangle) = \sum_{v\in S} \delta_S(v) = \sum_{v\in S} \delta_{\overline{S}}(v) + k|S| = c + k|S|.$$

Moreover, as S is a global boundary offensive $(k + 2)$-alliance in G,

$$c = \sum_{v\in \overline{S}} \delta_S(v) = \sum_{v\in \overline{S}} \delta_{\overline{S}}(v) + (n - |S|)(k + 2) = 2m(\langle \overline{S}\rangle) + (n - |S|)(k + 2).$$

Now, as $m = m(\langle S\rangle) + m(\langle \overline{S}\rangle) + c$, we obtain the value of $|S|$. □

Corollary 3.9. *Let $G = (V, E)$ be a δ-regular graph and let $S \subset V$. Let c be the number of edges of G with one endpoint in S and the other endpoint outside of S. If S is a global boundary powerful k-alliance in G, with $k \neq -1$, then*

(i) $|S| = \frac{n(\delta+k+2)-4c}{2k+2}$,

(ii) $c = \frac{n(\delta^2+2\delta-k^2-2k)}{4(\delta+1)}$.

Proof. (i) is trivial and (ii) is a direct consequence of Corollary 3.5 and Theorem 3.8. □

3.3 Powerful k-alliances in Cartesian product graphs

As we mention at the beginning of the present chapter, the study of powerful alliances in Cartesian product of graphs was first studied by Brigham, Dutton and Hedetniemi in [8], where it was studied the Cartesian product of cycle graphs. In this section we study general relationships between (global) powerful k-alliances in Cartesian product graphs and (global) powerful k-alliances in its factors.

Theorem 3.10. *Let $G_i = (V_i, E_i)$ be a graph of maximum degree Δ_i. If $S_i \subset V_i$ is a powerful k_i-alliance in G_i, $i \in \{1, 2\}$, then $S_1 \times S_2$ is a powerful k-alliance in $G_1 \times G_2$, for every $k \in \{-\Delta_1 - \Delta_2, ..., \min\{k_1 - \Delta_2, k_2 - \Delta_1\}\}$.*

Proof. If S_i is a defensive k_i-alliance in G_i, then for every $v \in S_i$ we have, $\delta_{S_i}(v) \geq \delta_{\overline{S_i}}(v) + k_i$, $i \in \{1, 2\}$. If $X = S_1 \times S_2$ and $(a, b) \in X$, then

$$\begin{aligned} \delta_X(a, b) &= \delta_{S_1}(a) + \delta_{S_2}(b) \\ &\geq \delta_{\overline{S_1}}(a) + \delta_{\overline{S_2}}(b) + k_1 + k_2 \\ &= \delta_{\overline{X}}(a, b) + k_1 + k_2. \end{aligned}$$

So, we obtain

$$\delta_X(a, b) \geq \delta_{\overline{X}}(a, b) + k_1 - \Delta_2,$$

and

$$\delta_X(a, b) \geq \delta_{\overline{X}}(a, b) + k_2 - \Delta_1.$$

Thus, for every $k \in \{-\Delta_1 - \Delta_2, ..., \min\{k_1 - \Delta_2, k_2 - \Delta_1\}\}$, X is a defensive k-alliance in $G_1 \times G_2$.

On the other hand, if S_i is an offensive $(k_i + 2)$-alliance in G_i, then for every $u \in \partial(S_i)$ we have, $\delta_{S_i}(u) \geq \delta_{\overline{S_i}}(u) + k_i + 2$, $i \in \{1, 2\}$. Now, let $(a, b) \in \partial(X)$, then either, $a \in S_1$ and $b \in \partial(S_2)$ or $a \in \partial(S_1)$ and $b \in S_2$. Let us suppose, for instance, $a \in S_1$ and $b \in \partial(S_2)$, hence we have

$$\begin{aligned} \delta_X(a, b) &= \delta_{S_2}(b) \\ &\geq \delta_{\overline{S_2}}(b) + k_2 + 2 \\ &= \delta_{\overline{X}}(a, b) - \delta(a) + k_2 + 2 \\ &\geq \delta_{\overline{X}}(a, b) + k_2 - \Delta_1 + 2. \end{aligned}$$

The case $a \in \partial(S_1)$ and $b \in S_2$ is analogous to the previous one, and we obtain $\delta_X(a,b) \geq \delta_{\overline{X}}(a,b) + k_1 - \Delta_2 + 2$.

Therefore, for every $k \in \{-\Delta_1 - \Delta_2, ..., \min\{k_1 - \Delta_2, k_2 - \Delta_1\}\}$, X is an offensive $(k+2)$-alliance in $G_1 \times G_2$ and, as a consequence, X is a powerful k-alliance in $G_1 \times G_2$. □

Corollary 3.11. *Let $G_i = (V_i, E_i)$ be a graph of maximum degree Δ_i, $i \in \{1,2\}$. If G_i contains powerful k_i-alliances, then*

$$a_k^p(G_1 \times G_2) \leq a_{k_1}^p(G_1)a_{k_2}^p(G_2),$$

for every $k \in \{-\Delta_1 - \Delta_2, ..., \min\{k_1 - \Delta_2, k_2 - \Delta_1\}\}$.

Theorem 3.12. *Let $G_i = (V_i, E_i)$ be a graph of maximum degree Δ_i, $i \in \{1,2\}$. If $S_1 \subset V_1$ is a global powerful k_1-alliance in G_1, then $S_1 \times V_2$ is a global powerful k-alliance in $G_1 \times G_2$, for every $k \in \{-\Delta_1 - \Delta_2, ..., k_1 - \Delta_2\}$.*

Proof. If S_1 is a dominating set in G_1, then $S_1 \times V_2$ is a dominating set in $G_1 \times G_2$. On the other hand, if S_1 is a defensive k_1-alliance in G_1, then for every $v \in S_1$, $\delta_{S_1}(v) \geq \delta_{\overline{S_1}}(v) + k_1$. Now, let $X = S_1 \times V_2$ and let $(a,b) \in X$. Hence,

$$\begin{aligned}
\delta_X(a,b) &= \delta_{S_1}(a) + \delta(b) \\
&\geq \delta_{\overline{S_1}}(a) + \delta(b) + k_1 \\
&= \delta_{\overline{X}}(a,b) + k_1 + \delta(b) \\
&\geq \delta_{\overline{X}}(a,b) + k_1 - \Delta_2.
\end{aligned}$$

Therefore, X is a global defensive $(k_1 - \Delta_2)$-alliance in $G_1 \times G_2$.

Now, if S_1 is a global offensive (k_1+2)-alliance in G_1, for every $u \in \overline{S_1}$,

$\delta_{S_1}(u) \geq \delta_{\overline{S_1}}(u) + k + 2$. If $(a, b) \in \overline{X}$, then

$$\begin{aligned}\delta_X(a,b) &= \delta_{S_1}(a)\\ &\geq \delta_{\overline{S_1}}(a) + k_1 + 2\\ &= \delta_{\overline{X}}(a,b) - \delta(b) + k_1 + 2\\ &\geq \delta_{\overline{X}}(a,b) - \Delta_2 + k_1 + 2.\end{aligned}$$

Therefore, X is a global offensive $(k_1 - \Delta_2 + 2)$-alliance in $G_1 \times G_2$. As a consequence, X is a global powerful $(k_1 - \Delta_2)$-alliance in $G_1 \times G_2$. □

Corollary 3.13. *Let $G_i = (V_i, E_i)$ be a graph of order n_i and maximum degree Δ_i, $i \in \{1, 2\}$. If G_1 contains global powerful k_1-alliances, then for every $k \in \{-\Delta_1 - \Delta_2, ..., k_1 - \Delta_2\}$,*

$$\gamma_k^p(G_1 \times G_2) \leq \gamma_{k_1}^p(G_1)n_2.$$

3.4 Partitions into powerful k-alliances

For any graph $G = (V, E)$, the (*global*) *powerful k-alliance partition number* of G, $(\psi_k^{gp}(G))$ $\psi_k^p(G)$, is defined to be the maximum number of sets in a partition of V such that each set is (global) powerful k-alliance. We say that a graph G is partitionable into (global) powerful k-alliances if $(\psi_k^{gp}(G) \geq 2)$ $\psi_k^p(G) \geq 2$.

3.4.1 Partitions into boundary powerful k-alliances

Remark 3.14. *Let $G = (V, E)$ be a graph.*

(i) *$S \subset V$ is a global boundary powerful (-1)-alliance in G, if and only if, $\overline{S}$ is a global boundary powerful (-1)-alliance in G.*

(ii) *If G can be partitioned into two global boundary powerful k-alliances, then $k = -1$.*

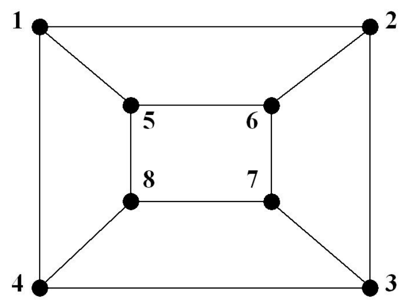

Figure 3.4: $\{\{1,3,5,7\},\{2,4,6,8\}\}$ is a partition of the graph into two powerful (-1)-alliances.

Proof. If S is a global boundary powerful k-alliance in G, then

$$\delta_S(v) = \delta_{\overline{S}}(v) + k, \quad \forall\, v \in S \tag{3.4}$$

and

$$\delta_S(v) = \delta_{\overline{S}}(v) + k + 2, \quad \forall\, v \in \overline{S}. \tag{3.5}$$

So, (i) follows immediately from (3.4) and (3.5). If $\overline{S}$ is a global boundary powerful k-alliance in G, then

$$\delta_{\overline{S}}(v) = \delta_S(v) + k, \quad \forall\, v \in \overline{S} \tag{3.6}$$

and

$$\delta_{\overline{S}}(v) = \delta_S(v) + k + 2, \quad \forall\, v \in S. \tag{3.7}$$

Hence, by (3.4) and (3.7) (or by (3.5) and (3.6)), we obtain that $k = -1$. $\square$

Theorem 3.15. *Let $G = (V, E)$ be a graph, if S is a global boundary powerful (-1)-alliance in G, then*

$$\left\lceil \frac{n(\delta+1)}{\Delta+\delta+2} \right\rceil \le |S| \le \left\lfloor \frac{n(\Delta+1)}{\Delta+\delta+2} \right\rfloor.$$

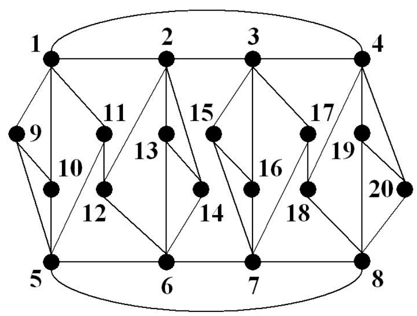

Figure 3.5: $S = \{1, ..., 8\}$ and $\overline{S}$ are global boundary powerful (-1)-alliances.

