DOCTORAL THESIS

DOROTA KUZIAK

STRONG RESOLVABILITY IN PRODUCT GRAPHS

UNIVERSITAT ROVIRA I VIRGILI

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DOCTORAL THESIS

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I STATE that the present study, entitled "Strong resolvability in product graphs", presented by Dorota Kuziak for the award of the degree of Doctor, has been carried out under my supervision at the Department of Computer Engineering and Mathematics of this university, and that it fulfils all the requirements to be eligible for the International Doctorate Award.

Tarragona, October 20th, 2014

Doctoral Thesis Supervisor
Dr. Juan Alberto Rodríguez Velázquez

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Contents

Introduction

Graph structures may be used to model computer networks. Servers, hosts or hubs in a network can be represented as vertices in a graph and edges could represent connections between them. Each vertex in a graph is a possible location for an intruder (fault in a computer network, spoiled device) and, in this sense, a correct surveillance of each vertex of the graph to control such a possible intruder would be worthwhile. According to this fact, it would be desirable to uniquely recognize each vertex of the graph. In order to solve this problem, Slater [97, 99] brought in the notion of locating sets and locating number of graphs. Also, Harary and Melter [47] introduced independently the same concept, but using the terms resolving sets and metric dimension to refer to locating sets and locating number, respectively. Moreover, in a more recent article, by Sebö and Tannier [96], the terminology of metric generators and metric dimension for the concepts mentioned above, began to be used. These terms arose from the notion of metric generators of metric spaces. In this thesis we follow this terminology and notation from [96].

Informally, a metric generator is an ordered subset S of vertices in a graph G, such that every vertex of G is uniquely determined by its vector of distances to the vertices in S. The cardinality of a minimum metric generator for G is called the metric dimension of G (formal definition is presented in the next chapter)[1].

Once the first papers on this topic were published, some authors have developed diverse theoretical works on this concept including for example, [13, 14, 17, 18, 32, 49, 50, 78, 104, 116]. Several applications of the metric generators have been also appearing. An interested example, as the authors of [18, 21] have described in their articles, is that the structure of some chemical compounds is frequently represented by a labeled graph where the

[1]For graph terminology not defined herein, we refer the reader to [25, 48] and to the next chapter.

vertex and edge labels specify the atom and bond types, respectively. Also, a lot of issues in the field of chemistry are related to obtaining a mathematical representation for chemical compounds, such that each one of these representations leads to different compounds. For instance, the author of [57, 58] rediscovered the concepts of metric generators while he was investigating some aspects about patrons recognition into a chemical compound in a pharmacy company. Furthermore, this topic has some applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [78]. Other applications to navigation of robots in networks and other areas appear in [18, 51, 60]. Some interesting connections between metric generators in graphs and the Mastermind game or coin weighing have been presented in [14]. Moreover, we refer the reader to the work [3], where it can be found some historical evolution, nonstandard terminologies and more references to this topic.

Given a metric generator S of a graph H, the following question was asked in [96]: whenever H is a subgraph of a graph G and the vectors of distances of the vertices of H relative to S agree in both H and G, is H an isometric subgraph of G? Even though the vectors of distances relative to a metric generator for a graph distinguish all pairs of vertices in the graph, they do not uniquely determine all distances in a graph as was first shown in [96]. Fig. 1 shows two graphs of order seven having the same vectors of distances relative to the metric generator $\{a, b\}$, but for which the distances between pairs of vertices having the same vector of distances are not the same. It was observed in [96] that, if "metric generator" is replaced by a stronger notion, namely that of "strong metric generator", then the question above can be answered in the affirmative.

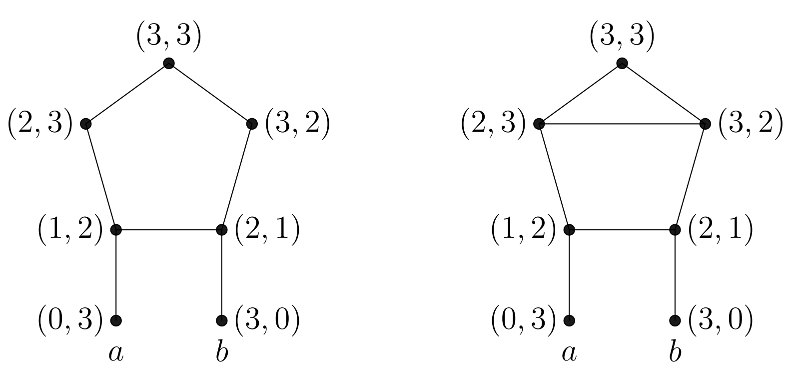

Figure 1: Nonisomorphic graphs with the same metric vectors.

Keeping the track of this new concept, some applications of strong metric generators to combinatorial searching have been presented in [96]. Specifically, there have been analyzed some problems on false coins arising from a connection between information theory and extremal combinatorics. Also, they have dealt with a combinatorial optimization problem related to finding "connected joins" in graphs.

Apart from the concept of strong metric generator, other variations of metric generators have been studied. In general the metric parameters can be classified into four types. Notice that we do not mention every instances of metric parameters, but just some of the most remarkable ones, from our point of view.

1) Metric generators which also satisfy other properties of the graph:

– resolving dominating set [12], when the metric generator has also to be a dominating set;

– independent resolving set [22], when the metric generator is also an independent set;

– connected resolving set [91, 92], when the metric generator is also a connected set.

2) Metric generators which have a modified condition of resolvability:

– strong metric generator [82, 96] - the subject of this thesis;

– local metric generator [85] - a set such that every two adjacent vertices of the graph have distinct vectors of distances to the vertices in this set;

– adjacency resolving set [53] - a set such that any two different vertices not belonging to the set have different neighborhood in this set;

– locating-dominating set [98, 99] - locating set (any two different vertices not belonging to the set have different neighbors in this set) which is a dominating set;

– identifying code [42, 59] - a set such that any two different vertices of the graph have different closed neighborhood in this set and is also a dominating set.

3) Partitions of the vertex set of a graph having some metric properties:

– resolving partitions [23, 41, 88] - a partition such that every two different vertices of the graph have distinct vectors of distances to the sets of the partition;

– strong resolving partition [115] - a partition where every two different vertices of the graph belonging to the same set of the partition are strongly

resolved by some set of the partition;
– metric coloring [20] - a partition such that every two adjacent vertices of the graph have distinct vectors of distances to the set of the partition.
4) Variants which are extensions of the metric generators:
– k-metric generator [28, 29] - a set such that any pair of vertices of the graph is distinguished by at least k vertices of this set;
– simultaneous metric generator [84] - a set which is simultaneously a metric generator for a given family of connected graphs with a common vertex set.
According to the amount of literature concerning all the variants of this topic, we restrict the references related to them only to those articles in which each variant was presented first and/or other ones closely related to this thesis.

On the other hand, studies about operations on graphs are being frequently presented and published in the last few decades. By an operation on graphs we mean, in this thesis, a binary operation, which generates a new graph starting with two initial graphs. Such binary operations may be divided into two or more categories according to the books [45, 52]. In the first category could be included the so-called product graphs and in the second one any other binary operations.

From [45, 52], a graph product of graphs G and H means a graph whose vertex set is defined on the cartesian product $V(G) \times V(H)$ of the vertex sets of G and H, and where, its edges are determined by a function on the edges of the factor graphs G and H. By these rules, there exist 256 possible products. Now, according to several properties of these products, like associativity, commutativity, complementarity, and some other ones, the most important and investigated products are the Cartesian product, the direct product, the strong product, and the lexicographic product, which are also called standard products [45, 52]. Nonetheless, there exist other less known graph products which are interesting for some investigations, for instance we could mention the Cartesian sum graph, the modular product, and the symmetric difference product [2].

As we mention before, in the second category of operations on graphs are included these ones which generate graphs whose vertex sets can have various different structures and the operation itself could satisfy or not typical properties, like commutativity or associativity. Hence, in this category we

[2] Notice that these products could be also known by other names. For more information we refer to [45, 52, 81] and to the next chapter.

can find for instance, the graph sum, the graph difference, the graph intersection[3], the graph join, the rooted product, and the corona product. These mentioned operations, even that they are not considered in the books [45, 52] as products, are frequently treated and called in the literature as such. Since the goal of this thesis is not related to study the structure of the product, but to study the behavior of some invariants of the product, from now on we will consider all these operations mentioned above as a product of graphs.

Operations on graphs are in many aspects natural constructions, and in several cases serve as a model of diverse realistic situations. Roughly speaking, we just mention some basic cases. The Cartesian product of graphs has a wide range of applications, like in coding theory, network designs and mathematical chemistry [100]. The most difficult in many aspects among standard products, the direct product graphs, also has various applications, for instance, it can be used as a model for concurrency in multiprocessor systems [74] and in automata theory [39]. Moreover, the molecular graph of some chemical compounds are obtained as a corona product graph. As an example are the cycloalkanes with a single ring, whose chemical formula is C_kH_{2k}, and whose molecular graph can be expressed as the corona product graph formed by the cycle graph of order n and n copies of the empty graph of order two [117]. Also, some other classes of chemical graphs can be considered as the rooted product [2].

Investigations on operations on graphs have two different styles of being developed. One of them is concerned with the structure and recognition of the operation, and the second one deals with deducing properties of a product with respect to properties of its factors. In the last years a rich theory involving the structure and recognition of graph products has popularly emerged. Also, the other standard approach in graph products is common. Nevertheless, for our second category of operations on graphs is more frequent to find researches addressed to study their properties with respect to their factor graphs. In this thesis we are not interested into problems related to the structure and recognition of operations on graphs.

Some typical properties which are studied on operations on graphs are for instance, domination, coloring, connectivity and independence related parameters. The metric dimension of product graphs is not an exception. Some

[3]Notice that these three first mentioned operations are defined on two graphs of the same order.

examples of these appear in the following published articles. The metric dimension of Cartesian product graphs, lexicographic product graphs, strong product graphs, rooted product graphs and corona product graphs has been studied in [14], [53, 93], [86], [33] and [116], respectively. Also, the strong metric dimension of the Cartesian product of two cycles has been obtained in [82] and the case of Hamming graphs has been investigated in [67]. This has also motivated us to study the strong metric dimension of some product graphs. On the other hand, it was shown in [82] that the problem of computing the strong metric dimension of a graph is NP-hard. This suggests obtaining closed formulaes for the strong metric dimension of special non-trivial families of graphs or bounding the value of this invariant as tight as possible.

The thesis is organized as follows. In the first chapter, we recall some basic definitions on graph theory, present the concepts of product graphs and the strong metric generators, and recall the transformation from the strong metric dimension problem to another well-known problem. The rest of the chapters are focused on the strong metric dimension of graphs obtained from several operations on graphs. Each of these chapters relates to one operation on graphs, and in every one of them we present the exact values of the strong metric dimension of some general classes of the corresponding operation on graphs, or tight lower and upper bounds. Chapters 2–5 are focused on the strong metric dimension in standard products: the Cartesian product graphs, the direct product graphs, the strong product graphs and the lexicographic product graphs, respectively. Chapter 6 deals with the corona product graphs and the join graphs, Chapter 7 with the Cartesian sum of graphs, and the last chapter with the rooted product graphs. We conclude the work with highlights of the principal studied issues, contributions of the thesis, and future works.

Chapter 1

Basic concepts and tools

We begin by establishing the basic terminology and notations which is used throughout the thesis. For the sake of completeness we refer the reader to the books [25, 110]. Graphs considered herein are undirected, finite and contain neither loops nor multiple edges. Let G be a graph of order $n = |V(G)|$. A graph is nontrivial if $n \geq 2$. We use the notation $u \sim v$ for two adjacent vertices u and v of G. For a vertex v of G, $N_G(v)$ denotes the set of neighbors that v has in G, *i.e.*, $N_G(v) = \{u \in V(G) : u \sim v\}$. The set $N_G(v)$ is called the *open neighborhood of a vertex* v in G and $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood of a vertex* v in G. The *degree* of a vertex v of G is denoted by $\delta_G(v)$, *i.e.*, $\delta_G(v) = |N_G(v)|$. The *open neighborhood of a set* S of vertices of G is $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the *closed neighborhood of* S is $N_G[S] = N_G(S) \cup S$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

We use the notation K_n, C_n, P_n, and N_n for the *complete graph*, *cycle*, *path*, and *empty graph*, respectively. Moreover, we write $K_{s,t}$ for the *complete bipartite graph* of order $s + t$ and in particular case $K_{1,n}$ for the *star* of order $n + 1$. Let T be a *tree*, a vertex of degree one in T is called a *leaf* and the number of leaves in T is denoted by $l(T)$.

The *distance* between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. The *diameter*, $D(G)$, of G is the longest distance between any two vertices in G. If G is not connected, then we assume that the distance between any two vertices belonging to different components of G is infinity and, thus, its diameter is $D(G) = \infty$.

We recall that the *complement* of G is a graph G^c has with the same vertex set as G and $uv \in E(G^c)$ if and only if $uv \notin E(G)$. The *subgraph* *induced by a set* X is denoted by $\langle X\rangle$. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph G, we denote by $\sigma(G)$ the set of simplicial vertices of G. We recall that a *clique* in a graph G is a set of pairwise adjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique in G. Two distinct vertices u, v are called *true twins* if $N_G[u] = N_G[v]$. In this sense, a vertex x is a *twin* if there exists $y \neq x$ such that they are true twins. We say that $X \subset V(G)$ is a *twin-free clique* in G if the subgraph induced by X is a clique and for every $u, v \in X$ it follows $N_G[u] \neq N_G[v]$, *i.e.*, the subgraph induced by X is a clique and it contains no true twins. We say that the *twin-free clique number* of G, denoted by $\varpi(G)$, is the maximum cardinality among all twin-free cliques in G. So, $\omega(G) \geq \varpi(G)$. We refer to a $\varpi(G)$-set in a graph G as a twin-free clique of cardinality $\varpi(G)$.

Figure 1.1 shows examples of basic concepts such as true twins and twin-free clique.

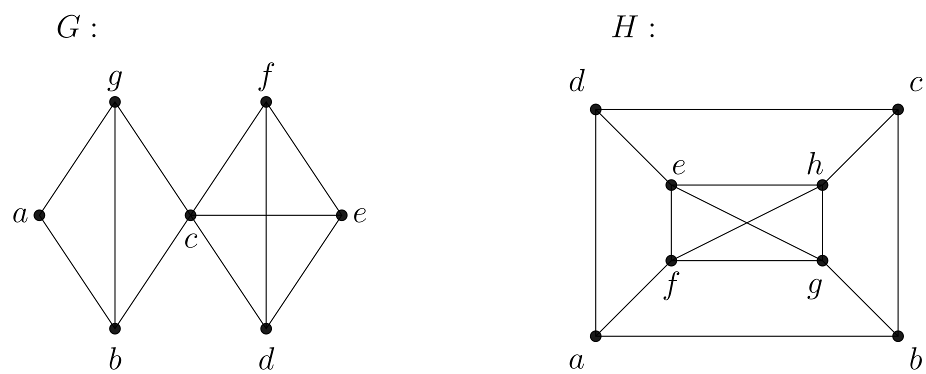

Figure 1.1: The set $\{d, e, f\} \subset V(G)$ is composed by true twin vertices in G. Notice that b and g are true twin vertices in G which are not simplicial, while f and d are true twin and simplicial vertices. The set $\{e, f, g, h\} \subset V(H)$ is a twin-free clique in H.

A graph G is 2-*antipodal* if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x, y) = D(G)$. For example even cycles are 2-antipodal graphs. Also, a *distance-regular* graph G is a regular connected graph of diameter $D(G)$, for which the following holds. There are natural numbers $b_0, b_1, ..., b_{D(G)-1}$, $c_1 = 1$, $c_2, ..., c_{D(G)}$ such that for each pair (u, v) of vertices satisfying $d_G(u, v) = j$ we have

(1) the number of vertices in $G_{j-1}(v)$ adjacent to u is c_j $(1 \leq j \leq D(G))$,

(2) the number of vertices in $G_{j+1}(v)$ adjacent to u is b_j $(0 \leq j \leq D(G)-1)$,

where $G_i(v) = \{u \in V(G) : d_G(u,v) = i\}$. Classes of distance-regular graphs include complete graphs, cycle graphs, and hypercube graphs.

We recall that a graph G is *vertex-transitive* if its automorphism group acts transitively on $V(G)$. Thus for any two distinct vertices of G there is an automorphism mapping one to the other. Vertex-transitive graphs include, for instance, cycle graphs, the Petersen graph, and the Cayley graphs.

Other remaining definitions not defined herein are given the first time that the concept appears in the text.

1.1 Products of graphs

This section is a brief overview on products of graphs. Here we are concerned with these products of graphs that we study after with respect to the strong metric dimension problem.

1.1.1 Cartesian product

The *Cartesian product* of two graphs G and H is the graph $G \Box H$, such that $V(G \Box H) = V(G) \times V(H)$ and two vertices $(a,b) \in V(G \Box H)$ and $(c,d) \in V(G \Box H)$ are adjacent in $G \Box H$ if and only if either

- $a = c$ and $bd \in E(H)$, or
- $ac \in E(G)$ and $b = d$.

The Cartesian product is a straightforward and natural construction, and is in many respects the simplest graph product [45, 52]. Hypercubes, Hamming graphs and grid graphs are some particular cases of this product. The *Hamming graph* $H_{k,n}$ is the Cartesian product of k copies of the complete graph K_n, *i.e.*,

$$H_{k,n} = \underbrace{K_n \Box K_n \Box \ldots \Box K_n}_{k \text{ times}}$$

Hypercube Q_n is defined as $H_{n,2}$. Moreover, the *grid graph* $P_k \Box P_n$ is the Cartesian product of the paths P_k and P_n, the *cylinder graph* $C_k \Box P_n$ is the Cartesian product of the cycle C_k and the path P_n, and the *torus graph* $C_k \Box C_n$ is the Cartesian product of the cycles C_k and C_n. Figure 1.2 shows two examples of Cartesian products.

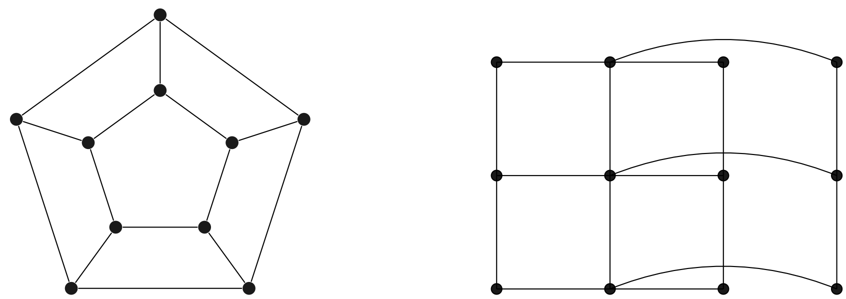

Figure 1.2: Cartesian products $C_5 \Box K_2$ and $K_{1,3} \Box P_3$.

This operation is commutative [45] in the sense that $G \Box H \cong H \Box G$, and is also associative, as the graphs $(F \Box G) \Box H$ and $F \Box (G \Box H)$ are naturally isomorphic. A Cartesian product of graphs is connected if and only if both of its factors are connected. The relation between distances in the Cartesian product of graphs and in its factors is presented in the following remark.

Remark 1.1. [45] *If (a, b) and (c, d) are vertices of a Cartesian product $G \Box H$, then*

$$d_{G \Box H}((a, b), (c, d)) = d_G(a, c) + d_H(b, d).$$

This product has been extensively investigated from various perspectives. For instance, the most popular open problem in the area of domination theory known as Vizing's conjecture [108]. Vizing suggested that the domination number of the Cartesian product of two graphs is at least as large as the product of domination numbers of its factors. Several researchers have worked on it, for instance, some partial results appears in [7, 45]. Moreover, Vizing [107] has investigated the independence number of Cartesian products. The chromatic number of this product has been completely studied in [90]. The connectivity and the hamiltonian properties of Cartesian products have been described in [100, 111] and [26], respectively. For more information on structure and properties of the Cartesian product of graphs we refer the reader to [45, 52].

1.1.2 Direct product

The *direct product* of two graphs G and H is the graph $G \times H$, such that $V(G \times H) = V(G) \times V(H)$ and two vertices $(a, b), (c, d)$ are adjacent in $G \times H$ if and only if

- $ac \in E(G)$ and
- $bd \in E(H)$.

The direct product is also known as the *Kronecker product*, the *tensor product*, the *categorical product*, the *cardinal product*, the *cross product*, the *conjunction*, the *relational product*, the *weak direct product*, or simply the *product*. This product is commutative and associative in a natural way [45, 52]. Figure 1.3 illustrates two examples of direct products. Notice that $K_{1,3} \times P_3$ is not connected (one component is bolded).

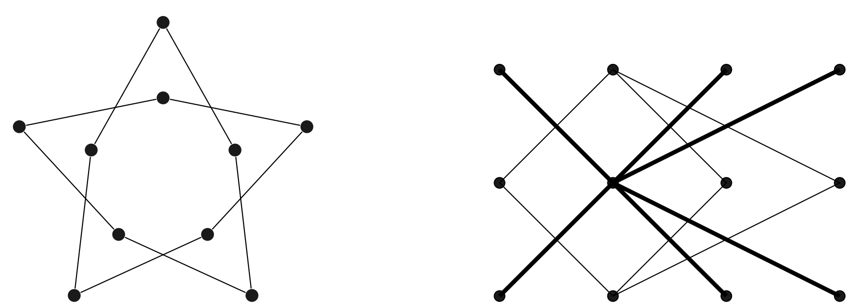

Figure 1.3: Direct products $C_5 \times K_2$ and $K_{1,3} \times P_3$.

The distance and connectedness in the direct product are more subtle than for the Cartesian product. The characterization of connectedness in the direct product of two graphs is presented in the next result.

Theorem 1.2. [109] *A direct product of nontrivial graphs is connected if and only if both factors are connected and at least one factor is nonbipartite.*

Many different properties of direct products have been investigated. The most well-known problem dealing with this product is the Hedetniemi's conjecture. Hedetniemi conjectured that the chromatic number of the direct product of two graphs is equal to the minimum of the chromatic numbers of its factors. We refer to [45, 103] as surveys on this open problem. The connectivity and the edge-connectivity is also a difficult problem in the case of a direct product graphs. Some partial results can be found in [11, 16].

The independence number, domination number and hamiltonicity have been studied, for instance, in [55, 102], [9, 81] and [4, 54], respectively. The direct product graphs is the most difficult in many aspects among standard products, what may confirm open problems concerning this product. For more information on this product we suggest [45, 52].

1.1.3 Strong product

The *strong product* of two graphs G and H is the graph $G \boxtimes H$, such that $V(G \boxtimes H) = V(G) \times V(H)$ and two vertices (a, b) and (c, d) are adjacent in $G \boxtimes H$ if and only if either

- $a = c$ and $bd \in E(H)$, or
- $ac \in E(G)$ and $b = d$, or
- $ac \in E(G)$ and $bd \in E(H)$.

Other known names for the strong product are the *strong direct product* or the *symmetric composition*. Notice that $G \Box H$ and $G \times H$ are subgraphs of $G \boxtimes H$. Figure 1.4 shows two examples of strong products.

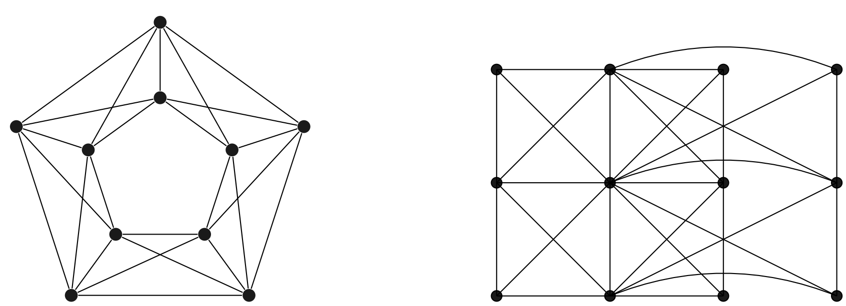

Figure 1.4: Strong products $C_5 \boxtimes K_2$ and $K_{1,3} \boxtimes P_3$.

The commutativity of the strong product follows from the symmetry of the definition of adjacency and for associativity see [45, 52]. A strong product of graphs is connected if and only if every one of its factors is connected. The formula on the vertex distances and the well-known result about the neighborhood in the strong product of graphs are presented in the next remarks.

Remark 1.3. [45, 52] *For any graphs G and H and any two vertices (a, b), (c, d) of $G \boxtimes H$,*

$$d_{G\boxtimes H}((a, b), (c, d)) = \max\{d_G(a, c), d_H(b, d)\}.$$

Remark 1.4. [45, 52] *Let G and H be two graphs. For every $u \in V(G)$ and $v \in V(H)$*

$$N_{G\boxtimes H}[(u, v)] = N_G[u] \times N_H[v].$$

As a direct consequence of the remark above the following result is obtained.

Corollary 1.5. *Let G and H be two graphs and let $u, u' \in V(G)$ and $v, v' \in V(H)$. The following assertions hold.*

(i) *If $(u', v') \in N_{G\boxtimes H}(u, v)$, then $u' \in N_G[u]$ and $v' \in N_G[v]$.*

(ii) *If $u' \in N_G(u)$ and $v' \in N_G(v)$, then $(u', v') \in N_{G\boxtimes H}(u, v)$.*

With the strong product is closely connected an important information theoretical parameter, which in general is very difficult to calculate - the Shannon capacity. The Shannon capacity of a graph G is defined as the limit of $\sqrt[k]{\alpha(G^k)}$ when n tends to infinity, and where $\alpha(G)$ denotes the independence number of the graph G and G^k is the strong product of G with itself k times. This problem has been attracted for several researchers and some partial results are presented in [1, 45].

Various properties of strong products have been also studied. The investigation encompasses, for instance, domination [45, 81], chromatic number [61, 106], connectivity [10, 101] and hamiltonian properties [30, 63]. For more information on the strong product we refer the reader to [45, 52].

1.1.4 Lexicographic product

The *lexicographic product* of two graphs G and H is the graph $G \circ H$ with the vertex set $V(G \circ H) = V(G) \times V(H)$ and two vertices (a, b) and (c, d) are adjacent in $G \circ H$ if either

- $ac \in E(G)$, or
- $a = c$ and $bd \in E(H)$.

In the literature we can also find the names the *composition* or the *substitution* for the lexicographic product. The lexicographic product is clearly not commutative, while it is associative [45, 52]. Figure 1.5 illustrates two examples of lexicographic products and at the same time emphasizes the fact that the lexicographic product is not commutative.

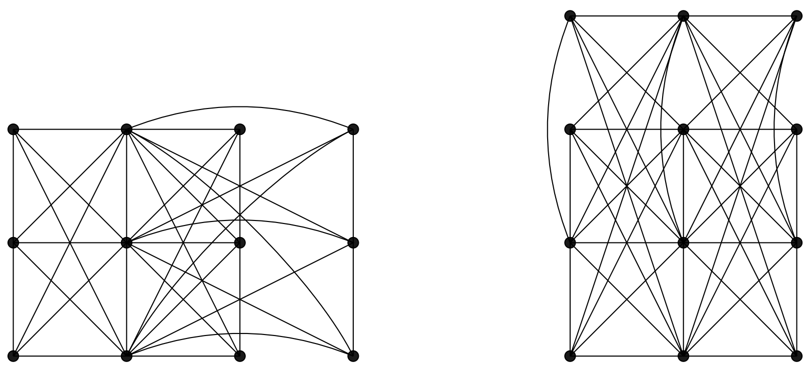

Figure 1.5: Lexicographic products $K_{1,3} \circ P_3$ and $P_3 \circ K_{1,3}$.

A lexicographic product $G \circ H$ is connected if and only if G is connected. The relation between distances in the lexicographic product of graphs and in its factors is presented in the following remark.

Remark 1.6. [45, 52] *If (a, b) and (c, d) are vertices of $G \circ H$, then*

$$d_{G\circ H}((a,b),(c,d)) = \begin{cases} d_G(a,c), & \text{if } a \neq c, \\ d_H(b,d), & \text{if } a = c \text{ and } \delta_G(a) = 0, \\ \min\{d_H(b,d), 2\}, & \text{if } a = c \text{ and } \delta_G(a) \neq 0. \end{cases}$$

The lexicographic product of graphs has been studied from several points of view. The investigation includes, for instance, the independence number [38], domination number [81], chromatic number [24, 38], connectivity [112], and hamiltonicity [5, 68]. For more details see [45, 52].

1.1.5 Corona product and join graphs

Let G and H be two graphs of order n_1 and n_2, respectively. Recall that the *corona product* $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. We denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $v_i \sim v$ for every $v \in V_i$.

Observe that $G \odot H$ is connected if and only if G is connected. Moreover, it is readily seen from the definition that this product is neither an associative nor a commutative operation. Figure 1.6 shows some examples of corona products and also underscores the fact that the corona product is not commutative.

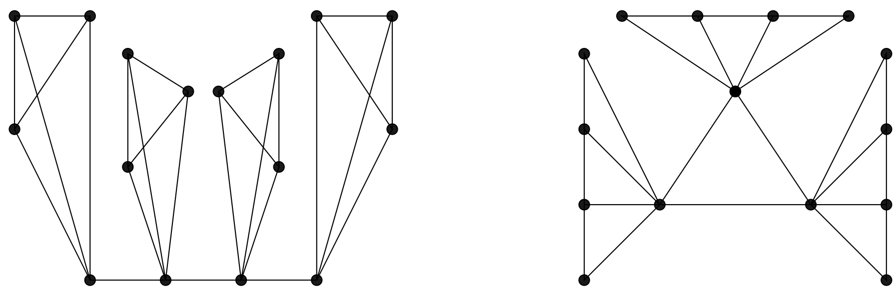

Figure 1.6: Corona products $P_4 \odot C_3$ and $C_3 \odot P_4$.

The concept of corona product of two graphs was first introduced by Frucht and Harary [35]. This product is not too much popular and widely investigated. One of the reason should be the fact, that corona product is a simple operation on two graphs and some mathematical properties could be directly consequences of its factors. Despite this, it is interesting to study metric dimension related parameters in this product. Moreover, there are works on some topological indices [114] and the equitable chromatic number [36] of corona product.

Other simple operation connected with the corona product is a join graph. The *join graph* $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H [46, 120]. It is a commutative operation. Notice that the corona product $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. Now, for the sake of completeness, Figure 1.7 illustrates two examples of join graphs.

Moreover, complete k-partite graphs are typical examples of the join graphs. A *complete k-partite graph* $K_{p_1,p_2,...,p_k}$ is the join graph of empty graphs on $p_1, p_2, ..., p_k$ vertices. Notice that $N_2 + N_2 + N_2$, illustrated in Figure 1.7, is none other than the complete 3-partite graph $K_{2,2,2}$.

