DOCTORAL THESIS

ALEJANDRO ESTRADA MORENO

ON THE (k, t)-METRIC DIMENSION OF A GRAPH

UNIVERSITAT ROVIRA I VIRGILI

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I STATE that the present study, entitled "On the (k,t)-metric dimension of a graph", presented by Alejandro Estrada Moreno for the award of the degree of Doctor, has been carried out under our supervision at the Department of Computer Engineering and Mathematics of this university.

Tarragona, March 29th, 2016

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Contents

Introduction

In everyday life we have several situations where the concept of distance is present. Suppose, for instance, that we wish to travel from Barcelona city to Madrid city. Thus, we may be interested in one or more of the following numbers.

1) The straight line distance, in kilometres, from Barcelona to Madrid.

2) The distance, in kilometres, from Barcelona to Madrid by road.

3) The time, in minutes, of the shortest journey from Barcelona to Madrid by train or bus.

4) The cost, in euros, of the cheapest journey from Barcelona to Madrid by train or bus.

Each of these numbers is of interest to someone and none of them is easily obtained from another. However, they do share some similar properties which can be used to establish a common terminology for altogether. Given a set X and a function $d : X \times X \to \mathbb{R}$ with the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

we say that d is a *metric* on X and that (X, d) is a *metric space*. Given a subset $M \subset X$, for any point $x \in X$, its metric M*-representation* is the set $\{(m, d(x, m)) : m \in M\}$ of its metric M*-coordinates* $(m, d(x, m))$. The set M was called by Blumenthal in [11], a *metric generator* for X, if distinct points $x \in X$ have distinct M-representations. A metric generator of minimum cardinality is called a *metric basis* of (X, d), and its cardinality the *metric dimension* of (X, d). For instance, it is shown in [9] that if U is any non-empty open subset of any one of the three classical n-dimensional geometries of constant curvature, namely Euclidean space $\mathbb{R}^n$, Spherical space $\mathbb{S}^n$ and Hyperbolic space $\mathbb{H}^n$, then $\dim(U) = n + 1$.

From now on we consider a simple and connected graph $G = (V, E)$ and the function $d_G : V \times V \to \mathbb{N} \cup \{0\}$, where $d_G(x, y)$ is the length of a shortest path between x and y in G, and $\mathbb{N}$ is the set of positive integers. Obviously (V, d_G) is a metric space, since d_G is a metric on V. On the other hand, since graph structures may be used to model computer networks, social networks, molecules or any structure which depends on objects and their relationships, it is therefore an interesting problem to study the metric space (V, d_G). Objects can be represented as vertices in a graph and edges could represent their relationships. For instance, the problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension of (V, d_G) by Slater in [116, 117], where the metric generators were called locating sets. The concept of metric dimension of (V, d_G) was also introduced independently by Harary and Melter in [59], where metric generators were called resolving sets. Moreover, the terminology of metric generators for the case of graphs was recently introduced by Sebö and Tannier in [113].

Given a positive integer t, we define the following function $d_{G,t} : V \times V \to \mathbb{N} \cup \{0\}$, where

$$d_{G,t}(x, y) = \min\{d_G(x, y), t\}. \tag{1}$$

Since $d_{G,t}$ is a metric on V, we also have that $(V, d_{G,t})$ is a metric space. Note that if t is at least the diameter of G, then the metric $d_{G,t}$ is equivalent to d_G. Any metric generator for $(V, d_{G,t})$ is a metric generator for $(V, d_{G,t+1})$ and, as a consequence, the metric dimension of $(V, d_{G,t+1})$ is at most the metric dimension of $(V, d_{G,t})$. In particular, the metric dimension of $(V, d_{G,1})$ is equal to $|V| - 1$, and as a consequence, it only deserves to study the metric dimension of $(V, d_{G,t})$ for $t \geq 2$. Notice that while using the metric $d_{G,t}$, the concept of metric generator needs not be restricted to the case of connected graphs, as for any pair of vertices x, y belonging to different connected components of G we can assume that $d_G(x, y) = +\infty > t$ and so $d_{G,t}(x, y) = t$.