Proof. From equations (3.4) and (3.5) we have,

$$\sum_{v\in S} \delta_S(v) = \sum_{v\in S} \delta_{\overline{S}}(v) - |S| \tag{3.8}$$

and

$$\sum_{v\in \overline{S}} \delta_S(v) = \sum_{v\in \overline{S}} \delta_{\overline{S}}(v) + n - |S|. \tag{3.9}$$

Hence, as $\sum_{v\in S} \delta_{\overline{S}}(v) = \sum_{v\in \overline{S}} \delta_S(v)$, we have

$$\sum_{v\in S} \delta_S(v) = \sum_{v\in \overline{S}} \delta_{\overline{S}}(v) + n - 2|S|. \tag{3.10}$$

Thus, by (3.10) we obtain the following inequalities,

$$|S|\frac{\Delta - 1}{2} \geq (n - |S|)\frac{\delta - 1}{2} + n - 2|S| \tag{3.11}$$

and

$$|S|\frac{\delta - 1}{2} \leq (n - |S|)\frac{\Delta - 1}{2} + n - 2|S|. \tag{3.12}$$

By solving the above inequalities for $|S|$ we obtain the bounds. $\square$

Notice that the bounds obtained in the above theorem are attained, for instance, in the case of the graph in Figure 3.5.

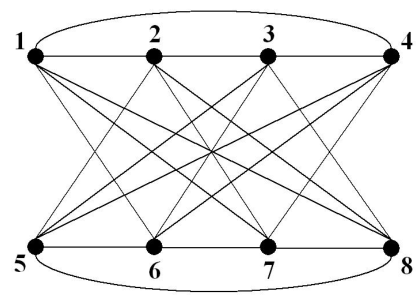

Figure 3.6: $S = \{1,2,3,4\}$ and $\overline{S}$ are global boundary powerful (-1)-alliances.

Corollary 3.16. *If S is a global boundary powerful (-1)-alliance in a δ-regular graph, then $|S| = \frac{n}{2}$.*

Figure 3.6 shows an example of a 5-regular graph, which can be partitioned into two global boundary powerful (-1)-alliances.

Theorem 3.17. *Let $S \subset V$ be a global boundary powerful (-1)-alliance in a graph $G = (V, E)$ and let $M \subset E$ be a cut set with one endpoint in S and the other endpoint outside of S. Then $\left\lceil \frac{2m+n}{2\Delta+2} \right\rceil \leq |S| \leq \left\lfloor \frac{2m+n}{2\delta+2} \right\rfloor$ and $|M| = \frac{2m+n}{4}$.*

Proof. Since S is a global boundary defensive (-1)-alliance in G, for every $v \in S$, $\delta(v) = 2\delta_{\overline{S}}(v) - 1$, therefore,

$$\sum_{v \in S} \delta(v) = 2|M| - |S|.$$

Moreover, as $\overline{S}$ is a global boundary offensive 1-alliance in G, for every $v \in \overline{S}$, $\delta(v) = 2\delta_S(v) - 1$, therefore,

$$\sum_{v \in \overline{S}} \delta(v) = 2|M| - n + |S|.$$

Hence, $2m = 4|M| - n$. So, the value of $|M|$ follows. The bounds on $|S|$ are obtained from the above equation by using that, for every $v \in S$, $\frac{\delta+1}{2} \leq \delta_{\overline{S}}(v) \leq \frac{\Delta+1}{2}$. □

Notice that the above result leads to the Corollary 3.16 for the case of regular graphs.

The following result shows the relationship between the algebraic connectivity (and the Laplacian spectral radius) of a graph and the cardinality of its global boundary powerful (-1)-alliances.

Theorem 3.18. *If $X \subset V$ is a global boundary powerful (-1)-alliance in $G = (V, E)$, then, without loss of generality,*

$$\frac{n}{2} + \left\lceil \sqrt{\frac{n^2(\mu - 1) - 2nm}{4\mu}} \right\rceil \leq |X| \leq \frac{n}{2} + \left\lfloor \sqrt{\frac{n^2(\mu_* - 1) - 2nm}{4\mu_*}} \right\rfloor$$

and

$$\frac{n}{2} - \left\lfloor \sqrt{\frac{n^2(\mu_* - 1) - 2nm}{4\mu_*}} \right\rfloor \leq |\overline{X}| \leq \frac{n}{2} - \left\lceil \sqrt{\frac{n^2(\mu - 1) - 2nm}{4\mu}} \right\rceil.$$

Proof. On one hand, by Theorem 3.17 we have,

$$\sum_{v \in X} \delta_{\overline{X}}(v) = \frac{2m + n}{4}. \tag{3.13}$$

On the other hand, by equations (2.5) and (1.4), taken $w \in \mathbb{R}^n$ defined as in (1.6) we have

$$\mu \leq \frac{n \displaystyle\sum_{v \in X} \delta_{\overline{X}}(v)}{|X|(n - |X|)} \leq \mu_*. \tag{3.14}$$

Now, by using equation (3.13) in (3.14) we obtain both bounds on $|X|$. Moreover, as $|\overline{X}| = n - |X|$, the bounds on $|\overline{X}|$ follows. □

By Corollary 3.16 and the above theorem we obtain the following consequence.

Theorem 3.19. *Let $G = (V, E)$ be a δ-regular graph. If G contains a global boundary powerful (-1)-alliance, then the algebraic connectivity of G is $\mu = \delta + 1$.*

The above theorem gives a necessary condition for the existence global boundary powerful (-1)-alliances. Thus we obtain, for instance, that the Icosahedron does not contain global boundary powerful (-1)-alliances; because its algebraic connectivity is $\mu = 5 - \sqrt{5}$. Notice that the same occurs for the Petersen graph because, in this case, $\delta = 3$ and $\mu = 2$.

3.4.2 Partitions into r powerful k-alliances

In this subsection we will study the partitions of an arbitrary graph into powerful k-alliances. To begin with, we consider the following example[3]. Let $\{v_1, v_2, ..., v_{3t}\}$ the vertex set of $CR(3t, 3)$. Then the sets $\{v_1, v_4, ..., v_{3t-2}\}$, $\{v_2, v_5, ..., v_{3t-1}\}$ and $\{v_3, v_6, ..., v_{3t}\}$ form a maximum partition of $CR(3t, 3)$ into three global powerful (-4)-alliances, therefore $\psi^{gp}_{-4}(CR(3t, 3)) = 3$.

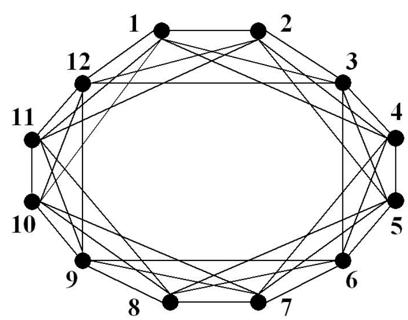

Figure 3.7: $\{\{1, 4, 7, 10\}, \{2, 5, 8, 11\}, \{3, 6, 9, 12\}\}$ is a partition of $CR(12, 3)$ into three global powerful (-4)-alliances.

Theorem 3.20. *Let $\Pi_r(G)$ be a partition of a graph G into r dominating sets. If there are two different sets in $\Pi_r(G)$ such that one of them is a defensive k-alliance and the other one is an offensive $(k + 2)$-alliance, then $k \leq 1 - r$.*

[3]See page 1.4.1 for the definition of circulant graph.

Proof. Let $\Pi_r(G) = \{S_1, S_2, ...S_r\}$ and let $S_i, S_j \in \Pi_r(G)$, $i \neq j$, such that S_i is an offensive $(k+2)$-alliance and S_j is a defensive k-alliance. For every $v \in S_j \subseteq \overline{S_i}$ we have $\delta_{S_i}(v) \geq \delta_{\overline{S_i}}(v) + k + 2$ and $\delta_{S_j}(v) \geq \delta_{\overline{S_j}}(v) + k$. Hence,

$$\begin{aligned}\delta_{S_i}(v) &\geq \delta_{\overline{S_i}}(v) + k + 2\\ &= \sum_{l=1, l\neq i}^{r} \delta_{S_l}(v) + k + 2\\ &= \sum_{l=1, l\neq i,j}^{r} \delta_{S_l}(v) + \delta_{S_j}(v) + k + 2\\ &\geq \sum_{l=1, l\neq i,j}^{r} \delta_{S_l}(v) + \delta_{\overline{S_j}}(v) + 2k + 2\end{aligned}$$

Moreover, since every set $S_l \in \Pi_r(G)$ is a dominating set, we have that $\sum_{l=1, l\neq i,j}^{r} \delta_{S_l}(v) \geq r - 2$. Thus,

$$\begin{aligned}\delta_{S_i}(v) &\geq \delta_{\overline{S_j}}(v) + 2k + r\\ &= \sum_{l=1, l\neq j}^{r} \delta_{S_l}(v) + 2k + r\\ &= \sum_{l=1, l\neq j,i}^{r} \delta_{S_l}(v) + \delta_{S_i}(v) + 2k + r\\ &\geq \delta_{S_i}(v) + 2k + 2r - 2.\end{aligned}$$

Therefore, $k + r - 1 \leq 0$. □

The above result has the following direct and useful consequences.

Corollary 3.21. *For $k \geq 0$, no graph is partitionable into global powerful k-alliances.*

Corollary 3.22. *If a graph G is partitionable into global powerful k-alliances, then $\psi_k^{gp}(G) \leq 1 - k$.*

Notice that this bound is achieved, for instance, for the complete graph, which can be partitioned into two global powerful (-1)-alliances.

Lemma 3.23. *Let G be a graph of maximum degree Δ and minimum degree δ. If S is a global powerful k-alliance in G, then*

$$(\delta + k + 2)|\overline{S}| \leq (\Delta - k)|S|.$$

Proof. If S is a defensive k-alliance, by equation (2.2) we have

$$\sum_{v \in S} \delta_{\overline{S}}(v) \leq \frac{\sum_{v \in S} \delta(v) - k|S|}{2} \leq \frac{(\Delta - k)|S|}{2}.$$

Moreover, if S is a global offensive $(k+2)$-alliance, by equation (1.2) we have

$$\sum_{v \in \overline{S}} \delta_S(v) \geq \frac{\sum_{v \in \overline{S}} \delta(v) + (k+2)|\overline{S}|}{2} \geq \frac{(\delta + k + 2)|\overline{S}|}{2}.$$

Now, as $\displaystyle\sum_{v \in S} \delta_{\overline{S}}(v) = \sum_{v \in \overline{S}} \delta_S(v)$, we obtain $(\delta + k + 2)|\overline{S}| \leq (\Delta - k)|S|$. □

Theorem 3.24. *Let G be a graph of minimum degree δ and maximum degree Δ. If G is partitionable into global powerful k-alliances, then*

$$\psi_k^{gp}(G) \leq \left\lfloor \frac{\Delta + \delta + 2}{\delta + k + 2} \right\rfloor.$$

Proof. Let $\Pi_r^{gp}(G) = \{S_1, S_2, ..., S_r\}$ be a partition of G into r global powerful k-alliances. By Lemma 3.23 we have that for every $S_i \in \Pi_r^{gp}(G)$, $(\delta + k + 2)|\overline{S_i}| \leq (\Delta - k)|S_i|$. Hence,

$$\begin{aligned}(\delta + k + 2)n(r-1) &= (\delta + k + 2)\sum_{i=1}^{r}(n - |S_i|) \\ &= (\delta + k + 2)\sum_{i=1}^{r}|\overline{S_i}| \\ &\leq (\Delta - k)\sum_{i=1}^{r}|S_i| \\ &= (\Delta - k)n.\end{aligned}$$

Thus, the bound follows. □

The above bound is achieved, for instance, for the complete graph $G = K_n$, for which $\psi_{-1}^{gp}(G) = 2$. Other example is the circulant graph $G = CR(3t, 3)$ where $\psi_{-4}^{gp}(G) = 3$ and the above bound leads to $\psi_{-4}^{gp}(G) \leq \lfloor \frac{14}{4} \rfloor = 3$.