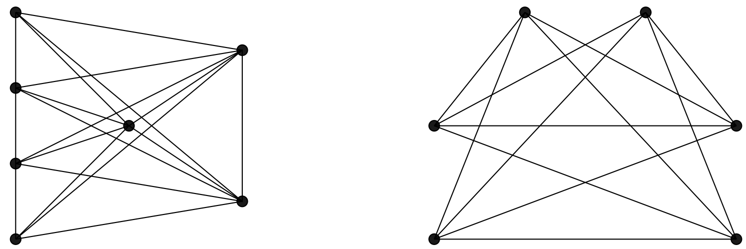

Figure 1.7: Join graphs $P_4 + C_3$ and $N_2 + N_2 + N_2$.

1.1.6 Cartesian sum

The *Cartesian sum* of two graphs G and H, denoted by $G \oplus H$, has as the vertex set $V(G \oplus H) = V(G) \times V(H)$ and two vertices (a, b) and (c, d) are adjacent in $G \oplus H$ if and only if

- $ac \in E(G)$, or
- $bd \in E(H)$.

This notion of a graph product was introduced by Ore [83]. The Cartesian sum is also known as the *disjunctive product* [94] and the *inclusive product* [31, 75]. This graph product is commutative and associative operation [45, 52]. Figure 1.8 shows two Cartesian sum graphs.

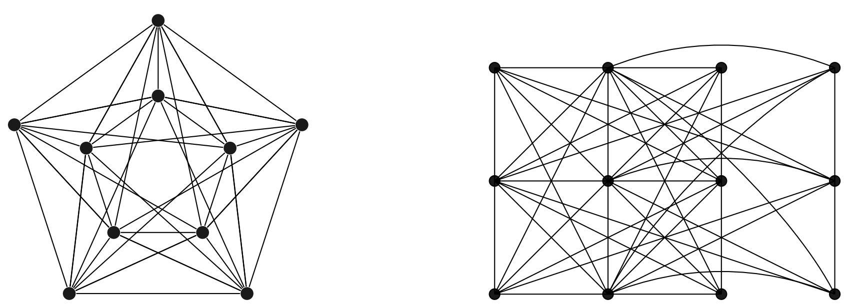

Figure 1.8: Cartesian sum graphs $C_5 \oplus K_2$ and $K_{1,3} \oplus P_3$.

Notice that $G \Box H$, $G \times H$, and $G \boxtimes H$ are subgraphs of $G \oplus H$. Moreover, there exists the following relation between the Cartesian sum graphs and the strong product of graphs, which is a reason of that Cartesian sum is called a complementary product. The connection between these products has appeared in some publications, nevertheless without a concrete proof. For the sake of completeness we present a proof below.

Lemma 1.7. [45, 81] *For any graphs G and H,*

$$(G \oplus H)^c = G^c \boxtimes H^c.$$

Proof. Two vertices (u, v) and (u', v') are adjacent in $(G \oplus H)^c$ if and only if u and u' are not adjacent in G and v and v' are not adjacent in H. *I.e.*, (u, v) and (u', v') are adjacent in $(G \oplus H)^c$ if and only if

- $u = u'$ and $v \sim v'$ in H^c, or
- $u \sim u'$ in G^c and $v = v'$, or
- $u \sim u'$ in G^c and $v \sim v'$ in H^c.

Therefore, $(G \oplus H)^c = G^c \boxtimes H^c$. □

The Cartesian sum graphs is not a popular and widely investigated product of graphs. A typical problem on graph theory, which have been studied extensively in the Cartesian sum graphs, is the chromatic number. Some results on this topic have been presented in [24, 76, 113]. Furthermore, the fact that the independence number of the Cartesian sum of graphs G and H is multiplicative has been proved in [31, 113].

1.1.7 Rooted product

A *rooted graph* is a graph in which one vertex is labeled in a special way so as to distinguish it from other vertices. The special vertex is called the *root* of the graph. Let G be a labeled graph on n vertices. Let $\mathcal{H}$ be a sequence of n rooted graphs H_1, H_2,...,H_n. The *rooted product graph* $G(\mathcal{H})$ is the graph obtained by identifying the root of H_i with the i^{th} vertex of G [40]. In this thesis we consider the particular case of rooted product graph where $\mathcal{H}$ consists of n isomorphic rooted graphs [95]. More formally, assuming that $V(G) = \{u_1, ..., u_n\}$ and that the root vertex of H is v, we define the rooted product graph $G \circ_v H = (V, E)$, where $V = V(G) \times V(H)$ and

$$E = \bigcup_{i=1}^{n} \{(u_i, b)(u_i, y) : \ by \in E(H)\} \cup \{(u_i, v)(u_j, v) : \ u_i u_j \in E(G)\}.$$

In the case when H is a vertex-transitive graph, we have that $G \circ_v H$ does not depend on the choice of v, up to isomorphism. In such a case we denote the

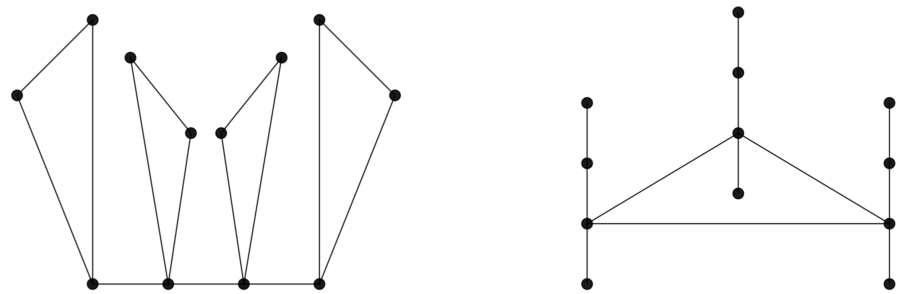

Figure 1.9: Rooted products $P_4 \circ_* C_3$ and $C_3 \circ_v P_4$, where v has degree two.

rooted product by $G \circ_* H$. Figure 1.9 shows the case of the rooted product graphs $P_4 \circ_* C_3$ and $C_3 \circ_v P_4$, where v has degree two.

A rooted product of graphs is connected if and only if both of its factors are connected. The formula on the vertex distances in this product is presented in the following remark.

Remark 1.8. *If (a, b) and (c, d) are vertices of $G \circ_v H$, then*

$$d_{G\circ_v H}((a,b),(c,d)) = \begin{cases} d_H(b,d), & \text{if } a = c, \\ d_H(b,v) + d_G(a,c) + d_H(v,d), & \text{if } a \neq c. \end{cases}$$

Observe that the corona product graph is a particular case of a rooted product graph. If G and H are connected graphs of order $n \geq 2$ and $t \geq 2$, respectively, then $G \odot H \cong G \circ_v (K_1 + H)$, where v denotes the vertex of K_1.

This product was recently redefined and renamed as hierarchical product in [6], where besides already known properties, have been also studied some new ones. Furthermore, there are some works on domination related parameters [69], some topological indices [2], and independence polynomials [89] of rooted product.

1.2 Strong metric generator

A *generator* of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of S. Given a simple and connected graph G, we consider the metric $d_G : V(G) \times V(G) \to \mathbb{R}^+$, where $d_G(x, y)$ is the length of a shortest path between x and y. The pair $(V(G), d_G)$ is readily seen to be a metric space. A vertex $v \in V(G)$ is said to distinguish two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V(G)$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. If $S = \{w_1, w_2, \ldots, w_k\}$ is an (ordered) set of vertices, then the *metric vector* of a vertex $v \in V(G)$ relative to S is the vector $(d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_k))$. Thus, S is a metric generator if distinct vertices have distinct metric vectors relative to S. A minimum metric generator is called a *metric basis* and its cardinality, the *metric dimension* of G, is denoted by $dim(G)$.

A vertex $w \in V(G)$ *strongly resolves* two different vertices $u, v \in V(G)$ if $d_G(w, u) = d_G(w, v) + d_G(v, u)$ or $d_G(w, v) = d_G(w, u) + d_G(u, v)$, *i.e.*, there exists some shortest $w - u$ path containing v or some shortest $w - v$ path containing u. A set S of vertices in a connected graph G is a *strong metric generator* for G if every two vertices of G are strongly resolved by some vertex of S. The smallest cardinality of a strong metric generator for G is called *strong metric dimension* and is denoted by $dim_s(G)$. A *strong metric basis* of G is a strong metric generator for G of cardinality $dim_s(G)$.

One can immediately see that a strong metric generator is also a metric generator, which leads to $dim(G) \leq dim_s(G)$. It was shown in [18] that $dim(G) = 1$ if and only if G is a path. It now readily follows that $dim_s(G) = 1$ if and only if G is a path. At the other extreme we see that $dim_s(G) = n-1$ if and only if G is the complete graph of order n. For the cycle C_n of order n, the strong metric dimension is $dim_s(C_n) = \lceil n/2 \rceil$, and if T is a tree with $l(T)$ leaves, then its strong metric dimension equals $l(T) - 1$ (see [96]).

The strong metric dimension is a relatively new parameter (defined in 2004). Until now only some classes of graphs have been studied in this regard. Furthermore, just a few known results are concerned with product graphs, exactly with the Cartesian product, and they are presented below.

- [67] For hypercubes $dim_s(Q_n) = 2^{n-1}$.
- [67] For Hamming graphs $dim_s(H_{k,n}) = (n-1)n^{k-1}$.
- [82] $dim_s(C_n \Box C_{2k}) = nk$.
- [82] $dim_s(C_{2n+1} \Box C_{2r+1}) = \min\{(2n+1)(r+1), (2r+1)(n+1)\}$.

Moreover, the Cayley graphs [82], distance-hereditary graphs [77], and convex polytopes [64] have been investigated with respect to the problem of finding the strong metric dimension. Also, some Nordhaus-Gaddum type results for the strong metric dimension of a graph and its complement are known [118]. Besides the theoretical results related to the strong metric dimension, a mathematical programming model [64] and metaheuristic approaches [65, 80] for finding this parameter have been developed. For more information we refer the reader to [66] as a short survey on the strong metric dimension.

1.2.1 Strong metric generator versus vertex cover

In [82], the authors have developed the approach of transforming the problem of finding the strong metric dimension of a graph to computing the vertex cover number of some other related graph. A vertex u of G is *maximally distant* from v if for every vertex $w \in N_G(u)$, $d_G(v,w) \leq d_G(u,v)$. We denote by $M_G(v)$ the set of vertices of G which are maximally distant from v. The collection of all vertices of G that are maximally distant from some vertex of the graph is called the *boundary* of the graph, see [8, 15], and is denoted by $\partial(G)$[1]. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are *mutually maximally distant.* If u is maximally distant from v, and v is not maximally distant from u, then v has a neighbor v_1, such that $d_G(v_1,u) > d_G(v,u)$, *i.e.*, $d_G(v_1,u) = d_G(v,u)+1$. It is easily seen that u is maximally distant from v_1. If v_1 is not maximally distant from u, then v_1 has a neighbor v_2, such that $d_G(v_2,u) > d_G(v_1,u)$. Continuing in this manner we construct a sequence of vertices $v_1, v_2, \ldots$ such that $d_G(v_{i+1},u) > d_G(v_i,u)$ for every i. Since G is finite this sequence terminates with some v_k. Thus for all neighbors x of v_k we have $d_G(v_k,u) \geq d_G(x,u)$, and so v_k is maximally distant from u and u is maximally distant from v_k. Hence every boundary vertex belongs to the set $S = \{u \in V(G) :$ there exists $v \in V(G)$ such that u,v are mutually maximally distant$\}$. Certainly every vertex of S is a boundary vertex. For some basic graph classes, such as complete graphs K_n, complete bipartite graphs $K_{r,s}$, cycles C_n and hypercube graphs Q_k, the boundary is simply the whole vertex set. It is not difficult to see that this property also holds for all 2-antipodal graphs and for all distance-regular graphs. Notice that

[1] In fact, the boundary $\partial(G)$ of a graph was defined first in [19] as the subgraph of G induced by the set mentioned in our work with the same notation. We follow the approach of [8, 15] where the boundary of the graph is just the subset of the boundary vertices defined in this article.

the boundary of a tree consists of its leaves. Also, it is readily seen that $\sigma(G) \subseteq \partial(G)$.

Figure 1.10 shows examples of basic concepts such as maximally distant vertices, mutually maximally distant vertices and boundary.

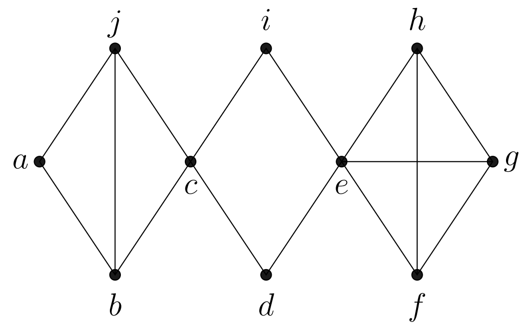

Figure 1.10: The set $\{a, f, g, h\}$ is composed by simplicial vertices and its elements are mutually maximally distant between them. Also, b and j (d and i) are mutually maximally distant. Thus, the boundary of G is $\partial(G) = \{a, b, d, f, g, h, i, j\}$. Now, $M_G(d) = \{a, f, g, h, i\}$ is the set of vertices which are maximally distant from d. Nevertheless, the vertex d is maximally distant only from the vertex i.

As a direct consequence of the definition of mutually maximally distant vertices, we have the following.

Remark 1.9. *For every pair of mutually maximally distant vertices x, y of a connected graph G and for every strong metric basis S of G, it follows that $x \in S$ or $y \in S$.*

We use the notion of "strong resolving graph" based on a concept introduced in [82]. The *strong resolving graph* of G is defined on the vertex set of G where two vertices u, v are adjacent if and only if u and v are mutually maximally distant in G. Clearly, the vertices of the set $V(G) - \partial(G)$ are isolated vertices in the strong resolving graph. According to this fact, in the present work we consider two different versions of this graph: G_{SR} and G_{SR+I}. That is, G_{SR} has vertex set $\partial(G)$ and G_{SR+I} has vertex set $V(G)$. Notice that the difference between G_{SR} and G_{SR+I} is the existence of isolated vertices in G_{SR+I}, when $V(G) - \partial(G) \neq \emptyset$. Figure 1.11 shows the strong resolving graphs G_{SR} and G_{SR+I} of the graph G illustrated in Figure 1.10.

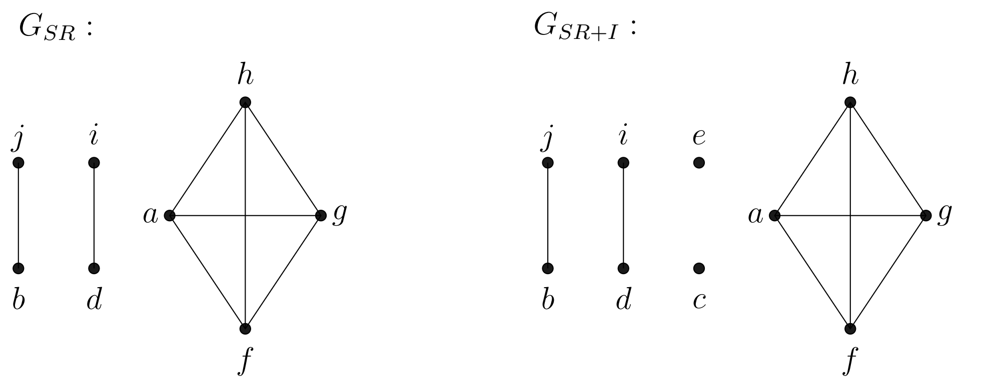

Figure 1.11: G_{SR} and G_{SR+I} of the graph G illustrated in Figure 1.10.

In this thesis we generally apply our concept of the strong resolving graph defined above. The main reason of this fact is related to have a simpler notation and more clarity while proving our results. We consider the strong resolving graph defined in [82] only in the case of the strong product graphs (Chapter 4), and there we also emphasize this fact. Moreover, we use different notation for these definition (as the reader can notice above), so there is no danger of confusion.

We now turn the attention to our definition of the strong resolving graph. There are some families of graphs for which the strong resolving graphs can be obtained relatively easily. We state some of these here. Moreover, we refer to these cases in other chapters.

Observation 1.10.

(a) *If $\partial(G) = \sigma(G)$, then $G_{SR} \cong K_{\partial(G)}$. In particular, $(K_n)_{SR} \cong K_n$ and for any tree T, $(T)_{SR} \cong K_{l(T)}$.*

(b) *For any 2-antipodal graph G of order n, $G_{SR} \cong \bigcup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$.*

(c) *For odd cycles $(C_{2k+1})_{SR} \cong C_{2k+1}$.*

(d) *For any complete k-partite graph $G = K_{p_1,p_2,...,p_k}$ such that $p_i \geq 2$, $i \in \{1, 2, ..., k\}$, $(G)_{SR} \cong \bigcup_{i=1}^{k} K_{p_i}$.*

(e) *For any grid graph $P_r \Box P_t$, $(P_r \Box P_t)_{SR} = K_2 \cup K_2$.*

A set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S. The *vertex cover number* of G, denoted by

$\beta(G)$, is the smallest cardinality of a vertex cover of G. We refer to a $\beta(G)$-set in a graph G as a vertex cover of cardinality $\beta(G)$. Oellermann and Peters-Fransen [82] showed that the problem of finding the strong metric dimension of a connected graph G can be transformed to the problem of finding the vertex cover number of G_{SR+I}.

Theorem 1.11. [82] *For any connected graph G,*

$$dim_s(G) = \beta(G_{SR+I}).$$

It is readily seen that $\beta(G_{SR+I}) = \beta(G_{SR})$. Therefore we present an analogous theorem.

Theorem 1.12. *For any connected graph G,*

$$dim_s(G) = \beta(G_{SR}).$$

Figure 1.12 illustrates this theorem.

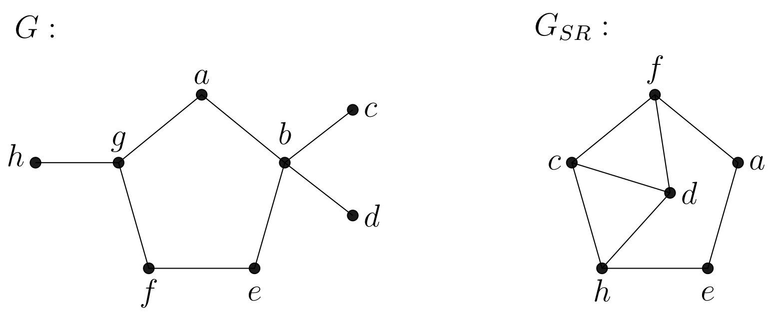

Figure 1.12: The set $\{a, c, d, h\} \subset V(G)$ forms a strong metric basis of G. Also, the set $\{a, c, d, h\} \subset V(G_{SR})$ is a vertex cover of G_{SR}. Thus, $dim_s(G) = \beta(G_{SR}) = 4$.

Recall that the largest cardinality of a set of vertices of G, no two of which are adjacent, is called the *independence number* of G and is denoted by $\alpha(G)$. We refer to an $\alpha(G)$-set in a graph G as an independent set of cardinality $\alpha(G)$. The following well-known result, due to Gallai [37], states the relationship between the independence number and the vertex cover number of a graph.

Theorem 1.13. [37](Gallai, 1959) *For any graph G of order n,*

$$\alpha(G) + \beta(G) = n.$$

Thus, for any graph G, by using Theorems 1.12 and 1.13, we immediately obtain that

$$dim_s(G) = |\partial(G)| - \alpha(G_{SR}) \tag{1.1}$$

Chapter 2

Strong metric dimension of Cartesian product graphs

2.1 Overview

This chapter is concerned with finding exact values of the strong metric dimension of some families of Cartesian product graphs, or general lower and upper bounds, and express these in terms of invariants of the factor graphs. In particular, we investigate the cases in which the strong resolving graph of one factor is a bipartite graph with a perfect matching, or is a regular bipartite graph, or in which the strong resolving graphs of both factors are vertex-transitive graphs. The strong metric dimension of Hamming graphs is also studied.

2.2 Main results

We begin this section by establishing an interesting connection between the strong resolving graph of the Cartesian product of two graphs and the direct product of the strong resolving graphs of its factors.

Theorem 2.1. *Let G and H be two connected graphs. Then*

$$(G\Box H)_{SR} \cong G_{SR} \times H_{SR}.$$

Proof. Let $(g,h),(g',h')$ be any two vertices of $G\Box H$. Then, we have

$$d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h').$$

Thus, if $g'' \sim g'$ and $d_G(g,g'') = d_G(g,g') + 1$, then $(g',h') \sim (g'',h')$ and $d_{G\Box H}((g,h),(g'',h')) = d_G(g,g') + d_H(h,h') + 1 = d_{G\Box H}((g,h),(g',h')) + 1$.

Using these observations, it is readily seen that (g,h) and (g',h') are mutually maximally distant if and only if g and g' are mutually maximally distant in G and h and h' are mutually maximally distant in H. Moreover, $(g,h)(g',h') \in E((G\Box H)_{SR})$ if and only if $gg' \in E(G_{SR})$ and $hh' \in E(H_{SR})$. Thus

$$V((G\Box H)_{SR}) = \partial(G\Box H) = \partial(G) \times \partial(H) = V(G_{SR} \times H_{SR}),$$

and

$$(G\Box H)_{SR} \cong G_{SR} \times H_{SR}.$$

□

Figure 2.1 illustrates Cartesian product of two cycles of order three and its strong resolving graph. Since the strong resolving graph of C_3 is isomorphic to C_3, we can easy observe that $(C_3\Box C_3)_{SR}$ is isomorphic to $(C_3)_{SR} \times (C_3)_{SR}$.

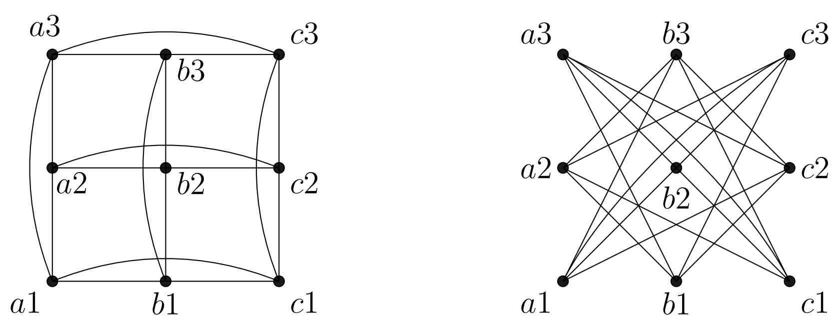

Figure 2.1: Cartesian product graph $C_3\Box C_3$ and its strong resolving graph $(C_3\Box C_3)_{SR}$.

The following result, which is obtained by using Theorem 1.12 and Theorem 2.1, is the main tool of this chapter.

Corollary 2.2. *Let G and H be two connected graphs. Then*

$$dim_s(G\Box H) = \beta(G_{SR} \times H_{SR}).$$

Now we consider some cases in which we can compute $\beta(G_{SR} \times H_{SR})$. To begin with, we recall the following well-known result of König and Egerváry.

In this sense, we need more terminology. A *matching* on a graph G is a set of edges of G such that no two edges share a vertex in common. A matching is *maximum* if it has the maximum possible cardinality. Moreover, if every vertex of the graph is incident to exactly one edge of the matching, then it is called a *perfect matching*.

Theorem 2.3. [27, 62](König, Egerváry, 1931) *For bipartite graphs the size of a maximum matching equals the size of a minimum vertex cover.*

Our next result deals with graphs whose strong resolving graphs are bipartite with a perfect matching. We use the theorem above as a tool for our purposes.

Theorem 2.4. *Let G and H be two connected graphs such that H_{SR} is bipartite with a perfect matching. Let G_i, $i \in \{1, ..., k\}$, be the connected components of G_{SR}. If for each $i \in \{1, ..., k\}$, G_i is Hamiltonian or G_i has a perfect matching, then*

$$dim_s(G\Box H) = \frac{|\partial(G)||\partial(H)|}{2}.$$

Proof. Since H_{SR} is bipartite, $G_{SR} \times H_{SR}$ is bipartite. We show next that $G_{SR} \times H_{SR}$ has a perfect matching. Let n_i be the order of G_i, $i \in \{1, ..., k\}$, and let $\{x_1y_1, x_2y_2, ..., x_{|\partial(H)|/2}y_{|\partial(H)|/2}\} \subset E(H_{SR})$ be a perfect matching of H_{SR}. We distinguish two cases.

Case 1: G_i has a perfect matching. If $\{u_1v_1, u_2v_2..., u_{n_i/2}v_{n_i/2}\} \subset E(G_i)$ is a perfect matching of G_i, then the set of edges

$$\begin{aligned}&\{(u_1, y_1)(v_1, x_1),\ (v_1, y_1)(u_1, x_1), ..., (u_{n_i/2}, y_1)(v_{n_i/2}, x_1),\\ &(v_{n_i/2}, y_1)(u_{n_i/2}, x_1), (u_1, y_2)(v_1, x_2),\ (v_1, y_2)(u_1, x_2), ...,\\ &(u_{n_i/2}, y_2)(v_{n_i/2}, x_2),\ (v_{n_i/2}, y_2)(u_{n_i/2}, x_2), \dots,\\ &(u_1, y_{|\partial(H)|/2})(v_1, x_{|\partial(H)|/2}),\ (v_1, y_{|\partial(H)|/2})(u_1, x_{|\partial(H)|/2}), \dots,\\ &(u_{n_i/2}, y_{|\partial(H)|/2})(v_{n_i/2}, x_{|\partial(H)|/2}),\ (v_{n_i/2}, y_{|\partial(H)|/2})(u_{n_i/2}, x_{|\partial(H)|/2})\}\end{aligned}$$

is a perfect matching of $G_i \times H_{SR}$.

Case 2: G_i is Hamiltonian. Let $v_1, v_2, ..., v_{n_i}, v_1$ be a Hamiltonian cycle of G_i. If n_i is even, then G_i has a perfect matching and this case coincides with Case 1. So we suppose that n_i is odd. In this case, the set of edges

$$\begin{aligned}&\{(v_1, x_1)(v_2, y_1),\ (v_2, x_1)(v_3, y_1), ..., (v_{n_i-1}, x_1)(v_{n_i}, y_1),\ (v_{n_i}, x_1)(v_1, y_1),\\ &(v_1, x_2)(v_2, y_2),\ (v_2, x_2)(v_3, y_2), ..., (v_{n_i-1}, x_2)(v_{n_i}, y_2),\ (v_{n_i}, x_2)(v_1, y_2), \dots,\\ &(v_1, x_{|\partial(H)|/2})(v_2, y_{|\partial(H)|/2}),\ (v_2, x_{|\partial(H)|/2})(v_3, y_{|\partial(H)|/2}), \dots,\\ &(v_{n_i-1}, x_{|\partial(H)|/2})(v_{n_i}, y_{|\partial(H)|/2}),\ (v_{n_i}, x_{|\partial(H)|/2})(v_1, y_{|\partial(H)|/2})\}\end{aligned}$$

is a perfect matching of $G_i \times H_{SR}$.

According to Cases 1 and 2 the graph $\bigcup_{i=1}^{k} G_i \times H_{SR} = G_{SR} \times H_{SR}$ has a perfect matching. Now, since $G_{SR} \times H_{SR}$ is bipartite and it has a perfect matching, by Theorem 2.3 we have $\beta(G_{SR} \times H_{SR}) = \frac{|\partial(G)||\partial(H)|}{2}$. By Corollary 2.2 the result now follows. □

Since 2-antipodal graphs have strong resolving graphs that are bipartite with a perfect matching, the next result follows from the previous theorem and Observation 1.10.

Corollary 2.5. *For any connected 2-antipodal graph G of order n, the following statements hold.*

(a) *If H is a connected 2-antipodal graph of order r, then*

$$dim_s(G\Box H) = \frac{nr}{2}.$$

(b) *If H is a connected graph where $|\partial(H)| = |\sigma(H)|$, then*

$$dim_s(G\Box H) = \frac{n|\sigma(H)|}{2}.$$

In particular, for any tree T,

$$dim_s(G\Box T) = \frac{nl(T)}{2},$$

and for any complete graph K_r,

$$dim_s(G\Box K_r) = \frac{nr}{2}.$$

On the other hand, by Observation 1.10 and Theorem 2.4, we obtain the following values of $dim_s(G\Box H)$ for some specific examples of graphs G and H.

Corollary 2.6.

(a) $dim_s(K_n\Box P_r) = n$.

(b) *For any tree T, $dim_s(T\Box P_r) = l(T)$.*

(c) $dim_s(C_n\Box P_r) = n$.

(d) $dim_s(K_n\Box C_{2k}) = nk$.

(e) *For any tree T, $dim_s(T\Box C_{2k}) = l(T)k$.*

(f) [82] $dim_s(C_n\Box C_{2k}) = nk$.

Our next tool is a well-known consequence of Hall's marriage theorem.

Lemma 2.7. [44](Hall, 1935) *Every regular bipartite graph has a perfect matching.*

The result above is particularly useful when we have a graph whose strong resolving graph is regular and bipartite. Notice that there are several classes of graphs satisfying this property, for instance, paths, cycles of even order, hypercubes, *etc.*

Theorem 2.8. *Let G and H be two connected graphs such that G_{SR} and H_{SR} are regular and at least one of them is bipartite. Then*

$$dim_s(G\Box H) = \frac{|\partial(G)||\partial(H)|}{2}.$$

Proof. Since G_{SR} and H_{SR} are regular graphs and at least one of them is bipartite, $G_{SR} \times H_{SR}$ is a regular bipartite graph. Hence, by Lemma 2.7, $G_{SR} \times H_{SR}$ has a perfect matching. Thus, by Theorem 2.3, $\beta(G_{SR} \times H_{SR}) = \frac{|\partial(G)||\partial(H)|}{2}$. The result follows by Corollary 2.2. $\Box$

Note that Corollary 2.6 can also be deduced from Theorem 2.8.

Our next result, which is obtained from Theorem 2.8, is derived from the fact that the strong resolving graph of a distance-regular graph is regular.

Corollary 2.9. *Let G be a distance-regular graph of order n and let H be a connected graph such that H_{SR} is a regular bipartite graph. Then*

$$dim_s(G\Box H) = \frac{n|\partial(H)|}{2}.$$

In particular, if H is a 2-antipodal graph of order r, then

$$dim_s(G\Box H) = \frac{nr}{2}.$$

By using Theorem 1.13 and Corollary 2.2 we obtain our mentioned useful tool, which relates $dim_s(G\Box H)$ with the independence number of $G_{SR} \times H_{SR}$.

Corollary 2.10. $dim_s(G\Box H) = |\partial(G)||\partial(H)| - \alpha(G_{SR} \times H_{SR})$.

We now state a recent result, from [119], on the independence number of the direct product of graphs, that is useful in establishing the subsequent theorem.