A vertex $v \in V$ is said to *distinguish* two different vertices x and y, if $d_{G,t}(v,x) \neq d_{G,t}(v,y)$. In this sense, a set $S \subset V$ is a metric generator for $(V, d_{G,t})$ if every pair of different vertices of G is distinguished by some element of S.

Despite the undeniable usefulness of a metric generator for $(V, d_{G,t})$, in its primary version, it has a weakness related with the possible uniqueness of the vertex identifying a pair of different vertices of the graph. Consider, for instance, some robots which are navigating, moving from node to node of a network. On a graph, however, there is neither the concept of direction nor that of visibility. We assume that robots have communication with a set of landmarks S (a subset of nodes) which provides them the distance to the landmarks in order to facilitate the navigation. Assume that all landmarks have a transmission range t. As a consequence, robots only have communication with those landmarks in S which are at distance at most t from them. Our aim is that the landmarks would uniquely determine the robot's position on the graph. A minimum set of landmarks which uniquely determines the robot's position is a metric basis of $(V, d_{G,t})$, and the minimum number of landmarks is the metric dimension of $(V, d_{G,t})$. Suppose that in a specific moment there are two robots x, y whose positions are only distinguished by one landmark $s \in S$. If the communication between x and s is unexpectedly blocked, then the robot x will get lost since it could assume to have the position of y. So, for more realistic settings it may be desirable to consider a set of landmarks where each pair of nodes is distinguished by at least k landmarks, for some $k \geq 2$.

A natural solution regarding that weakness is the location of one landmark in every node of the graph. But, such a solution, would have a very high cost. Thus, the choice of a correct set of landmarks is convenient for a satisfiable performance of the navigation system. That is, in order to achieve a reasonable efficiency, it would be convenient to have a set of as few landmarks as possible, always having the guarantee that every object of the network will be properly distinguished. In this sense, we introduce the concept of (k,t)-metric generator for $(V, d_{G,t})$, which is a natural extension of the concept of metric generator. A set $S \subseteq V$ is said to be a (k,t)-*metric generator* for a graph G if and only if any pair of different vertices of G is distinguished by at least k elements of S, *i.e.*, for any pair of different vertices

$u, v \in V$, there exist at least k distinct vertices $w_1, w_2, \ldots, w_k \in S$ such that

$$d_{G,t}(u, w_i) \neq d_{G,t}(v, w_i), \text{ for every } i \in \{1, \ldots, k\}. \tag{2}$$

A (k,t)-metric generator of minimum cardinality in G is called a (k,t)-*metric basis* and its cardinality the (k,t)-*metric dimension* of G, which is denoted by $\dim_{k,t}(G)$.

As an example, we take a graph G obtained from the cycle graph C_5 and the path P_r of order $r \geq 2$, by identifying one of the vertices of the cycle, say u_1, and one of the extremes of P_r, as we show in Figure 1. Let $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_2, u_r\}$, $S_3 = \{v_1, v_2, v_3, u_r\}$ and $S_4 = \{v_1, v_2, v_3, v_4, u_r\}$. For $k \in \{1, 2, 3, 4\}$ and t at least the diameter of G, the set S_k is a (k,t)-metric basis of G.

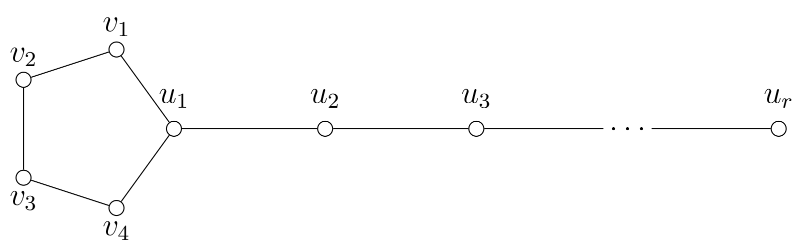

Figure 1: For $k \in \{1, 2, 3, 4\}$ and t at least the diameter of G, $\dim_{k,t}(G) = k + 1$.