Next we obtain other bound on $\psi_k^{gp}(G)$ in terms of n and k.

Theorem 3.25. *Let G be a graph of order n. If G is partitionable into global powerful k-alliances, then*

$$\psi_k^{gp}(G) \leq \left\lfloor \frac{\sqrt{8n + (2k-1)^2} - 2k + 1}{4} \right\rfloor.$$

Proof. Let $\Pi_r^{gp}(G) = \{S_1, ...S_r\}$ be a partition of G into r global powerful k-alliances, and let $S_i \in \Pi_r^{gp}(G)$ such that $|S_i| = \min\{|S_l| : S_l \in \Pi_r^{gp}(G)\}$. If $v \in S_j$, $j \neq i$, then, analogously to the proof of Theorem 3.20 we obtain $\delta_{S_i}(v) \geq \delta_{\overline{S_j}}(v) + 2k + r$. Thus,

$$\frac{n}{r} \geq |S_i| \geq \delta_{S_i}(v) \geq \delta_{\overline{S_j}}(v) + 2k + r = \sum_{l=1, l\neq j}^{r} \delta_{S_l}(v) + 2k + r \geq 2r + 2k - 1.$$

Therefore, the bound follows by solving the inequality $\frac{n}{r} \geq 2r + 2k - 1$ for r. □

The above bound is achieved, for instance, for the circulant graph $G = CR(10, 2)$, for which $\psi_{-4}^{gp}(G) = 5$. Now on we will study the relationship that exists between the powerful k-alliance partition number of Cartesian product graph and the powerful k-alliance partition number of its factors.

If $\Pi_{r_i}^{p}(G_i)$ is a partition of G_i into r_i powerful k_i-alliances, $i \in \{1, 2\}$, then by Theorem 3.10 we obtain a partition of $G_1 \times G_2$ into $r = r_1 r_2$ powerful k-alliances, with $k \in \{-\Delta_1 - \Delta_2, ..., \min\{k_1 - \Delta_2, k_2 - \Delta_1\}\}$. So, we obtain the following result.

Corollary 3.26. *Let $G_i = (V_i, E_i)$ be a graph of maximum degree Δ_i, $i \in \{1, 2\}$. If G_i is partitionable into r_i powerful k_i-alliances, then the graph $G_1 \times G_2$ is partitionable into $r = r_1 r_2$ powerful k-alliances, for every $k \in \{-\Delta_1 - \Delta_2, ..., \min\{k_1 - \Delta_2, k_2 - \Delta_1\}\}$. Moreover,*

$$\psi_k^p(G_1 \times G_2) \geq \psi_{k_1}^p(G_1)\psi_{k_2}^p(G_2).$$

Now, if $\Pi_{r_i}^{gp}(G_i)$ is a partition of G_i into r_i global powerful k_i-alliances, $i \in \{1, 2\}$, then by Theorem 3.12 we obtain the following result.

Corollary 3.27. *Let $G_i = (V_i, E_i)$ be a graph of order n_i and maximum degree Δ_i, $i \in \{1, 2\}$. If G_1 is partitionable into global powerful k_1-alliances, then for every $k \in \{-\Delta_1 - \Delta_2, ..., k_1 - \Delta_2\}$,*

$$\psi_k^{gp}(G_1 \times G_2) \geq \psi_{k_1}^{gp}(G_1).$$

For instance, if $G_1 = CR(3t, 3)$ and $G_2 = K_2$, then we have $\psi_{-5}^{gp}(G_1 \times G_2) = 3 = \psi_{-4}^{gp}(G_1)$.

Chapter 4

Alliance Free Sets and Alliance Cover Sets

Abstract

We investigate some mathematical properties of alliance free sets and alliance cover sets of a graph and its relationship with other structures of the graph like alliances and dominating set. Moreover, we study the closed relationships that exist between the (defensive, offensive, powerful) k-alliance free sets of Cartesian product graph and the (defensive, offensive, powerful) k-alliance free sets of its factors.

4.1 Introduction

A nonempty set $X \subseteq V$ is a *defensive* (respectively, *offensive* or *powerful*) *k-alliance free set*, k-daf (respectively, k-oaf or k-paf) of a graph $G = (V, E)$, if for all defensive (respectively, offensive or powerful) k-alliance S, $S \setminus X \neq \emptyset$, i.e., X does not contain any defensive (respectively, offensive or powerful) k-alliance as a subset. A (defensive offensive, powerful) k-alliance free set X is *maximal* if for every (defensive, offensive, powerful) k-alliance free set Y, $X \not\subset Y$. A *maximum* (k-daf, k-oaf, k-paf) set is a maximal (k-daf, k-oaf, k-paf) set of largest cardinality.

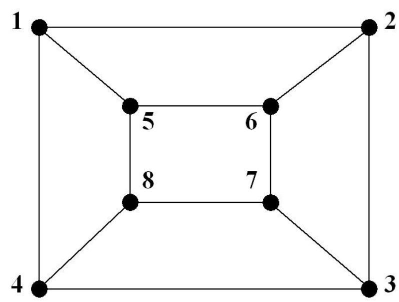

Figure 4.1: $S_1 = \{1, 3, 6, 8\}$ is a (-1)-daf, $\overline{S_1}$ is a (-1)-dac, $S_2 = \{1, 2, 3, 4\}$ is a (0)-oaf and $\overline{S_2}$ is a (0)-oac.

A nonempty set $Y \subseteq V$ is a *defensive* (respectively, *offensive* or *powerful*) *k-alliance cover*, k-dac (respectively, k-oac or k-pac) of G, if for all defensive (respectively, offensive or powerful) k-alliances S, $S \cap Y \neq \emptyset$, i.e., Y contains at least one vertex from each defensive (respectively, offensive or powerful) k-alliance in G. A (k-dac, k-oac, k-pac) set Y is *minimal* if no proper subset of Y is a (defensive, offensive, powerful) k-alliance cover set. A *minimum* (k-dac, k-oac, k-pac) set is a minimal cover set of smallest cardinality. For short, in the case of a global (offensive, powerful) k-alliance cover (respectively, free) set we will write (k-goac, k-gpac) (respectively, k-goaf, k-gpaf). Associated with the characteristic sets defined above we have the following invariants:

$\phi_k^d(G)$: cardinality of a maximum k-daf set in G.

$\phi_k^o(G)$: cardinality of a maximum k-oaf set in G.

$\phi_k^p(G)$: cardinality of a maximum k-paf set in G.

$\phi_k^{go}(G)$: cardinality of a maximum k-goaf set in G.

$\phi_k^{gp}(G)$: cardinality of a maximum k-gpaf set in G.

$\zeta_k^d(G)$: cardinality of a minimum k-dac set in G.

$\zeta_k^o(G)$: cardinality of a minimum k-oac set in G.

$\zeta_k^p(G)$: cardinality of a minimum k-pac set in G.

$\zeta_k^{go}(G)$: cardinality of a minimum k-goac set in G.

$\zeta_k^{gp}(G)$: cardinality of a minimum k-gpac set in G.

Here we present some of the principal known results about alliance free sets and alliance cover sets. We begin by presenting the following straightforward duality between alliance cover sets and alliance free sets showed in [64, 68].

Theorem 66. [64, 68] *X is a (defensive, offensive) k-alliance cover set if and only if $\overline{X}$ is (defensive, offensive) k-alliance free set.*

Corollary 67. [64, 68] $\phi_k^d(G) + \zeta_k^d(G) = \phi_k^o(G) + \zeta_k^o(G) = n$.

Corollary 68. [64, 68]

(i) *If X is a minimal (k-dac, k-oac) set, then, for all $v \in X$, there exists a (defensive, offensive) k-alliance S_v for which $S_v \cap X = \{v\}$.*

(ii) *If X is a maximal (k-daf, k-oaf) set, then, for all $v \in \overline{X}$, there exists $S_v \subseteq X$ such that $S_v \cup \{v\}$ is a (defensive, offensive) k-alliance.*

About alliance free sets and alliance cover sets there are just a few investigations and, in general, the principal results in this topic are frequently centered into obtaining lower and upper bound for the maximum alliance free sets or minimum alliance cover sets. In this sense, in [65] was showed the following upper bound for the maximum k-daf set of a graph.

Theorem 69. [65] *For every connected graph G and $0 \leq k \leq \Delta$,*

$$\phi_k^d(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor .$$

On the other hand, in [66] was studied the case of partitioning a graph into two defensive k-alliance free sets and there were characterized those graphs having such a partition. In this sense, in [66] the authors defined a graph G to be *partitionable* if it contains a partition into two defensive k-alliance free sets and there were obtained the following result.

Theorem 70. [66] *A connected graph G is partitionable if and only if G has a block that is other than an odd clique or an odd cycle.*

Moreover, in [66] it is defined the concept of defensive 0-alliance free cover as a set of vertices S, which is both a defensive 0-alliance free set and a defensive 0-alliance cover set. Equivalently, S is a defensive 0-alliance free cover if for all alliances X, $X \cap S \neq \emptyset$ and $X \cap (V - S) \neq \emptyset$. Thus, it is satisfied the following:

Lemma 71. [66] *A set S is a defensive 0-alliance free cover if and only if $V - S$ is a defensive 0-alliance free cover.*

Thus, from the above lemma and Theorem 66, in [66] was obtained the following result.

Theorem 72. [66] *A graph G is partitionable if and only if G has a defensive 0-alliance free cover.*

Other investigations in this topic have been centered into obtaining relationships between k-alliance free sets (k-alliance cover sets) and other structures of a graph. For instance, in [69] appeared some results about the relationship that exist among k-alliance free sets, k-alliance cover sets, defensive k-alliances, offensive k-alliances and dominating sets in a graph. Moreover, there were obtained the following bounds for the cardinality of maximum k-daf sets and maximum k-oaf sets of graphs in terms of order, minimum degree, algebraic connectivity and Laplacian spectral radius.

Theorem 73. [69] *For any connected graph G of order n, minimum degree δ and algebraic connectivity μ,*

$$\left\lceil \frac{n(k+\mu)-\mu}{n+\mu} \right\rceil \le \phi_k^d(G) \le \left\lfloor \frac{2n+k-\delta-1}{2} \right\rfloor .$$

Theorem 74. [69] *For any connected graph G of order n, minimum degree δ and Laplacian spectral radius μ_*,*

$$\zeta_k^d(G) \le \frac{n}{\mu_*}\left(\mu_* - \left\lceil \frac{\delta+k}{2} \right\rceil\right).$$

Theorem 75. [69] *For any graph G of order n and minimum degree δ,*

$$\left\lceil \frac{\delta+k-2}{2} \right\rceil \le \phi_k^o(G) \le \left\lfloor \frac{2n-\delta+k-3}{2} \right\rfloor .$$

For a more detailed study about alliance free sets and alliance cover sets and its application to data clustering we refer to the Ph. D. Thesis [67] and [69].

4.2 Alliance free sets and alliance cover sets

We begin by presenting the following relationship between the defensive k-alliance cover sets and dominating sets of a graph.