Lemma 2.11. [119] *Let G and H be two vertex-transitive graphs of order n_1, n_2, respectively. Then*

$$\alpha(G \times H) = \max\{n_1\alpha(H), n_2\alpha(G)\}.$$

Theorem 2.12. *Let G and H be two connected graphs such that G_{SR} and H_{SR} are vertex-transitive graphs. Then*

$$dim_s(G\Box H) = \min\{|\partial(G)|dim_s(H), |\partial(H)|dim_s(G)\}.$$

Proof. Since G_{SR} and H_{SR} are vertex-transitive graphs, it follows from Lemma 2.11 that $\alpha(G_{SR} \times H_{SR}) = \max\{|\partial(G)|\alpha(H), |\partial(H)|\alpha(G)\}$. So, by Corollary 2.10 we have

$$\begin{aligned} dim_s(G\Box H) &= |\partial(G)||\partial(H)| - \alpha(G_{SR} \times H_{SR}) \\ &= |\partial(G)||\partial(H)| - \max\{|\partial(G)|\alpha(H_{SR}), |\partial(H)|\alpha(G_{SR})\} \\ &= \min\{|\partial(G)|dim_s(H), |\partial(H)|dim_s(G)\}. \end{aligned}$$

□

With this theorem in hand we deduce, by Observation 1.10, the following values of $dim_s(G\Box H)$ for other specific families of graphs G and H.

Corollary 2.13.

(a) $dim_s(K_n\Box K_r) = \min\{n(r-1), r(n-1)\}$.

(b) *For any trees T_1 and T_2,*

$$dim_s(T_1\Box T_2) = \min\{l(T_1)(l(T_2)-1), l(T_2)(l(T_1)-1)\}.$$

(c) [82] $dim_s(C_{2n+1}\Box C_{2r+1}) = \min\{(2n+1)(r+1), (2r+1)(n+1)\}$.

(d) $dim_s(K_n\Box C_{2r+1}) = \min\{n(r+1), (2r+1)(n-1)\}$.

(e) $dim_s(T\Box C_{2r+1}) = \min\{l(T)(r+1), (2r+1)(l(T)-1)\}$.

(f) *For any tree T,*

$$dim_s(K_n\Box T) = \min\{l(T)(n-1), n(l(T)-1)\}.$$

We continue with an easily verified bound relating the strong metric dimension of a graph with the number of its simplicial vertices and the number of its mutually maximally distant vertices. It is clear that, if both sets (simplicial vertices and mutually maximally distant vertices) are equal, then we have equality, which is useful when studying $dim_s(G\Box H)$.

Lemma 2.14. *For every graph G,*

$$|\sigma(G)|-1 \leq dim_s(G) \leq |\partial(G)|-1.$$

Proof. Each simplicial vertex of G is mutually maximally distant with every other simplicial vertex of G. So, G_{SR} has a subgraph isomorphic to $K_{|\sigma(G)|}$. Thus, $\beta(G_{SR}) \geq |\sigma(G)|-1$. Hence, by Theorem 1.12, $dim_s(G) \geq |\sigma(G)|-1$.

On the other hand, notice that for any graph H, $\beta(H) \leq |V(H)|-1$. Since $V(G_{SR}) = \partial(G)$, it follows that $\beta(G_{SR}) \leq |\partial(G)|-1$. Thus, by Theorem 1.12, the upper bound follows. $\square$

Note that if $\sigma(G) = \partial(G)$, then by Lemma 2.14, $dim_s(G) = |\partial(G)|-1$. Hence, as a particular case of Theorem 2.12 we obtain the following result.

Corollary 2.15. *Let G and H be two connected graphs. If $\partial(G) = \sigma(G)$ and $\partial(H) = \sigma(H)$, then*

$$dim_s(G\Box H) = \min\{|\partial(G)|(|\partial(H)|-1), |\partial(H)|(|\partial(G)|-1)\}.$$

2.3 Tight bounds

Again we use the matching of a graph to give a result for $dim_s(G\Box H)$. In this case, we also give a relationship with the matchings of the graph $G\times K_2$. To this end, we consider the *matching number* of a graph G (*i.e.*, the cardinality of a maximum matching of G), which is denoted by $\nu(G)$, and we first present the following useful facts.

Lemma 2.16. *For any nontrivial nonempty graphs G and H,*

$$\beta(G\times H) \geq \nu(H)\beta(G\times K_2) = \nu(H)\nu(G\times K_2) \geq 2\nu(G)\nu(H).$$

Proof. We consider a maximum matching $M = \{u_iv_i :\ i \in \{1,...,k\}\}$ of H, and a minimum vertex cover A of $G\times H$. Now, for every $i \in \{1,,...,k\}$, let $A_i = A\cap(V(G)\times\{u_i,v_i\})$. Notice that $A_i \neq \emptyset$ for every $i \in \{1,...,k\}$.

Also, since $A_i \cap A_j = \emptyset$, with $i \neq j$, it follows $|A_1| + |A_2| + ... + |A_k| \leq |A|$. Moreover, for every $i \in \{1, ..., k\}$ we have that A_i is a vertex cover of $G \times \langle\{u_i, v_i\}\rangle \cong G \times K_2$. Thus, $k\,\beta(G \times K_2) \leq \sum_{i=1}^{k} |A_i| \leq |A| = \beta(G \times H)$. As a result,

$$\beta(G \times H) \geq \nu(H)\beta(G \times K_2).$$

Since $G \times K_2$ is a bipartite graph, from Theorem 2.3 it follows that

$$\beta(G \times K_2) = \nu(G \times K_2).$$

Finally, every matching $\{x_i y_i : \ i \in \{1, ..., k'\}\}$ of G induces a matching $\{(x_i, a)(y_i, b), (y_i, a)(x_i, b) : \ i \in \{1, ..., k'\}\}$ of $G \times K_2$, where $\{a, b\}$ is the vertex set of K_2. Thus, $\nu(G \times K_2) \geq 2\nu(G)$. This completes the proof. $\square$

Observation 2.17. *Let G and H be two graphs of orders n_1 and n_2, respectively. If G and H have perfect matchings and at least one of them is bipartite, then $G \times H$ is bipartite and*

$$\frac{n_1 n_2}{2} \geq \nu(G \times H) = \beta(G \times H) \geq 2\nu(G)\nu(H) = \frac{n_1 n_2}{2}.$$

Moreover,

$$\beta(G \times H) = \nu(H)\beta(G \times K_2) = \nu(H)\nu(G \times K_2) = 2\nu(G)\nu(H) = \frac{n_1 n_2}{2}.$$

Once described the relations above, we are able to give a bound on $dim_s(G \Box H)$ relating $\nu(H_{SR})$, $dim_s(G \Box K_2)$ and $\nu(G_{SR})$. Note that this result is obtained from Lemma 2.16 and Corollary 2.2.

Corollary 2.18. *Let G and H be two connected graphs.*

$$dim_s(G \Box H) \geq \nu(H_{SR}) dim_s(G \Box K_2) \geq 2\nu(G_{SR})\nu(H_{SR}).$$

Examples of graphs where

$$dim_s(G \Box H) = \nu(H_{SR}) dim_s(G \Box K_2) = 2\nu(G_{SR})\nu(H_{SR}) = \frac{|\partial(G)||\partial(H)|}{2}$$

are given in Corollary 2.6.

Now we give sharp upper and lower bounds on the strong metric dimension of Cartesian products of graphs. We begin by stating a useful relationship between the independence numbers of the direct product of two graphs and that of its factors.

Lemma 2.19. [56] *For any graphs G and H of orders n_1 and n_2, respectively,*

$$\alpha(G \times H) \geq \max\{n_2\alpha(G), n_1\alpha(H)\}.$$

The next result gives a sharp upper bound on the strong metric dimension of the Cartesian product of two graphs in terms of the strong metric dimension of its factors and the cardinality of their boundaries.

Theorem 2.20. *For any connected graphs G and H,*

$$dim_s(G\Box H) \leq \min\{dim_s(G)|\partial(H)|, |\partial(G)|dim_s(H)\}.$$

Moreover, this bound is sharp.

Proof. By using Lemma 2.19 we deduce

$$\alpha(G_{SR} \times H_{SR}) \geq \max\{|\partial(H)|\alpha(G_{SR}), |\partial(G)|\alpha(H_{SR})\}.$$

Thus, by Theorem 1.13,

$$\beta(G_{SR} \times H_{SR}) \leq \min\{|\partial(H)|\beta(G_{SR}), |\partial(G)|\beta(H_{SR})\}.$$

The result now follows from Corollary 2.2. Several examples of pairs of graphs where the bound above is attained are given in Corollary 2.13. □

To prove a lower bound on the strong metric dimension of the Cartesian product of two graphs we use the following.

Lemma 2.21. [102] *For any graphs G and H of orders n_1 and n_2, respectively,*

$$\alpha(G \times H) \leq n_2\alpha(G) + n_1\alpha(H) - \alpha(G)\alpha(H).$$

Theorem 2.22. *For any connected graphs G and H,*

$$dim_s(G\Box H) \geq dim_s(G)dim_s(H).$$

Proof. Notice that Lemma 2.21 leads to

$$\alpha(G_{SR} \times H_{SR}) \leq |\partial(H)|\alpha(G_{SR}) + |\partial(G)|\alpha(H_{SR}) - \alpha(G_{SR})\alpha(H_{SR}).$$

Hence, from Theorem 1.13,

$$\beta(G_{SR} \times H_{SR}) \geq \beta(G_{SR})\beta(H_{SR}).$$

This inequality together with Corollary 2.2 gives the desired result. □

With respect to the sharpness of the lower bound of Theorem 2.22, it is necessary to observe that this bound is sharp if and only if the bound of Lemma 2.21 is also sharp. It was shown in [102] that there is a sequence of direct products $G_n \times H_n$ such that

$$\frac{\alpha(G_n \times H_n)}{|V(H_n)|\alpha(G_n) + |V(G_n)|\alpha(H_n) - \alpha(G_n)\alpha(H_n)} \to 1$$

as $n \to \infty$. Thus to show that the bound above is asymptotically sharp one needs to find sequences of graphs $G'_1, G'_2, \ldots$ and $H'_1, H'_2, \ldots$ such that $(G'_n)_{SR} = G_n$ and $(H'_n)_{SR} = H_n$ for every n. No specific graph or family of graphs was described in [102] where the bound of Lemma 2.21 is achieved. It appears to be a nontrivial task to describe such sequences of graphs. We do observe that there is an infinite family of Cartesian products for which the strong dimension is "close" to the bound. In particular, $dim_s(K_n \Box K_2) = n = dim_s(K_n)dim_s(K_2) + 1$.

2.4 Pairs of graphs for which the Cartesian product has strong metric dimension two

Even though those graphs for which the strong metric dimension is two are not yet fully understood, in this section we characterize those pairs of graphs for which the Cartesian product has strong metric dimension two. In order to present our results we need to introduce some more terminology. For two vertices $u, v \in V(G)$, the interval $I_G[u, v]$ between u and v is defined as the collection of all vertices that belong to some shortest $u - v$ path. Notice that vertex w strongly resolves two vertices u and v if $v \in I_G[u, w]$ or $u \in I_G[v, w]$.

Lemma 2.23. *Let $a, x, c \in V(G)$ and $b, y, d \in V(H)$. Then, $(a, b) \in I_{G\Box H}[(x, y), (c, d)]$ if and only if $a \in I_G[x, c]$ and $b \in I_H[y, d]$.*

Proof. Suppose first that $a \in I_G[x, c]$ and $b \in I_H[y, d]$. Then, $d_G(x, c) = d_G(x, a) + d_G(a, c)$ and $d_H(y, d) = d_H(y, b) + d_H(b, d)$. Hence

$$\begin{aligned} d_{G\Box H}((x, y), (c, d)) &= d_G(x, c) + d_H(y, d) \\ &= (d_G(x, a) + d_G(a, c)) + (d_H(y, b) + d_H(b, d)) \\ &= (d_G(x, a) + d_H(y, b)) + (d_G(a, c) + d_H(b, d)) \\ &= d_{G\Box H}((x, y), (a, b)) + d_{G\Box H}((a, b), (c, d)). \end{aligned}$$

Thus $(a,b) \in I_{G\square H}[(x,y),(c,d)]$.

Conversely, if $(a,b) \in I_{G\square H}[(x,y),(c,d)]$, then

$$\begin{aligned} d_{G\square H}((x,y),(c,d)) &= d_{G\square H}((x,y),(a,b)) + d_{G\square H}((a,b),(c,d)) \\ &= (d_G(x,a) + d_H(y,b)) + (d_G(a,c) + d_H(b,d)) \\ &= (d_G(x,a) + d_G(a,c)) + (d_H(y,b) + d_H(b,d)). \end{aligned}$$

Now, if $a \notin I_G[x,c]$ or $b \notin I_G[y,d]$, then $d_{G\square H}((x,y),(c,d)) > d_G(x,c) + d_H(y,d)$, a contradiction. $\square$

Given two graphs G, H and a subset S of vertices of $G\square H$, the *projections* of S onto the graphs G and H, respectively, are the following ones $P_G(S) = \{u \in V(G) : (u,v) \in S, \text{ for some vertex } v \in V(H)\}$ and $P_H(S) = \{v \in V(H) : (u,v) \in S, \text{ for some vertex } u \in V(G)\}$.

Proposition 2.24. *Let G and H be two connected graphs of order at least 2. Then, $dim_s(G\square H) = 2$ if and only if G and H are both paths.*

Proof. If G and H are paths, then, by Corollary 2.2,

$$dim_s(G\square H) = \beta(K_2 \times K_2) = 2.$$

On the other hand, let $S = \{(a,x),(b,y)\}$ be a strong metric basis of $G\square H$. If $a \neq b$ and $x \neq y$. Let c be a neighbor of b on a $a-b$ path (it might be that $a = c$). Let z be a neighbor of y on a $x-y$ path (notice that could be $x = z$). So, we have $d_{G\square H}((b,z),(a,x)) = d_G(a,b)+d_H(z,x) = d_G(a,c)+1+d_H(x,y)-1 = d_G(a,c) + d_H(x,y) = d_{G\square H}((c,y),(a,x))$. Thus, $(b,z) \notin I_{G\square H}[(c,y),(a,x)]$ and $(c,y) \notin I_{G\square H}[(b,z),(a,x)]$. Moreover, $d_{G\square H}((b,z),(b,y)) = d_H(z,y) = 1 = d_G(b,c) = d_{G\square H}((c,y),(b,y))$. Thus, $(b,z) \notin I_{G\square H}[(c,y),(b,y)]$ and $(c,y) \notin I_{G\square H}[(b,z),(b,y)]$. Therefore, $S = \{(a,x),(b,y)\}$ does not strongly resolve (b,z) and (c,y), and so either $a = b$ or $x = y$.

If $a = b$, then the projection of S onto G is a single vertex. By Lemma 2.23, the projection of S onto G strongly resolves G. As observed in Section 1.2, G is a path. Similarly, if $x = y$, then H is a path. Therefore either G or H is a path. We assume, without loss of generality, that G is a path. By Corollary 2.2 and Observation 2.17 it follows that $2 = dim_s(G\square H) = \beta(K_2 \times H_{SR})$. Thus, either H_{SR} is isomorphic to K_2 or $\beta(H_{SR}) = 1$ which implies that $dim_s(H) = 1$. Therefore, as observed in Section 1.2, H is a path. $\square$

2.5 Hamming graphs

Now we study a particular case of Cartesian products of graphs, the so-called Hamming graphs. The strong metric dimension of Hamming graphs was obtained in [67] where the authors gave a long and complicated proof. Here we give a simple proof for this result, using Theorem 2.1 and the next result due to Valencia-Pabon and Vera [105].

Lemma 2.25. [105] *For any positive integers,* $n_1, n_2, ...n_r$*,*

$$\alpha(K_{n_1} \times K_{n_2} \times ... \times K_{n_r}) = \max_{1\leq i\leq r} \left\{ \frac{n_1 n_2 ... n_r}{n_i} \right\}.$$

By Theorem 2.1, it follows that for any positive integers, $n_1, n_2, ...n_r$,

$$(K_{n_1} \Box K_{n_2} \Box ... \Box K_{n_r})_{SR} \cong K_{n_1} \times K_{n_2} \times ... \times K_{n_r}.$$

Therefore, Corollary 2.10 and Lemma 2.25 give the following result.

Theorem 2.26. *For any positive integers,* $n_1, n_2, ...n_r$*,*

$$dim_s(K_{n_1} \Box K_{n_2} \Box ... \Box K_{n_r}) = n_1 n_2 ... n_r - \max_{1\leq i\leq r} \left\{ \frac{n_1 n_2 ... n_r}{n_i} \right\}.$$

As a consequence of the result above we obtain an expression for the strong metric dimension of Hamming graphs.

Corollary 2.27. [67] *For any Hamming graph* $H_{k,n}$*,*

$$dim_s(H_{k,n}) = (n-1)n^{k-1}.$$

Chapter 3

Strong metric dimension of direct product graphs

3.1 Overview

In this chapter we study the problem of finding the strong metric dimension of several families of direct products of graphs. Specifically, we obtain closed formulae for the strong metric dimension of the direct products of odd cycles of the same order, and the direct product of a complete graph with either a complete graph, a path or a cycle, in terms of the orders of their factors.

3.2 Formulae for some families of direct product graphs

In concordance with Theorem 1.2, we emphasize the fact that the strong metric dimension is not defined for the direct product graphs $C_r \times C_t$ with r, t even, $P_r \times P_t$, and $P_r \times C_t$ with t even. We focus mainly on the problem of finding the strong metric dimension of the direct products of odd cycles of the same order and the direct product of a complete graph with either a complete graph, a path or a cycle. The case where one factor is an odd cycle and the other is an even cycle or a path or an odd cycle of a different order appears to be computationally quite tedious and is not considered here, with the exception of the case $C_{2k+1} \times C_{2k+1}$, which is obtained directly from some other known results according to the following fact.

Lemma 3.1. [79] *Let G and H be two connected graphs. Then, $G\Box H \cong G \times H$ if and only if $G \cong H \cong C_{2k+1}$ for some positive integer k.*

The characterization above, and the results from the previous chapter, allow us to immediately determine the strong metric dimension of the direct product of such pairs of graphs. Specifically, the lemma above and Corollary 2.13 (c) give the following result.

Corollary 3.2. *For any positive integer k,*

$$dim_s(C_{2k+1} \times C_{2k+1}) = (2k+1)(k+1).$$

For the remainder of this chapter we focus on the strong metric dimension of the direct product of two graphs one of which is complete. In Chapter 2 we showed that the strong resolving graph of the Cartesian product of two graphs is the direct product of the strong resolving graphs of the factors. No such result is known for the direct product of two graphs, but the next result gives a relationship between the strong resolving graph of the direct product of complete graphs and their Cartesian product.

Lemma 3.3. *For any positive integers $r, t \geq 3$,*

$$(K_r \times K_t)_{SR} \cong K_r \Box K_t.$$

Proof. Let V_1 and V_2 be the vertex sets of K_r and K_t, respectively. Let (u_1, v_1) and (u_2, v_2) be two distinct vertices of $K_r \times K_t$. If $u_1 = u_2$ or $v_1 = v_2$, then $d_{K_r\times K_t}((u_1, v_1), (u_2, v_2)) = 2$. On the other hand, if $u_1 \neq u_2$ and $v_1 \neq v_2$, then $d_{K_r\times K_t}((u_1, v_1), (u_2, v_2)) = 1$. Thus, any two distinct vertices (u_1, v_1) and (u_2, v_2) are mutually maximally distant in $K_r \times K_t$ if and only if either $u_1 = u_2$ or $v_1 = v_2$. So, every vertex (x, y) is adjacent in $(K_r \times K_t)_{SR}$ to all the vertices of the sets $\{(x, v_i) : v_i \in V_2 - \{y\}\}$ and $\{(u_i, y) : u_i \in V_1 - \{x\}\}$ and thus, $(K_r \times K_t)_{SR}$ is isomorphic to the Cartesian product $K_r\Box K_t$. $\square$

A well-known result of Vizing is used to find the strong metric dimension of the direct product of complete graphs. Furthermore, the upper bound of the following lemma is also helpful to prove the lower bound on the strong metric dimension of strong products of graphs in Chapter 4.

Lemma 3.4. [107](Vizing, 1963) *For any graphs G and H of order r and t, respectively,*

$$\alpha(G)\alpha(H) + \min\{r - \alpha(G), t - \alpha(H)\} \leq \alpha(G\Box H) \leq \min\{t\alpha(G), r\alpha(H)\}.$$

Corollary 3.5. *For any positive integers $r, t \geq 3$,*

$$dim_s(K_r \times K_t) = \max\{r(t-1), t(r-1)\}.$$

Proof. By Theorem 1.12, Lemma 3.3, and Theorem 1.13, $dim_s(K_r \times K_t) = rt - \alpha(K_r \Box K_t)$. By Lemma 3.4, $\alpha(K_r \Box K_t) = \min\{r, t\}$. Thus $dim_s(K_r \times K_t) = rt - \min\{r, t\} = \max\{r(t-1), t(r-1)\}$. □

We now introduce a well-known class of graphs that is used in deriving a formula for the strong metric dimension of the direct product of cycles and complete graphs. Let $\mathbb{Z}_n$ be the additive group of integers modulo n and let $M \subset \mathbb{Z}_n$, such that, $i \in M$ if and only if $-i \in M$. We can construct a graph G as follows: the vertices of $V(G)$ are the elements of $\mathbb{Z}_n$ and (i, j) is an edge in $E(G)$ if and only if $j - i \in M$. This graph is a *circulant of order* n and is denoted by $CR(n, M)$. With this notation, a cycle is the same as $CR(n, \{-1, 1\})$ and the complete graph is $CR(n, \mathbb{Z}_n)$. In order to simplify the notation we use $CR(n, t)$, $0 < t \leq \frac{n}{2}$, instead of $CR(n, \{-t, -t+1, ..., -1, 1, 2, ..., t\})$. This is also the t^{th} power of C_n.

Lemma 3.6. *For any circulant graph $CR(n, 2)$,*

$$\alpha(CR(n, 2)) = \left\lfloor \frac{n}{3} \right\rfloor.$$

Proof. Let $V(CR(n, 2)) = \{u_0, u_1, ..., u_{n-1}\}$ be the set of vertices of $CR(n, 2)$, where two vertices u_i, u_j are adjacent if and only if $i - j \in \{-2, -1, 1, 2\}$. Notice that every vertex u_i is adjacent to the vertices $u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}$, where the operations with the subscripts i are expressed modulo n. Let S be the set of vertices of $CR(n, 2)$ satisfying the following.

- If $n \equiv 0 \bmod 3$, then $S = \{u_0, u_3, u_6, ..., u_{n-6}, u_{n-3}\}$.
- If $n \equiv 1 \bmod 3$, then $S = \{u_0, u_3, u_6, ..., u_{n-7}, u_{n-4}\}$.
- If $n \equiv 2 \bmod 3$, then $S = \{u_0, u_3, u_6, ..., u_{n-8}, u_{n-5}\}$.

Notice that S is an independent set. Thus, $\alpha(CR(n, 2)) \geq |S| = \lfloor \frac{n}{3} \rfloor$. Now, let us suppose that $\alpha(CR(n, 2)) > \lfloor \frac{n}{3} \rfloor$ and let S' be an independent set of maximum cardinality in $CR(n, 2)$. Hence there exist two vertices $u_i, u_j \in S'$ such that either $i = j+1$, $i = j-1$, $i = j+2$ or $i = j-2$, where the operations with the subscripts i, j are expressed modulo n. Thus, $i - j \in \{-2, -1, 1, 2\}$ and, hence, u_i and u_j are adjacent, which is a contradiction. Therefore, $\alpha(CR(n, 2)) = \lfloor \frac{n}{3} \rfloor$ and the proof is complete. □

The lemma above is particularly useful for our study, as we can see in the next result, since the strong resolving graph of $C_r \times K_t$ contains several subgraphs which are isomorphic to a circulant graph.

Theorem 3.7. *For any positive integers $r \geq 4$ and $t \geq 3$,*

$$dim_s(C_r \times K_t) = \begin{cases} t(r-1), & \text{if } r \in \{4,5\}, \\ \frac{tr}{2}, & \text{if } r \text{ is even and } r \geq 6, \\ t(r - \lfloor \frac{r}{3} \rfloor), & \text{otherwise.} \end{cases}$$

Proof. Let $V_1 = \{u_0, u_1, ..., u_{r-1}\}$ and $V_2 = \{v_1, v_2, ..., v_t\}$ be the vertex sets of C_r and K_t, respectively. We assume $C_r : u_0u_1u_2 \cdots u_{r-1}u_0$. Hereafter all the operations with the subscript of a vertex u_i of C_r are expressed modulo r. Let $(u_i, v_j), (u_l, v_k)$ be two distinct vertices of $C_r \times K_t$.

Case 1: Let $r = 4$ or 5.

Subcase 1.1: $u_i = u_l$. Hence, $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$. Since $(u_i, v_j) \sim (u_{i-1}, v_k)$, if $k \neq j$ and $d_{C_r \times K_t}((u_{i-1}, v_k), (u_l, v_k)) = 3$, then it follows that (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $C_r \times K_t$.

Subcase 1.2: $v_j = v_k$. If $l = i+1$ or $i = l+1$, then without loss of generality we suppose $l = i+1$ and we have that $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 3 = D(C_r \times K_t)$. Thus, (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $C_r \times K_t$. On the other hand, if $l \neq i+1$ and $i \neq l+1$, then $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$. Since for every vertex $(u, v) \in N_{C_r \times K_t}(u_i, v_j)$ we have that $d_{C_r \times K_t}((u, v), (u_l, v_k)) \leq 2$ and also for every vertex $(u, v) \in N_{C_r \times K_t}(u_l, v_k)$ we have that $d_{C_r \times K_t}((u, v), (u_i, v_j)) \leq 2$, we obtain that (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $C_r \times K_t$.

Subcase 1.3: $u_i \neq u_l$, $v_j \neq v_k$ and $(u_i, v_j) \sim (u_l, v_k)$. So, there exists a vertex $(u, v) \in N_{C_r \times K_t}(u_l, v_k)$ such that $d_{C_r \times K_t}((u, v), (u_i, v_j)) = 2$ and, as a consequence, (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $C_r \times K_t$.

Subcase 1.4: $u_i \neq u_l$, $v_j \neq v_k$ and $(u_i, v_j) \not\sim (u_l, v_k)$. Hence, we have $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$. We can suppose, without loss of generality, that $l = i+2$. Since

- $(u_i, v_j) \sim (u_{l-1}, v_k)$ and $(u_l, v_k) \sim (u_{l-1}, v_j)$ and also,
- $d_{C_r \times K_t}((u_i, v_j), (u_{l-1}, v_j)) = 3$ and $d_{C_r \times K_t}((u_l, v_k), (u_{l-1}, v_k)) = 3$,

we obtain that (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $C_r \times K_t$. Hence the strong resolving graph $(C_r \times K_t)_{SR}$ is isomorphic to $\bigcup_{i=1}^{t} K_r$. Thus, by Theorem 1.12,

$$dim_s(C_r \times K_t) = \beta((C_r \times K_t)_{SR}) = \beta\left(\bigcup_{i=1}^{t} K_r\right) = \sum_{i=1}^{t} \beta(K_r) = t(r-1).$$

Case 2: $r \geq 6$. Let $(u_i, v_j), (u_l, v_k)$ be two different vertices of $C_r \times K_t$.
Subcase 2.1: $u_i = u_l$. As in Subcase 1.1 it can be shown that $(u_i, v_j), (u_l, v_k)$ are not mutually maximally distant.
Subcase 2.2: $v_j = v_k$. We consider the following further subcases.

(a) $l = i+1$ or $i = l+1$. Without loss of generality we assume $l = i+1$. Hence, it follows $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = 3$. Notice that

$$N_{C_r \times K_t}(u_i, v_j) = \{u_{i-1}, u_{i+1}\} \times (V_2 - \{v_j\})$$

and

$$N_{C_r \times K_t}(u_l, v_k) = \{u_i, u_{i+2}\} \times (V_2 - \{v_k\}).$$

Thus, we have $d_{C_r \times K_t}((u, v), (u_l, v_k)) \leq 2$ for every vertex $(u, v) \in N_{C_r \times K_t}(u_i, v_j)$, and it follows $d_{C_r \times K_t}((u, v), (u_i, v_j)) \leq 2$ for every vertex $(u, v) \in N_{C_r \times K_t}(u_l, v_k)$. Hence, (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $C_r \times K_t$.

(b) $l \neq i+1$, $i \neq l+1$ and $d_{C_r}(u_i, u_l) < D(C_r)$. So, $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = \min\{l-i, i-l\}$. Since we have $(u_i, v_j) \sim (u_{i-1}, v_q)$ with $q \neq j$ and $d_{C_r \times K_t}((u_{i-1}, v_q), (u_l, v_k)) = \min\{l-i+1, i-l+1\}$, then (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $C_r \times K_t$.

(c) $l \neq i+1$, $i \neq l+1$ and $d_{C_r}(u_i, u_l) = D(C_r)$. Thus, it follows $d_{C_r \times K_t}((u_i, v_j), (u_l, v_k)) = \min\{l-i, i-l\} = D(C_r) = \left\lfloor \frac{r}{2} \right\rfloor$ and, as a consequence, we have that (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $C_r \times K_t$.

Subcase 2.3: $u_i \neq u_l$, $v_j \neq v_k$ and $d_{C_r}(u_i, u_l) < D(C_r)$. As in Subcase 2.2 (b) it can be shown that (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $C_r \times K_t$.
Subcase 2.4: $u_i \neq u_l$, $v_j \neq v_k$ and $d_{C_r}(u_i, u_l) = D(C_r)$. As in Subcase 2.2 (c) it can be shown that (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $C_r \times K_t$.

From the cases above it follows that the strong resolving graph $(C_r \times K_t)_{SR}$ has vertex set $V_1 \times V_2$ and two vertices $(u_i, v_j), (u_l, v_k)$ are adjacent in this graph if and only if either, $(\min\{l-i, i-l\} = 1$ and $j = k)$ or $(\min\{l-i, i-l\} = D(C_r) = \left\lfloor \frac{r}{2} \right\rfloor$ and $1 \leq j, k \leq t)$. Next we obtain the vertex cover number of $(C_r \times K_t)_{SR}$.