Note that the $(2,1)$-metric dimension of any graph is equal to $|V|$ and that there are no $(k,1)$-metric bases for $k \geq 3$. Thus, from now on we assume that $t \geq 2$. It can also be noted that every (k,t)-metric generator S satisfies that $|S| \geq k$ and, if $k > 1$, then S is also a $(k-1,t)$-metric generator. Moreover, $(1,t)$-metric generators, where t is at least the diameter of G, are the standard metric generators (resolving sets or locating sets as defined in [59] or [116], respectively). Notice that if $k = 1$, then the problem of checking if a set S is a metric generator reduces to check condition (2) only for those vertices $u, v \in V - S$, as every vertex in S is distinguished at least by itself. Also, if $k = 2$, then condition (2) must be checked only for those pairs having at most one vertex in S, since two vertices of S are distinguished at least by themselves. Nevertheless, if $k \geq 3$, then condition (2) must be checked for every pair of different vertices of the graph.

We also must remark that a vertex $v \in V(G)$ distinguish two different vertices $x, y \in V(G)$ with regard to the distance $d_{G,2}$, if v is exactly adjacent to one of x, y. In some sense, one could say that v distinguishes x, y in connection with the neighbourhood of v (or according to adjacencies between these vertices). Therefore, from now on we shall refer to the $(k, 2)$-metric generators (bases) as *k-adjacency generators (bases)*, and the $(k, 2)$-metric dimension shall be called *k-adjacency dimension* and shall be denoted by $\mathrm{adim}_k(G)$. On the other hand, the (k, t)-metric generators (bases) of a graph G, where t is greater than or equal to the diameter of G, had been previously referred to as *k-metric generators (bases)*, and its (k, t)-metric dimension had been called *k-metric dimension* and had been denoted by $\dim_k(G)$. As a consequence, we keep this nomenclature and notation. In this thesis we focus on the study of the k-adjacency dimension and the k-metric dimension of graphs.

The literature about the 1-metric generators for graphs shows its highly significant potential to be used for solving a representative number of real life problems, which has been described in several works. For instance, some applications to the navigation of robots in networks are discussed in [71, 72, 89, 90, 114]; to chemistry in [21, 22, 68, 69, 75]; to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures, in [93]; to multiprocessor interconnection networks in [91] and to the network discovery(verification) problem in [10]. In addition, interesting connections with the Mastermind game were presented in [16, 25, 51, 52, 55, 70], throughout the development of an strategy for such game which precisely needs the uniquely recognition of some "elements" of the game. Finally, 1-metric generators have been also used in studies concerning some coin weighing problems in [4, 15, 16, 17, 18, 31, 46, 56, 113]. There could probably exist some other possible applications of 1-metric generators to some real problems. However, to the best of our knowledge, we have collected here the main part of them. Moreover, this invariant was further theoretically studied in a number of other papers including [6, 16, 20, 21, 24, 37, 38, 61, 96, 111, 118, 132, 134].

The study of this invariant in graphs is more interesting if we consider that the problem of finding the 1-metric dimension of graphs is NP-hard [72], even when restricted to planar graphs [27]. However, there exist a linear-time and a polynomial-time algorithm for determining the 1-metric dimension for trees [72] and outerplanar graphs [27], respectively. For these reasons, many efforts have been made to computationally solve the problem of finding a 1-metric generator of a graph in the last few years. For instance, an increasing interest into algorithmic questions on this topic has been raised (see [30, 40, 60] as some examples). In this direction we can also highlight some papers of Kratica *et al.* in [77, 78, 79, 80], where some of them are using some interesting heuristic on genetic algorithms to solve such problem for some families of graphs. We finally suggest to consult the paper [6], which contains an important collection of distinct ways of presenting the 1-metric generators from different points of view that exists nowadays in the literature. This article mentioned above is also an attempt to unifying the terminology associated to this parameter in different areas of mathematics. In this thesis we also try to contribute with this unification.