Theorem 4.1. *If X is a minimal k-dac set, then $\overline{X}$ is a dominating set.*

Proof. By Theorem 66, if X is a minimal k-dac set, then $\overline{X}$ is a maximal k-daf set. Therefore, for all $v \in X$, there exists $X_v \subseteq \overline{X}$ such that $X_v \cup \{v\}$ is a defensive k-alliance. So, for every $u \in X_v$, $\delta_{X_v}(u) + \delta_{\{v\}}(u) = \delta_{X_v \cup \{v\}}(u) \geq \delta_{\overline{X_v \cup \{v\}}}(u) + k = \delta_{\overline{X_v}}(u) - \delta_{\{v\}}(u) + k$. On the other hand, as X_v is not a defensive k-alliance, there exists $w \in X_v$ such that $\delta_{X_v}(w) < \delta_{\overline{X_v}}(w) + k$. Hence, by the above inequalities, $\delta_{\overline{X_v}}(w)+k+\delta_{\{v\}}(w) > \delta_{\overline{X_v}}(w)-\delta_{\{v\}}(w)+k$. Thus, $2\delta_{\{v\}}(w) > 0$ and, as a consequence, v is adjacent to w. □

Corollary 4.2. *Let G be a graph of order n. Then $\gamma(G) \leq n - \zeta_k^d(G)$.*

Notice that there exist minimal k-oac sets such that their complement sets are not dominating sets. For instance we consider the graph obtained from the cycle graph $C_8 = v_1v_2, ..., v_8v_1$ by adding the edge $\{v_1, v_3\}$ and the edge $\{v_5, v_7\}$. In this graph the set $S = \{v_2, v_3, v_5, v_6, v_7\}$ is a minimal 0-oac but $\bar{S}$ is not a dominating set.

Now, if one vertex $v \in V$ belongs to any offensive k-alliance, then $V \setminus \{v\}$ is a k-oaf set. Hence, $\delta(v) < k$. So, if $k \leq \delta$ and X is a minimal k-oac set, then $|X| \geq 2$.

Theorem 4.3. *For every $k \in \{2 - \Delta, ..., \Delta\}$, if X is a minimal k-goac set such that $|X| \geq 2$, then $\overline{X}$ is an offensive $(k-2)$-alliance. Moreover, if $k \in \{3, ..., \Delta\}$, then $\overline{X}$ is a global offensive $(k-2)$-alliance.*

Proof. If $X \subset V$ is a minimal k-goac set, then for all $v \in X$ there exists a global offensive k-alliance, S_v, such that $S_v \cap X = \{v\}$. Hence, for every $u \in \overline{S_v}$, $1 + \delta_{\overline{X}}(u) \geq \delta_{S_v}(u) \geq \delta_{\overline{S_v}}(u) + k \geq \delta_X(u) + k - 1$. As $X \setminus \{v\} \subset \overline{S_v}$, we have $\delta_{\overline{X}}(u) \geq \delta_X(u) + k - 2$ for every $u \in X \setminus \{v\}$. Now we take a vertex $w \in X \setminus \{v\}$ and by the above procedure, taking the vertex w instead of v, we obtain that $\delta_{\overline{X}}(v) \geq \delta_X(v) + k - 2$. Therefore, $\overline{X}$ is an offensive $(k-2)$-alliance. Moreover, if $k > 2$, $\overline{X}$ is a dominating set. So, in such a case, it is a global offensive $(k-2)$-alliance. □

Corollary 4.4. *For every $k \in \{3, ..., \delta\}$, $\phi_k^{go}(G) \geq \gamma_{k-2}^{o}(G)$ and $\zeta_k^{go}(G) \leq n - \gamma_{k-2}^{o}(G)$.*

Now we will present a result characterizing some classes of graphs which are defensive k-alliance free, i.e., the set of vertices of these graphs do not contain any defensive k-alliance.

Proposition 4.5. *Let G be a graph of order n and maximum degree Δ. Then $\phi_k^{d}(G) = n$, for each of the following cases:*

(i) *G is a tree of maximum degree $\Delta \geq 2$ and $k \in \{2, ..., \Delta\}$.*

(ii) *G is a planar graph of maximum degree $\Delta \geq 6$ and $k \in \{6, ..., \Delta\}$.*

(iii) *G is a planar triangle-free graph of maximum degree $\Delta \geq 4$ and $k \in \{4, ..., \Delta\}$.*

Proof. Suppose S is a defensive k-alliance in $G = (V, E)$. That is, for every $v \in S$, it follows

$$2\delta_S(v) \geq \delta(v) + k. \tag{4.1}$$

If some vertex $v \in S$ satisfies $\delta(v) < k$, then equation (4.1) leads to $\delta_S(v) > \delta(v)$, a contradiction. Hence, for every $v \in S$ we have $\delta(v) \geq k$ and, as a consequence, equation (4.1) leads to $\delta_S(v) \geq k$. Now, let m_s be the size of the subgraph induced by S. Then we have

$$2m_s = \sum_{v \in S} \delta_S(v) \geq k|S|. \tag{4.2}$$

Case (i). Since G is a tree, we obtain $2(|S| - 1) \geq 2m_s \geq k|S| \geq 2|S|$, a contradiction.

For the cases (ii) and (iii) we have $|S| \geq 3$, due to that if $|S| \leq 2$, then equation (4.1) leads to $2 \geq \delta(v) + k$, a contradiction. It is well-known that the size of a planar graph of order $n' \geq 3$ is bounded above by $3(n' - 2)$.

Moreover, in the case of triangle-free graphs the bound is $2(n'-2)$. Therefore, in case (ii) we have $m_s \leq 3(|S|-2)$ and, as a consequence, equation (4.2) leads to $6(|S|-2) \geq k|S| \geq 6|S|$, a contradiction. Analogously, in case (iii) we have $m_s \leq 2(|S|-2)$ and, as a consequence, equation (4.2) leads to $4(|S|-2) \geq k|S| \geq 4|S|$, a contradiction. □

Theorem 4.6. *If X is a k-goaf set, $k \in \{1, ..., \Delta-2\}$, such that $|X| \leq n-2$, then there exists $v \in \overline{X}$ such that $X \cup \{v\}$ is a $(k+2)$-goaf set.*

Proof. Let us suppose that for every $x \in \overline{X}$, $X \cup \{x\}$ is not a $(k+2)$-goaf set. Let $v \in \overline{X}$ and let $S_v \subset X$, such that $S_v \cup \{v\}$ is a global offensive $(k+2)$-alliance in G. Then for every $u \in \overline{S_v \cup \{v\}} = \overline{S_v} \setminus \{v\}$ we have $\delta_{S_v}(u) = \delta_{S_v \cup \{v\}}(u) - \delta_{\{v\}}(u) \geq \delta_{\overline{S_v \cup \{v\}}}(u) - \delta_{\{v\}}(u) + k + 2 = \delta_{\overline{S_v}}(u) - 2\delta_{\{v\}}(u) + k + 2 \geq \delta_{\overline{S_v}}(u) + k$. So, for every $u \in \overline{X} \setminus \{v\} \subset \overline{S_v} \setminus \{v\}$, $\delta_X(u) \geq \delta_{S_v}(u) \geq \delta_{\overline{S_v}}(u) + k \geq \delta_{\overline{X}}(u) + k$. Now we take a vertex $w \in \overline{X} \setminus \{v\}$ and by the above procedure, taking the vertex w instead of v, we obtain that $\delta_X(v) \geq \delta_{\overline{X}}(v) + k$. So, X is a global offensive k-alliance, a contradiction. □

If X is a k-goaf for $k \leq \delta$, then $|X| \leq n-2$, as a consequence, the above result can be simplified as follows.

Corollary 4.7. *If X is a k-goaf set, $k \in \{1, ..., \delta\}$, then there exists $v \in \overline{X}$ such that $X \cup \{v\}$ is a $(k+2)$-goaf set.*

It is easy to check the monotony of ϕ_k^{go}, i.e., $\phi_k^{go}(G) \leq \phi_{k+1}^{go}(G)$. As we can see below, Theorem 4.6 leads to an interesting property about the monotony of ϕ_k^{go}.

Corollary 4.8. *For every $k \in \{1, ..., \min\{\delta, \Delta-2\}\}$ and $r \in \left\{1, ..., \lfloor \frac{\Delta-k}{2} \rfloor\right\}$,*

$$\phi_k^{go}(G) + r \leq \phi_{k+2r}^{go}(G).$$

Theorem 4.9. *If X is a k-daf set and $v \in \overline{X}$, then $X \cup \{v\}$ is $(k+2)-daf$.*

Proof. Let us suppose that there exists a defensive $(k+2)$-alliance A such that $A \subseteq X \cup \{v\}$. If $v \notin A$, then $A \subset X$, a contradiction because every defensive (k+2)-alliance is a defensive k-alliance. If $v \in A$, let $B = A \setminus \{v\}$. For every $u \in B$ we have, $\delta_B(u) = \delta_A(u) - \delta_{\{v\}}(u) \geq \delta_{\overline{A}}(u) + k + 2 - \delta_{\{v\}}(u) \geq \delta_{\overline{B}}(u) + k + 2\left(1 - \delta_{\{v\}}(u)\right) \geq \delta_{\overline{B}}(u) + k$. So, $B \subseteq X$ is a defensive k-alliance, a contradiction. □

Corollary 4.10. *For every* $k \in \{-\Delta, ..., \Delta - 2\}$ *and* $r \in \left\{1, ..., \lfloor \frac{\Delta - k}{2} \rfloor\right\}$,

$$\phi_k^d(G) + r \leq \phi_{k+2r}^d(G).$$

From Theorem 69 we have a lower bound for the maximum defensive k-alliance free sets of a graph in terms of the order. In order to obtain a similar result for maximum global offensive k-alliance free sets of a graph we present at next the following lemma.

Lemma 4.11. *If* $\{X, Y\}$ *is a vertex partition of a graph* G *into two global boundary offensive* 0*-alliances, then* X *and* Y *are minimal global offensive* 0*-alliances in* G.

Proof. Let us suppose, for instance, that X is not a minimal global offensive 0-alliances. Hence, there exists $A \subset X$, such that, $X \setminus A \neq \emptyset$ and A is a global offensive 0-alliance. Thus, for every $v \in \overline{A}$, $\delta_X(v) \geq \delta_A(v) \geq \delta_{\overline{A}}(v) \geq \delta_Y(v)$.

As $Y \subset \overline{A}$ and $\{X, Y\}$ is a vertex partition of the graph into two global boundary offensive 0-alliances, then for every $v \in Y$, $\delta_Y(v) = \delta_X(v) \geq \delta_A(v) \geq \delta_{\overline{A}}(v) \geq \delta_Y(v)$. Thus, for every $v \in Y$ we have $\delta_A(v) = \delta_{\overline{A}}(v) = \delta_Y(v) + \delta_{X\setminus A}(v) = \delta_X(v) + \delta_{X\setminus A}(v) = \delta_A(v) + 2\delta_{X\setminus A}(v)$. Hence, we have that Y is a dominating set and for every $v \in Y$, $\delta_{X\setminus A}(v) = 0$, a contradiction. So, X and Y are minimal global offensive 0-alliances. □

Theorem 4.12. *For every* $k \in \{0, ..., \Delta\}$, $\phi_k^{go}(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor - 1$.