If r is even, then every vertex (u_i, v_j) has t neighbors of type $(u_{i+r/2}, v_l)$, $1 \leq l \leq t$ and two neighbors $(u_{i-1}, v_j), (u_{i+1}, v_j)$. So, $\beta((C_r \times K_t)_{SR}) \geq t\beta(C_r) = t\frac{r}{2}$. On the other hand, if we take the set of vertices $A = \{(u_i, v_j) : i \in \{0, 2, 4, ..., r-2\},\ j \in \{1, ..., t\}\}$, then every edge of $(C_r \times K_t)_{SR}$ is incident to some vertex of A. So, A is a vertex cover and $\beta((C_r \times K_t)_{SR}) \leq |A| = t\frac{r}{2}$. Hence $\beta((C_r \times K_t)_{SR}) = t\frac{r}{2}$. Therefore

$$dim_s(C_r \times K_t) = \beta((C_r \times K_t)_{SR}) = t\frac{r}{2}.$$

If r is odd, then every vertex (u_i, v_j) has t neighbors of type $(u_{i+(r-1)/2}, v_l)$, t neighbors of type $(u_{i+(r+1)/2}, v_l)$, $1 \leq l \leq t$, and the two neighbors (u_{i-1}, v_j), (u_{i+1}, v_j). Thus for every $k \in \{1, ..., t\}$ it follows that

$$\begin{aligned}(u_0, v_k) \sim (u_{\frac{r-1}{2}}, v_k) \sim (u_{r-1}, v_k) \sim (u_{\frac{r-1}{2}-1}, v_k) \sim (u_{r-2}, v_k) \sim \cdots \\ \sim (u_1, v_k) \sim (u_{\frac{r+1}{2}}, v_k) \sim (u_0, v_k).\end{aligned} \tag{3.1}$$

Also, since $(u_0, v_k) \sim (u_1, v_k) \sim \cdots \sim (u_{r-1}, v_k) \sim (u_0, v_k)$, the graph G' formed from t disjoint copies of a circulant graph $CR(r, 2)$ is a subgraph of $(C_r \times K_t)_{SR}$. By Lemma 3.6

$$\beta((C_r \times K_t)_{SR}) \geq t\beta(CR(r,2)) = t(r - \alpha(CR(r,2))) = t\left(r - \left\lfloor \frac{r}{3} \right\rfloor\right).$$

Now, we rename the vertices of C_r according to the adjacencies in (3.1), *i.e.*, $u'_0 = u_0$, $u'_1 = u_{\frac{r-1}{2}}$, $u'_2 = u_{r-1}$, $u'_3 = u_{\frac{r-1}{2}-1}$, $u'_4 = u_{r-2}$, ..., $u'_{r-2} = u_1$ and $u'_{r-1} = u_{\frac{r+1}{2}}$. With this notation, we define a set B, of vertices of $(C_r \times K_t)_{SR}$, as follows:

- $B = \{(u'_i, v_j) : i \in \{0, 1, 3, 4, 6, 7, ..., r-3, r-2\},\ j \in \{1, ..., t\}\}$, if $r \equiv 0 \pmod 3$.
- $B = \{(u'_i, v_j) : i \in \{0, 1, 3, 4, 6, 7, ..., r-4, r-3, r-1\},\ j \in \{1, ..., t\}\}$, if $r \equiv 1 \pmod 3$.
- $B = \{(u'_i, v_j) : i \in \{0, 1, 3, 4, 6, 7, ..., r-5, r-4, r-2, r-1\},\ j \in \{1, ..., t\}\}$, if $r \equiv 2 \pmod 3$.

Note that if $(u,v),(x,y) \notin B$, then $(u,v) \not\sim (x,y)$ and, thus B is a vertex cover of $(C_r \times K_t)_{SR}$. Hence, $\beta((C_r \times K_t)_{SR}) \le |B| = t(r - \lfloor \frac{r}{3} \rfloor)$, which leads to $\beta((C_r \times K_t)_{SR}) = t(r - \lfloor \frac{r}{3} \rfloor)$. Therefore, we have the following

$$dim_s(C_r \times K_t) = \beta((C_r \times K_t)_{SR}) = t\left(r - \left\lfloor \frac{r}{3} \right\rfloor\right).$$

□

Figure 3.1 shows an example regarding Theorem 3.7.

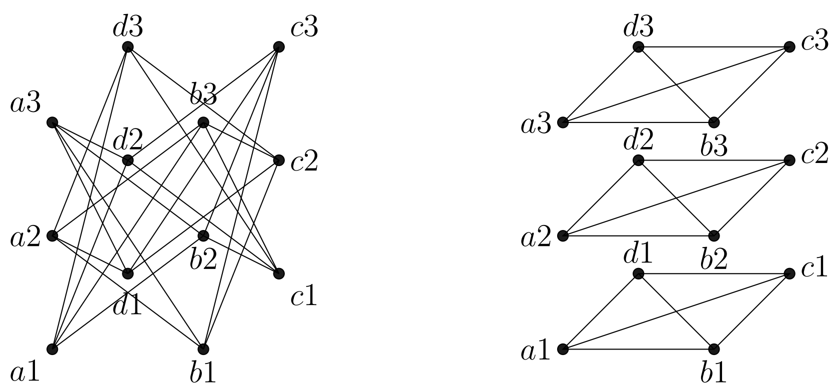

Figure 3.1: Direct product $C_4 \times K_3$ and its strong resolving graph $(C_4 \times K_3)_{SR}$.

We finish the exposition of our results throughout the study of the strong metric dimension of the direct product of a path with a complete graph.

Theorem 3.8. *For any positive integers $r \ge 2$ and $t \ge 3$,*

$$dim_s(P_r \times K_t) = t\left\lceil \frac{r}{2} \right\rceil.$$

Proof. Let $V_1 = \{u_1, u_2, ..., u_r\}$ and $V_2 = \{v_1, v_2, ..., v_t\}$ be the vertex sets of P_r and K_t, respectively. We assume $u_1 \sim u_2 \sim u_3 \sim \cdots \sim u_r$ in P_r. If $r = 2$, then a vertex (u_i, v_j) in $P_2 \times K_t$ is mutually maximally distant only with the vertex (u_l, v_j), where $i \ne l$. So, $(P_2 \times K_t)_{SR} \cong \bigcup_{m=1}^{t} K_2$. Thus, by Theorem 1.12,

$$dim_s(P_2 \times K_t) = \beta((P_2 \times K_t)_{SR}) = \beta\left(\bigcup_{i=1}^{t} K_2\right) = \sum_{i=1}^{t} \beta(K_2) = t.$$

If $r = 3$, then a vertex (u_i, v_j) in $P_3 \times K_t$ is mutually maximally distant only with those vertices (u_l, v_j), where $i \ne l$. Thus, $(P_3 \times K_t)_{SR} \cong \bigcup_{m=1}^{t} K_3$ and, by Theorem 1.12,

$$dim_s(P_3 \times K_t) = \beta((P_3 \times K_t)_{SR}) = \beta\left(\bigcup_{i=1}^{t} K_3\right) = \sum_{i=1}^{t} \beta(K_3) = t\left\lceil \frac{r}{2} \right\rceil.$$

From now on we suppose $r \geq 4$. Let $(u_i, v_j), (u_l, v_k)$ be two different vertices of $P_r \times K_t$. We consider the following cases.
Case 1: $u_i = u_l$. Hence, it is satisfied that $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = 2$. If $i \neq 1$, then $(u_i, v_j) \sim (u_{i-1}, v_k)$ and $d_{P_r \times K_t}((u_{i-1}, v_k), (u_l, v_k)) = 3$. Also, if $i = 1$, then $(u_i, v_j) \sim (u_{i+1}, v_k)$ and $d_{P_r \times K_t}((u_{i+1}, v_k), (u_l, v_k)) = 3$. Thus, (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $P_r \times K_t$.
Case 2: $v_j = v_k$ and, without loss of generality, $i < l$. We have the following cases.

(a) If $u_i \sim u_l$ in P_r, then $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = 3$. Let (u_a, v_b) be a vertex such that $(u_i, v_j) \sim (u_a, v_b)$. So, ($a = i - 1$ or $a = l$) and $b \neq j$. Thus, for every (u_a, v_b) we have that $d_{P_r \times K_t}((u_a, v_b), (u_l, v_k)) = 2 < 3 = d_{P_r \times K_t}((u_i, v_j), (u_l, v_k))$. Now, let (u_c, v_d) be a vertex such that $(u_l, v_k) \sim (u_c, v_d)$. So, ($c = i$ or $c = l + 1$) and $d \neq j$. Thus, for every (u_c, v_d) we have that $d_{P_r \times K_t}((u_c, v_d), (u_i, v_j)) = 2 < 3 = d_{P_r \times K_t}((u_i, v_j), (u_l, v_k))$. Therefore, (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $P_r \times K_t$.

(b) If $u_i \not\sim u_l$ in P_r, then $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = |i - l|$. Now, if $u_i \neq u_1$, then for every vertex (u_{i-1}, v_p), $p \neq j$, we have that $(u_i, v_j) \sim (u_{i-1}, v_p)$ and $d_{P_r \times K_t}((u_{i-1}, v_p), (u_l, v_k)) = |i - l + 1|$. Similarly, if $u_l \not\sim u_r$, then for every vertex (u_{l+1}, v_p), $p \neq j$, we have that $(u_l, v_k) \sim (u_{l+1}, v_p)$ and $d_{P_r \times K_t}((u_{l+1}, v_p), (u_i, v_j)) = |i - l + 1|$. Thus, we obtain that (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $P_r \times K_t$.

(c) If $u_i = u_1$ and $u_l = u_r$, then $d_{P_r \times K_t}((u_i, v_j), (u_l, v_k)) = r - 1 = D(P_r \times K_t)$. Thus, (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $P_r \times K_t$.

Case 3: $u_i \neq u_l$, $v_j \neq v_k$ and we consider, without loss of generality, $i < l$. If $u_i \neq u_1$ or $u_l \neq u_r$, then as in Case 2 (b) it follows that (u_i, v_j) and (u_l, v_k) are not mutually maximally distant in $P_r \times K_t$. On the other hand, if $u_i = u_1$ and $u_l = u_r$, then as in Case 2 (c) it follows that (u_i, v_j) and (u_l, v_k) are mutually maximally distant in $P_r \times K_t$.

Therefore, $(P_r \times K_t)_{SR}$ is isomorphic to a graph with vertex set $V_1 \times V_2$ and such that two vertices $(u_i, v_j), (u_l, v_k)$ are adjacent if and only if either, ($|l - i| = 1$ and $j = k$) or ($|l - i| = r - 1$ and $1 \leq j, k \leq r$). Notice that every vertex (u_i, v_j), where $1 < i < r$, has only two neighbors (u_{i-1}, v_j) and

(u_{i+1}, v_j), while every vertex (u_1, v_j) has a neighbor (u_2, v_j) and r neighbors of type (u_r, v_l), $1 \leq l \leq t$. Also, every vertex (u_r, v_j) has a neighbor (u_{r-1}, v_j) and r neighbors of type (u_1, v_l), $1 \leq l \leq t$. So, $(P_r \times K_t)_{SR}$ has a subgraph G' isomorphic to the disjoint union of t cycles of order r and, as a consequence, $\beta((P_r \times K_t)_{SR}) \geq t\beta(C_r) = t\left\lceil \frac{r}{2} \right\rceil$.

On the other hand, let r be an even number. If we take the set of vertices $A = \{(u_i, v_j) : i \in \{1, 3, 5, ..., r-1\},\ j \in \{1, ..., t\}\}$, then every edge of $(P_r \times K_t)_{SR}$ is incident to some vertex of A. Thus, A is a vertex cover of $(P_r \times K_t)_{SR}$ and we have that $\beta((P_r \times K_t)_{SR}) \leq |A| = t\left\lceil \frac{r}{2} \right\rceil$. Now, suppose r odd. If we take the set of vertices $B = \{(u_i, v_j) : i \in \{1, 3, 5, ..., r\},\ j \in \{1, ..., t\}\}$, then every edge of $(P_r \times K_t)_{SR}$ is incident to some vertex of B. So, B is a vertex cover of $(P_r \times K_t)_{SR}$ and thus $\beta((P_r \times K_t)_{SR}) \leq |B| = t\left\lceil \frac{r}{2} \right\rceil$. Hence $\beta((P_r \times K_t)_{SR}) = t\left\lceil \frac{r}{2} \right\rceil$. Therefore, from Theorem 1.12,

$$dim_s(P_r \times K_t) = \beta((P_r \times K_t)_{SR}) = t\left\lceil \frac{r}{2} \right\rceil.$$

□

Chapter 4

Strong metric dimension of strong product graphs

4.1 Overview

The current chapter is concerned with finding some relationships between the strong resolving graph of strong product graphs and that of its factor graphs. Furthermore, we give general lower and upper bounds on the strong metric dimension of the strong product of graphs in terms of the order and the strong metric dimension of its factors. We also describe some classes of graphs where these bounds are achieved.

4.2 Main results

In this chapter we use the concept of the strong resolving graph defined in [82]. We recall, according to [82], the strong resolving graph G_{SR+I} of a graph G has vertex set $V(G_{SR+I}) = V(G)$ and two vertices u, v are adjacent in G_{SR+I} if and only if u and v are mutually maximally distant in G.

For any graph G of order n, by using Theorems 1.11 and 1.13, we immediately obtain a very useful tool of this chapter. Notice that this equality is analogous to this one in (1.1), where it is considered the strong resolving graph G_{SR} instead of the original strong resolving graph G_{SR+I}.

$$dim_s(G) = n - \alpha(G_{SR+I}) \tag{4.1}$$

We now describe the structure of the strong resolving graph of $G \boxtimes H$.

Lemma 4.1. *Let G and H be two connected nontrivial graphs. Let u, x be two vertices of G and let v, y be two vertices of H. Then (u, v) and (x, y) are mutually maximally distant vertices in $G \boxtimes H$ if and only if one of the following conditions holds:*

(i) *u, x are mutually maximally distant in G and v, y are mutually maximally distant in H;*

(ii) *u, x are mutually maximally distant in G and $v = y$;*

(iii) *v, y are mutually maximally distant in H and $u = x$;*

(iv) *u, x are mutually maximally distant in G and $d_G(u, x) > d_H(v, y)$;*

(v) *v, y are mutually maximally distant in H and $d_G(u, x) < d_H(v, y)$.*

Proof. (Sufficiency) Let $(u', v') \in N_{G\boxtimes H}(u, v)$ and $(x', y') \in N_{G\boxtimes H}(x, y)$. By Corollary 1.5 we have $u' \in N_G[u]$, $x' \in N_G[x]$, $v' \in N_H[v]$ and $y' \in N_H[y]$.

(i) If u, x are mutually maximally distant in G and v, y are mutually maximally distant in H, then

$$\begin{aligned} d_{G\boxtimes H}((u', v'), (x, y)) &= \max\{d_G(u', x), d_H(v', y)\} \\ &\leq \max\{d_G(u, x), d_H(v, y)\} \\ &= d_{G\boxtimes H}((u, v), (x, y)) \end{aligned}$$

and

$$\begin{aligned} d_{G\boxtimes H}((u, v), (x,' y')) &= \max\{d_G(u, x'), d_H(v, y')\} \\ &\leq \max\{d_G(u, x), d_H(v, y)\} \\ &= d_{G\boxtimes H}((u, v), (x, y)). \end{aligned}$$

Thus, (u, v) and (x, y) are mutually maximally distant vertices in $G \boxtimes H$.

(ii) If u, x are mutually maximally distant in G and $v = y$, then

$$\begin{aligned} d_{G\boxtimes H}((u', v'), (x, y)) &= \max\{d_G(u', x), d_H(v', y)\} \\ &= d_G(u', x) \\ &\leq d_G(u, x) \\ &= d_{G\boxtimes H}((u, v), (x, y)) \end{aligned}$$

and

$$\begin{aligned} d_{G\boxtimes H}((u,v),(x',y')) &= \max\{d_G(u,x'),d_H(v,y')\} \\ &= d_G(u,x') \\ &\leq d_G(u,x) \\ &= d_{G\boxtimes H}((u,v),(x,y)). \end{aligned}$$

Thus, (u,v) and (x,y) are mutually maximally distant vertices in $G \boxtimes H$.

(iii) According to the commutativity of the strong product of graphs, the result follows directly from (ii).

(iv) If u, x are mutually maximally distant in G and $d_G(u,x) > d_H(v,y)$, then

$$\begin{aligned} d_{G\boxtimes H}((u',v'),(x,y)) &= \max\{d_G(u',x),d_H(v',y)\} \\ &\leq \max\{d_G(u,x),d_H(v,y)+1\} \\ &= \max\{d_G(u,x),d_H(v,y)\} \\ &= d_{G\boxtimes H}((u,v),(x,y)) \end{aligned}$$

and

$$\begin{aligned} d_{G\boxtimes H}((u,v),(x,'y')) &= \max\{d_G(u,x'),d_H(v,y')\} \\ &\leq \max\{d_G(u,x),d_H(v,y)+1\} \\ &= \max\{d_G(u,x),d_H(v,y)\} \\ &= d_{G\boxtimes H}((u,v),(x,y)). \end{aligned}$$

Thus, (u,v) and (x,y) are mutually maximally distant vertices in $G \boxtimes H$.

(v) According to the commutativity of the strong product of graphs, the result follows directly from (iv).

(Necessity) Let (u,v) and (x,y) be two mutually maximally distant vertices in $G \boxtimes H$. Let $u' \in N_G(u)$, $x' \in N_G(x)$, $v' \in N_H(v)$ and $y' \in N_H(y)$. Notice that, by Corollary 1.5 $(u',v') \in N_{G\boxtimes H}(u,v)$ and $(x',y') \in N_{G\boxtimes H}(x,y)$. So, we have that

$$d_{G\boxtimes H}((u,v),(x,y)) \geq d_{G\boxtimes H}((u',v'),(x,y))$$

and

$$d_{G\boxtimes H}((u,v),(x,y)) \geq d_{G\boxtimes H}((u,v),(x',y')).$$

We differentiate two cases.

Case 1: $d_G(u,x) \geq d_H(v,y)$. Hence,

$$d_{G\boxtimes H}((u,v),(x,y)) = \max\{d_G(u,x), d_H(v,y)\} = d_G(u,x).$$

Thus,

$$d_G(u,x) \geq \max\{d_G(u',x), d_H(v',y)\}$$

and

$$d_G(u,x) \geq \max\{d_G(u,x'), d_H(v,y')\}.$$

So, we obtain four inequalities:

$$d_G(u,x) \geq d_G(u',x), \tag{4.2}$$

$$d_G(u,x) \geq d_H(v',y), \tag{4.3}$$

$$d_G(u,x) \geq d_G(u,x'), \tag{4.4}$$

$$d_G(u,x) \geq d_H(v,y'). \tag{4.5}$$

From (4.2) and (4.4) we have, that u and x are mutually maximally distant in G. If v and y are mutually maximally distant in H, then (i) holds and, if $v = y$, then (ii) holds. Suppose that there exists a vertex $v'' \in N_H(v)$ such that $d_H(v'',y) > d_H(v,y)$ or there exists a vertex $y'' \in N_H(y)$ such that $d_H(v,y'') > d_H(v,y)$. In such a case,

$$d_H(v'',y) \geq d_H(v,y) + 1 \tag{4.6}$$

or

$$d_H(v,y'') \geq d_H(v,y) + 1. \tag{4.7}$$

Since $v'' \in N_H(v)$, for any $u'' \in N_G(u)$ we have $(u'',v'') \in N_{G\boxtimes H}(u,v)$ and following the procedure above, taking (u'',v'') instead of (u',v') we obtain two inequalities equivalent to (4.3) and (4.5). Thus,

$$d_G(u,x) \geq d_H(v'',y) > d_H(v,y) \tag{4.8}$$

and

$$d_G(u,x) \geq d_H(v,y'') > d_H(v,y). \tag{4.9}$$

So, u,x are mutually maximally distant in G and $d_G(u,x) > d_H(v,y)$. Hence, (iv) is satisfied.

Case 2: $d_G(u,x) < d_H(v,y)$. By using analogous procedure we can prove that v,y are mutually maximally distant in H and $u = x$ or $d_G(u,x) < d_H(v,y)$, showing that (iii) and (v) hold. Therefore, the result follows. $\square$

Notice that Lemma 4.1 leads to the following relationship. To begin with, we need to introduce more notation. Let $G=(V,E)$ and $G'=(V',E')$ be two graphs. If $V'\subseteq V$ and $E'\subseteq E$, then G' is a subgraph of G and we denote that by $G'\sqsubseteq G$.

Theorem 4.2. *For any connected graphs G and H,*

$$G_{SR+I}\boxtimes H_{SR+I}\sqsubseteq (G\boxtimes H)_{SR+I}\sqsubseteq G_{SR+I}\oplus H_{SR+I}.$$

Proof. Notice that

$$V(G_{SR+I}\boxtimes H_{SR+I})=V((G\boxtimes H)_{SR+I})=V(G_{SR+I}\oplus H_{SR+I})=V(G)\times V(H).$$

Let (u,v) and (x,y) be two vertices adjacent in $G_{SR+I}\boxtimes H_{SR+I}$. So, either

- $u=x$ and $vy\in E(H_{SR+I})$, or
- $ux\in E(G_{SR+I})$ and $v=y$, or
- $ux\in E(G_{SR+I})$ and $vy\in E(H_{SR+I})$.

Hence, by using respectively the condition (iii), (ii) and (i) of Lemma 4.1 we have that (u,v) and (x,y) are also adjacent in $(G\boxtimes H)_{SR+I}$.

Now, let (u',v') and (x',y') be two vertices adjacent in $(G\boxtimes H)_{SR+I}$. From Lemma 4.1 we obtain that $u'x'\in E(G_{SR+I})$ or $v'y'\in E(H_{SR+I})$. Thus, (u',v') and (x',y') are also adjacent in $G_{SR+I}\oplus H_{SR+I}$. $\square$

Corollary 4.3. *For any connected graphs G and H,*

$$\alpha(G_{SR+I}\boxtimes H_{SR+I})\geq\alpha((G\boxtimes H)_{SR+I})\geq\alpha(G_{SR+I}\oplus H_{SR+I}).$$

In order to better understand what the strong resolving graph $(G\boxtimes H)_{SR+I}$ looks like, by using Lemma 4.1, we prepare a kind of "graphical representation" of $(G\boxtimes H)_{SR+I}$ which we present in Figure 4.1. According to the conditions (i), (ii) and (iii) of Lemma 4.1 the solid lines represents those edges of $(G\boxtimes H)_{SR+I}$ which always exists. Also, from the conditions (iv) and (v) of Lemma 4.1, two vertices belonging to different rounded rectangles with identically filled areas could be adjacent or not in $(G\boxtimes H)_{SR+I}$.

The following known result is useful for our purposes.

Theorem 4.4. [56] *For any graphs G and H,*

$$\alpha(G)\alpha(H)\leq\alpha(G\boxtimes H)\leq\alpha(G\Box H).$$

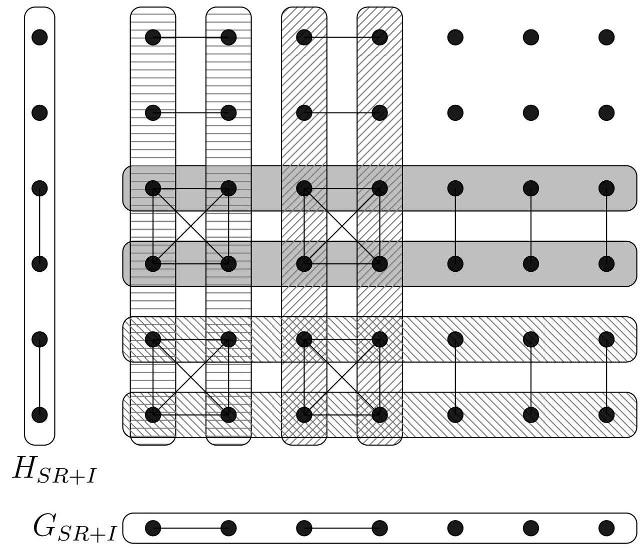

Figure 4.1: Sketch of a representation of $(G \boxtimes H)_{SR+I}$.

Next we present the following lemma, from [24], about the independence number of Cartesian sum graphs.

Lemma 4.5. [24] *For any graphs G and H,*

$$\alpha(G \oplus H) = \alpha(G)\alpha(H).$$

The result below gives general lower and upper bounds on the strong metric dimension of the strong product of two graphs in terms of the order and the strong metric dimension of its factors.

Theorem 4.6. *Let G and H be two connected nontrivial graphs of order n_1, n_2, respectively. Then*

$$dim_s(G \boxtimes H) \geq \max\{n_2 dim_s(G), n_1 dim_s(H)\}$$

and

$$dim_s(G \boxtimes H) \leq n_2 dim_s(G) + n_1 dim_s(H) - dim_s(G)dim_s(H).$$

Proof. By using Corollary 4.3 we have that

$$\alpha(G_{SR+I} \boxtimes H_{SR+I}) \geq \alpha((G \boxtimes H)_{SR+I}).$$

Hence, from equality (4.1), Theorem 4.4 and Lemma 3.4 we obtain

$$\begin{aligned}
dim_s(G\boxtimes H) &= n_1n_2-\alpha((G\boxtimes H)_{SR+I})\\
&\geq n_1n_2-\alpha(G_{SR+I}\boxtimes H_{SR+I})\\
&\geq n_1n_2-\alpha(G_{SR+I}\Box H_{SR+I})\\
&\geq n_1n_2-\min\{n_2\alpha(G_{SR+I}),n_1\alpha(H_{SR+I})\}\\
&= \max\{n_2(n_1-\alpha(G_{SR+I})),n_1(n_2-\alpha(H_{SR+I}))\}\\
&= \max\{n_2dim_s(G),n_1dim_s(H)\}.
\end{aligned}$$

On the other hand, from Corollary 4.3 it follows

$$\alpha((G\boxtimes H)_{SR+I})\geq \alpha(G_{SR+I}\oplus H_{SR+I}).$$

So, by using (4.1) and Lemma 4.5 we have

$$\begin{aligned}
dim_s(G\boxtimes H) &= n_1n_2-\alpha((G\boxtimes H)_{SR+I})\\
&\leq n_1n_2-\alpha(G_{SR+I}\oplus H_{SR+I})\\
&= n_1n_2-\alpha(G_{SR+I})\alpha(H_{SR+I})\\
&= n_1n_2-(n_1-dim_s(G))(n_2-dim_s(H))\\
&= n_2dim_s(G)+n_1dim_s(H)-dim_s(G)dim_s(H).
\end{aligned}$$

□

4.3 The strong product of graphs where one factor is a $\mathcal{C}$-graph or a $\mathcal{C}_1$-graph

We define a *$\mathcal{C}$-graph* as a graph G whose vertex set can be partitioned into $\alpha(G)$ cliques. Notice that there are several graphs which are $\mathcal{C}$-graphs. For instance, we emphasize the following cases: complete graphs and cycles of even order. In order to prove the next result we also need to introduce the following notation. Given two graphs G, H and a subset X of vertices of $G\boxtimes H$, the *projections* of X onto the graphs G and H, respectively, are the following ones $P_G(X)=\{u\in V(G):\ (u,v)\in X,\ \text{for some } v\in V(H)\}$ and $P_H(X)=\{v\in V(H):\ (u,v)\in X,\ \text{for some } u\in V(G)\}$.

Lemma 4.7. *For any $\mathcal{C}$-graph G and any graph H,*

$$\alpha(G\boxtimes H)=\alpha(G)\alpha(H).$$

Proof. Let $A_1, A_2, ..., A_{\alpha(G)}$ be a partition of $V(G)$ such that A_i is a clique for every $i \in \{1, 2, ..., \alpha(G)\}$. Let S be an $\alpha(G \boxtimes H)$-set and let $S_i = S \cap (A_i \times V(H))$ for $i \in \{1, 2, ..., \alpha(G)\}$. First we show that $P_H(S_i)$ is an independent set in H. If $|P_H(S_i)| = 1$, then $P_H(S_i)$ is an independent set in H. If $|P_H(S_i)| \geq 2$, then for any two vertices $x, y \in P_H(S_i)$ there exist $u, v \in A_i$ such that $(u, x), (v, y) \in S_i$. We suppose that $x \sim y$. If $u = v$, then $(u, x) \sim (v, y)$, which is a contradiction. Thus, $u \neq v$. Since $(u, x) \not\sim (v, y)$, we have that $u \not\sim v$, which is a contradiction with the fact that A_i is a clique. Therefore, for every $i \in \{1, 2, ..., \alpha(G)\}$ the projection $P_H(S_i)$ is an independent set in H and $\alpha(H) \geq |P_H(S_i)|$.

Now, if $|S_i| > |P_H(S_i)|$ for some $i \in \{1, 2, ..., \alpha(G)\}$, then there exists a vertex $z \in P_H(S_i)$ and two different vertices $a, b \in A_i$ such that $(a, z), (b, z) \in S_i$, and this is a contradiction with the facts that A_i is a clique and S_i is an independent set. Thus, $|S_i| = |P_H(S_i)|$, $i \in \{1, 2, ..., \alpha(G)\}$, and we have the following

$$\alpha(G \boxtimes H) = |S| = \sum_{i=1}^{\alpha(G)} |S_i| = \sum_{i=1}^{\alpha(G)} |P_H(S_i)| \leq \alpha(G)\alpha(H).$$

Therefore, by using Theorem 4.4 we conclude the proof. □

The lemma above is particularly useful for our study, as we can see in the next result.

Theorem 4.8. *Let G and H be two connected nontrivial graphs of order n_1, n_2, respectively. If G_{SR+I} is a $\mathcal{C}$-graph, then*

$$dim_s(G \boxtimes H) = n_2 dim_s(G) + n_1 dim_s(H) - dim_s(G) dim_s(H).$$

Proof. By using Corollary 4.3 we have that

$$\alpha(G_{SR+I} \boxtimes H_{SR+I}) \geq \alpha((G \boxtimes H)_{SR+I}).$$

Hence, from equality (4.1) and Lemma 4.7 we have

$$\begin{aligned} dim_s(G \boxtimes H) &= n_1 n_2 - \alpha((G \boxtimes H)_{SR+I}) \\ &\geq n_1 n_2 - \alpha(G_{SR+I} \boxtimes H_{SR+I}) \\ &= n_1 n_2 - \alpha(G_{SR+I})\alpha(H_{SR+I}) \\ &= n_1 n_2 - (n_1 - dim_s(G))(n_2 - dim_s(H)) \\ &= n_2 dim_s(G) + n_1 dim_s(H) - dim_s(G) dim_s(H). \end{aligned}$$

The result now follows from Theorem 4.6. □

At next we give examples of graphs for which its strong resolving graphs are $\mathcal{C}$-graphs. To do so we need some additional terminology and notations.

We recall that a *cut vertex* in a graph is a vertex whose removal increases the number of connected component. Also, a *block* is a maximal biconnected subgraph of the graph. Now, let $\mathfrak{F}$ be the family of sequences of connected graphs $G_1, G_2, ..., G_k$, $k \geq 2$, such that G_1 is a complete graph K_{n_1}, $n_1 \geq 2$, and G_i, $i \geq 2$, is obtained recursively from G_{i-1} by adding a complete graph K_{n_i}, $n_i \geq 2$, and identifying a vertex of G_{i-1} with a vertex in K_{n_i}.

From this point we say that a connected graph G is a *generalized tree*[1] if and only if there exists a sequence $\{G_1, G_2, ..., G_k\} \in \mathfrak{F}$ such that $G_k = G$ for some $k \geq 2$. Notice that in these generalized trees every vertex is either, a cut vertex or a simplicial vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every G_i is isomorphic to K_2, then G_k is a tree, justifying the terminology used.