On the other hand, in order to gain more insight into the metric properties of graphs, several variations of 1-metric generators have been introduced and studied. Such variations has become more or less known and popular in connection to their applicability or according to how much challenging problems they are raising up. Among them we could remark resolving dominating sets [14], independent resolving sets [22], local metric sets [8, 41, 96, 103, 104], strong resolving sets [32, 86, 83, 84, 92, 95, 105, 107, 113, 135], resolving partitions [23, 24, 38, 53, 106, 108], strong resolving partitions [130, 131], simultaneous metric dimension [32, 99, 100, 101] and k-antiresolving sets [119]. About this last one variation, it is maybe interesting to point out that has been applied to generate a privacy measure for social graphs. Besides, the concept of 1-adjacency generator[1] was introduced by Jannesari and Omoomi in [67] as a tool to study the 1-metric dimension of lexicographic product graphs. This concept has been also studied by Fernau and Rodríguez-Velázquez in [41, 42], where it was shown that the 1-metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the 1-adjacency dimension of H. As a consequence of this strong relationship was obtained that the problem of computing the 1-adjacency dimension of graphs is NP-hard. The identifying codes are another variation of 1-metric generators, and they are nothing else than dominating 1-adjacency generators. Little is known about the robustness notions and other variants discussed in connection with wireless networks for identifying codes [43, 44, 45, 64, 87, 102].

[1] 1-adjacency generators were called adjacency resolving sets in [67]

The 1-metric generators has been widely studied in the last decade. In the database of MathSciNet, for instance, we have found 190 papers related to the metric dimension of graphs since 2000 to date. Likewise, in Google Scholar, for the same time interval, we have found 1260 documents about the metric dimension of graphs. According to TDX (Theses and Dissertations Online), a digital cooperative repository of doctoral theses presented at some Spanish universities, there are 59 doctoral theses that have been defended since 2000 to date, where the metric dimension of graphs is mentioned. However, extensions of the 1-metric dimension to k-metric dimension for any $k \geq 2$ had not been taken into account before our work would had began. To the best of our knowledge, the only work in this direction was presented in [63], where was weakly studied the case $k = 2$ through the study of some parameter they called fault tolerant metric dimension. More recently, and in parallel with this work, a similar approach to k-metric dimension has appeared in [1, 2], although the direction of such works are more going in the algorithmic and computer science sense. Considering all these previous facts, it is of high importance the study of the (k,t)-metric dimension of graphs.

The thesis is organized as follows. In the first chapter, we recall some basic definitions on graph theory. The rest of the chapters are focused on the (k,t)-metric dimension of graphs with special emphasis on the k-metric dimension and the k-adjacency dimension. Chapter 2 is focused on finding the largest integer k such that there exists a (k,t)-metric basis for a given graph. Chapter 3 deals with finding formulae and bounds for the (k,t)-metric dimension of some graphs. In Chapter 4 we study some complexity issues concerning the problems of computing the largest integer k such that there exists a (k,t)-metric basis for a graph, as well as computing the k-metric dimension of a graph. We conclude the work with highlights of the principal studied issues, contributions of the thesis, and future works.

Chapter 1

Basic concepts and tools

1.1 Basic concepts and notations

We begin by establishing the basic terminology and notations which is used throughout the thesis. For the sake of completeness we refer the reader to the books [28, 126]. Graphs considered herein are undirected, finite and contain neither loops nor multiple edges. From now on G represents a graph with vertex set $V(G)$, edge set $E(G)$, and order $n = |V(G)|$. A graph is nontrivial if $n \geq 2$. If two graphs G and H are isomorphic, then we say that $G \cong H$. We use the notation $u \sim v$ for two adjacent vertices u and v of G. For a vertex v of G, $N_G(v)$ denotes the set of neighbours that v has in G, *i.e.*, $N_G(v) = \{u \in V(G) : \ u \sim v\}$. The set $N_G(v)$ is called the *open neighbourhood of a vertex* v in G and $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighbourhood of a vertex* v in G. The *degree* of a vertex v of G is denoted by $\delta_G(v)$, *i.e.*, $\delta_G(v) = |N_G(v)|$. The *open neighbourhood of a set* S of vertices of G is $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the *closed neighbourhood of* S is $N_G[S] = N_G(S) \cup S$. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *girth* $\mathsf{g}(G)$ of G is the length of a shortest cycle contained in G. A set S is a dominating set in G if every vertex not in S is adjacent to a vertex in S. As usual, we denote by $A \nabla B = (A \cup B) - (A \cap B)$ the *symmetric difference* of two sets A and B.