Proof. First, we will prove the case $k = 0$. Let $\{X, Y\}$ be a partition of the vertex set, such that $|X| = \lfloor \frac{n}{2} \rfloor$, $|Y| = \lceil \frac{n}{2} \rceil$ and there is a minimum number of edges between X and Y. If X (or Y) is a 0-goaf set, then $\phi_0^{go}(G) \geq \lfloor \frac{n}{2} \rfloor - 1$. We suppose there exist $A \subseteq X$ and $B \subseteq Y$, such that A and B are global offensive 0-alliances. Hence $\delta_X(v) \geq \delta_A(v) \geq \delta_{\overline{A}}(v) \geq \delta_Y(v), \quad \forall v \in \overline{A}$, and $\delta_Y(v) \geq \delta_B(v) \geq \delta_{\overline{B}}(v) \geq \delta_X(v), \quad \forall v \in \overline{B}$. As $Y \subset \overline{A}$ and $X \subset \overline{B}$ we have, for every $v \in Y$, $\delta_X(v) \geq \delta_Y(v)$ and for every $v \in X$, $\delta_Y(v) \geq \delta_X(v)$.

For any $y \in Y$ and $x \in X$, let us take $X' = X \setminus \{x\} \cup \{y\}$ and $Y' = Y \setminus \{y\} \cup \{x\}$. If $\delta_X(y) > \delta_Y(y)$ or $\delta_Y(x) > \delta_X(x)$ then, the edge cutset between X' and Y' is lesser than the other one between X and Y, a contradiction. Therefore $\delta_X(y) = \delta_Y(y)$ and $\delta_Y(x) = \delta_X(x)$ and, as a consequence, $\{X, Y\}$ is a partition of the vertex set into two global boundary offensive 0-alliances. Now, by using Lemma 4.11 we obtain that X and Y are minimal global offensive 0-alliances. As a consequence, $\phi_0^{go}(G) \geq \lfloor \frac{n}{2} \rfloor - 1$.

Now, let us prove the case $k > 0$. Case 1: $\phi_k^{go}(G) \geq n - 2$. Since $n-1 \geq \lfloor \frac{2n-1}{2} \rfloor \geq \lfloor \frac{n+\Delta}{2} \rfloor \geq \lfloor \frac{n+k}{2} \rfloor \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor$, we have $\phi_k^{go}(G) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor - 1$. Case 2: $\phi_k^{go}(G) < n - 2$. As every k-goaf set is also a $(k+1)$-goaf set, $\phi_1^{go}(G) \geq \phi_0^{go}(G) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{1}{2} \rfloor - 1$, then the statement is true for $k = 1$. Hence, we will proceed by induction on k. Let us assume that the statement is true for an arbitrary $k \in \{2, ..., \Delta - 2\}$, that is, there exists a maximal k-goaf set X in G such that, $|X| = \phi_k^{go}(G) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor - 1$. Now, by Theorem 4.6 there exists $v \in \overline{X}$, such that $X \cup \{v\}$ is a $(k+2)$-goaf set. Therefore, $\phi_{k+2}^{go}(G) \geq |X \cup \{v\}| \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k}{2} \rfloor = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{k+2}{2} \rfloor - 1$. So, the proof is complete. □

The above bound is attained, for instance, in the case of the complete graph if n and k are both even or if n and k have different parity: $\phi_k^{go}(K_n) = \lfloor \frac{n+k-2}{2} \rfloor$. At next we obtain a general bound for the global powerful k-alliance free (cover) sets of a graph in terms of its order.

Theorem 4.13. *For any graph $G=(V,E)$ of order n,*

$$\zeta_k^{gp}(G) \le \left\lfloor \frac{n^2-n(k-1)-2}{n+2} \right\rfloor \text{ and } \phi_k^{gp}(G) \ge \left\lceil \frac{n(k-3)+2}{n+2} \right\rceil .$$

Proof. Let S be a minimal k-gpac in a graph. Hence, for every $v \in S$ there exists a global powerful k-alliance S_v, such that $S_v \cap S = \{v\}$. Thus, for every $u \in S_v$, $\delta_{S_v}(u) \ge \delta_{\overline{S_v}}(u)+k$ and for every $u \in \overline{S_v}$, $\delta_{S_v}(u) \ge \delta_{\overline{S_v}}(u)+k+2$ and we have

$$\sum_{u\in S_v} \delta_{S_v}(u) \ge \sum_{u\in S_v} \delta_{\overline{S_v}}(u) + \sum_{u\in S_v} k,$$

$$\sum_{u\in \overline{S_v}} \delta_{S_v}(u) \ge \sum_{u\in \overline{S_v}} \delta_{\overline{S_v}}(u) + \sum_{u\in \overline{S_v}} (k+2).$$

So, we obtain that

$$\begin{aligned} n(n-|S|+1) &\ge n|S_v| \\ &\ge \sum_{u\in V} \delta_{S_v}(u) \\ &\ge \sum_{u\in V} \delta_{\overline{S_v}}(u) + \sum_{u\in V} k + \sum_{u\in \overline{S_v}} 2 \\ &\ge nk + 2(n-|S_v|) \\ &\ge nk + 2(n-(|\overline{S}|+1)) \\ &= nk + 2(|S|-1). \end{aligned}$$

Therefore, $n(n-|S|+1) \ge nk+2(|S|-1)$ and by solving this inequality for $|S|$ we obtain the bound for $\zeta_k^{gp}(\Gamma)$. Since $\phi_k^{gp}(\Gamma) = n - \zeta_k^{gp}(\Gamma)$ we obtain the other bound. □

4.3 k-daf sets in Cartesian product graphs

To begin with the study we present the following straightforward result, where $\alpha(G)$ represents the independence number of G.

Remark 4.14. *Let G_i be a graph of order n_i, minimum degree δ_i and maximum degree Δ_i, $i \in \{1,2\}$. Then, for every $k \in \{1-\delta_1-\delta_2, ..., \Delta_1+\Delta_2\}$,*

$$\phi^d_k(G_1 \times G_2) \geq \alpha(G_1)\alpha(G_2) + \min\{n_1 - \alpha(G_1), n_2 - \alpha(G_2)\}.$$

Proof. For every graph G of minimum degree δ and maximum degree Δ, any independent set in G is a k-daf set for $k \in \{1-\delta, ..., \Delta\}$. Hence, $\phi^d_k(G_1 \times G_2) \geq \alpha(G_1 \times G_2)$, for every $k \in \{1-\delta_1-\delta_2, ..., \Delta_1+\Delta_2\}$, and by the Vizing's inequality, $\alpha(G_1 \times G_2) \geq \alpha(G_1)\alpha(G_2) + \min\{n_1 - \alpha(G_1), n_2 - \alpha(G_2)\}$, we obtain the result. □

Let G_1 be the star graph of order $t+1$ and let G_2 be the path graph of order 3. In this case, $\phi^d_k(G_1 \times G_2) = 2t+1$ for $k \in \{-1, 0\}$. Therefore, the above bound is tight. Even so, Corollary 4.16 (ii) improves the above bound for the cases where $\phi^d_{k_i}(G_i) > \alpha(G_i)$, for some $i \in \{1,2\}$.

Theorem 4.15. *Let $G_i = (V_i, E_i)$ be a simple graph of maximum degree Δ_i, $i \in \{1,2\}$, and let $S \subseteq V_1 \times V_2$. Then the following assertions hold.*

(i) *If $P_{V_i}(S)$ is a k_i-daf set in G_i, then S is a $(k_i + \Delta_j)$-daf set in $G_1 \times G_2$, where $j \in \{1,2\}$, $j \neq i$.*

(ii) *If for every $i \in \{1,2\}$, $P_{V_i}(S)$ is a k_i-daf set in G_i, then S is a $(k_1 + k_2 - 1)$-daf set in $G_1 \times G_2$.*

Proof. Let $A \subseteq S$ and we suppose $P_{V_1}(S)$ is a k_1-daf set in G_1. Since $P_{V_1}(A) \subseteq P_{V_1}(S)$, there exists $a \in P_{V_1}(A)$ such that $\delta_{P_{V_1}(A)}(a) < \delta_{\overline{P_{V_1}(A)}}(a) + k_1$. If we take $b \in V_2$ such that $(a,b) \in A$, then

$$\delta_A(a,b) \leq \delta_{P_{V_1}(A)}(a) + \delta_{P_{V_2}(A)}(b) < \delta_{\overline{P_{V_1}(A)}}(a) + k_1 + \delta(b) \leq \delta_{\overline{P_{V_1}(A)}}(a) + k_1 + \Delta_2.$$

Thus, A is not a defensive $(k_1 + \Delta_2)$-alliance in $G_1 \times G_2$. Therefore, (i) follows.

In order to prove (ii), let $x \in X = P_{V_1}(A)$ such that $\delta_X(x) < \delta_{\overline{X}}(x) + k_1$. Let $A_x \subseteq A$ be the set composed by the elements of A whose first component is x. On the other hand, since $P_{V_2}(S)$ is a k_2-daf set and $Y = P_{V_2}(A_x) \subseteq P_{V_2}(S)$, there exists $y \in Y$ such that $\delta_Y(y) < \delta_{\overline{Y}}(y) + k_2$. Notice that $(x, y) \in A$. Let $A_y \subseteq A$ be the set composed by the elements of A whose second component is y. Hence,

$$\begin{aligned}
\delta_A(x,y) &= \delta_{A_x}(x,y) + \delta_{A_y}(x,y) \\
&\le \delta_Y(y) + \delta_X(x) \\
&< \delta_{\overline{Y}}(y) + \delta_{\overline{X}}(x) + k_1 + k_2 - 1 \\
&\le \delta_{\overline{A_x}}(x,y) - \delta(x) + \delta_{\overline{A_y}}(x,y) - \delta(y) + k_1 + k_2 - 1 \\
&\le \delta_{\overline{A_x}}(x,y) + \delta_{\overline{A_y}}(x,y) + k_1 + k_2 - 1 \\
&= \delta_{\overline{A}}(x,y) + k_1 + k_2 - 1.
\end{aligned}$$

Thus, A is not a defensive $(k_1 + k_2 - 1)$-alliance in $G_1 \times G_2$ and, as a consequence, (ii) follows. □

Corollary 4.16. *Let G_l be a graph of order n_l, maximum degree Δ_l and minimum degree δ_l, with $l \in \{1, 2\}$. Then the following assertions hold.*

(i) *For every $k \in \{\Delta_j - \Delta_i, ..., \Delta_i + \Delta_j\}$ $(i, j \in \{1, 2\},\ i \neq j)$,*

$$\phi_k^d(G_1 \times G_2) \ge n_j \phi_{k-\Delta_j}^d(G_i).$$

(ii) *For every $k_i \in \{1 - \delta_i, ..., \Delta_i\}$, $i \in \{1, 2\}$,*

$$\phi_{k_1+k_2-1}^d(G_1 \times G_2) \ge \phi_{k_1}^d(G_1)\phi_{k_2}^d(G_2) + \min\{n_1 - \phi_{k_1}^d(G_1), n_2 - \phi_{k_2}^d(G_2)\}.$$

Proof. By Theorem 4.15 (i) we conclude that for every k_i-daf set S_i in G_i, $i \in \{1, 2\}$, the sets $S_1 \times V_2$ and $V_1 \times S_2$ are $(k_1 + \Delta_2)$-daf and $(k_2 + \Delta_1)$-daf, respectively, in $G_1 \times G_2$. Therefore, (i) follows.