- $(K_n)_{SR+I}$ is isomorphic to K_n.
- For any complete k-partite graph such that at least all but one $p_i \geq 2$, $i \in \{1, 2, ..., k\}$, $(K_{p_1,p_2,...,p_k})_{SR+I}$ is isomorphic to the graph $\bigcup_{i=1}^{k} K_{p_i}$.
- If G is a generalized tree of order n and c cut vertices, then G_{SR+I} is isomorphic to the graph $K_{n-c} \cup (\bigcup_{i=1}^{c} K_1)$.
- For any 2-antipodal graph G of order n, G_{SR+I} is isomorphic to the graph $\bigcup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR+I} \cong \bigcup_{i=1}^{k} K_2$.
- For any grid graph, $(P_n \Box P_r)_{SR+I}$ is isomorphic to the graph $\left(\bigcup_{i=1}^{2} K_2\right) \cup \left(\bigcup_{i=1}^{nr-4} K_1\right)$.

By using the examples above and Theorem 4.8 we have the following corollary.

Corollary 4.9. *Let G and H be two connected nontrivial graphs of order n_1 and n_2, respectively.*

(i) $dim_s(K_{n_1} \boxtimes H) = n_2(n_1 - 1) + n_1 dim_s(H) - (n_1 - 1) dim_s(H)$.

(ii) *If G is a complete k-partite graph, then*

$$dim_s(G \boxtimes H) = n_2(n_1 - k) + n_1 dim_s(H) - (n_1 - k) dim_s(H).$$

[1]In some works those graphs are called block graphs.

(iii) *If G is a generalized tree with c cut vertices, then*

$$dim_s(G \boxtimes H) = n_2(n_1 - c - 1) + n_1 dim_s(H) - (n_1 - c - 1)dim_s(H).$$

Particularly, if G is a tree with $l(G)$ leaves, then

$$dim_s(G \boxtimes H) = n_2(l(G) - 1) + n_1 dim_s(H) - (l(G) - 1)dim_s(H).$$

(iv) *If G is a 2-antipodal graph, then*

$$dim_s(G \boxtimes H) = \frac{n_2 n_1}{2} + n_1 dim_s(H) - \frac{n_1}{2} dim_s(H).$$

(v) *If G is a grid graph, then*

$$dim_s(G \boxtimes H) = 3n_2 + n_1 dim_s(H) - 3dim_s(H).$$

Notice that Corollary 4.9 (iv) gives the value of the strong metric dimension of $C_r \boxtimes H$ for any graph H and r even. Next we study separately the strong product graphs $C_r \boxtimes H$ for any graph H and r odd. In order to prove the next result we need to introduce the following notation. We define a $\mathcal{C}_1$-*graph* as a graph G whose vertex set can be partitioned into $\alpha(G)$ cliques and one isolated vertex. Notice that cycles of odd order are $\mathcal{C}_1$-graphs.

Lemma 4.10. *For any $\mathcal{C}_1$-graph G and any graph H,*

$$\alpha(G \boxtimes H) \leq \alpha(H)(\alpha(G) + 1).$$

Proof. Let $A_1, A_2, ..., A_{\alpha(G)}, B$ be a partition of $V(G)$ such that A_i is a clique for every $i \in \{1, 2, ..., \alpha(G)\}$ and $B = \{b\}$, where b is isolated vertex. Let S be an $\alpha(G \boxtimes H)$-set and let $S_i = S \cap (A_i \times V(H))$ and $i \in \{1, 2, ..., \alpha(G)\}$. Let $S_B = S \cap (B \times V(H))$. By using analogous procedures as in proof of Lemma 4.7 we can show that for every $i \in \{1, 2, ..., \alpha(G)\}$, $P_H(S_i)$ is an independent set in H and $|S_i| = |P_H(S_i)|$. Moreover, since $|B| = 1$ we have that $P_H(S_B)$ is an independent set in H and $|S_B| = |P_H(S_B)|$. Thus, we obtain the following

$$\begin{aligned} \alpha(G \boxtimes H) = |S| &= \sum_{i=1}^{\alpha(G)} |S_i| + |S_B| = \sum_{i=1}^{\alpha(G)} |P_H(S_i)| + |P_H(S_B)| \\ &\leq \alpha(G)\alpha(H) + \alpha(H) = \alpha(H)(\alpha(G) + 1). \end{aligned}$$

□

Theorem 4.11. *Let G and H be two connected nontrivial graphs of order n_1, n_2, respectively. If G_{SR+I} is a $\mathcal{C}_1$-graph, then*

$$dim_s(G \boxtimes H) \geq n_2(dim_s(G) - 1) + dim_s(H)(n_1 - dim_s(G) + 1).$$

Proof. By using Corollary 4.3 we have that

$$\alpha(G_{SR+I} \boxtimes H_{SR+I}) \geq \alpha((G \boxtimes H)_{SR+I}).$$

Hence, from equality (4.1) and Lemma 4.10 we have

$$\begin{aligned} dim_s(G \boxtimes H) &= n_1 n_2 - \alpha((G \boxtimes H)_{SR+I}) \\ &\geq n_1 n_2 - \alpha(G_{SR+I} \boxtimes H_{SR+I}) \\ &\geq n_1 n_2 - \alpha(H_{SR+I})(\alpha(G_{SR+I}) + 1) \\ &= n_1 n_2 - (n_2 - dim_s(H))(n_1 - dim_s(G) + 1) \\ &= n_2(dim_s(G) - 1) + dim_s(H)(n_1 - dim_s(G) + 1). \end{aligned}$$

□

Since $dim_s(C_{2r+1}) = r + 1$, Theorems 4.6 and 4.11 lead to the following result.

Theorem 4.12. *Let H be a connected nontrivial graph of order n and $r \geq 1$. Then*

$$nr + dim_s(H)(r + 1) \leq dim_s(C_{2r+1} \boxtimes H) \leq n(r + 1) + r dim_s(H).$$

The next theorem on the independence number of strong products of odd cycles is obtained in [43].

Theorem 4.13. [43] *For $1 \leq r \leq t$,*

$$\alpha(C_{2r+1} \boxtimes C_{2t+1}) = rt + \left\lfloor \frac{r}{2} \right\rfloor.$$

By using the result above we obtain the following.

Theorem 4.14. *For $1 \leq r \leq t$,*

$$3rt + 2r + 2t + 1 - \left\lfloor \frac{r}{2} \right\rfloor \leq dim_s(C_{2r+1} \boxtimes C_{2t+1}) \leq 3rt + 2r + 2t + 1.$$

Proof. By using Theorem 4.2 we have that $G_{SR+I} \boxtimes H_{SR+I} \sqsubseteq (G \boxtimes H)_{SR+I}$. Thus, $\alpha(G_{SR+I} \boxtimes H_{SR+I}) \geq \alpha((G \boxtimes H)_{SR+I})$. Hence, from equality (4.1) and Theorem 4.13 we have

$$\begin{aligned} dim_s(C_{2r+1} \boxtimes C_{2t+1}) &= (2r+1)(2t+1) - \alpha((C_{2r+1} \boxtimes C_{2t+1})_{SR+I}) \\ &\geq (2r+1)(2t+1) - \alpha((C_{2r+1})_{SR+I} \boxtimes (C_{2t+1})_{SR+I}) \\ &= (2r+1)(2t+1) - \alpha(C_{2r+1} \boxtimes C_{2t+1}) \\ &= (2r+1)(2t+1) - rt - \left\lfloor \frac{r}{2} \right\rfloor \\ &= 3rt + 2r + 2t + 1 - \left\lfloor \frac{r}{2} \right\rfloor . \end{aligned}$$

The upper bound is a direct consequence of Theorem 4.12. $\square$

Notice that for $r = 1$ the lower bound is equal to the upper bound of the theorem above. Thus, $dim_s(C_3 \boxtimes C_{2t+1}) = 5t + 3$ for every $t \geq 1$.

Chapter 5

Strong metric dimension of lexicographic product graphs

5.1 Overview

This chapter is concerned with establishing the strong resolving graph of lexicographic product graphs, and with finding closed formulae for the strong metric dimension of some families of this product of graphs and express these in terms of invariants of the factor graphs.

5.2 Main results

To begin with the study we would point out the following known result.

Claim 5.1. [45] *Let G and H be two nontrivial graphs such that G is connected. Then the following assertions hold for any $a, c \in V(G)$ and $b, d \in V(H)$ such that $a \neq c$.*

(i) $N_{G\circ H}(a, b) = (\{a\} \times N_H(b)) \cup (N_G(a) \times V(H))$.

(ii) $d_{G\circ H}((a, b),(c, d)) = d_G(a, c)$

(iii) $d_{G\circ H}((a, b),(a, d)) = \min\{d_H(b, d), 2\}$.

By using lemmas presented below we can describe the structure of the strong resolving graph of $G \circ H$.

Lemma 5.2. *Let G be a connected nontrivial graph and let H be a nontrivial graph. Let $a, b \in V(G)$ such that they are not true twin vertices and let*

$x, y \in V(H)$. Then (a,x) and (b,y) are mutually maximally distant in $G \circ H$ if and only if a and b are mutually maximally distant in G.

Proof. Let $x, y \in V(H)$. We assume that $a, b \in V(G)$ are mutually maximally distant in G and that they are not true twins. First of all, notice that $d_G(a,b) \geq 2$ (if $d_G(a,b) = 1$, then to be mutually maximally distant in G, they must be true twins). Hence, by Claim 5.1 (i) we have that if $(c,d) \in N_{G\circ H}(b,y)$, then either $c = b$ or $c \in N_G(b)$. In both cases, by Claim 5.1 (ii) we obtain $d_{G\circ H}((a,x),(c,d)) = d_G(a,c) \leq d_G(a,b) = d_{G\circ H}((a,x),(b,y))$. So, (b,y) is maximally distant from (a,x) and, by symmetry, we conclude that (b,y) and (a,x) are mutually maximally distant in $G \circ H$.

Conversely, assume that (a,x) and (b,y), $a \neq b$, are mutually maximally distant in $G \circ H$. If $c \in N_G(b)$, then for any $z \in V(H)$ we have $(c,z) \in N_{G\circ H}(b,y)$. Now, by Claim 5.1 (ii) we obtain $d_G(a,c) = d_{G\circ H}((a,x),(c,z)) \leq d_{G\circ H}((a,x),(b,y)) = d_G(a,b)$. So, b is maximally distant from a and, by symmetry, we conclude that b and a are mutually maximally distant in G. $\square$

Lemma 5.3. *Let G be a connected nontrivial graph, let H be a graph of order $n \geq 2$, let $a, b \in V(G)$ be two different true twin vertices and let $x, y \in V(H)$. Then (a,x) and (b,y) are mutually maximally distant in $G \circ H$ if and only if both, x and y, have degree $n-1$.*

Proof. If $x \in V(H)$ has degree $n-1$, then for any $y \in V(H)$ of degree $n-1$ we have that (a,x) and (b,y) are true twins in $G \circ H$. Hence, (a,x) and (b,y) are mutually maximally distant in $G \circ H$.

Now, suppose that there exists $z \in V(H) - N_H(x)$. Hence, Claim 5.1 (iii) leads to $d_{G\circ H}((a,x),(a,z)) = 2$. Also, for every $y \in V(H)$, Claim 5.1 (ii) leads to $d_{G\circ H}((a,x),(b,y)) = 1$. Thus, we conclude that (a,x) and (b,y) are not mutually maximally distant in $G \circ H$. $\square$

In order to present our results we need to introduce some more terminology. Given a graph G, we define G^* as the graph with vertex set $V(G^*) = V(G)$ such that two vertices u, v are adjacent in G^* if and only if either $d_G(u,v) \geq 2$ or u, v are true twins. If a graph G has at least one isolated vertex, then we denote by G_- the graph obtained from G by removing all its isolated vertices. In this sense, G^*_- is obtained from G^* by removing all its isolated vertices. Notice that G^* satisfies the following straightforward properties.

Remark 5.4. *Let G be a connected graph of diameter $D(G)$, order n and maximum degree $\Delta(G)$.*

(i) *If $\Delta(G) \leq n-2$, then $G^* \cong (K_1 + G)_{SR}$.*

(ii) *If $D(G) \leq 2$, then $G^*_- \cong G_{SR}$.*

(iii) *If G has no true twins, then $G^* \cong G^c$.*

Lemma 5.5. *Let G be a connected nontrivial graph. Let $x, y \in V(H)$ be two distinct vertices of a graph H and let $a \in V(G)$. Then (a, x) and (a, y) are mutually maximally distant vertices in $G \circ H$ if and only if x and y are adjacent in H^*.*

Proof. By Claim 5.1 (iii), $d_{G\circ H}((a,x),(a,y)) \leq 2$ and, by Claim 5.1 (i), if $c \neq a$, then $(c, w) \in N_{G\circ H}(a, x)$ if and only if $c \in N_G(a)$. Hence, (a, x) and (a, y) are mutually maximally distant if and only if either (a, x) and (a, y) are true twins in $G \circ H$ or (a, x) and (a, y) are not adjacent in $G \circ H$.

On one hand, by definition of lexicographic product, (a, x) and (a, y) are not adjacent in $G \circ H$ if and only if x and y are not adjacent in H.

On the other hand, by Claim 5.1 (i), (a, x) and (a, y) are true twins in $G \circ H$ if and only if x and y are true twins in H.

Therefore, the result follows. □

Proposition 5.6. *Let G be a connected graph of order $n \geq 2$ and let H be a noncomplete graph of order $n' \geq 2$. If G has no true twin vertices, then*

$$(G \circ H)_{SR} \cong (G_{SR} \circ H^*) \cup \bigcup_{i=1}^{n-|\partial(G)|} H^*_-.$$

Proof. We assume that G has no true twin vertices. By Lemmas 5.2 and 5.5, we have the following facts.

- For any $a \notin \partial(G)$ it follows that $(G \circ H)_{SR}$ has a subgraph, say H_a, induced by $(\{a\} \times V(H)) \cap \partial(G \circ H)$ which is isomorphic to H^*_-

- For any $b \in \partial(G)$, we have that $(G \circ H)_{SR}$ has a subgraph, say H_b, induced by $(\{b\} \times V(H)) \cap \partial(G \circ H)$ which is isomorphic to H^*.

- The set $(\partial(G) \times V(H)) \cap \partial(G \circ H)$ induces a subgraph in $(G \circ H)_{SR}$ which is isomorphic to $G_{SR} \circ H^*$.

- For any $a \notin \partial(G)$ and any $b \in \partial(G)$ there are no edges of $(G \circ H)_{SR}$ connecting vertices belonging to H_a with vertices belonging to H_b.

- For any different vertices $a_1, a_2 \notin \partial(G)$ there are no edges of $(G \circ H)_{SR}$ connecting vertices belonging to H_{a_1} with vertices belonging to H_{a_2}.

Therefore, the result follows. $\square$

Figure 5.1 shows the graph $P_4 \circ P_3$ and its strong resolving graph. Notice that $(P_3)^*_- \cong K_2$, $(P_3)^* \cong K_2 \cup K_1$ and $(P_4)_{SR} \cong K_2$. So, $(P_4 \circ P_3)_{SR} \cong K_2 \circ (K_2 \cup K_1) \cup K_2 \cup K_2$.

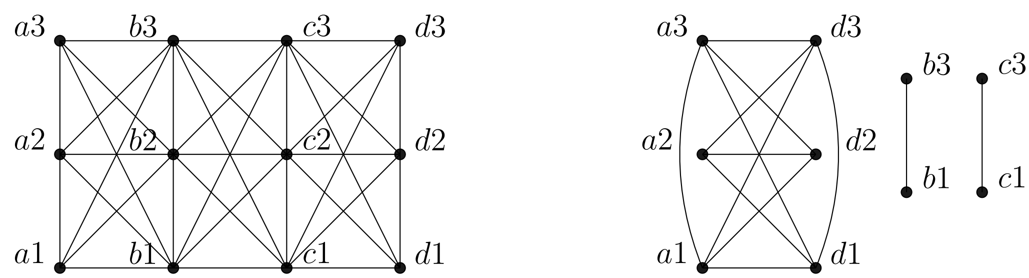

Figure 5.1: The graph $P_4 \circ P_3$ and its strong resolving graph.

The following well-known result is a useful tool in determining the strong metric dimension of lexicographic product graphs.

Theorem 5.7. [38] *For any graphs G and H of order n and n', respectively,*

$$\beta(G \circ H) = n\beta(H) + n'\beta(G) - \beta(G)\beta(H).$$

Theorem 5.8. *Let G be a connected graph of order $n \geq 2$ and let H be a graph of order $n' \geq 2$. If G has no true twin vertices, then the following assertions hold:*

(i) *If $D(H) \leq 2$, then*

$$dim_s(G \circ H) = n \cdot dim_s(H) + n' \cdot dim_s(G) - dim_s(G)dim_s(H).$$

(ii) *If $D(H) > 2$, then*

$$dim_s(G \circ H) = n \cdot dim_s(K_1 + H) + n' \cdot dim_s(G) - dim_s(G)dim_s(K_1 + H).$$

Proof. By Theorem 1.12 and Proposition 5.6 we have,

$$dim_s(G \circ H) = \beta(G_{SR} \circ H^*) + (n - |\partial(G)|)\beta(H^*_-)$$

and, by Theorem 5.7 we have

$$\begin{aligned} dim_s(G \circ H) = |\partial(G)|\beta(H^*) + n'\beta(G_{SR}) - \beta(G_{SR})\beta(H^*) \\ + (n - |\partial(G)|)\beta(H^*_-). \end{aligned} \tag{5.1}$$

Now, if $D(H) \leq 2$, then $\beta(H^*) = \beta(H^*_-) = \beta(H_{SR})$ and, as a result,

$$dim_s(G \circ H) = n\beta(H_{SR}) + n'\beta(G_{SR}) - \beta(G_{SR})\beta(H_{SR}).$$

Also, and if $D(H) > 2$, then $\beta(H^*) = \beta(H^*_-) = \beta((K_1 + H)_{SR})$, so

$$dim_s(G \circ H) = n\beta((K_1 + H)_{SR}) + n'\beta(G_{SR}) - \beta(G_{SR})\beta((K_1 + H)_{SR}).$$

Therefore, by Theorem 1.12 we conclude the proof. □

Note that the case where H is nonconnected is also considered in Theorem 5.8, because we assume that if H is nonconnected, then $D(H) = \infty > 2$.

Now we show some particular examples of graphs G without true twin vertices where is easy to compute $dim_s(G)$ by using Observation 1.10.

(1) For any complete k-partite graph $G = K_{p_1,p_2,...,p_k}$ such that $p_i \geq 2$, $i \in \{1, 2, ..., k\}$, we have $dim_s(G) = \sum_{i=1}^{k}(p_i - 1)$.

(2) For any tree T, $dim_s(T) = l(T) - 1$.

(3) For any even cycle, $dim_s(C_{2k}) = k$ and for any odd cycle, we have $dim_s(C_{2k+1}) = k + 1$.

(4) For any grid graph $P_r \Box P_t$, $dim_s(P_r \Box P_t) = 2$.

Notice that by using Theorem 5.8 (or other ones given throughout the chapter), and the known values above for a few families of graphs, we can obtain directly the strong metric dimension of several combinations of lexicographic product of two graphs. We omit these calculations and leave it to the reader.

According to Theorem 5.8 (i), for any connected graph G without true twin vertices it holds $dim_s(G \circ K_{n'}) = n(n' - 1) + dim_s(G)$. Now we show that this formula holds for any connected graph G.

Proposition 5.9. *For any connected nontrivial graph G of order $n \geq 2$ and any integer $n' \geq 2$,*

$$(G \circ K_{n'})_{SR} \cong (G_{SR} \circ K_{n'}) \cup \bigcup_{i=1}^{n-|\partial(G)|} K_{n'}.$$

Proof. Notice that $(K_{n'})^* \cong K_{n'}$ and, by Lemma 5.5, for any $a \in V(G)$, the subgraph of $(G \circ K_{n'})_{SR}$ induced by $(\{a\} \times V(K_{n'})) \cap \partial(G \circ K_{n'})$ is isomorphic to $K_{n'}$. Also, from Lemmas 5.2 and 5.3, the subgraph of $(G \circ K_{n'})_{SR}$ induced by $(\partial(G) \times V(K_{n'})) \cap \partial(G \circ K_{n'})$ is isomorphic to $G_{SR} \circ K_{n'}$. Moreover, for $a \notin \partial(G)$ and $b \in \partial(G)$ there are not edges of $(G \circ K_{n'})_{SR}$ connecting vertices belonging to $\{a\} \times V(K_{n'})$ with vertices belonging to $\{b\} \times V(K_{n'})$. Therefore, the result follows. □

Theorem 5.10. *For any connected nontrivial graph G of order $n \geq 2$ and any integer $n' \geq 2$,*

$$dim_s(G \circ K_{n'}) = n(n'-1) + dim_s(G).$$

Proof. From Theorem 1.12 and Proposition 5.9 we have,

$$dim_s(G \circ K_{n'}) = \beta(G_{SR} \circ K_{n'}) + (n - |\partial(G)|)(n'-1)$$

and, by using Theorem 5.7 and again Theorem 1.12 we obtain that

$$\begin{aligned} dim_s(G \circ K_{n'}) &= |\partial(G)|(n'-1) + n'\beta(G_{SR}) - \beta(G_{SR})(n'-1) \\ &\quad + (n - |\partial(G)|)(n'-1) = n(n'-1) + dim_s(G). \end{aligned}$$

□

We have studied the case in which the second factor in the lexicographic product is a complete graph. Since this product is not commutative, it remains to study the case in which the first factor is a complete graph, which we do at next.

Proposition 5.11. *Let $n \geq 2$ be an integer and let H be a graph of order $n' \geq 2$. If H has maximum degree $\Delta(H) \leq n'-2$, then*

$$(K_n \circ H)_{SR} \cong \bigcup_{i=1}^{n} H^*.$$

Proof. We assume that H has maximum degree $\Delta(H) \leq n' - 2$. Notice that H^* has no isolated vertices and, by Lemma 5.5, for any $a \in V(K_n)$, the subgraph $(K_n \circ H)_{SR}$ induced by $(\{a\} \times V(H)) \cap \partial(K_n \circ H)$ is isomorphic to H^*.

Also, by Lemma 5.3, for any different $a, b \in V(K_n)$ and any $x, y \in V(H)$, the vertices (a, x) and (b, y) are not mutually maximally distant in $K_n \circ H$. Therefore, the result follows. □

Theorem 5.12. *Let $n \geq 2$ be an integer and let H be a graph of order $n' \geq 2$ and maximum degree $\Delta(H) \leq n' - 2$.*

(i) *If $D(H) = 2$, then*

$$dim_s(K_n \circ H) = n \cdot dim_s(H).$$

(ii) *If $D(H) > 2$, then*

$$dim_s(K_n \circ H) = n \cdot dim_s(K_1 + H).$$

Proof. By Theorems 1.12 and 5.11 we have, $dim_s(K_n \circ H) = n \cdot \beta(H^*)$. Hence, if $D(H) = 2$, then $dim_s(K_n \circ H) = n \cdot \beta(H_{SR})$ and if $D(H) > 2$, then $dim_s(K_n \circ H) = n \cdot \beta((K_1 + H)_{SR})$. Therefore, by Theorem 1.12 we conclude the proof. □

For the particular case of empty graphs $H = N_{n'}$, Theorem 5.12 leads to the next corollary, which is straightforward because $K_n \circ N_{n'} \cong K_{n',n',\ldots,n'}$, is a complete n-partite graph, and so $(K_n \circ N_{n'})_{SR} \cong \bigcup_{i=1}^{n} K_{n'}$.

Corollary 5.13. *For any integers $n, n' \geq 2$,*

$$dim_s(K_n \circ N_{n'}) = n(n' - 1).$$

We define the *TF-boundary* of a noncomplete graph G as a set $\partial_{TF}(G) \subseteq \partial(G)$, where $x \in \partial_{TF}(G)$ whenever there exists $y \in \partial(G)$, such that x and y are mutually maximally distant in G and $N_G[x] \neq N_G[y]$ (which means that x, y are not true twins). The *strong resolving TF-graph* of G is a graph G_{SRS} with vertex set $V(G_{SRS}) = \partial_{TF}(G)$, where two vertices u, v are adjacent in G_{SRS} if and only if u and v are mutually maximally distant in G and $N_G[x] \neq N_G[y]$. Since the strong resolving TF-graph is a subgraph of the strong resolving graph, an instance of the problem of transforming a graph into its strong resolving TF-graph forms part of the general problem of transforming a graph into its strong resolving graph. From [82], it is known that this general transformation is polynomial. Thus, the problem of transforming a graph into its strong resolving TF-graph is also polynomial.

An interesting example of a strong resolving TF-graph is obtained from the corona graph $G \odot K_{n'}$, $n' \geq 2$, where G has order $n \geq 2$. Notice that any two different vertices belonging to any two copies of the complete graph $K_{n'}$ are mutually maximally distant, but if they are in the same copy, then they are also true twins. Thus, in this case $\partial_{TF}(G \odot K_{n'}) = \partial(G \odot K_{n'})$, while we have have that $(G \odot K_{n'})_{SR} \cong K_{nn'}$ and $(G \odot K_{n'})_{SRS}$ is isomorphic to a complete n-partite graph $K_{n',n',\ldots,n'}$.

Proposition 5.14. *Let G be a connected noncomplete graph of order $n \geq 2$ and let H be a graph of order $n' \geq 2$. If H has maximum degree $\Delta(H) \leq n' - 2$, then*

$$(G \circ H)_{SR} \cong (G_{SRS} \circ H^*) \cup \bigcup_{i=1}^{n-|\partial_{TF}(G)|} H^*.$$

Proof. We assume that H has maximum degree $\Delta(H) \leq n' - 2$. Notice that H^* has no isolated vertices and, by Lemma 5.5, for any $a \in V(G)$, the subgraph $(G \circ H)_{SR}$ induced by $(\{a\} \times V(H)) \cap \partial(G \circ H)$ is isomorphic to H^*.

Also, by Lemma 5.3, if two different vertices a, b are true twins in G and $x, y \in V(H)$, then (a, x) and (b, y) are not mutually maximally distant in $G \circ H$. So, from Lemmas 5.2 and 5.5 we deduce that the subgraph of $(G \circ H)_{SR}$ induced by $(\partial_{TF}(G) \times V(H)) \cap \partial(G \circ H)$ is isomorphic to $G_{SRS} \circ H^*$. Moreover, for $a \notin \partial_{TF}(G)$ and $b \in \partial_{TF}(G)$ there are no edges of $(G \circ H)_{SR}$ connecting vertices belonging to $\{a\} \times V(H)$ with vertices belonging to $\{b\} \times V(H)$. Therefore, the result follows. □

Figure 5.2 shows the graph $(K_1 + (K_1 \cup K_2)) \circ P_4$ and its strong resolving graph. Notice that $(P_4)^* \cong P_4$ and $(K_1 + (K_1 \cup K_2))_{SRS} \cong P_3$. So, $((K_1 + (K_1 \cup K_2)) \circ P_4)_{SR} \cong (P_3 \circ P_4) \cup P_4$.

Theorem 5.15. *Let G be a connected noncomplete graph of order $n \geq 2$ and let H be a graph of order $n' \geq 2$ and maximum degree $\Delta(H) \leq n' - 2$.*

(i) *If $D(H) = 2$, then*

$$dim_s(G \circ H) = n \cdot dim_s(H) + n' \cdot \beta(G_{SRS}) - \beta(G_{SRS}) dim_s(H).$$

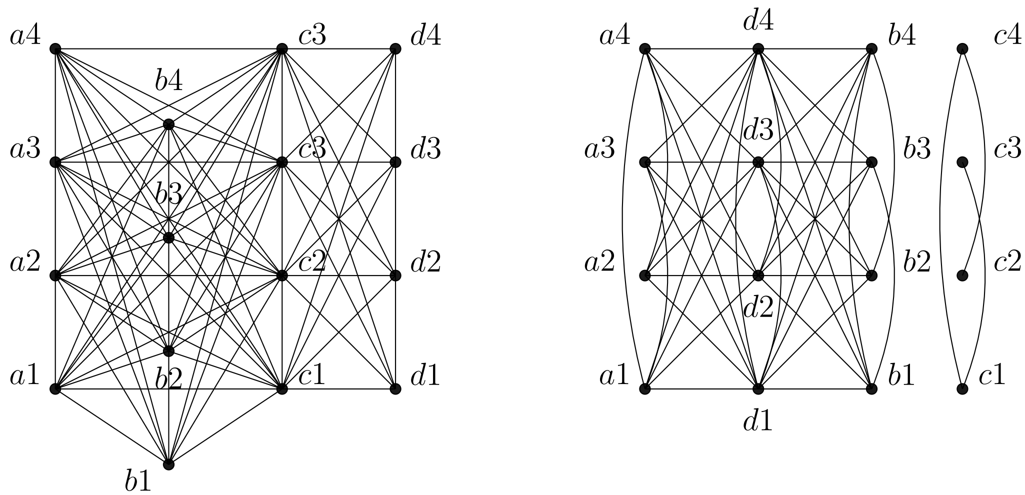

Figure 5.2: The graph $(K_1 + (K_1 \cup K_2)) \circ P_4$ and its strong resolving graph.

(ii) *If* $D(H) > 2$, *then*

$$\begin{aligned} dim_s(G \circ H) &= n \cdot dim_s(K_1 + H) \\ &\quad + n' \cdot \beta(G_{SRS}) - \beta(G_{SRS}) dim_s(K_1 + H). \end{aligned}$$

Proof. By Theorem 1.12 and Proposition 5.14 we have,

$$dim_s(G \circ H) = \beta(G_{SRS} \circ H^*) + (n - |\partial_{SR}(G)|)\beta(H^*)$$

and, by Theorem 5.7, we have

$$\begin{aligned} dim_s(G \circ H) &= |\partial(G)|\beta(H^*) + n'\beta(G_{SRS}) - \beta(G_{SRS})\beta(H^*) \\ &\quad + (n - |\partial_{SR}(G)|)\beta(H^*). \end{aligned} \tag{5.2}$$

Now, if $D(H) = 2$, then $\beta(H^*) = \beta(H_{SR})$ and, if $D(H) > 2$, then $\beta(H^*) = \beta((K_1 + H)_{SR})$. Hence, if $D(H) = 2$, then

$$dim_s(G \circ H) = n\beta(H_{SR}) + n'\beta(G_{SRS}) - \beta(G_{SRS})\beta(H_{SR}),$$

and if $D(H) > 2$, then

$$dim_s(G \circ H) = n\beta((K_1 + H)_{SR}) + n'\beta(G_{SRS}) - \beta(G_{SRS})\beta((K_1 + H)_{SR}).$$

Therefore, by Theorem 1.12 we conclude the proof. □

Now we consider the case in which the second factor is a empty graph.