We use the notation K_n, C_n, P_n, and N_n for the *complete graph*, *cycle graph*, *path graph*, and *empty graph*, respectively. Moreover, we write $K_{s,t}$ for the *complete bipartite graph* of order $s+t$ and, particularly, $K_{1,n}$ for the *star graph* of order $n+1$. An *end vertex* is a vertex of degree one while a *support vertex* is a vertex adjacent to an end vertex. Let T be a *tree*, an end vertex in T is called a *leaf*.

The *diameter*, $D(G)$, of G is the longest distance between any two vertices in G, *i.e.*, $D(G) = \max_{u,v \in V(G)}\{d_G(u,v)\}$. If G is not connected, then we assume that the distance between any two vertices belonging to different components of G is infinity and, thus, its diameter is $D(G) = \infty$.

The *complement* of a graph G is a graph $\overline{G}$ with vertex set $V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The *subgraph induced by a set* X is denoted by $\langle X \rangle$. A *clique* in G is a set of pairwise adjacent vertices. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique in G. We will say that S is an $\omega(G)$-*clique* if $|S| = \omega(G)$.

Two vertices x, y are called *false twins* if $N(x) = N(y)$, and x, y are called *true twins* if $N[x] = N[y]$. In particular, if G contains more than one isolated vertex, then they are false twin vertices. Figure 1.1 shows examples of basic concepts such as true twins. Two different vertices x, y are *twins* if they are either false twin vertices or true twin vertices. We also say that a vertex x is a twin, if there exists other vertex y such that x, y are twins. In concordance with that, we define the *twin equivalence relation* $\mathcal{R}$ on $V(G)$ as follows:

$$x\mathcal{R}y \longleftrightarrow N_G(x) - \{y\} = N_G(y) - \{x\}.$$

We have three possibilities for each twin equivalence class U:

(a) U is a singleton twin equivalence class, or

(b) U is a false twin equivalence class, *i.e.*, $N_G(x) = N_G(y)$, for any $x, y \in U$ (and case (a) does not apply), or

(c) U is a true twin equivalence class, *i.e.*, $N_G[x] = N_G[y]$, for any $x, y \in U$ (and case (a) does not apply).

If all twin equivalence classes of a graph G are singletons, then we say that G is a *twins free graph.* If G does not have any true (false) twin equivalence class, then we say that G is a *true* (*false*) *twins free graph.* Given a vertex $x \in V(G)$, we denote the *true twin equivalence class* to which x belongs by $TT(x)$, and the *false twin equivalence class* to which x belongs by $FT(x)$. We also denote by $S(G)$, $FT(G)$ and $TT(G)$ the union of the singletons, the false, and the true twin equivalence classes of a graph G, respectively.

An example of a graph where every equivalence class is a true twin equivalence class is $K_r + (K_s \cup K_t)$, $r, s, t \geq 2$ (see Subsection 1.2.1 and

Section 1.2 for the concepts of join graph $G + H$ and disjoint union $G \cup H$, respectively). In this case, there are three equivalence classes composed by r, s and t true twin vertices, respectively. As an example where no class is composed by true twin vertices we take the complete bipartite graph $K_{r,s}$, $r, s \geq 2$. Finally, the graph $K_r + N_s$, $r, s \geq 2$, has two equivalence classes and one of them is composed by r true twin vertices. On the other hand, $K_1 + (K_r \cup N_s)$, $r, s \geq 2$, is an example where one class is a singleton, one class is composed by true twin vertices and the other one is composed by false twin vertices.