In order to prove (ii), let $V_1 = \{u_1, u_2, ..., u_{n_1}\}$ and $V_2 = \{v_1, v_2, ..., v_{n_2}\}$. Moreover, let S_i be a k_i-daf set of maximum cardinality in G_i, $i \in \{1, 2\}$. We suppose $S_1 = \{u_1, ..., u_r\}$ and $S_2 = \{v_1, ..., v_s\}$. By Theorem 4.15 (ii) we deduce that $S_1 \times S_2$ is a $(k_1 + k_2 - 1)$-daf set in $G_1 \times G_2$. Now let $X = \{(u_{r+i}, v_{s+i}), i = 1, ..., t\}$, where $t = \min\{n_1 - r, n_2 - s\}$ and let $S = X \cup (S_1 \times S_2)$. Since, for every $x \in X$, $\delta_S(x) = 0$ and $k_i > -\delta_i$, $i \in \{1, 2\}$, we obtain that S is a (k_1+k_2-1)-daf set in $G_1 \times G_2$. Thus, $\phi^d_{k_1+k_2-1}(G_1 \times G_2) \geq |S| = \phi^d_{k_1}(G_1)\phi^d_{k_2}(G_2) + \min\{n_1 - \phi^d_{k_1}(G_1), n_2 - \phi^d_{k_2}(G_2)\}$. □

We emphasize that Corollary 4.16 and Proposition 4.5 lead to infinite families of graphs whose Cartesian product satisfies $\phi^d_k(G_1 \times G_2) = n_1 n_2$. For instance, if G_1 is a tree of order n_1 and maximum degree $\Delta_1 \geq 2$, G_2 is a graph of order n_2 and maximum degree Δ_2, and $k \in \{2 + \Delta_2, ..., \Delta_1 + \Delta_2\}$, we have $\phi^d_k(G_1 \times G_2) = \phi^d_{k-\Delta_2}(G_1) n_2 = n_1 n_2$. In particular, if G_2 is a cycle graph, then $\phi^d_4(G_1 \times G_2) = n_1 n_2$.

Another example of equality in Corollary 4.16 (ii) is obtained, for instance, taking the Cartesian product of the star graph S_t of order $t+1$ and the path graph P_r of order r. In that case, for $G_1 = S_t$ we have $\delta_1 = 1$, $n_1 = t+1$ and $\phi^d_0(G_1) = t$, and, for $G_2 = P_r$ we have $\delta_2 = 1$, $n_2 = r$ and $\phi^d_1(G_2) = r-1$. Therefore, $\phi^d_0(G_1)\phi^d_1(G_2) + \min\{n_1 - \phi^d_0(G_1), n_2 - \phi^d_1(G_2)\} = t(r-1)+1$. On the other hand, it is not difficult to check that, if we take all leaves belonging to the copies of S_t corresponding to the first $r-1$ vertices of G_2 and we add the vertex of degree t belonging to the last copy of S_t, we obtain a maximum defensive 0-alliance free set of cardinality $t(r-1)+1$ in the graph $G_1 \times G_2$, that is, $\phi^d_0(G_1 \times G_2) = t(r-1)+1$. This example also shows that this bound is better than the bound obtained in Remark 4.14 which is $t\left\lceil \frac{r}{2} \right\rceil + 1$. In this particular case, both bounds are equal if and only if $r = 2$ or $r = 3$.

Theorem 4.17. *Let $G_i = (V_i, E_i)$ be a graph and let $S_i \subseteq V_i$, $i \in \{1, 2\}$. If $S_1 \times S_2$ is a k-daf set in $G_1 \times G_2$ and S_2 is a defensive k'-alliance in G_2,*

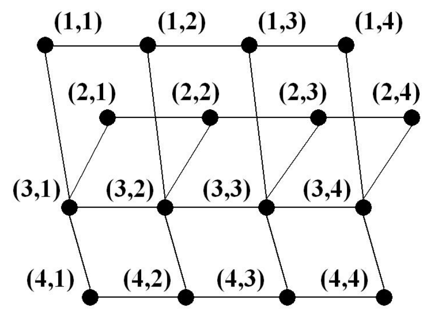

Figure 4.2: This graph is the Cartesian product $S_3 \times P_4$ where $\{(1,1),(2,1),(4,1),(1,2),\ (2,2),(4,2),(1,3),(2,3),(4,3),(3,4)\}$ is a maximum defensive 0-alliance free set.

then S_1 is a $(k-k')$-daf set in G_1.

Proof. If $S \subseteq S_1$, then $S \times S_2 \subseteq S_1 \times S_2$ is a k-daf set in $G_1 \times G_2$. So, there exists $(a,b) \in S \times S_2$ such that $\delta_{S\times S_2}(a,b) < \delta_{\overline{S\times S_2}}(a,b) + k$. Thus, we have

$$\delta_S(a) + \delta_{S_2}(b) = \delta_{S\times S_2}(a,b) < \delta_{\overline{S\times S_2}}(a,b) + k = \delta_{\overline{S}}(a) + \delta_{\overline{S_2}}(b) + k. \quad (4.3)$$

As S_2 is a defensive k'-alliance in G_2, for every $b \in S_2$ we have, $\delta_{S_2}(b) \geq \delta_{\overline{S_2}}(b) + k'$. Hence, from equation (4.3) we obtain $\delta_S(a) < \delta_{\overline{S}}(a) + k - k'$. Therefore, S is not a defensive $(k-k')$-alliance in G_1 and, as a consequence, S_1 is a $(k-k')$-daf set. □

Taking into account that V_2 is a defensive δ_2-alliance in G_2 we obtain the following result.

Corollary 4.18. *Let $G_i = (V_i, E_i)$ be a graph, $i \in \{1,2\}$. Let δ_2 be the minimum degree of G_2 and let $S_1 \subseteq V_1$. If $S_1 \times V_2$ is a k-daf set in $G_1 \times G_2$, then S_1 is a $(k-\delta_2)$-daf set in G_1.*

By Theorem 4.15 (i) and Corollary 4.18 we obtain the following result.

Proposition 4.19. *Let G_1 be a graph of maximum degree Δ_1 and let G_2 be a δ_2-regular graph. For every $k \in \{\delta_2 - \Delta_1, ..., \Delta_1 + \delta_2\}$, $S_1 \times V_2$ is a k-daf set in $G_1 \times G_2$ if and only if S_1 is a $(k - \delta_2)$-daf set in G_1.*

4.4 k-oaf sets in Cartesian product graphs

Theorem 4.20. *Let $G_i = (V_i, E_i)$ be a graph, $i \in \{1, 2\}$, and let $S \subset V_1 \times V_2$. If $P_{V_i}(S)$ is a k-oaf set in G_i, then S is a $(k - \delta_j)$-oaf set in $G_1 \times G_2$, where δ_j denotes the minimum degree of G_j and $j \in \{1, 2\}$, $i \neq j$.*

Proof. If $P_{V_1}(S)$ is a k-oaf set in G_1 and $A \subseteq S$, then $P_{V_1}(A) \subseteq P_{V_1}(S)$ is a k-oaf set in G_1. So, there exists $a \in \partial(P_{V_1}(A))$, such that $\delta_{P_{V_1}(A)}(a) < \delta_{\overline{P_{V_1}(A)}}(a) + k$. Let $a' \in P_{V_1}(A)$ such that a and a' are adjacent, and let $Y_{a'}$ be the set of elements of A whose first component is a'. Thus, if $b \in P_{V_2}(Y_{a'})$, then $(a, b) \in \partial(A)$, so we have

$$\delta_A(a, b) \leq \delta_{P_{V_1}(A)}(a) < \delta_{\overline{P_{V_1}(A)}}(a) + k \leq \delta_{\overline{A}}(a, b) - \delta(b) + k \leq \delta_{\overline{A}}(a, b) + k - \delta_2.$$

Therefore, A is not an offensive $(k - \delta_2)$-alliance in $G_1 \times G_2$. The proof of the other case is completely analogous. □

From Theorem 4.20 we conclude that for every k_i-oaf set S_i in G_i, $i \in \{1, 2\}$, the sets $S_1 \times V_2$ and $V_1 \times S_2$ are $(k_1 - \delta_2)$-oaf and $(k_2 - \delta_1)$-oaf, respectively, in $G_1 \times G_2$. Therefore, we obtain the following result.

Corollary 4.21. *Let G_l be a graph of order n_l, maximum degree Δ_l and minimum degree δ_l, $l \in \{1, 2\}$. Then, for every $k \in \{2 - \delta_j - \Delta_i, ..., \Delta_i - \delta_j\}$, $\phi_k^o(G_1 \times G_2) \geq n_j \phi_{k+\delta_j}^o(G_i)$, where $i, j \in \{1, 2\}$, $i \neq j$.*

Example of equality in the above result is the following. If we take $G_1 = C_4$, $G_2 = P_3$ and $k_2 = 2$, then $\phi_0^o(C_4 \times P_3) = 8 = 4\phi_2^o(P_3)$.

Theorem 4.22. *Let $G_i = (V_i, E_i)$ be a graph of minimum degree δ_i and maximum degree Δ_i. If S_i is a k_i-oaf set in G_i, $i \in \{1, 2\}$, then for every $k \in \{k', ..., \Delta_1 + \Delta_2\}$, $(S_1 \times V_2) \cup (V_1 \times S_2)$ is a k-oaf set in $G_1 \times G_2$, where $k' = \max\{k_1 - \delta_2, k_2 - \delta_1, \min\{k_2 + \Delta_1, k_1 + \Delta_2\}\}$.*

Proof. Let $A \subseteq (S_1 \times V_2) \cup (V_1 \times S_2)$. By Theorem 4.20 we deduce that, if $A \subseteq S_1 \times V_2$, then A is a $(k_1 - \delta_2)$-oaf set in $G_1 \times G_2$. Analogously, if $A \subseteq V_1 \times S_2$, then A is a $(k_2 - \delta_1)$-oaf set in $G_1 \times G_2$.

Now we suppose $A \nsubseteq S_1 \times V_2$ and $A \nsubseteq V_1 \times S_2$. Let $B = A \setminus (S_1 \times V_2)$. For every $a \in P_{V_1}(B)$, the set Y_a, composed by the elements of B whose first component is a, satisfies that $P_{V_2}(Y_a)$ is a k_2-oaf set in G_2. Then, there exists $b \in \partial(P_{V_2}(Y_a))$ such that $\delta_{P_{V_2}(Y_a)}(b) < \delta_{\overline{P_{V_2}(Y_a)}}(b) + k_2$. Also, notice that $(a, b) \in \partial(A)$. Thus,

$$\delta_A(a, b) \le \delta_{P_{V_2}(Y_a)}(b) + \delta(a) < \delta_{\overline{P_{V_2}(Y_a)}}(b) + k_2 + \delta(a) \le \delta_{\overline{A}}(a, b) + k_2 + \Delta_1.$$

We conclude that A is not an offensive $(k_2 + \Delta_1)$-alliance in $G_1 \times G_2$. Analogously, A is not an offensive $(k_1 + \Delta_2)$-alliance in $G_1 \times G_2$. Therefore, the result follows. □

Corollary 4.23. *Let G_i be a graph of order n_i, minimum degree δ_i and maximum degree Δ_i, $i \in \{1, 2\}$. Let $k' = \max\{k_1 - \delta_2, k_2 - \delta_1, \min\{k_2 + \Delta_1, k_1 + \Delta_2\}\}$, where $k_i \in \{2 - \Delta_i, ..., \Delta_i\}$. Then, for every $k \in \{k', ..., \Delta_1 + \Delta_2\}$,*

$$\phi^o_k(G_1 \times G_2) \ge n_1 \phi^o_{k_2}(G_2) + n_2 \phi^o_{k_1}(G_1) - \phi^o_{k_1}(G_1)\phi^o_{k_2}(G_2).$$

For instance, if we take $G_1 = C_3$, $G_2 = P_3$, $k_1 = 1$ and $k_2 = 2$, then $\phi^o_3(C_3 \times P_3) = 7 = 3\phi^o_2(P_3) + 3\phi^o_1(C_3) - \phi^o_1(C_3)\phi^o_2(P_3)$.