Corollary 5.16. *Let G be a connected noncomplete graph of order $n \geq 2$ and let $n' \geq 2$ be an integer. Then*

$$dim_s(G \circ N_{n'}) = n(n'-1) + \beta(G_{SRS}).$$

In particular, if G has no true twin vertices, then

$$dim_s(G \circ N_{n'}) = n(n'-1) + dim_s(G).$$

As we can expect, if G has no true twin vertices and H has maximum degree $\Delta(H) \leq n'-2$, then both, Theorem 5.8 and Theorem 5.15, lead to the same result.

Theorem 5.17. *Let G be a connected graph of order $n \geq 2$ and let H be a graph of order $n' \geq 2$ and maximum degree $\Delta(H) \leq n'-2$. Then the following assertions hold:*

(i) *If H has no true twin vertices, then*

$$dim_s(G \circ H) = (n - \beta(G_{SRS}))(n' - \omega(H)) + n'\beta(G_{SRS}).$$

(ii) *If neither G nor H have true twin vertices, then*

$$dim_s(G \circ H) = (n - dim_s(G))(n' - \omega(H)) + n' dim_s(G).$$

Proof. First of all, notice that Theorem 1.13 leads to $\beta(H^c) = n' - \alpha(H^c) = n' - \omega(H)$. Also, from $\Delta(H) \leq n'-2$ we have $H^* = H^*_-$ and, if H has no true twin vertices, then $H^* = H^c$. Hence, (5.2) leads to (i). Moreover, if G has no true twin vertices, then (5.1) leads to (ii). $\square$

Chapter 6

Strong metric dimension of corona product graphs and join graphs

6.1 Overview

In this chapter we show that the problem of computing the strong metric dimension of the corona product of two graphs can be transformed to the problem of finding certain clique number of the second factor. Moreover, we prove that if the second factor is not connected or its diameter is greater than two, then the strong metric dimension of corona product is obtained from the strong metric dimension of some other related graph. The strong metric dimension of join graphs is also studied.

6.2 Main results

We start this section with a relationship between the strong metric dimension of a connected graph and its twin-free clique number. Furthermore, the following result is also an important tool of Chapter 7. In order to present our results we need to recall the terminology introduced in Chapter 2. For two vertices $u, v \in V(H)$, the interval $I_H[u, v]$ between u and v is defined as the collection of all vertices that belong to some shortest $u - v$ path. Note that vertex w strongly resolves two vertices u and v if $v \in I_H[u, w]$ or $u \in I_H[v, w]$.

Theorem 6.1. *Let H be a connected graph of order $n \geq 2$. Then*

$$dim_s(H) \leq n - \varpi(H).$$

Moreover, if H has diameter two, then

$$dim_s(H) = n - \varpi(H).$$

Proof. Let W be a maximum twin-free clique in H. We show that $V(H)-W$ is a strong metric generator for H. Since W is a twin-free clique, for any two distinct vertices $u, v \in W$ there exists $s \in V(H) - W$ such that either ($s \in N_H(u)$ and $s \notin N_H(v)$) or ($s \in N_H(v)$ and $s \notin N_H(u)$). Without loss of generality, we consider $s \in N_H(u)$ and $s \notin N_H(v)$. Thus, $u \in I_H[v, s]$ and, as a consequence, s strongly resolves u and v. Therefore, $dim_s(H) \leq n - \varpi(H)$.

Now, suppose that H has diameter two. Let X be a strong metric basis of H and let u, v be two distinct vertices of H. If $d_H(u, v) = 2$ or $N_H[u] = N_H[v]$, then u and v are mutually maximally distant vertices of H, so $u \in X$ or $v \in X$. Hence, for any two distinct vertices $x, y \in V(H) - X$ we have $x \sim y$ and $N_H(x) \neq N_H(y)$. As a consequence, $|V(H) - X| \leq \varpi(H)$. Therefore, $dim_s(H) \geq n - \varpi(H)$ and the result follows. □

Corollary 6.2. *Let H be a graph of diameter two and order n. Let $c(H)$ be the number of vertices of H having degree $n-1$. If the only true twins of H are vertices of degree $n-1$, then*

$$dim_s(H) = n + c(H) - \omega(H) - 1.$$

Moreover, if H has no true twins, then

$$dim_s(H) = n - \omega(H).$$

The twin-free clique number of any join graph satisfies one of the following relationships.

Lemma 6.3. *Let G and H be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively.*

(i) *If $\Delta(G) \neq n_1 - 1$ or $\Delta(H) \neq n_2 - 1$, then*

$$\varpi(G + H) = \varpi(G) + \varpi(H).$$

(ii) *If* $\Delta(G) = n_1 - 1$ *and* $\Delta(H) = n_2 - 1$, *then*

$$\varpi(G + H) = \varpi(G) + \varpi(H) - 1.$$

Proof. Given a $\varpi(G+H)$-set Z we have that for every $u_1, u_2 \in U = Z \cap V(G)$ it follows $N_{G+H}[u_1] \neq N_{G+H}[u_2]$. So, $N_G[u_1] \neq N_G[u_2]$ and, as a consequence, U is a twin-free clique in G. Analogously we show that $W = Z \cap V(H)$ is a twin-free clique in H. Hence, $\varpi(G+H) = |Z| = |U| + |W| \leq \varpi(G) + \varpi(H)$.

Now, if $\Delta(G) = n_1 - 1$ and $\Delta(H) = n_2 - 1$, then every $\varpi(G)$-set ($\varpi(H)$-set) contains exactly one vertex of degree $\Delta(G) = n_1 - 1$ ($\Delta(H) = n_2 - 1$) and every $\varpi(G + H)$-set contains exactly one vertex of degree $n_1 + n_2 - 1$. Hence, in this case $|U| < \varpi(G)$ or $|W| < \varpi(H)$ and, as a consequence, $\varpi(G + H) = |Z| = |U| + |W| \leq \varpi(G) + \varpi(H) - 1$.

On the other hand, let U' be a $\varpi(G)$-set and let W' be a $\varpi(H)$-set.

In order to complete the proof of (i), we assume, without loss of generality, that $\Delta(G) \neq n_1 - 1$. Let $u \in U'$ and $w \in W'$. Since $\delta_G(u) \neq n_1 - 1$, there exists a vertex $x \in V(G) - U'$ such that $u \not\sim x$. From the definition of $G + H$ we have $w \sim x$ and the subgraph induced by $U' \cup W'$ is a clique in $G + H$. So, u and w are not true twins in $G + H$ and, as a consequence, $U' \cup W'$ is a twin-free clique in $G + H$. Hence, $\varpi(G + H) \geq |U' \cup W'| = \varpi(G) + \varpi(H)$. The proof of (i) is complete.

Now, if $\Delta(G) = n_1 - 1$, then we take $x \in U'$ such that $\delta_G(x) = n_1 - 1$ and as above we see that two vertices $v, w \in U' \cup W' - \{x\}$ are not true twins in $G + H$. Hence, $U' \cup W' - \{x\}$ is a twin-free clique in $G + H$. So, $\varpi(G + H) \geq |U'| + |W'| - 1 = \varpi(G) + \varpi(H) - 1$. Therefore, the proof of (ii) is complete. $\square$

If G and H are two complete graphs of order n_1 and n_2, respectively, then $G + H = K_{n_1+n_2}$ and $dim_s(G + H) = dim_s(K_{n_1+n_2}) = n_1 + n_2 - 1$. From Theorem 6.1 and Lemma 6.3 we obtain the following results.

Theorem 6.4. *Let G and H be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively.*

(i) *If* $\Delta(G) \neq n_1 - 1$ *or* $\Delta(H) \neq n_2 - 1$, *then*

$$dim_s(G + H) = n_1 + n_2 - \varpi(G) - \varpi(H) \geq dim_s(G) + dim_s(H).$$

(ii) *If G and H are graphs of diameter two where $\Delta(G) \neq n_1 - 1$ or $\Delta(H) \neq n_2 - 1$, then*

$$dim_s(G + H) = dim_s(G) + dim_s(H).$$

(iii) *If $\Delta(G) = n_1 - 1$ and $\Delta(H) = n_2 - 1$, then*

$$dim_s(G + H) = dim_s(G) + dim_s(H) + 1.$$

The following lemma shows that the problem of finding the strong metric dimension of a corona product graph can be transformed to the problem of finding the strong metric dimension of a graph of diameter two.

Lemma 6.5. *Let G be a connected graph of order n and let H be a graph. Let H_i be the subgraph of $G \odot H$ corresponding to the i^{th}-copy of H. Then*

$$dim_s(G \odot H) = dim_s(K_1 + \bigcup_{i=1}^{n} H_i).$$

Proof. As the result is obvious for $n = 1$, we take $n \geq 2$. Let v be the vertex of K_1 and let S' be a strong metric generator for $G \odot H$. We show that $S = \bigcup_{i=1}^{n}(S' \cap V(H_i))$ is a strong metric generator for $K_1 + \bigcup_{i=1}^{n} H_i$. We consider x, y are two different vertices of $K_1 + \bigcup_{i=1}^{n} H_i$ not belonging to S. We differentiate the following cases.

Case 1: $x = v$ and $y \in V(H_i)$, for some i. For any $u \in V(H_j)$, $j \neq i$, we have $x \in I_{K_1 + \bigcup_{i=1}^{n} H_i}[u, y]$ and since y and u are mutually maximally distant in $G \odot H$, we have $y \in S$ or $u \in S$.

Case 2: $x, y \in V(H_i)$. Let u be a vertex of S' which strongly resolves x and y in $G \odot H$. As no vertex of $G \odot H$ not belonging to $V(H_i)$ strongly resolves x and y, we have that $u \in V(H_i)$ and $u \in S$. Hence, u strongly resolves x and y in $K_1 + \bigcup_{i=1}^{n} H_i$.

Note that in the case $x \in V(H_i)$ and $y \in V(H_j)$, $i \neq j$, we have that x and y are mutually maximally distant in $G \odot H$. Thus, we have $x \in S$ or $y \in S$. Hence, S is a strong metric generator for $K_1 + \bigcup_{i=1}^{n} H_i$ and, as a consequence, $dim_s(G \odot H) \geq dim_s(K_1 + \bigcup_{i=1}^{n} H_i)$.

Now, given a strong metric generator for $K_1 + \bigcup_{i=1}^{n} H_i$ denoted by W', let us show that $W = W' - \{v\}$ is a strong metric generator for $G \odot H$. Let x, y be two different vertices of $G \odot H$ not belonging to W. We denote by $V(G) = \{v_1, v_2, ..., v_n\}$ the vertex set of G, where v_i is the vertex of G

adjacent to every vertex of $V(H_i)$ in $G \odot H$, $i \in \{1, ..., n\}$. We differentiate the following cases.
Case 1': $x = v_i \in V(G)$ and $y \in V(H_i)$. Let $u \in V(H_j)$, $j \neq i$. In this case we have $x \in I_{G\odot H}[u, y]$ and, since y and u are mutually maximally distant in $K_1 + \bigcup_{i=1}^{n} H_i$, we have $y \in W$ or $u \in W$.
Case 2': $x = v_i \in V(G)$ and $y \in V(H_j)$, $j \neq i$. For every $u \in V(H_i)$ we have $x \in I_{G\odot H}[u, y]$ and, since y and u are mutually maximally distant in $K_1 + \bigcup_{i=1}^{n} H_i$, we have $y \in W$ or $u \in W$.
Case 3': $x, y \in V(G)$. Let $x = v_i$, $y = v_j$, $u_i \in V(H_i)$ and $u_j \in V(H_j)$. We have $x \in I_{G\odot H}[u_i, y]$ and $y \in I_{G\odot H}[u_j, x]$. As u_i and u_j are mutually maximally distant in $K_1 + \bigcup_{i=1}^{n} H_i$, we have $u_i \in W$ or $u_j \in W$.

Finally, note that the case $x \in V(H_i)$ and $y \in V(H_j)$, where $i, j \in \{1, 2, ..., n\}$, leads to $x \in W$ or $y \in W$. Therefore, W is a strong metric generator for $G \odot H$ and, as a consequence, $dim_s(G \odot H) \leq dim_s(K_1 + \bigcup_{i=1}^{n} H_i)$. □

Figure 6.1 illustrates the theorem above.

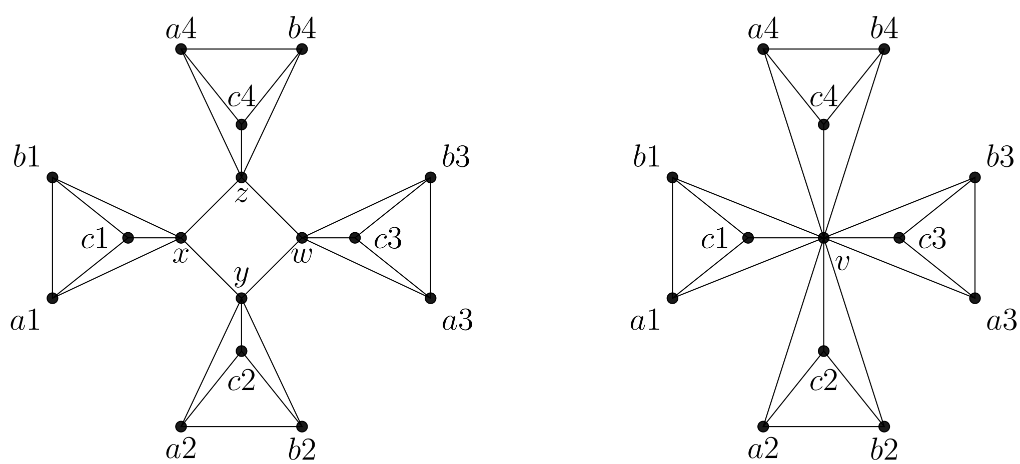

Figure 6.1: Corona product $C_4 \odot K_3$ and join graph $K_1 + \bigcup_{i=1}^{4} K_3$, where v is the vertex of K_1.

Corollary 6.6. *For any connected graph G of order n,*

$$dim_s(G \odot K_1) = n - 1.$$

Proof. For $H \cong K_1$ Lemma 6.5 leads to $dim_s(G \odot K_1) = dim_s(K_1 + \bigcup_{i=1}^{n} K_1) = dim_s(K_{1,n}) = n - 1$. □

Our next result is obtained from Lemma 6.5 and Theorem 6.1.

Theorem 6.7. *Let G be a connected graph of order n_1. Let H be a graph of order n_2.*

(i) *If $\Delta(H) = n_2 - 1$, then*

$$dim_s(K_1 + H) = n_2 + 1 - \varpi(H).$$

(ii) *If $\Delta(H) \leq n_2 - 2$ or $n_1 \geq 2$, then*

$$dim_s(G \odot H) = n_1 n_2 - \varpi(H).$$

Proof. Since (i) is trivial, we prove (ii). For $\Delta(H) = n_2 - 1$ we have

$$\varpi\left(K_1 + \bigcup_{i=1}^{n_1} H_i\right) \overset{n_1 \geq 1}{=} \varpi(K_1 + H) + 1 = \varpi(H) + 1,$$

while for $\Delta(H) \leq n_2 - 2$ we have

$$\varpi\left(K_1 + \bigcup_{i=1}^{n_1} H_i\right) = \varpi(K_1 + H) = \varpi(H) + 1.$$

So, by Lemma 6.5 and Theorem 6.1 we conclude the proof. □

Let us derive some consequences of the result above.

Corollary 6.8. *Let G be a connected graph of order n_1 and let H be a graph of order n_2. Let $c(H)$ be the number of vertices of H having degree $n_2 - 1$.*

(i) *If H has no true twins and $\Delta(H) = n_2 - 1$, then*

$$dim_s(K_1 + H) = n_2 + 1 - \omega(H).$$

(ii) *If H has no true twins and $\Delta(H) \leq n_2 - 2$,*

$$dim_s(K_1 + H) = n_2 - \omega(H).$$

(iii) *If H has no true twins and $n_1 \geq 2$, then*

$$dim_s(G \odot H) = n_1 n_2 - \omega(H).$$

(iv) *If the only true twins of H are vertices of degree $n_2 - 1$, then*

$$dim_s(K_1 + H) = n_2 + c(H) - \omega(H)$$

(v) *If the only true twins of H are vertices of degree $n_2 - 1$ and $n_1 \geq 2$, then*

$$dim_s(G \odot H) = n_1 n_2 + c(H) - 1 - \omega(H).$$

As our next result shows, when H is a triangle-free graph we obtain the exact value of the strong metric dimension of $G \odot H$.

Corollary 6.9. *Let G be a connected graph of order n_1 and let H be a triangle-free graph of order $n_2 \geq 3$. If $n_1 \geq 2$ or $\Delta(H) \leq n_2 - 2$, then*

$$dim_s(G \odot H) = n_1 n_2 - 2.$$

Our next result is an interesting consequence of Theorem 6.1 and Theorem 6.7.

Theorem 6.10. *Let G be a connected graph of order n_1. Let H be a graph of order n_2.*

(i) *If $\Delta(H) = n_2 - 1$, then*

$$dim_s(K_1 + H) = dim_s(H) + 1.$$

(ii) *If H has diameter two and either $\Delta(H) \leq n_2 - 2$ or $n_1 \geq 2$, then*

$$dim_s(G \odot H) = (n_1 - 1)n_2 + dim_s(H).$$

(iii) *If H is not connected or its diameter is greater than two, then*

$$dim_s(G \odot H) = (n_1 - 1)n_2 + dim_s(K_1 + H).$$

Note that the theorem above allows us to derive results on the strong metric dimension of some join graphs. Also, observe that Corollary 6.11 can be obtained from Theorem 6.1.

Corollary 6.11. *Let H be a graph of order n.*

(i) *If $\Delta(H) = n - 1$, then*

$$dim_s(K_r + H) = dim_s(H) + r.$$

(ii) *If* $\Delta(H) \leq n-2$ *and* H *has diameter two, then*

$$dim_s(K_r + H) = dim_s(H) + r - 1.$$

(iii) *If* H *is not connected or its diameter is greater than two, then*

$$dim_s(K_r + H) = dim_s(K_1 + H) + r - 1.$$

6.3 Strong metric dimension versus algebraic connectivity

It is well-known that the second smallest Laplacian eigenvalue of a graph is probably the most important information contained in the spectrum. This eigenvalue, frequently called *algebraic connectivity*, is related to several important graph invariants and imposes reasonably good bounds on the values of several parameters of graphs which are very hard to compute.

The following theorem shows the relationship between the algebraic connectivity of a graph and the clique number.

Theorem 6.12. *Let* G *be a connected noncomplete graph of order* n *and algebraic connectivity* μ*. The clique number of* $\omega(G)$ *is bounded by*

$$\omega(G) \leq \frac{n(\Delta(G) - \mu + 1)}{n - \mu}.$$

Proof. The algebraic connectivity of $G = (V, E)$, satisfies the following equality shown by Fiedler [34],

$$\mu = 2n \min \left\{ \frac{\sum_{v_i \sim v_j} (w_i - w_j)^2}{\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2} \right\}, \tag{6.1}$$

where not all the components of the vector $(w_1, w_2, ..., w_n) \in \mathbb{R}^n$ are equal. Let S be a clique in G of cardinality $\omega(G)$. The vector $w \in \mathbb{R}^n$ associated to S is defined as,

$$w_i = \begin{cases} 1 & \text{if} \quad v_i \in S; \\ 0 & \text{otherwise}, \end{cases} \tag{6.2}$$

Considering the 2-partition $\{S, V - S\}$ of the vertex set V we have $(w_i - w_j)^2 = 1$ if v_i and v_j are in different sets of the partition, and 0 if they are in the same set. Then,

$$\sum_{v_i \in V} \sum_{v_j \in V} (w_i - w_j)^2 = 2\,|S|\,(n - |S|). \tag{6.3}$$

By (6.1) and (6.3) we have

$$\mu \leq \frac{n \sum_{v_i \sim v_j} (w_i - w_j)^2}{|S|\,(n - |S|)}. \tag{6.4}$$

Moreover, since $\sum\limits_{v_i \sim v_j} (w_i - w_j)^2$ is the number of edges of G having one end-point in S and the other one in $V-S$, we have $\sum\limits_{v_i \sim v_j} (w_i - w_j)^2 = \sum\limits_{v \in S} |N_{V-S}(v)|$, where $N_{V-S}(v)$ denotes the set of neighbors that v has in $V-S$. Thus, since S is a clique in G, we have that for every $v \in S$, $|N_{V-S}(v)| = \delta_G(v) - (|S|-1)$. Hence,

$$\mu \leq \frac{n \sum_{v \in S} (\delta_G(v) - |S| + 1)}{|S|\,(n - |S|)} \leq \frac{n(\Delta(G) - |S| + 1)}{n - |S|}. \tag{6.5}$$

The result follows directly by inequality (6.5). □

The bound above is tight, it is achieved, for instance, for the Cartesian product graph $G = K_r \Box K_2$, where $\mu = 2$, $n = 2r$, $\Delta(G) = r$ and $\omega(G) = r$.

Notice that the result above and the inequality $\omega(H) \geq \varpi(H)$ combined with Theorem 6.1, Theorem 6.4 or Theorem 6.7, lead to lower bounds on the strong metric dimension. For instance, by Theorem 6.1 we derive the following tight bound on the strong metric dimension of graphs of diameter two. An example for the tightness is the graph $K_r \Box K_2$, where $dim_s(K_r \Box K_2) = r$, $\mu = 2$, $n = 2r$ and $\Delta(K_r \Box K_2) = r$.

Theorem 6.13. *Let H be a connected graph of diameter two, order $n \geq 2$, and algebraic connectivity μ. Then*

$$dim_s(H) \geq \left\lceil \frac{n(n - \Delta(H) - 1)}{n - \mu} \right\rceil.$$

Chapter 7

Strong metric dimension of Cartesian sum graphs

7.1 Overview

The current chapter is primarily concerned with finding several relationships between the strong metric dimension of Cartesian sum graphs and the strong metric dimension, clique number or twin-free clique number of its factor graphs. Specifically, we obtain general lower and upper bounds on the strong metric dimension of the Cartesian sum of graphs and give some classes of graphs where these bounds are tight.

7.2 Primary results

We begin this section by establishing a direct consequence of the definition of Cartesian sum graph.

Remark 7.1. *A graph $G \oplus H$ is complete if and only if both, G and H, are complete graphs.*

In concordance with the remark above, from now on we continue with the Cartesian sum of two graphs G and H, such that G or H is not complete. Moreover, the fact that the Cartesian sum is a commutative operation is very useful and in several results, symmetric cases are omitted without specific mentioning of this fact. The next result gives the diameter of Cartesian sum graphs.

Proposition 7.2. *Let G and H be two nontrivial graphs such that at least one of them is noncomplete and let $n \geq 2$ be an integer. Then the following assertion hold.*

(i) $D(G \oplus N_n) = \max\{2, D(G)\}$.

(ii) *If G and H have isolated vertices, then $D(G \oplus H) = \infty$.*

(iii) *If neither G nor H has isolated vertices, then $D(G \oplus H) = 2$.*

(iv) *If $D(H) \leq 2$, then $D(G \oplus H) = 2$.*

(v) *If $D(H) > 2$, H has no isolated vertices and G is a nonempty graph having at least one isolated vertex, then $D(G \oplus H) = 3$.*

Proof. Note that since G and H are two graphs such that at least one of them is noncomplete, by Remark 7.1 we have that $D(G \oplus H) \geq 2$.

(i) If G is connected, then we have $d_{G\oplus N_n}((a,b),(c,d)) = d_G(a,c)$ for $a \neq c$, and $d_{G\oplus N_n}((a,b),(a,d)) = 2$. Thus, $D(G \oplus N_n) = \max\{2, D(G)\}$.

On the other hand, if G_1 and G_2 are two connected components of G, then for any $u \in V(G_1)$, $x \in V(G_2)$ and $v, y \in V(N_n)$, we have that $(u,v) \not\sim (x,y)$, so $G \oplus N_n$ is not connected and, as a result, $D(G \oplus N_n) = \infty$.

(ii) If $u \in V(G)$ and $v \in V(H)$ are isolated vertices, then $(u,v) \in V(G \oplus H)$ is an isolated vertex, so (ii) follows.

(iii) Assume that neither G nor H has isolated vertices. We consider the following cases for two different vertices $(u,v),(x,y) \in V(G \oplus H)$.

Case 1: $v = y$. Since H has no isolated vertices, then there exists a vertex $w \in N_H(v)$. So, $(u,v) \sim (u,w) \sim (x,y)$ and, as a consequence, $d_{G\oplus H}((u,v),(x,y)) \leq 2$.

Case 2: $u = x$. This case is symmetric to Case 1.

Case 3: $v \neq y$ and $u \neq x$. Since G and H have no isolated vertices, there exist vertices $z \in N_G(x)$ and $w \in N_H(v)$. Hence, $(u,v) \sim (z,w) \sim (x,y)$ and, as a result, $d_{G\oplus H}((u,v),(x,y)) \leq 2$

According to the cases above the proof of (iii) is complete.

(iv) Let $D(H) \leq 2$. If v and y are two adjacent vertices of H, then for any $u,x \in V(G)$ we have $d_{G\oplus H}((u,v),(x,y)) = 1$, while if $v \not\sim y$, then for any $w \in N_H(v) \cap N_H(y)$ we have $(u,v) \sim (x,w) \sim (x,y)$. Thus, $d_{G\oplus H}((u,v),(x,y)) \leq 2$ and so (iv) follows.

(v) Assume that G has an isolated vertex, H has no isolated vertices and $D(H) > 2$. If u and x are not isolated vertices in G, then we proceed like in the proof of (iii) to show that $d_{G\oplus H}((u,v),(x,y)) \leq 2$. If u or x is an isolated vertex of G and $d_H(v,y) \leq 2$, then we proceed like in the proof of (iv). So, we consider that u or x is an isolated vertex and $d_H(v,y) \geq 3$.

Case 1': u is an isolated vertex and x is not an isolated vertex. In this case there exists $t \in N_G(x)$ and, since H has no isolated vertices, there exists $w \in N_H(v)$. Hence, $(u,v) \sim (t,w) \sim (x,y)$ and, as a consequence, $d_{G\oplus H}((u,v),(x,y)) \leq 2$.

Case 2': u and x are isolated vertices (u and x are not necessarily different). Since H has no isolated vertices and $d_H(v,y) \geq 3$, for every two vertices $w \in N_H(v)$ and $z \in N_H(y)$ it follows that $w \neq z$. Moreover, since G is not empty, there exist two different vertices $s,t \in V(G)$ such that $s \sim t$. Hence, $(u,v) \sim (t,w) \sim (s,z) \sim (x,y)$. Thus, $d_{G\oplus H}((u,v),(x,y)) \leq 3$. On the other hand, since $N_{G\oplus H}(u,v) = V(G) \times N_H(v)$, $N_{G\oplus H}(x,y) = V(G) \times N_H(y)$ and $N_H(v) \cap N_H(y) = \emptyset$, we obtain that $N_{G\oplus H}(u,v) \cap N_{G\oplus H}(x,y) = \emptyset$. Therefore, we have $d_{G\oplus H}((u,v),(x,y)) = 3$ and the proof of (v) is complete.

$\square$

Corollary 7.3. *The graph $G \oplus H$ is not connected if and only if both G and H have isolated vertices or G is an empty graph and H is not connected.*

Now we would point out a relationship between the Cartesian sum graphs and the lexicographic product of graphs.

Remark 7.4. *For any graph G and any nonnegative integer n,*

$$G \oplus N_n \cong G \circ N_n.$$

Notice that the strong metric dimension of $G \circ N_n$ have been studied in Chapter 5.

In order to present the next results we need to recall the terminology and notation introduced in Chapter 5. Given a graph G, by G^* we mean a graph with vertex set $V(G^*) = V(G)$ such that two vertices u, v are adjacent in G^* if and only if either $d_G(u, v) \geq 2$ or u, v are true twins. If a graph G has at least one isolated vertex, then we denote by G_- the graph obtained from G by removing all its isolated vertices. In this sense, G^*_- is obtained from G^* by removing all its isolated vertices. Moreover, if G has no true twins, then $G^* \cong G^c$.

Proposition 7.5. *Let G and H be two nontrivial graphs such that at least one of them is noncomplete. If $D(G) \leq 2$ or neither G nor H has isolated vertices, then*

$$(G \oplus H)_{SR} \cong (G \oplus H)^*_-.$$

Proof. We assume that $D(G) \leq 2$ or neither G nor H has isolated vertices. Then, by Proposition 7.2 we have $D(G \oplus H) = 2$ and, as a consequence, two vertices are mutually maximally distant in $G \oplus H$ if and only if they are true twins or they are not adjacent. Hence, $(G \oplus H)_{SR} \cong (G \oplus H)^*_-$. □

Our next result is derived from Theorem 1.12 and Proposition 7.5.

Proposition 7.6. *Let G and H be two nontrivial graphs such that at least one of them is noncomplete. If $D(G) \leq 2$ or neither G nor H has isolated vertices, then*

$$dim_s(G \oplus H) = \beta((G \oplus H)^*_-).$$

Theorem 7.7. *Let G and H be two graphs of order n and n', respectively, and let $\Delta(G) \leq n - 2$ and $\Delta(H) \leq n' - 2$. If (neither G nor H has true twin vertices) and ($D(G) \leq 2$ or neither G nor H has isolated vertices), then*

$$dim_s(G \oplus H) = \beta(G^c \boxtimes H^c).$$

Proof. If $D(G) \leq 2$ or neither G nor H has isolated vertices, then by Proposition 7.6 we have $dim_s(G \oplus H) = \beta((G \oplus H)^*_-)$.

Now, for any $(u, v) \in V(G \oplus H)$ we have

$$N_{G \oplus H}[(u, v)] = N_G(u) \times V(H) \cup V(G) \times N_H(v) \cup \{(u, v)\},$$

Hence, if neither G nor H has true twins, $\Delta(G) \leq n - 2$ and $\Delta(H) \leq n' - 2$, then $G \oplus H$ have no true twins and, as a result, $(G \oplus H)^*_- = (G \oplus H)^c_-$. Therefore, we conclude the proof by Lemma 1.7, *i.e.*, $dim_s(G \oplus H) = \beta((G \oplus H)^*_-) = \beta((G \boxtimes H)^c_-) = \beta((G^c \boxtimes H^c)_-) = \beta(G^c \boxtimes H^c)$. □

7.3 Strong metric dimension and (twin-free) clique number

From now we present several relationships between the strong metric dimension of Cartesian sum graphs and clique number or twin-free clique number of its factor graphs. We begin with the connection between the twin-free clique number of a Cartesian sum graphs and the twin-free clique number of its factors, which is useful in this section.