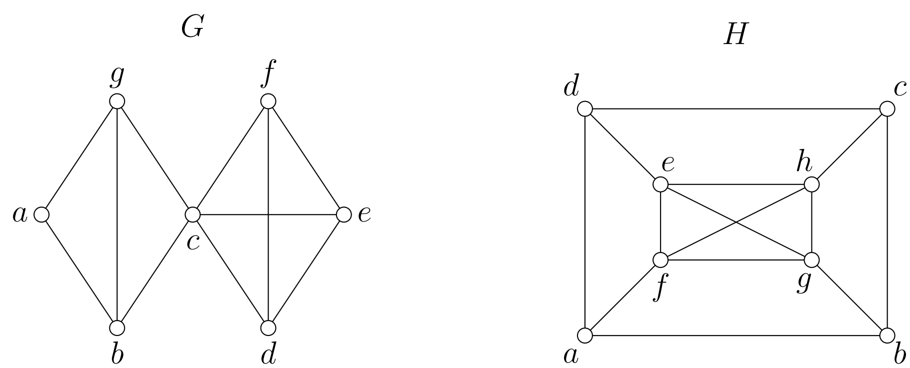

Figure 1.1: The set $\{d, e, f\} \subset V(G)$ is composed by true twin vertices in G. Notice that b and g are true twin vertices in G and f and d are also true twins. The set $\{e, f, g, h\} \subset V(H)$ is a twin-free clique in H.

A graph G is 2-*antipodal* if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x, y) = D(G)$. For example even cycles are 2-antipodal graphs.

Other remaining definitions not defined herein are given the first time that the concept appears in the text.

1.2 Graph operations

This section is a brief overview on some graphs operations. The *union* $G \cup H$ of two graphs G and H with disjoint vertex sets V_1 and V_2, respectively, and edge sets E_1 and E_2, respectively, is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. This operation is sometimes also known explicitly as the graph disjoint union.

A product of graphs is an operation made with two or more graphs in order to generate another graph. The graphs used in the operation are called the factors of the product. In the last few years a rich theory involving the structure and recognition of classes of product graphs has emerged [57]. The most studied graph products are the Cartesian product, the strong product, the direct product and the lexicographic product, which are also called standard products. However, there are also several other non standard styles of operations with graphs, which have been intensively studied. In this work we center our attention in one case of the standard products, and other case of the non standard ones. Specifically, we focus on the lexicographic product and the corona product of graphs. Even so, here we present two standard products that we also use in this thesis, the so called Cartesian product and strong product of graphs.

1.2.1 Lexicographic product

The *lexicographic product* of two graphs G and H is the graph $G \circ H$ with vertex set $V(G \circ H) = V(G) \times V(H)$ and two vertices (a, b) and (c, d) are adjacent in $G \circ H$ if either

- $ac \in E(G)$, or
- $a = c$ and $bd \in E(H)$.

In the literature we can also find the names *composition* or *substitution* for the lexicographic product. This product is clearly not commutative, while it is associative [57, 65]. Figure 1.2 illustrates two examples of lexicographic products and, at the same time, emphasizes the fact that the lexicographic product is not commutative.

A lexicographic product $G \circ H$ is connected if and only if G is connected. The relation between distances in the lexicographic product of graphs and in its factors is presented in the following remark, for which is necessary to recall (1).

Remark 1.1. [57, 65] *If (a, b) and (c, d) are vertices of $G \circ H$, then*

$$d_{G\circ H}((a,b),(c,d)) = \begin{cases} d_G(a,c), & \text{if } a \neq c, \\ d_H(b,d), & \text{if } a = c \text{ and } \delta_G(a) = 0, \\ d_{H,2}(b,d), & \text{if } a = c \text{ and } \delta_G(a) \neq 0. \end{cases}$$

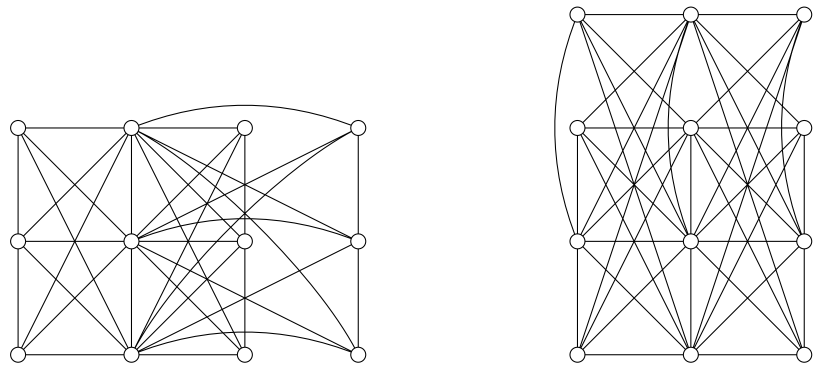

Figure 1.2: Two lexicographic product graphs: $K_{1,3} \circ P_3$ and $P_3 \circ K_{1,3}$.