4.5 k-paf sets in Cartesian product graphs

Since for every graph G, $\phi^p_k(G) \ge \max\{\phi^d_k(G), \phi^o_{k+2}(G)\}$, we have that lower bounds on $\phi^d_k(G)$ and $\phi^o_{k+2}(G)$ lead to lower bounds on $\phi^p_k(G)$. So, by the

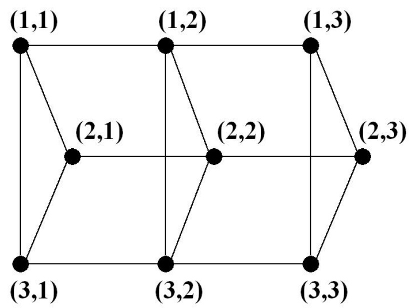

Figure 4.3: The graph $G = (V, E)$ is the Cartesian product of the cycle graph C_3 by the path graph P_3 where $S = V \setminus \{(1,3),(2,3)\}$ is a maximum offensive 3-alliance free set.

results obtained in the above sections on $\phi_k^d(G_1 \times G_2)$ and $\phi_{k+2}^o(G_1 \times G_2)$ we deduce lower bounds on $\phi_k^p(G_1 \times G_2)$.

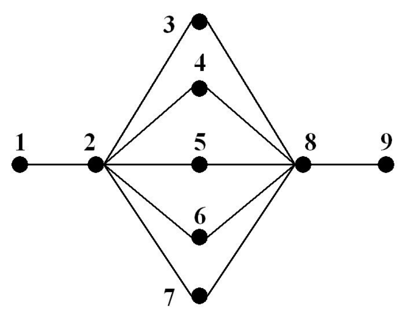

Figure 4.4: A graph $G = (V, E)$ where V is a powerful 2-alliance free set, although $\{2,3,4,5,6,8\}$ is a defensive 2-alliance and $\{3,4,5,6,7\}$ is an offensive 4-alliance.

We emphasize that there are graphs where $\phi_k^p(G) > \max\{\phi_k^d(G), \phi_{k+2}^o(G)\}$. For instance, the graph of Figure 4.4 satisfies $\phi_2^p(G) = 9$ while $\phi_2^d(G) = 8$ and $\phi_4^o(G) = 7$.

Theorem 4.24. *Let $G_i = (V_i, E_i)$ be a simple graph of maximum degree Δ_i*

and minimum degree δ_i, $i \in \{1,2\}$, and let $S \subseteq V_1 \times V_2$. Then the following assertions hold.

(i) *If $P_{V_i}(S)$ is a k_i-paf set in G_i, then, for every $k \in \{k_i + \Delta_j, ..., \Delta_i + \Delta_j - 2\}$, S is a k-paf set in $G_1 \times G_2$, where $j \in \{1,2\}$, $j \neq i$.*

(ii) *If for every $i \in \{1,2\}$, $P_{V_i}(S)$ is a k_i-paf set in G_i, then, for every $k \in \{k', ..., \Delta_1 + \Delta_2 - 2\}$, S is a k-paf set in $G_1 \times G_2$, where $k' = \max\{k_1 + k_2 - 1, \min\{k_2 - \delta_1, k_1 - \delta_2\}\}$.*

Proof. Let $A \subseteq S$. We suppose $P_{V_i}(S)$ is a k_i-paf set in G_i for some $i \in \{1,2\}$. Since $P_{V_i}(A) \subseteq P_{V_i}(S)$, it follows that $P_{V_i}(A)$ is not a powerful k_i-alliance in G_i. If $P_{V_i}(A)$ is not a defensive k_i-alliance, by analogy to the proof of Theorem 4.15 (i) we obtain that A is not a defensive $(k_i + \Delta_j)$-alliance in $G_1 \times G_2$, $j \neq i$. If $P_{V_i}(A)$ is not an offensive $(k_i + 2)$-alliance in G_i, then by analogy to the proof of Theorem 4.20 we obtain that A is not an offensive $(k_i - \delta_j + 2)$-alliance in $G_1 \times G_2$, $j \neq i$. Since, $k_i + \Delta_j > k_i - \delta_j$, we obtain that A is not a powerful $(k_i + \Delta_j)$-alliance in $G_1 \times G_2$. Therefore, (i) follows.

If for every $l \in \{1,2\}$, $P_{V_l}(S)$ is a k_l-paf set in G_l, then $P_{V_l}(A)$ is not a powerful k_l-alliance in G_l. Hence, we differentiate two cases.

Case 1: For some $l \in \{1,2\}$, $P_{V_l}(A)$ is not a defensive k_l-alliance. We suppose $P_{V_1}(A)$ is not a defensive k_1-alliance. Hence, there exists $x \in P_{V_1}(A)$ such that $\delta_{P_{V_1}(A)}(x) < \delta_{\overline{P_{V_1}(A)}}(x) + k_1$. Let $A_x \subseteq A$ be the set composed by the elements of A whose first component is x. If $P_{V_2}(A_x) \subset P_{V_2}(S)$ is not a defensive k_2-alliance, then by analogy to the proof of Theorem 4.15 (ii) we obtain that A is not a defensive $(k_1 + k_2 - 1)$-alliance in $G_1 \times G_2$. On the other hand, if $P_{V_2}(A_x)$ is a defensive k_2-alliance, then it is not an offensive $(k_2 + 2)$-alliance. Thus, there exists $y \in \partial(P_{V_2}(A_x))$ such that

$\delta_{P_{V_2}(A_x)}(y) < \delta_{\overline{P_{V_2}(A_x)}}(y) + (k_2+2)$. We note that $(x,y) \in \partial(A)$. Hence,

$$\begin{aligned}\delta_A(x,y) &\leq \delta_{P_{V_1}(A)}(x) + \delta_{P_{V_2}(A_x)}(y)\\ &< \delta_{\overline{P_{V_1}(A)}}(x) + \delta_{\overline{P_{V_2}(A_x)}}(y) + k_1 + k_2 + 1\\ &\leq \delta_{\overline{A}}(x,y) + k_1 + k_2 + 1.\end{aligned}$$

As a consequence, A is not an offensive (k_1+k_2+1)-alliance in $G_1 \times G_2$. Thus, in this case, A is not a powerful (k_1+k_2-1)-alliance in $G_1 \times G_2$.

Case 2: For every $i \in \{1,2\}$, $P_{V_i}(A)$ is not an offensive (k_i+2)-alliance in G_i. In this case, as we have shown in the proof of (i), A is not an offensive $(k_i - \delta_j + 2)$-alliance in $G_1 \times G_2$, $j \in \{1,2\}$, $j \neq i$.

As a consequence, for $k = \max\{k_1+k_2-1, k_1-\delta_2, k_2-\delta_1\}$, A is not a powerful k-alliance in $G_1 \times G_2$. Hence, S is a k-paf set in $G_1 \times G_2$. Therefore, (ii) follows. □

Corollary 4.25. *Let G_l be a graph of order n_l, maximum degree Δ_l and minimum degree δ_l, $l \in \{1,2\}$. Let $k_l \in \{1-\delta_l, ..., \Delta_l - 2\}$. Then the following assertions hold.*

(i) *For every $k \in \{\Delta_j - \Delta_i, ..., \Delta_i + \Delta_j - 2\}$, $(i,j \in \{1,2\}, i \neq j)$*

$$\phi_k^p(G_1 \times G_2) \geq n_j \phi_{k-\Delta_j}^p(G_i).$$

(ii) *For every $k \in \{k_1+k_2-1, ..., \Delta_1+\Delta_2-2\}$,*

$$\phi_k^p(G_1 \times G_2) \geq \phi_{k_1}^p(G_1)\phi_{k_2}^p(G_2) + \min\{n_1 - \phi_{k_1}^p(G_1), n_2 - \phi_{k_2}^p(G_2)\}.$$

Proof. By Theorem 4.24 (i) we conclude that for every k_i-paf set S_i in G_i, $i \in \{1,2\}$, the sets $S_1 \times V_2$ and $V_1 \times S_2$ are, respectively, $(k_1+\Delta_2)$-paf and $(k_2+\Delta_1)$-paf in $G_1 \times G_2$. Therefore, (i) follows.

In order to prove (ii), let $V_1 = \{u_1, u_2, ..., u_{n_1}\}$ and $V_2 = \{v_1, v_2, ..., v_{n_2}\}$. Let S_i be a k_i-paf set of maximum cardinality in G_i, $i \in \{1,2\}$. We suppose

$S_1 = \{u_1, ..., u_r\}$ and $S_2 = \{v_1, ..., v_s\}$. By Theorem 4.24 (ii) we deduce that, for $k \geq k_1 + k_2 - 1$, $S_1 \times S_2$ is a k-paf set in $G_1 \times G_2$. Now let $X = \{(u_{r+i}, v_{s+i}), i = 1, ..., t\}$, where $t = \min\{n_1 - r, n_2 - s\}$ and let $S = X \cup (S_1 \times S_2)$. Since, for every $x \in X$, $\delta_S(x) = 0$ and $k_i > -\delta_i$, $i \in \{1, 2\}$, we obtain that for every $A \subseteq S$, such that $A \cap X \neq \emptyset$, A is not a defensive (k_1+k_2-1)-alliance in $G_1 \times G_2$. Hence, S is a k-paf set for $k \geq k_1+k_2-1$. As a consequence, $\phi^p_k(G_1 \times G_2) \geq |S| = \phi^p_{k_1}(G_1)\phi^p_{k_2}(G_2) + \min\{n_1 - \phi^p_{k_1}(G_1), n_2 - \phi^p_{k_2}(G_2)\}$. □

If $G_1 = C_{n_1}$ is the cycle graph of order n_1 and G_2 is the graph in Figure 4.4, then, by Corollary 4.25 (i), we deduce $\phi^p_4(G_1 \times G_2) = n_1 n_2$, that is, $\phi^p_4(G_1 \times G_2) \geq n_1 \phi^p_2(G_2) = n_1 n_2$. Moreover, if $G_1 = T_{n_1}$ is a tree of order n_1 and maximum degree $\Delta_1 \geq 4$ and G_2 is the graph in Figure 4.4, then $\phi^p_2(G_1) = n_1$ and $\phi^p_2(G_2) = n_2 = 9$. Therefore, by Corollary 4.25 (ii) we deduce $\phi^p_3(G_1 \times G_2) = 9n_1$.