Lemma 7.8. *Let G and H be two graphs. Then,*

$$\varpi(G \oplus H) \geq \varpi(G)\varpi(H).$$

Proof. If all the components of G and H are isomorphic to a complete graph, then $\varpi(G \oplus H) \geq 1 = \varpi(G)\varpi(H)$. If G or H, say G, is an empty graph, then for any twin-free clique S in H, and any $x \in V(G)$, the set $\{x\} \times S$, is also a twin-free clique in $G \oplus H$, since the adjacencies in each copy of H remains equal and, as a consequence, the inequality $\varpi(G \oplus H) \geq \varpi(G)\varpi(H)$ holds.

From now on, we assume G and H are nonempty graphs and we consider the case that at least one component of G or H is not isomorphic to a complete graph (notice that if at least one component of a graph is not isomorphic to a complete graph, then its twin-free clique number is greater than one). Let W be a $\varpi(G)$-set and let Z be a $\varpi(H)$-set. From the definition of Cartesian sum graphs, we have that the subgraph induced by $W \times Z$ is a clique in $G \oplus H$. We consider the following cases.

Case 1: either G or H, say G, has every component isomorphic to a complete graph. Hence, W is a singleton set, $W = \{u\}$, and the set Z is included in a component of H which is not isomorphic to a complete graph (if not, then $\varpi(H) = 1$, which is not possible). So, there exist $v, y \in Z$, $z \notin Z$, such that $z \in N_H(v) - N_H[y]$. By the definition of Cartesian sum graphs, we obtain that $(u, z) \sim (u, v)$ and $(u, z) \not\sim (u, y)$. Thus, $W \times Z$ is a twin-free clique.

Case 2: neither G nor H have every component isomorphic to a complete graph. Thus, as above, there exist $u, x \in W$ and $w \notin W$ such that $w \in N_G(u) - N_G[x]$. Also, there exist $v, y \in Z$ and $z \notin Z$ such that $z \in N_H(v) - N_H[y]$. Again, from the definition of Cartesian sum graphs, we have that

$$(w, z), (u, z), (x, z), (w, v), (w, y) \in N_{G \oplus H}[(u, v)],$$

$$(u,z),(w,v)\in N_{G\oplus H}[(x,y)] \text{ and } (w,z),(x,z),(w,y)\notin N_{G\oplus H}[(x,y)],$$

$$(w,z),(x,z),(w,v),(w,y)\in N_{G\oplus H}[(u,y)] \text{ and } (u,z)\notin N_{G\oplus H}[(u,y)],$$

$$(w,z),(u,z),(x,z),(w,y)\in N_{G\oplus H}[(x,v)] \text{ and } (w,v)\notin N_{G\oplus H}[(x,v)].$$

Therefore, $W\times Z$ is a twin-free clique in $G\oplus H$, which completes the proof. □

Notice that there are cases of Cartesian sum graphs not satisfying the equality in the result above. One example is obtained as a consequence of Corollary 7.14 considering the graph $K_{1,n}\oplus K_{n'}$.

The clique number of any Cartesian sum graph satisfies the following relationship.

Lemma 7.9. *For any graphs G and H,*

$$\omega(G\oplus H)=\omega(G)\omega(H).$$

Proof. Let W be an $\omega(G)$-set and let Y be an $\omega(H)$-set. From the definition of Cartesian sum graphs, we have that the subgraph induced by $W\times Y$ is a clique in $G\oplus H$. So, $\omega(G\oplus H)\ge\omega(G)\omega(H)$. Let Z be an $\omega(G\oplus H)$-set and let $(u,v)\in Z$. Thus, by using definition of Cartesian sum graphs, Z must be of the form $R\times S$, where R is maximum clique in G containing u and S is maximum clique in H containing v. Hence, $\omega(G\oplus H)=|R||S|\le\omega(G)\omega(H)$ and the equality holds. □

The following results give relationships between the strong metric dimension of the Cartesian sum graphs and the clique number or the twin-free clique number of the factor graphs. Notice that the graphs $G\oplus H$ having diameter two are described in Proposition 7.2.

Proposition 7.10. *Let G and H be two graphs of order n and n', respectively, such that $G\oplus H$ is connected. Then,*

$$dim_s(G\oplus H)\le nn'-\varpi(G)\varpi(H).$$

Moreover, if $D(G)\le 2$ or neither G nor H has isolated vertices, then

$$nn'-\omega(G)\omega(H)\le dim_s(G\oplus H)\le nn'-\varpi(G)\varpi(H).$$

Proof. From Theorem 6.1, Lemma 7.9 and the fact that $\omega(H) \geq \varpi(H)$, we have the lower bound. On the other hand, the upper bounds hold because of Theorem 6.1 and Lemma 7.8. $\square$

Corollary 7.11. *Let G and H be two graphs of order n and n', respectively, such that $D(G) \leq 2$ or neither G nor H has isolated vertices. If $\omega(G) = \varpi(G)$ and $\omega(H) = \varpi(H)$, then*

$$dim_s(G \oplus H) = nn' - \omega(G)\omega(H).$$

We recall that the *fan graph* $F_{1,n}$ is defined as the graph join $K_1 + P_n$ and the *wheel graph* of order $n+1$ is defined as $W_{1,n} = K_1 + C_n$. There are some families of graph, as the ones above, which have no true twin vertices. In this sense, its twin-free clique number is equal to its clique number *i.e.*,

- $\varpi(T_n) = \omega(T_n) = 2$, where T_n is a tree of order $n \geq 3$.
- $\varpi(C_n) = \omega(C_n) = 2$, where $n \geq 4$.
- $\varpi(F_{1,n}) = \omega(F_{1,n}) = 3$, where $n \geq 4$.
- $\varpi(W_{1,n}) = \omega(W_{1,n}) = 3$, where $n \geq 4$.
- $\varpi(P_n \Box P_{n'}) = \omega(P_n \Box P_{n'}) = 2$, where $n, n' \geq 2$.

By using the examples above, Corollary 7.11 leads to the following.

Remark 7.12. *The following assertions hold.*

(i) *If G and H are trees, cycles or grid graphs of order n and n', respectively, then*

$$dim_s(G \oplus H) = nn' - 4.$$

(ii) *If G and H are fans or wheels of order $n+1$ and $n'+1$, respectively, then*

$$dim_s(G \oplus H) = nn' + n + n' - 8.$$

(iii) *If G is a tree, a cycle or a grid graph of order n and H is a fan or a wheel of order $n'+1$, then*

$$dim_s(G \oplus H) = nn' + n - 6.$$

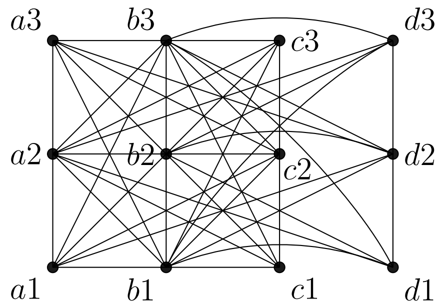

Figure 7.1: The set $\{a3, b3, c1, c2, c3, d1, d2, d3\}$ forms a strong metric basis of $K_{1,3} \oplus P_3$. Thus, $dim_s(K_{1,3} \oplus P_3) = 8$.

Figure 7.1 shows an example regarding Remark 7.12 (i).

Lemma 7.8 gives a general lower bound on $\varpi(G \oplus H)$ in terms of $\varpi(G)$ and $\varpi(H)$. Next we give another lower bound, which in some cases behaves better than this one from Lemma 7.8.

Lemma 7.13. *Let G and H be two nontrivial graphs where G has order n. Then*

$$\varpi(G \oplus H) \geq (\varpi(G) - 1)\omega(H) + 1.$$

Moreover, if there exists a $\varpi(G)$-set without vertices of degree $n-1$, then

$$\varpi(G \oplus H) \geq \varpi(G)\omega(H).$$

Proof. Let W be a $\varpi(G)$-set without vertices of degree $n-1$ and let Z be a $\omega(H)$-set. From the definition of Cartesian sum graphs, we have that the subgraph induced by $W \times Z$ is a clique in $G \oplus H$. Let (u, v) and (x, y) be two different vertices belonging to $W \times Z$. In order to show that $W \times Z$ is a twin-free clique, we consider the following cases.

Case 1: $v = y$. Since $u, x \in W$, then without loss of generality, there exists vertex $w \in N_G(u) - N_G[x]$. Hence, $(u, v) \sim (w, v) \not\sim (x, y)$.

Case 2: $v \neq y$. Since u has degree less than or equal to $n-2$, there exists vertex $z \in V(G)$ such that $u \not\sim z$. Thus, $(u, v) \not\sim (z, v) \sim (x, y)$.

Thus, $W \times Z$ is a twin-free clique and so $\varpi(G \oplus H) \geq |W \times Z| = \varpi(G)\omega(H)$.

On the other hand, let Y be $\varpi(G)$-set having a vertex a of degree $n-1$. Notice that Y cannot contain other vertex of degree $n-1$. Now, let b be a vertex belonging to Z. Observe that $S = ((Y - \{a\}) \times Z) \cup \{(a, b)\}$ is also a clique in $G \oplus H$ since $Y \times Z$ is a clique. We claim that S is a twin-free clique. To see this, we differentiate the following cases for two different vertices $(c, d), (e, f) \in S$.

Case 1': $d = f$. Proceeding like in the Case 1, we have that (c, d) and (e, f) are not true twins.

Case 2': $d \neq f$. If $c \neq a$, then c has degree less than or equal to $n-2$ and there exists a vertex $g \in V(G)$ such that $c \not\sim g$. Thus, $(c, d) \not\sim (g, d) \sim (e, f)$. Now, suppose that $c = a$. In this case $d = b$ and $e \neq a$. Since there exists $a' \in V(H)$ such that $a' \in N_H(a) - N_H[e]$, we have $(c, d) = (a, b) \sim (a', f) \not\sim (e, f)$.

Therefore, S is a twin-free clique, which leads to

$$\varpi(G \oplus H) \geq |S| = (\varpi(G) - 1)\omega(H) + 1.$$

□

The following result is a direct consequence of the lemma above and the well-known fact that the Cartesian sum of graphs is a commutative operation.

Corollary 7.14. *Let G and H be two nontrivial graphs of order n and n', respectively. Then the following assertions hold.*

(i) $\varpi(G \oplus H) \geq \max\{(\varpi(G) - 1)\omega(H), \omega(G)(\varpi(H) - 1)\} + 1.$

(ii) *If there exists a $\varpi(G)$-set without a vertex of degree $n - 1$ and there exists a $\varpi(H)$-set without a vertex of degree $n' - 1$, then*

$$\varpi(G \oplus H) \geq \max\{\varpi(G)\omega(H), \omega(G)\varpi(H)\}.$$

(iii) *If there exists a $\varpi(G)$-set without a vertex of degree $n - 1$, then*

$$\varpi(G \oplus H) \geq \max\{\varpi(G)\omega(H), \omega(G)(\varpi(H) - 1) + 1\}.$$

By using Theorem 6.1 and Corollary 7.14 we obtain another bounds on $dim_s(G \oplus H)$.

Proposition 7.15. *Let G and H be two nontrivial graphs of order n and n', respectively such that $G \oplus H$ is connected. Then the following assertions hold.*

(i) $dim_s(G \oplus H) \leq nn' - \max\{(\varpi(G) - 1)\omega(H), \omega(G)(\varpi(H) - 1)\} - 1.$

(ii) *If there exists a $\varpi(G)$-set without a vertex of degree $n-1$ and there exists a $\varpi(H)$-set without a vertex of degree $n'-1$, then*

$$dim_s(G \oplus H) \leq nn' - \max\{\varpi(G)\omega(H), \omega(G)\varpi(H)\}.$$

(iii) *If there exists a $\varpi(G)$-set without a vertex of degree $n-1$, then*

$$dim_s(G \oplus H) \leq nn' - \max\{\varpi(G)\omega(H), \omega(G)(\varpi(H)-1)+1\}.$$

Corollary 7.16. *Let G be a nontrivial graph of order n. If G has no true twins and $\Delta(G) \leq n-2$, then*

$$dim_s(G \oplus K_{n'}) = nn' - n'\omega(G).$$

Proof. First of all, note that $G \oplus K_{n'}$ is connected, as stated in Corollary 7.3. On the other hand, since G has no true twins, it follows $\omega(G) = \varpi(G)$. Now, from Proposition 7.10 we have that $dim_s(G \oplus K_{n'}) \geq nn' - n'\omega(G)$. Moreover, by using Proposition 7.15 (iii) we obtain $dim_s(G \oplus H) \leq nn' - \max\{\varpi(G)\omega(H), \omega(G)(\varpi(H)-1)+1\} = nn' - n'\omega(G)$. Therefore, the equality holds. □

Corollary 7.17. *For any integers, $n, n' \geq 2$,*

$$(n+1)n' - 2n' \leq dim_s(K_{1,n} \oplus K_{n'}) \leq (n+1)n' - n' - 1.$$

Proof. The lower bound is a direct consequence of Proposition 7.10 while the upper bound is a direct consequence of Proposition 7.15 (i). □

Chapter 8

Strong metric dimension of rooted product graphs

8.1 Overview

In this chapter we study the problem of computing exact values of the strong metric dimension of some classes of rooted product graphs and express these in terms of invariants of the factor graphs. Moreover, we present sharp lower and upper bounds on the strong metric dimension of the rooted product of graphs and give some families of graphs where these bounds are attained.

8.2 Formulae for some families of rooted product graphs

We start this section with stating the following easily verified lemmas, which allow us to derive the structure of the strong resolving graph of rooted product graphs.

Lemma 8.1. *Let G and H be two connected graphs. Let the vertices $a, b \in V(G)$, $a \neq b$ and $x, y, v \in V(H)$. Then (a, x) and (b, y) are mutually maximally distant vertices in $G \circ_v H$ if and only if $x, y \in M_H(v)$.*

Proof. (Sufficiency) Suppose that (a, x) and (b, y) are not mutually maximally distant vertices in $G \circ_v H$. So, there exists a vertex $(a, x') \in N_{G\circ_v H}(a, x)$ such that

$$d_{G\circ_v H}((a, x'), (b, y)) > d_{G\circ_v H}((a, x), (b, y)),$$

or there exists $(b, y') \in N_{G\circ_v H}(b, y)$ such that

$$d_{G\circ_v H}((a, x), (b, y')) > d_{G\circ_v H}((a, x), (b, y)).$$

We consider, without loss of generality, that $(a, x') \in N_{G\circ_v H}(a, x)$ and

$$d_{G\circ_v H}((a, x'), (b, y)) > d_{G\circ_v H}((a, x), (b, y)).$$

So we have,

$$\begin{aligned} d_H(x', v) &= d_{G\circ_v H}((a, x'), (b, y)) - d_G(a, b) - d_H(v, y) \\ &> d_{G\circ_v H}((a, x), (b, y)) - d_G(a, b) - d_H(v, y) \\ &= d_H(x, v). \end{aligned}$$

Thus, $d_H(x', v) > d_H(x, v)$. Since $x' \in N_H(x)$ and $x \in M_H(v)$, we have a contradiction.

(Necessity) Let us suppose that $x \notin M_H(v)$. So, there exists $x'' \in N_H(x)$ such that $d_H(x'', v) > d_H(x, v)$. Thus,

$$\begin{aligned} d_{G\circ_v H}((a, x), (b, y)) &= d_H(x, v) + d_G(a, b) + d_H(v, y) \\ &< d_H(x'', v) + d_G(a, b) + d_H(v, y) \\ &= d_{G\circ_v H}((a, x''), (b, y)). \end{aligned}$$

Hence, there exists a vertex $(a, x'') \in N_{G\circ_v H}((a, x))$ such that

$$d_{G\circ_v H}((a, x), (b, y)) < d_{G\circ_v H}((a, x''), (b, y)),$$

which is a contradiction since (a, x) and (b, y) are mutually maximally distant. □

Lemma 8.2. *Let G and H be two connected nontrivial graphs. Let v, x, y be vertices of H such that $x, y \neq v$. For every vertex a of G we have that (a, x) and (a, y) are mutually maximally distant vertices in $G \circ_v H$ if and only if the vertices x and y are mutually maximally distant in H.*

Proof. The result follows directly from the fact that for every vertex c of G and every vertex $z \neq v$ of H we have that $w \in N_H(z)$ if and only if $(c, w) \in N_{G\circ_v H}(c, z)$ and also that $d_{G\circ_v H}((a, x), (a, y)) = d_H(x, y)$ for every x, y of H. □

The following result deals with the boundary of rooted product graphs and is a very useful tool for our purposes.

Proposition 8.3. *Let G be a connected graph of order $n \geq 2$ and let H be a connected graph.*

(i) *If $v \in \partial(H)$, then*

$$\partial (G \circ_v H) = V(G) \times (\partial(H) - \{v\}).$$

(ii) *If $v \notin \partial(H)$, then*

$$\partial (G \circ_v H) = V(G) \times \partial(H).$$

Proof. Let (x, y) and (x', y') be two mutually maximally distant vertices in $G \circ_v H$. Since $(V(G) \times \{v\}) \cap \partial (G \circ_v H) = \emptyset$, it follows $y, y' \neq v$. We differentiate two cases.
Case 1: $x = x'$. By Lemma 8.2 we conclude that (x, y) and (x', y') are mutually maximally distant in $G \circ_v H$ if and only if y and y' are mutually maximally distant in H.
Case 2: $x \neq x'$. By Lemma 8.1 the vertices (x, y) and (x', y') are mutually maximally distant in $G \circ_v H$ if and only if $y, y' \in M_H(v)$. Note that, by definition of boundary, $y, y' \in \partial(H)$.

According to the cases above we conclude that if $(x, y) \in \partial(G \circ_v H)$, then $y \in \partial(H) - \{v\}$. Moreover, if $y \in \partial(H) - \{v\}$, then for every $x \in V(G)$ we have $(x, y) \in \partial(G \circ_v H)$.

Therefore, if $v \in \partial(H)$, then $\partial (G \circ_v H) = V(G) \times (\partial(H) - \{v\})$ and if $v \notin \partial(H)$, then $\partial (G \circ_v H) = V(G) \times \partial(H)$. □

In order to present the next proposition about the set of simplicial vertices of rooted product graphs we need to introduce some notation. Given a vertex $x \in V(G)$, we define H_x as the subgraph $H_x = \langle \{x\} \times V(H) \rangle$ of $G \circ_v H$. Note that for any vertex x the subgraph H_x of $G \circ_v H$ is isomorphic to H.

Proposition 8.4. *Let G be a connected graph of order $n \geq 2$ and let H be a connected graph.*

(i) *If $v \in \sigma(H)$, then*

$$\sigma (G \circ_v H) = V(G) \times (\sigma(H) - \{v\}).$$

(ii) *If $v \notin \sigma(H)$, then*

$$\sigma\left(G \circ_v H\right) = V(G) \times \sigma(H).$$

Proof. Notice that (x, v) is not simplicial in $G \circ_v H$. Since the following assertions are equivalent, the result immediately follows.

- The vertex $(x, y) \in V(G) \times (V(H) - \{v\})$ is simplicial in $G \circ_v H$.
- For $x \in V(G)$ and $y \neq v$ the vertex (x, y) is simplicial in H_x.
- The vertex $y \in V(H) - \{v\}$ is simplicial in H.

□

Observe that if $\partial(H) = \sigma(H)$, then by using the propositions above we obtain the exact values of the strong metric dimension of rooted product graphs. Hence, we have the following result.

Theorem 8.5. *Let G be a connected graph of order $n \geq 2$ and let H be a connected graph such that $\partial(H) = \sigma(H)$.*

(i) *If $v \in \partial(H)$, then*

$$dim_s(G \circ_v H) = n(|\partial(H)| - 1) - 1.$$

(ii) *If $v \notin \partial(H)$, then*

$$dim_s(G \circ_v H) = n|\partial(H)| - 1.$$

Proof. Since $\partial(H) = \sigma(H)$, as a direct consequence of Proposition 8.3 and Proposition 8.4 we obtain that if $v \notin \partial(H)$, then $\partial\left(G \circ_v H\right) = V(G) \times \partial(H) = \sigma\left(G \circ_v H\right)$ and if $v \in \partial(H)$, then $\partial\left(G \circ_v H\right) = V(G) \times (\partial(H) - \{v\}) = \sigma\left(G \circ_v H\right)$. Hence, if $v \notin \partial(H)$, then $(G \circ_v H)_{SR} \cong K_{n|\partial(H)|}$ and, if $v \in \partial(H)$, then $(G \circ_v H)_{SR} \cong K_{n(|\partial(H)|-1)}$. Therefore, the result follows by Theorem 1.12. □

We emphasize the following particular cases of Theorem 8.5.

Corollary 8.6. *Let G be a connected graph of order $n \geq 2$.*

(i) *For any complete graph of order n',*

$$dim_s(G \circ_* K_{n'}) = n(n' - 1) - 1.$$

(ii) *For any tree T,*

$$dim_s(G \circ_v T) = \begin{cases} n(l(T)-1)-1, & \text{if } v \text{ is a leaf of } T, \\ \\ n \cdot l(T) - 1, & \text{if } v \text{ is an inner vertex of } T. \end{cases}$$

(iii) *Let G' be a connected graph of order n' and let $H = G' \odot (\bigcup_{i=1}^{r} K_{t_i})$, where $r \geq 2$, $t_i \geq 1$. Then*

$$dim_s(G \circ_v H) = \begin{cases} n\sum_{i=1}^{r} t_i - n - 1, & \text{if } v \in \bigcup_{i=1}^{r} V(K_{t_i}), \\ \\ n\sum_{i=1}^{r} t_i - 1, & \text{if } v \in V(G'). \end{cases}$$

The next theorem gives a bound on $dim_s(G \circ_v H)$ with respect to the cardinality of the set of vertices which are maximally distant from v in H, that is $|M_H(v)|$.

Theorem 8.7. *Let G be a connected graph of order $n \geq 2$ and let H be a connected graph such that $H_{SR} \cong \bigcup_{i=1}^{\frac{|\partial(H)|}{2}} K_2$. Let $v \in V(H)$ and let $i(v)$ be the set of isolated vertices of the subgraph of H_{SR} induced by $M_H(v)$.*

(i) *If $v \notin \partial(H)$, then*

$$dim_s(G \circ_v H) = \frac{n(|\partial(H)| + |M_H(v)| - |i(v)|) - |M_H(v)| + |i(v)|}{2}.$$

(ii) *If $v \in \partial(H)$, then*

$$dim_s(G \circ_v H) = \frac{n(|\partial(H)| + |M_H(v)| - |i(v)|) - |M_H(v)| + |i(v)| - 2}{2}.$$

Proof. Let $V(G) = \{x_1, x_2, ..., x_n\}$ be the vertex set of G and let B be a vertex cover of $(G \circ_v H)_{SR}$. First we note that by premiss for every $a \in \partial(H)$ there exists exactly one vertex $a' \in \partial(H)$ such that a and a' are adjacent in H_{SR}. We consider the set $i'(v) \subset \partial(H)$ defined in the following way: $a' \in i'(v)$ if and only if there exists $a \in i(v)$ such that a and a' are mutually maximally distant in H. Note that $|i'(v)| = |i(v)|$ and, if $v \in \partial(H)$ and v, v' are mutually maximally distant, then $v \in i'(v)$ and $v' \in i(v)$. Also, since there are no edges in H_{SR} connecting vertices belonging to $M_H(v) \cup i'(v)$ to vertices belonging to $\partial(H) - M_H(v) \cup i'(v)$, by Lemmas 8.1 and 8.2 we conclude that there are no edges in $(G \circ_v H)_{SR}$ connecting vertices belonging to $V(G)\times(\partial(H)-(M_H(v)\cup i'(v))$ to vertices belonging to $V(G)\times(M_H(v)\cup i'(v))$. With this idea in mind, we proceed to prove the results.

In order to prove (i) we consider that $v\notin\partial(H)$. Note that in this case by Proposition 8.3 (ii), $\partial(G\circ_v H)=V(G)\times\partial(H)$. By Lemma 8.2 we have that for every mutually maximally distant vertices $a,a'\in\partial(H)-(M_H(v)\cup i'(v))$ and every $j\in\{1,...,n\}$ the vertices (x_j,a) and (x_j,a') are mutually maximally distant in $G\circ_v H$ and, as a consequence, $(x_j,a)\notin B$ if and only if $(x_j,a')\in B$. Thus, the subgraph of $(G\circ_v H)_{SR}$ induced by $V(G)\times(\partial(H)-M_H(v)\cup i'(v))$ is composed by $\frac{n}{2}(|\partial(H)|-|M_H(v)|-|i'(v)|)$ components isomorphic to K_2.

On the other hand, by Lemma 8.1 we have that $(x_j,a),(x_k,a)$ are mutually maximally distant in $G\circ_v H$, for every $a\in M_H(v)$ and $j\neq k$. Thus, if $(x_j,a)\notin B$ for some j, then $(x_k,a)\in B$ for every $k\neq j$. Moreover, as above, Lemma 8.2 allows us to conclude that given two mutually maximally distant vertices $a,a'\in M_H(v)\cup i'(v)$ it follows that $(x_j,a)\notin B$ if and only if $(x_j,a')\in B$. Thus, B contains exactly $(n-1)|M_H(v)+\frac{|M_H(v)\cup i'(v)|}{2}$ vertices belonging to $V(G)\times(M_H(v)\cup i'(v))$. Therefore,

$$\begin{aligned}|B| &= \frac{n(|\partial(H)|-|M_H(v)|-|i(v)|)}{2}+(n-1)|M_H(v)|+\frac{|M_H(v)|+|i(v)|}{2}\\ &= \frac{n(|\partial(H)|+|M_H(v)|-|i(v)|)-|M_H(v)|+|i(v)|}{2}\end{aligned}$$

The proof of (i) is complete.

From now on we suppose $v\in\partial(H)$. Note that in this case by Proposition 8.3 (i) we have $\partial(G\circ_v H)=V(G)\times(\partial(H)-\{v\})$. To prove (ii) we proceed by analogy to the proof of (i). In this case we obtain that the subgraph of $(G\circ_v H)_{SR}$ induced by $V(G)\times(\partial(H)-(M_H(v)\cup i'(v))$ is composed by $\frac{n}{2}|\partial(H)-M_H(v)\cup i'(v)|=\frac{n}{2}(|\partial(H)|-|M_H(v)|-|i(v)|)$ components isomorphic to K_2 and B contains exactly $(n-1)|M_H(v)|+\frac{|(M_H(v)-\{v'\})\cup(i'(v)-\{v\})|}{2}=(n-1)|M_H(v)|+\frac{|M_H(v)|+|i(v)|-2}{2}$ vertices of $G\circ_v H$ belonging to $V(G)\times(M_H(v)\cup(i'(v)-\{v\}))$. Thus,

$$\begin{aligned}|B| &= \frac{n(|\partial(H)|-|M_H(v)|-|i(v)|)}{2}+(n-1)|M_H(v)|+\frac{|M_H(v)|+|i(v)|-2}{2}\\ &= \frac{n(|\partial(H)|+|M_H(v)|-|i(v)|)-|M_H(v)|+|i(v)|-2}{2}.\end{aligned}$$

The proof of (ii) is complete. □

We conjecture that if $v\notin\partial(H)$, then $i(v)=i'(v)=\emptyset$. In order to show a particular case of Theorem 8.7 where $i(v)\neq\emptyset$ we consider the graph H

shown in the left hand side of Figure 8.1 where $\partial(H) = \{a, a', b, b', v, v'\}$, $M_H(v) = i(v) = \{a, v'\}$ and $i'(v) = \{a', v\}$. In the case of the graph H shown in the right hand side of Figure 8.1 we have $\partial(H) = \{a, a', b, b', v, v'\}$, $M_H(v) = \{a, a', v'\}$, $i(v) = \{v'\}$ and $i'(v) = \{v\}$. In both cases

$$B = (V(G) - \{u_n\}) \times (M_H(v) \cup \{b\}) \cup \{(u_n, a), (u_n, b)\}$$

is a strong metric basis of $G \circ_v H$ for any graph G with vertex set $V(G) = \{u_1, u_2, ..., u_n\}$.

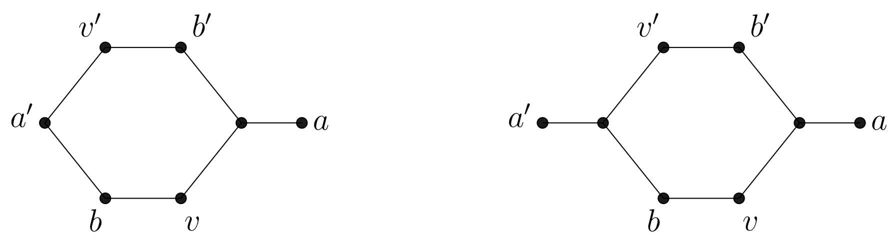

Figure 8.1: In left hand side graph $i(v) = \{a, v'\}$ and $i'(v) = \{a', v\}$. In right hand side graph $i(v) = \{v'\}$ and $i'(v) = \{v\}$.

Corollary 8.8. *Let G be a connected graph of order $n \geq 2$ and let H be a connected 2-antipodal graph of order n'. Then*

$$dim_s(G \circ_* H) = \frac{nn'}{2} - 1.$$

Now we study the strong metric dimension of $G \circ_* C_t$ for any nontrivial graph G and t greater than or equal to three.

Theorem 8.9. *Let C_t be a cycle of order $t \geq 3$. For any connected graph G of order $r \geq 2$,*

$$dim_s(G \circ_* C_t) = r \left\lceil \frac{t}{2} \right\rceil - 1.$$

Proof. Let $V(G) = \{x_1, x_2, ..., x_r\}$ and $V(C_t) = \{y_0, y_1, ..., y_{t-1}\}$ be the vertex sets of G and C_t, respectively. We assume $y_0 \sim y_1 \sim ... \sim y_{t-1} \sim y_0$ in C_t and from now on all the operations with the subscripts of y_i are done modulo t. Since C_t is a vertex-transitive graph, we can take without loss of generality $v = y_0$ as the root of C_t.

If t be an even number, then C_t is 2-antipodal. So the result follows by Corollary 8.8. Now let t be an odd number. Note that exactly two vertices

$y_{\lceil \frac{t}{2} \rceil}$ and $y_{\lfloor \frac{t}{2} \rfloor}$ are maximally distant from v in C_t. So, from Lemma 8.1 we have that every vertex (x_i, y_l) is mutually maximally distant from (x_j, y_k) in $G \circ_* C_t$, with $j \neq i$ and $l, k \in \left\{ \left\lceil \frac{t}{2} \right\rceil, \left\lfloor \frac{t}{2} \right\rfloor \right\}$. Moreover, from Lemma 8.2 we have that for every $i \in \{1, 2, ..., r\}$, (x_i, y_k) is mutually maximally distant from $(x_i, y_{k+\lfloor \frac{t}{2} \rfloor})$ and $(x_i, y_{k+\lceil \frac{t}{2} \rceil})$ in $G \circ_* C_t$ with $k \in \{1, 2, ..., \lfloor \frac{t}{2} \rfloor - 1, \lceil \frac{t}{2} \rceil + 1, ..., t-1\}$. Also, the vertex $(x_i, y_{\lfloor \frac{t}{2} \rfloor})$ is mutually maximally distant from (x_i, y_{t-1}) and the vertex $(x_i, y_{\lceil \frac{t}{2} \rceil})$ is mutually maximally distant from (x_i, y_1). Thus, we obtain that the graph $(G \circ_* C_t)_{SR}$ is isomorphic to a graph with set of vertices $U \cup (\bigcup_{i=1}^{r} V_i)$ where $\langle U \rangle$ is isomorphic to a complete r-partite graph $K_{2,2,...,2}$ and for every $i \in \{1, ..., r\}$, $\langle V_i \rangle$ is isomorphic to a path graph P_{t-1}. Notice that the leaves of P_{t-1} belong to U, so for every $i \in \{1, ..., r\}$, $|V_i \cap U| = 2$. Thus, we have the following:

$$\begin{aligned}
dim_s(G \circ_* C_t) &= \beta((G \circ_* C_t)_{SR}) \\
&= \beta(\langle U \rangle) + (r-1)\beta(P_{t-3}) + \beta(P_{t-1}) \\
&= 2(r-1) + (r-1)\frac{t-3}{2} + \frac{t-1}{2} \\
&= r\left\lceil \frac{t}{2} \right\rceil - 1.
\end{aligned}$$

The proof is complete. □

As we mention in Chapter 1, there exists a relationship between rooted product graphs and corona product graphs. Given a vertex v of a graph H, we denote by $H - v$ the graph obtained by removing v from H. Now, if v is a vertex of H of degree $n-1$, then the rooted product graph $G \circ_v H$ is isomorphic to the corona product graph $G \odot (H - v)$.

According to this connection above mentioned, from some results of Chapter 6 we can deduce some direct consequences. The next corollary is obtained from Theorem 6.7 (ii).

Corollary 8.10. *Let G be a connected graph of order $r \geq 2$. Let H be a connected graph of order $t \geq 2$ and let v be a vertex of H of degree $t-1$. Then*

$$dim_s(G \circ_v H) = r(t-1) - \varpi(H - v).$$

Corollary 6.9 gives the exact value of the strong metric dimension of $G \odot H$ when H is a triangle-free graph. As a direct consequence of this result we have the following.

Corollary 8.11. *Let G be a connected graph of order $r \geq 2$. Let H be a connected graph of order $t \geq 2$ and let v be a vertex of H of degree $t-1$. If $H-v$ is a triangle-free graph. Then*

$$dim_s(G \circ_v H) = r(t-1) - 2.$$

Theorem 6.10 (ii) and (iii) shows that the strong metric dimension of $G \odot H$ depends on the diameter of H. Therefore, by using Theorem 6.10 we obtain the following result for $G \circ_v H$.

Corollary 8.12. *Let G be a connected graph of order $r \geq 2$. Let H be a graph of order $t \geq 2$ and let v be a vertex of H of degree $t-1$.*

(i) *If $H-v$ has diameter two, then*

$$dim_s(G \circ_v H) = (r-1)(t-1) + dim_s(H-v).$$

(ii) *If $H-v$ has diameter greater than two, then*

$$dim_s(G \circ_v H) = (r-1)(t-1) + dim_s(H).$$

The strong metric dimension of $G \odot H$ depends on the existence or not of true twins in H. Hence, the strong metric dimension of rooted product graphs also depends on the existence or not of true twins in the second factor. Our next result is an interesting consequence of Corollary 6.8 (iii) and (v).

Corollary 8.13. *Let G be a connected graph of order $r \geq 2$. Let H be a connected graph of order $t \geq 2$ and let v be a vertex of H of degree $t-1$. Let $c(H-v)$ be the number of vertices of $H-v$ having degree $t-2$.*

(i) *If $H-v$ has no true twins, then*

$$dim_s(G \circ_v H) = r(t-1) - \omega(H-v).$$

(ii) *If the only true twins of $H-v$ are vertices of degree $t-2$, then*

$$dim_s(G \circ_v H) = r(t-1) + c(H-v) - 1 - \omega(H-v).$$

8.3 Tight bounds

In this section we present sharp lower and upper bounds on the strong metric dimension of the rooted product graphs and give some families of graphs where these bounds are attained. To begin with, we need to introduce more notation. Given $x \in V(G)$, $v \in V(H)$ and $B \subset V(G) \times V(H)$ we denote by B_x the set of element of B whose first component is x, *i.e.*, $B_x = B \cap (\{x\} \times V(H))$.

Lemma 8.14. *Let G and H be two connected graphs. Let $x \in V(G)$, $v \in V(H)$, and B be a strong metric basis of $G \circ_v H$. Then the following assertions hold.*

(i) $|B_x| \geq dim_s(H) - 1$.

(ii) *If $B_x \supset \{x\} \times M_H(v)$, then $|B_x| \geq dim_s(H)$.*

(iii) *If v does not belong to any strong metric basis of H, then $|B_x| \geq dim_s(H)$.*

Proof. First we consider a pair $(x, y), (x, y')$ of adjacent vertices in $(H_x)_{SR}$, where $y, y' \neq v$. Since B is a vertex cover of $(G \circ_v H)_{SR}$, either $(x, y) \in B_x$ or $(x, y') \in B_x$. Thus, $B_x \cup \{(x, v)\}$ is a vertex cover of $(H_x)_{SR}$. Note that $(x, v) \notin \partial(G \circ_v H)$ and, as a consequence, $(x, v) \notin B_x$. Hence, $|B_x| + 1 = |B_x \cup \{(x, v)\}| \geq dim_s(H_x) = dim_s(H)$. Therefore, (i) follows.

Now we suppose $B_x \supset \{x\} \times M_H(v)$. If (x, y) and (x, v) are adjacent in $(H_x)_{SR}$, then $y \in M_H(v)$. So the edge $\{(x, y), (x, v)\}$ of $(H_x)_{SR}$ is covered by $(x, y) \in B_x$. Thus, B_x is a vertex cover of $(H_x)_{SR}$ and, as a result, $|B_x| \geq dim_s(H)$. Therefore, (ii) follows.

Finally, suppose that v does not belong to any strong metric basis of H. Since the function $f : \{x\} \times V(H) \to V(H)$, where $f(x, y) = y$, is a graph isomorphism and $B_x \cup \{(x, v)\}$ is a strong metric generator for H_x, the set

$$A = f(B_x \cup \{(x, v)\}) = \{v\} \cup \{u : (x, u) \in B_x\}$$

is a strong metric generator for H. Thus, since v does not belong to any strong metric basis of H, $|A| > dim_s(H)$. Taking into account that $(x, v) \notin B_x$ we obtain $|B_x| = |B_x \cup \{(x, v)\}| - 1 = |A| - 1 \geq dim_s(H)$. The proof is complete. □

Theorem 8.15. *Let G be a connected graph of order $n \geq 2$ and let H be a connected graph.*

(i) *If $v \in V(H)$ belongs to a strong metric basis of H, then*

$$n \cdot dim_s(H) - 1 \leq dim_s(G \circ_v H) \leq (|\partial(H)| - 1)(n-1) + dim_s(H) - 1.$$

(ii) *If $v \in V(H)$ does not belong to any strong metric basis of H, then*

$$dim_s(G \circ_v H) \geq n \cdot dim_s(H)$$

and

$$dim_s(G \circ_v H) \leq \begin{cases} |\partial(H)|(n-1) + dim_s(H), & \text{if } v \notin \partial(H), \\ \\ (|\partial(H)| - 1)(n-1) + dim_s(H), & \text{if } v \in \partial(H). \end{cases}$$

Proof. Let W be a strong metric basis of H such that $v \in W$ and let B be a strong metric basis of $G \circ_v H$. Since v belongs to a metric basis of H, we have $v \in \partial(H)$. Suppose there exists $x \in V(G)$ such that $(x,u) \notin B_x$ for some $u \in M_H(v)$. By Lemma 8.14 (i) we obtain $|B_x| \geq dim_s(H) - 1$. Moreover, by Lemma 8.1 we have that for $x' \in V(G) - \{x\}$ and $u' \in M_H(v)$ the vertices (x,u) and (x',u') are mutually maximally distant in $G \circ_v H$. Hence, since $(x,u) \notin B_x$ and B is a vertex cover of $(G \circ_v H)_{SR}$, for every $x' \in V(G) - \{x\}$ we have $B_{x'} \supset \{x'\} \times M_H(v)$. So, according to Lemma 8.14 (ii) we have $|B_{x'}| \geq dim_s(H)$. Therefore,

$$dim_s(G \circ_v H) = |B| = |B_x| + \sum_{x' \in V(G) - \{x\}} |B_{x'}| \geq n \cdot dim_s(H) - 1.$$

On the other hand, since $v \in \partial(H)$, Proposition 8.3 (ii) leads to $\partial\left(G \circ_v H\right) = V(G) \times (\partial(H) - \{v\})$. We show that $S = \partial\left(G \circ_v H\right) - P$ is a vertex cover of $(G \circ_v H)_{SR}$, where $P = \{a\} \times (\partial(H) - W \cup \{v\})$ and $a \in V(G)$. Let (x,y) and (x',y') be two adjacent vertices in $(G \circ_v H)_{SR}$. If $x \neq a$ or $x' \neq a$, then $(x,y) \in S$ or $(x',y') \in S$. Now let, $x = x' = a$. Since $H_a \cong H$ and W is a vertex cover of H, $\{a\} \times W$ is a vertex cover of H_a and, as a consequence, $(x,y) \in \{a\} \times W \subset S$ or $(x',y') \in \{a\} \times W \subset S$. Hence, S is a vertex cover of $(G \circ_v H)_{SR}$. Therefore,

$$dim_s(G \circ_v H) \leq |S| = (|\partial(H)| - 1)(n-1) + dim_s(H) - 1.$$

The proof of (i) is complete.

From now on we assume that v does not belong to any strong metric basis of H. The lower bound of (ii) is a direct consequence of Lemma 8.14 (iii). Suppose $v \notin \partial(H)$. In this case, by Proposition 8.3 (i) we conclude $\partial(G \circ_v H) = V(G) \times \partial(H)$. By analogy with the proof of the upper bound of (i) we show that $S' = \partial(G \circ_v H) - P'$ is a vertex cover of $(G \circ_v H)_{SR}$, where $P' = \{a\} \times (\partial(H) - W')$, $a \in V(G)$ and W' is a strong metric basis of H. Hence,

$$dim_s(G \circ_v H) \leq |S'| = |\partial(H)|(n-1) + dim_s(H).$$

Finally, for the case $v \in \partial(H)$ we have $\partial(G \circ_v H) = V(G) \times (\partial(H) - \{v\})$ and proceeding by analogy with the proof of the upper bound of (i) we show that $S'' = \partial(G \circ_v H) - P''$ is a vertex cover of $(G \circ_v H)_{SR}$, where $P'' = \{a\} \times (\partial(H) - W'')$, $a \in V(G)$ and W'' is a strong metric basis of H. Thus, in this case

$$dim_s(G \circ_v H) \leq |S''| = (|\partial(H)| - 1)(n-1) + dim_s(H).$$

The proof of (ii) is complete. □

As Corollary 8.6 shows, the bounds of Theorem 8.15 (i) are tight and the upper bound $dim_s(G \circ_v H) \leq |\partial(H)|(n-1) + dim_s(H)$ of Theorem 8.15 (ii) is tight. To show the tightness of the upper bound $dim_s(G \circ_v H) \leq (|\partial(H)| - 1)(n-1) + dim_s(H)$ we consider the graph J shown in Figure 8.2. Notice that any strong metric basis of J is formed by the vertices y_2, y_4 and three vertices of the set $\{y_1, y_3, y_5, x_6\}$.

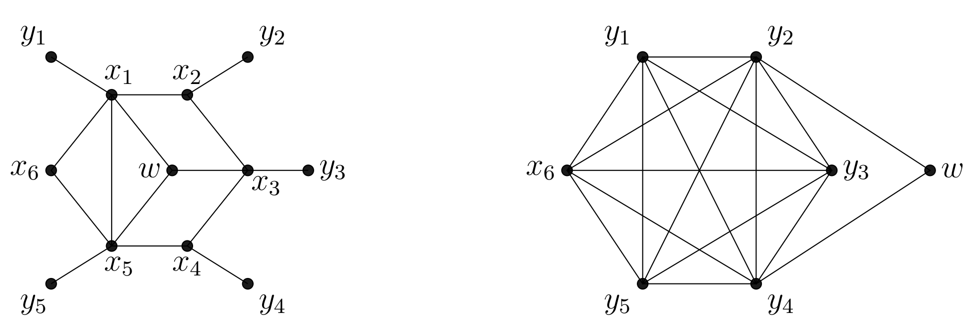

Figure 8.2: The graph J and its strong resolving graph J_{SR}.

Remark 8.16. *Let G be a connected graph of order n. Let v be the vertex of the graph J denoted by w. Then*

$$dim_s(G \circ_v J) = (\partial(J) - 1)(n - 1) + dim_s(J).$$

Proof. Let $V = \{u_1, u_2, ..., u_n\}$ be the set of vertices of G. From Figure 8.2 we have that there exits six vertices y_1, y_2, y_3, y_4, y_5 and x_6 which are maximally distant from v. So, by using Lemma 8.1, we have that every two vertices $(u_i, y), (u_j, y') \in V \times \{y_1, y_2, y_3, y_4, y_5, x_6\}$, where $i \neq j$, are mutually maximally distant. Moreover, by Lemma 8.2 for every two mutually maximally distant vertices z, z' in J we have that $(u_i, z), (u_i, z')$ are mutually maximally distant in $G \circ_v J$ for every vertex u_i of G. Thus, $(G \circ_v J)_{SR}$ is isomorphic to K_{6n}. Therefore,

$$dim_s(G \circ_v J) = 6n - 1 = (\partial(J) - 1)(n - 1) + dim_s(J).$$

□

To see the tightness of the lower bound of Theorem 8.15 (ii) we define the family $\mathcal{F}$ of graphs H containing a vertex of degree one not belonging to any strong metric basis of H. We begin with the cycle C_t, where t is an odd number such that $t \geq 5$, with set of vertices $X = \{x_1, x_2, ..., x_t\}$. To obtain a graph $H_{t,p,r} \in \mathcal{F}$ we add the sets of vertices $Y = \{y\}$, $W = \{w_1, w_2, ..., w_p\}$ and $Z = \{z_1, z_2, ..., z_r\}$, where $p, r \geq 1$, and edges yx_t, x_1x_{t-1}, $x_{\lfloor \frac{t}{2} \rfloor}w_i$, for every $i \in \{1, 2, ..., p\}$, and $x_{\lceil \frac{t}{2} \rceil}z_j$, for every $j \in \{1, 2, ..., r\}$. Notice that vertices of $Y \cup W \cup Z$ have degree one in $H_{t,p,r}$ and they are mutually maximally distant between them. Also, for any vertex $a \in N_{H_{t,p,r}}(x_1)$, $d_{H_{t,p,r}}(a, z_j) \leq d_{H_{t,p,r}}(x_1, z_j)$, where $j \in \{1, 2, ..., r\}$. Similarly, for any vertex $b \in N_{H_{t,p,r}}(x_{t-1})$, $d_{H_{t,p,r}}(b, w_i) \leq d_{H_{t,p,r}}(x_{t-1}, w_i)$, where $i \in \{1, 2, ..., p\}$. Moreover, we can observe that x_k and $x_{k+\lfloor \frac{t}{2} \rfloor}$ are mutually maximally distant for every $k \in 2, 3, ..., \lfloor \frac{t}{2} \rfloor - 1$. So, $(H_{t,p,r})_{SR}$ is formed by $\lfloor \frac{t}{2} \rfloor - 1$ connected components, that is, $\lfloor \frac{t}{2} \rfloor - 2$ connected components isomorphic to K_2 and also, a connected component isomorphic to a graph with set of vertices $Y \cup W \cup Z \cup \{x_1, x_{t-1}\}$ where $\langle Y \cup W \cup Z \rangle$ is isomorphic to $K_{|Y \cup W \cup Z|}$, x_1 is adjacent to every vertex z_j, $j \in \{1, 2, ..., r\}$, and x_{t-1} is adjacent to every vertex w_i, $i \in \{1, 2, ..., p\}$. Notice that every $\beta((H_{t,p,r})_{SR})$-set is formed only by the vertices of $W \cup Z$ and one vertex from each subgraph isomorphic to K_2. Therefore,

$$dim_s(H_{t,p,r}) = \frac{t - 5}{2} + p + r$$

and y is a vertex of degree one not belonging to any strong metric basis of $H_{t,p,r}$. The graphs $H_{9,3,4}$ and $(H_{9,3,4})_{SR}$ are shown in Figure 8.3.

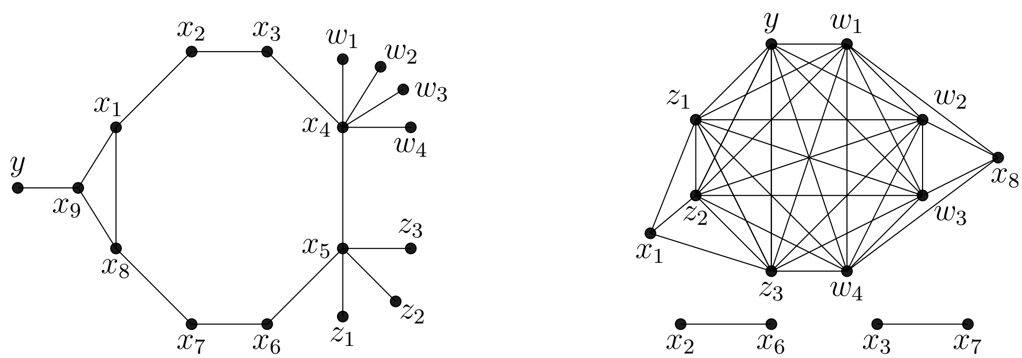

Figure 8.3: The graphs $H_{9,3,4}$ and $(H_{9,3,4})_{SR}$. The set $S = \{w_1, w_2, w_3, w_4, z_1, z_2, z_3, x_2, x_3\}$ is a strong metric basis of $H_{9,3,4}$.

Remark 8.17. *Let G be a connected graph of order n. Let v be the vertex of degree one not belonging to any strong metric basis of the graph $H_{t,p,r} \in \mathcal{F}$. Then*

$$dim_s(G \circ_v H_{t,p,r}) = n\left(\frac{t-5}{2} + p + r\right) = n \cdot dim_s(H_{t,p,r}).$$

Proof. Let V be the vertex set of G and let $H_{t,p,r} \in \mathcal{F}$ with set of vertices $W \cup X \cup Y \cup Z$, where $W = \{w_1, w_2, ..., w_p\}$, $X = \{x_1, x_2, ..., x_t\}$, $Y = \{y\}$ and $Z = \{z_1, z_2, ..., z_r\}$. Since every vertex $u \in W \cup Z$ is maximally distant from v, by Lemma 8.1, we have that every two different vertices $(x, y), (x', y') \in V \times (W \cup Z)$, $x \neq x'$, are mutually maximally distant. Moreover, by Lemma 8.2 for every two mutually maximally distant vertices v_i, v_j in $H_{t,p,r}$ we have that $(u, v_i), (u, v_j)$ are mutually maximally distant in $G \circ_v H_{t,p,r}$ for every vertex u of G. Thus, $(G \circ_v H_{t,p,r})_{SR}$ is formed by $n\frac{t-5}{2} + 1$ connected components, *i.e.*, $n\frac{t-5}{2}$ connected components isomorphic to K_2 and one connected component isomorphic to a graph G_1 with set of vertices $V \times (W \cup Z \cup \{x_1, x_{t-1})\}$ where $\langle V \times (W \cup Z)\rangle$ is isomorphic to $K_{n|W \cup Z|}$ and for every $u \in V$, (u, x_1) is adjacent to every vertex (u, z_j), $j \in \{1, 2, ..., r\}$, and (u, x_{t-1}) is adjacent to every vertex (u, w_i), $i \in \{1, 2, ..., p\}$. Since in G_1 every vertex of $\langle V \times (W \cup Z)\rangle$ has a neighbor not belonging to $V \times (W \cup Z)$ we have that $\beta(G_1) = n|W \cup Z|$.

Therefore, we obtain that

$$\begin{aligned} dim_s(G \circ_v H_{t,p,r}) &= \beta((G \circ_v H_{t,p,r})_{SR}) \\ &= n|W \cup Z| + n\frac{t-5}{2} \\ &= n\left(\frac{t-5}{2} + p + r\right). \end{aligned}$$

□

According to the Remark 8.17 we have that for every graph $H \in \mathcal{F}$ and any connected graph G of order n, $dim_s(G \circ_v H) = n \cdot dim_s(H)$ where v is the vertex of degree one not belonging to any strong metric basis of the graph H.

Proposition 8.18. *Let G be a connected graph of order $n \geq 2$ and let v be a vertex of a graph H. If v does not belong to the boundary of H and there exists a vertex different from v, of degree one in H, not belonging to any strong metric basis of H, then*

$$dim_s(G \circ_v H) \geq n(dim_s(H) + 1) - 1.$$

Proof. Let w be a vertex of degree one in H not belonging to any strong metric basis of H. Notice that the vertices of the set $A = \{(u_i, w) : i \in \{1, 2, ..., n\}\}$ are also vertices of degree one in $G \circ_v H$. Thus, they are simplicial vertices and from Lemma 2.14 we have that at least all but one vertices of A belongs to every strong metric basis of $G \circ_v H$. Thus,

$$\begin{aligned} dim_s(G \circ_v H) &= \beta((G \circ_v H)_{SR}) \\ &\geq n\,\beta(\langle \partial(H) \rangle) + |A| - 1 \\ &= n\,\beta(H_{SR}) + n - 1 \\ &= n(dim_s(H) + 1) - 1. \end{aligned}$$

□

As the following remark shows, the bound above is tight.

Remark 8.19. *Let G be a connected graph of order n. Let v be the vertex of the graph $H_{t,p,r} \in \mathcal{F}$ adjacent to the vertex of degree one not belonging to any strong metric basis of $H_{t,p,r}$. Then*

$$dim_s(G \circ_v H_{t,p,r}) = n\left(\frac{t-5}{2} + p + r + 1\right) - 1 = n(dim_s(H_{t,p,r}) + 1) - 1.$$

Proof. Let V be the vertex set of G. Now, according to the construction of the family $\mathcal{F}$, let the graph $H_{t,p,r}$ with set of vertices $W \cup X \cup Y \cup Z$, where $W = \{w_1, w_2, ..., w_p\}$, $X = \{x_1, x_2, ..., x_t\}$, $Y = \{y\}$ and $Z = \{z_1, z_2, ..., z_r\}$. Since every vertex $y \in W \cup Y \cup Z$ is maximally distant from v, by Lemma 8.1, we have that every two different vertices $(x, y), (x', y') \in V \times (W \cup Y \cup Z)$, $x \neq x'$, are mutually maximally distant. Moreover, by Lemma 8.2 for every two mutually maximally distant vertices v_i, v_j in $H_{t,p,r}$ we have that $(u, v_i), (u, v_j)$ are mutually maximally distant in $G \circ_v H_{t,p,r}$ for every vertex u of G. Thus, $(G \circ_v H_{t,p,r})_{SR}$ is formed by $n\frac{t-5}{2} + 1$ connected components, that is, $n\frac{t-5}{2}$ connected components isomorphic to K_2 and one connected component isomorphic to a graph G_1 with set of vertices $V \times (W \cup Y \cup Z \cup \{x_1, x_{t-1})\}$ where $\langle V \times (W \cup Y \cup Z)\rangle$ is isomorphic to $K_{n|W\cup Y \cup Z|}$ and for every $u \in V$, (u, x_1) is adjacent to every vertex (u, z_j), $j \in \{1, 2, ..., r\}$, and (u, x_{t-1}) is adjacent to every vertex (u, w_i), $i \in \{1, 2, ..., p\}$. Notice that $\beta(G_1) = n|W \cup Y \cup Z| - 1$. Therefore, we obtain that

$$\begin{aligned} dim_s(G \circ_v H_{t,p,r}) &= \beta((G \circ_v H_{t,p,r})_{SR}) \\ &= n|W \cup Y \cup Z| - 1 + n\frac{t-5}{2} \\ &= n\left(\frac{t-5}{2} + p + r + 1\right) - 1. \end{aligned}$$

□

Conclusion

In this thesis we study the strong metric dimension of product graphs. The central results of the thesis are focused on finding relationships between the strong metric dimension of product graphs and that of its factors together with other invariants of these factors. We have studied the following products: Cartesian product graphs, direct product graphs, strong product graphs, lexicographic product graphs, corona product graphs, join graphs, Cartesian sum graphs, and rooted product graphs, from now on "product graphs".

We have obtained closed formulaes for the strong metric dimension of several nontrivial families of product graphs involving, for instance, bipartite graphs, vertex-transitive graphs, Hamiltonian graphs, trees, cycles, complete graphs, *etc.*, or we have given general lower and upper bounds, and have expressed these in terms of invariants of the factor graphs like, for example, order, independence number, vertex cover number, matching number, algebraic connectivity, clique number, and twin-free clique number. We have also described some classes of product graphs where these bounds are achieved.

Oellermann and Peters-Fransen [82] showed that the problem of finding the strong metric dimension of a connected graph can be transformed to the problem of finding the vertex cover number of its strong resolving graph. In the thesis we have strongly exploited this tool. We have found several relationships between the strong resolving graph of product graphs and that of its factor graphs. For instance, it is remarkable that the strong resolving graph of the Cartesian product of two graphs is isomorphic to the direct product of the strong resolving graphs of its factors.

In addition to the strong resolving graphs, for some product graphs we have also developed a transformation of the problem of computing the strong metric dimension of these product graphs to the problem of finding the clique number or twin-free clique number of its factor graphs.

Contributions of the thesis

The results presented in this work led to elaborate several papers, which have been either published or submitted to ISI-JCR journals. Furthermore, some of the principal results have been presented in international conferences or in recognized foreign seminaries.

Publications in ISI-JCR journals

- J. A. Rodríguez-Velázquez, I. G. Yero, D. Kuziak and O. R. Oellermann, On the strong metric dimension of Cartesian and direct products of graphs, *Discrete Mathematics* **335** (2014) 8–19.
- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of the strong products of graphs, *Open Mathematics* (formerly Central European Journal of Mathematics) **13** (2015) 64–74.
- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of corona product graphs and join graphs, *Discrete Applied Mathematics* **161** (7–8) (2013) 1022–1027.

Publications in conference proceedings

- D. Kuziak, J. A. Rodríguez-Velázquez, I. G. Yero, On the strong metric dimension of product graphs, Proceedings of "IX Jornadas de Matemática Discreta y Algorítmica". *Electronic Notes in Discrete Mathematics* **46** (0) (2014) 169–176.
- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Resolvability in rooted product graphs, Proceedings of "VIII Encuentro Andaluz de Matemática Discreta". *Avances en Matemática Discreta en Andalucía*, Edited by: M. Cera López, P. García Vázquez, R. Moreno Casablanca, and J. C. Valenzuela Tripodoro, vol. III (2013) 197–204. ISBN: 978-84-15881-46-9.

Papers submitted to journals

- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of Cartesian sum graphs. Submitted to Fundamenta Informaticae (2014).

- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Closed formulae for the strong metric dimension of lexicographic product graphs. Submitted to Electronic Journal of Combinatorics (2014).

- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Strong metric dimension of rooted product graphs. Submitted to International Journal of Computer Mathematics (2013).

Participations in specialized conferences

- D. Kuziak, J. A. Rodríguez-Velázquez, I. G. Yero, On the strong metric dimension of product graphs, IX Jornadas de Matemática Discreta y Algorítmica, Tarragona, Spain (2014).

- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Resolvability in rooted product graphs, VIII Encuentro Andaluz de Matemática Discreta, Sevilla, Spain (2013).

- D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, Strong metric generators of Cartesian product graphs, Gdańsk Workshop on Graph Theory, Gdańsk, Poland (2013).

- I. G. Yero, D. Kuziak, J. A. Rodríguez-Velázquez, Resolvability of corona graphs, Colourings, Independence and Domination: 14^{th} Workshop on Graph Theory, Szklarska Poręba, Poland (2011).

Talks in seminaries

- Strong resolving sets for products of graphs. Seminary of the Mathematics, Physics and Informatics Department, University of Gdańsk, Gdańsk, Poland (December 19^{th}, 2013).

- On the strong metric dimension of Cartesian and direct products of graphs. Seminary of the Mathematics and Computer Science Department, University of Maribor, Maribor, Slovenia (June 3^{rd}, 2013).

Future works

- The strong resolving graph of a graph.

  It can be noticed the very important role that plays the strong resolving graph of a graph into computing its strong metric dimension. According to this interesting usefulness of the strong resolving graph we propose to describe the strong resolving graph of other (some) families of graphs. This problem was already mentioned (but not remarked) in the article [82], where was open the question of characterizing the class of all graphs having a strong resolving graph isomorphic to a bipartite graph. The motivation for this question is related to the fact that, in this case, the vertex cover number can be computed in polynomial time and, in concordance with Theorem 1.11, also the strong metric dimension. Moreover, is there another interesting application of the strong resolving graph?

- Strong metric dimension in product graphs.

  We have studied the strong metric dimension of Cartesian sum graphs $G \oplus H$ for all the possibilities but the situation (and the equivalent one, according to the commutativity of this product) in which G has an isolated vertex, H has no isolated vertices and $D(H) > 2$. Also, it remains to study the strong metric dimension of lexicographic product graphs $G \circ H$ when G is any connected graph and H is a nontrivial graph having maximum degree equal to its order minus one. For the case of the direct product of graphs, only specific families of graphs have been studied. In this sense, it would be desirable to obtain some relationships for the strong metric dimension of general direct product graphs.

- Metric dimension related parameters in product graphs.

  As we mention in Introduction, there are several variations of metric generators. Not all of them have been studied in product graphs. Our objective is to obtain mathematical properties of other variations of metric generators in product graphs.

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Symbol Index

The symbols are arranged in the order of the first appearance in the work. Page numbers refer to definitions.

$P_G(S)$	projection of the set S onto the factor G of a product graph, 35
$CR(n, M)$	circulant graph of order n, 39
$G' \sqsubseteq G$	G' as a subgraph of G, 51
G^*	graph obtained from $V(G)$, where two vertices u, v are adjacent if and only if either $d_G(u, v) \geq 2$ or u, v are true twins, 60
G_-	graph obtained from G by removing all its isolated vertices, 60
G^*_-	graph obtained from G^* by removing all its isolated vertices, 60
$\partial_{TF}(G)$	TF-boundary of the graph G, 65
G_{SRS}	strong resolving TF-graph of G, 65
μ	algebraic connectivity, 76.
$F_{1,n}$	fan graph of order $n+1$, 85
$W_{1,n}$	wheel graph order $n+1$, 85