Conclusion

In this work we studied mathematical properties of alliances in graphs. Particularly, we have studied the following subjects:

- The relationships that exist between the alliances in Cartesian product graphs and the alliances in its factors. In this sense we proved that the existence of alliances (alliance free sets) in two graphs leads to the existence of alliances (alliance free sets) in the Cartesian product graph of these two graphs and viceversa.

- Partitions of a graphs into alliances. Particularly, we obtained some estimations for the maximum number of sets in a partition of a graph into k-alliances. Also, we studied the relationships that exist between this maximum number and other invariants of the graph like chromatic number, isoperimetric number and bipartition width.

- Alliance free sets and alliance cover sets. We obtained some bounds for the cardinality of alliance free sets and alliance cover sets and a relationship between alliance free sets and dominating sets. We also characterized some classes of graphs which are defensive alliance free.

- Mathematical properties of boundary alliances. In particular, we obtained several bounds on the cardinality of every boundary alliance and we gave a necessary condition for the existence of a partition of a regular graph into two boundary alliances.

- Relationships between global offensive k-alliances and some characteristic sets of a graph including r-dependent sets, τ-dominating sets and standard dominating sets. Also, we obtained a closed formula for the global offensive k-alliance number of complete bipartite graphs.

Contributions of the Thesis

The volume and quantity of results obtained in this work have been possible to elaborate some papers, which have been either published or submitted to ISI-JCR journals. Moreover, some results have been presented in a specialized conference or presented in an invited talk.

Publications into ISI-JCR journals

- I. G. Yero, J. A. Rodríguez-Velázquez, Boundary defensive k-alliances in graphs, *Discrete Applied Mathematics* **158** (2010) 1205–1211.

- I. G. Yero, S. Bermudo, J. A. Rodríguez-Velázquez, J. M. Sigarreta, Partitioning a graph into defensive k-alliances. *Acta Mathematica Sinica-English Series* **26** (11) (2010) DOI: 10.1007/s10114-010-9075-6.

- S. Bermudo, J. A. Rodríguez-Velázquez, J. M. Sigarreta, I. G. Yero, On global offensive k-alliances in graphs, *Applied Mathematics Letters* **23** (2010) 1454–1458.

- I. G. Yero, J. A. Rodríguez-Velázquez, Boundary powerful k-alliances in graphs, *Ars Combinatoria*. In press.

- J. A. Rodríguez Velázquez, J. M. Sigarreta, I. G. Yero, S. Bermudo, Alliance free and alliance cover sets, *Acta Mathematica Sinica-English Series* **27** (6) (2011). In press.

Papers submitted to ISI-JCR journals

- I. G. Yero, J. A. Rodríguez-Velázquez, S. Bermudo, Alliance free in Cartesian product graphs, submitted to *Applied Mathematics and Computation* (2010).

- J. M. Sigarreta, I. G. Yero, S. Bermudo, J. A. Rodríguez-Velázquez, Partitioning a graphs into offensive k-alliances, submitted to *Discrete Applied Mathematics* (2009).

- I. G. Yero, J. A. Rodríguez-Velázquez, Partitioning a graph into powerful k-alliances, submitted to *Graphs and Combinatorics* (2009).

Contribution to a specialized conference

- J. M. Sigarreta, I. G. Yero, S. Bermudo, J. A. Rodríguez Velázquez, On decomposition of graphs into offensive k-alliances. *Cologne-Twente Workshop on Graphs and Combinatorial Optimization.* (CTW-09), Paris, France. Abstracts 297–300.

Invited talk

- Alliances in graphs. Mathematics, Physics and Informatics Department, University of Gdańsk, Gdańsk, Poland. January 20th, 2010.

Future works

In order to continue developing the topic of alliances in graphs we propose the following subjects.

- Alliances in product graphs.
  Graphs are basic combinatorial structures, and product of structures is

a fundamental construction in mathematics, for which results abound in several areas of research like category theory, set theory and algebra. Product of graphs occur naturally in discrete mathematics as tools in combinatorial constructions. They give rise to important classes of graphs and deep structural problems. The most popular product of graphs is the Cartesian product, which we have studied here. The study of relationships between invariants of Cartesian product graphs and invariants of its factors appears frequently in researches about graph theory [44]. In this sense, there are important open problems which are being investigated now. For instance, the Vizing's conjecture [74], which is one of the most known open problems in graph theory. Our main objective is to contribute to the study of mathematical properties of alliances in product graphs. We pretend to focus our attention in the properties of corona product graphs and strong product graphs.

- Secure sets.

  Since it was defined in [52], defensive alliances are related to the defense of a single vertex. But, in a general realistic settings, alliances should be formed so that any attack on the entire alliance or any subset of the alliance can be defended. In this sense, the authors of [7] presented an attempt to develop a model of this situation.

  - For any $S = \{v_1, v_2, ..., v_r\} \subset V$, an attack A on S is formed by any r mutually disjoint sets $\{A_1, A_2, ..., A_r\}$, for which $A_i \subset N_{\overline{S}}[v_i]$, with $i \in \{1, ..., r\}$.
  - A defense D of the set S is formed by any r mutually disjoint sets $\{D_1, D_2, ..., D_r\}$ for which $D_i \subset N_S[v_i]$, with $i \in \{1, ..., r\}$.
  - Attack A is defendable if there exists a defense D such that for every $i \in \{1, ..., r\}$ it follows, $|D_i| \geq |A_i|$.
  - Set S is secure if and only if every attack on S is defendable.

Until now, there are just a few works about secure sets of a graph [21, 22, 51]. Our main objective is to obtain mathematical properties of secure sets in graphs.

- Extremal graphs.

  Extremal graph theory studies extremal graphs which satisfy a certain property. Extremality can be taken with respect to different graph invariants, such as girth, chromatic number, domination number, etc. In this work we have obtained several properties of alliances in graphs. Thus, our main goal in future is to try to characterize the families of graphs in which its alliances satisfy the obtained properties.

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Glossary

G,	Simple graph, 1.
V,	Set of vertices of G, 1.
E,	Set of edges of G, 1.
$n(G)$,	Order of the graph G, 1.
$m(G)$,	Size of the graph G, 1.
f,	Number of faces of a planar graph G, 1.
$u \sim v$,	Vertex u is adjacent to v.
$G \cong H$	Graphs G and H are isomorphic.
Δ,	Maximum degree of a graph, 1.
δ,	Minimum degree of a graph, 1.
$N(v)$,	Open neighborhood of a vertex v, 1.
$N_S(v)$,	Open neighborhood of a vertex v in a set S, 1.
$N[v]$,	Closed neighborhood of a vertex v, 1.
$N_S[v]$,	Closed neighborhood of a vertex v in a set S, 1.
$\overline{S}$,	Complement of the set S, 1.
$\overline{G}$,	Complement of the graph G, 1.
$\delta(v)$,	Degree of a vertex v, 1.
$\delta_S(v)$,	Degree of a vertex v in a set S, 1.
μ,	Algebraic connectivity, 40, 42, 52, 62, 72, 94.
μ_*,	Laplacian spectral radius, 11, 33, 40, 52, 61, 62, 94.
λ,	Spectral radius, 40, 43, 52.

$\partial(S)$,	Neighborhood of the set S, 1.
$\langle S\rangle$,	Subgraph induced by the set S, 1.
$\mathcal{L}(G)$,	Line graph of a graph G, 1.
K_n,	Complete graph on n vertices.
P_n,	Path graph on n vertices.
C_n,	Cycle graph on n vertices.
Q_t,	Hipercube graph on 2^{t-1} vertices.
$CR(n,t)$,	Circulant graph on n vertices and $2t$ generators.
$K_{r,t}$,	Complete bipartite graph on $r+t$ vertices.
$P_r \times P_t$	Grid graph on rt vertices.
$G_1 \times G_2$,	Cartesian product graph of G_1 and G_2, 1, 19, 56, 87, 111, 116, 117.
$P_{G_1}(S)$	Projection of the set S over G_1 in $G_1 \times G_2$.
$a_k^d(G)$,	Defensive k-alliance number of G, 39.
$a_k^o(G)$,	Offensive k-alliance number of G, 7.
$a_k^p(G)$,	Powerful k-alliance number of G, 77.
$\gamma_k^d(G)$,	Global defensive k-alliance number of G, 39.
$\gamma_k^o(G)$,	Global offensive k-alliance number of G, 7.
$\gamma_k^p(G)$,	Global powerful k-alliance number of G, 77.
$\gamma_1^{io}(G)$,	Global independent offensive 1-alliance number of G, 12.
$\psi_k^d(G)$,	Defensive k-alliance partition number of G, 59.
$\psi_k^o(G)$,	Offensive k-alliance partition number of G, 22.
$\psi_k^p(G)$,	Powerful k-alliance partition number of G, 90.
$\psi_k^d(G)$,	Global defensive k-alliance partition number of G, 59.
$\psi_k^o(G)$,	Global offensive k-alliance partition number of G, 22.
$\psi_k^p(G)$,	Global powerful k-alliance partition number of G, 90.
$\phi_k^d(G)$,	Cardinality of a maximum k-daf set of G, 102
$\phi_k^o(G)$,	Cardinality of a maximum k-oaf set of G, 102.
$\phi_k^p(G)$,	Cardinality of a maximum k-paf set of G, 102.
$\phi_k^{go}(G)$,	Cardinality of a maximum k-goaf set of G, 102.
$\phi_k^{gp}(G)$,	Cardinality of a maximum k-gpaf set of G, 102.
$\zeta_k^d(G)$,	Cardinality of a minimum k-dac set of G, 102.

$\zeta_k^o(G)$, Cardinality of a minimum k-oac set of G, 102.

$\zeta_k^p(G)$, Cardinality of a minimum k-pac set of G, 102.

$\zeta_k^{go}(G)$, Cardinality of a minimum k-goac set of G, 102.

$\zeta_k^{gp}(G)$, Cardinality of a minimum k-gpac set of G, 102.

$\Pi_r^d(G)$, Partition of G into r defensive k-alliances, 59.

$\Pi_r^o(G)$, Partition of G into r offensive k-alliances, 22.

$\Pi_r^p(G)$, Partition of G into r powerful k-alliances, 99.

$\Pi_r^{gd}(G)$, Partition of G into r global defensive k-alliances, 59.

$\Pi_r^{go}(G)$, Partition of G into r global offensive k-alliances, 22.

$\Pi_r^{gp}(G)$, Partition of G into r global powerful k-alliances, 97.

$\Pi_r(G)$, Partition of G into r dominating sets, 95.

$C_{(r,k)}^{go}(G)$, Minimum number of edges having its endpoints in different sets of a partition of G into r global offensive k-alliances, 35.

$C_{(r,k)}^{gd}(G)$, Minimum number of edges having its endpoints in different sets of a partition of G into r global defensive k-alliances, 35.

$\gamma(G)$, Domination number of G, 1, 9,106.

$\gamma_k(G)$, k-domination number of G, 10.

$\gamma_{rt}(G)$, Total r-domination number of G, 81.

$i(G)$, Independence domination number of G, 11 12, 47,.

$\beta_0(T)$, Independence number of G, 11, 12, 46.

$\alpha_r(G)$, r-dependence number of G, 15.

$bw(G)$, Bipartition width of G, 70.

$\chi(G)$, Chromatic number of G, 28.

$\mathbf{i}(\mathbf{G})$, Isoperimetric number of G, 70.