We now define an extended lexicographic product between a graph G of order n and a family composed by n graphs. The *lexicographic product* of a graph G of order n and a family composed by graphs $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$, which is denoted by $G \circ \mathcal{H}$, is the graph with vertex set $\bigcup_{v_i \in V(G)} \{v_i\} \times V(H_i)$, where (a, v) is adjacent to (b, w) whenever

- $ab \in E(G)$, or
- $a = b$ and $vw \in E(H_i)$ for every $H_i \in \mathcal{H}$.

Note that this approach of the lexicographic product is a natural generalization of the standard lexicographic product of graphs, and therefore of its properties too. For instance, Remark 1.1 holds. Figure 1.3 shows the lexicographic product between P_3 and a family composed by $\{P_4, K_2, P_3\}$.

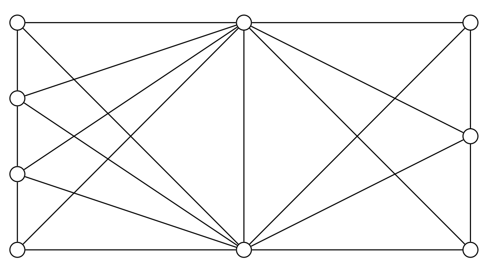

Figure 1.3: The lexicographic product graph $P_3 \circ \{P_4, K_2, P_3\}$.

If every $H_i \in \mathcal{H}$ holds that $H_i \cong H$, then we use the notation $G \circ H$ (as in the standard case) instead of $G \circ \mathcal{H}$ and we refer to H_i as the ith copy of H. In general, we can construct the graph $G \circ \mathcal{H}$ by taking one copy of each $H_i \in \mathcal{H}$ and, for every $u_i u_j \in E(G)$, we join by an edge every vertex of H_i

with every vertex of H_j. Note that $G \circ \mathcal{H}$ is connected if and only if G is connected.

A particular case of lexicographic product is the join. The *join graph* $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H [58, 136]. Note that $G + H \cong K_2 \circ \{G, H\}$. It is a commutative and associative operation. Now, for the sake of completeness, Figure 1.4 illustrates two examples of join graphs.

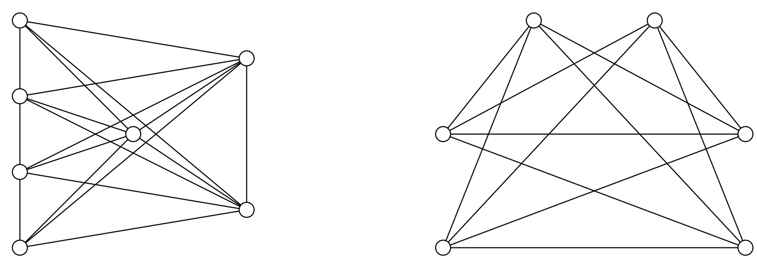

Figure 1.4: Two join graphs: $P_4 + C_3$ and $N_2 + N_2 + N_2$.

Moreover, complete k-partite graphs are typical examples of join graphs. A *complete k-partite graph* $K_{p_1,p_2,\ldots,p_k}$ is the join graph of empty graphs on $p_1, p_2, \ldots, p_k$ vertices. Notice that $N_2 + N_2 + N_2$, illustrated in Figure 1.4, is none other than the complete 3-partite graph $K_{2,2,2}$.

The lexicographic product has been studied from different points of view in the literature. One of the most common researches focuses on finding relationships between the value of some invariant in the product and that of its factors. In this sense, we can find in the literature a large number of investigations on diverse topics, like for instance, independence number [3, 50], domination number [3, 88, 94, 115], chromatic number [3, 26, 50, 73, 98], connectivity [128], hamiltonicity [7, 81], and metric dimension [39, 67, 85, 112]. For more information on the research on product graphs we suggest the books [57, 65].

1.2.2 Corona product graphs

Let G and H be two graphs of order n_1 and n_2, respectively. The *corona product* $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n_1 copies of H and joining by an edge each vertex from the ith copy of H with the ith vertex of G. We denote by $V = \{v_1, v_2, \ldots, v_n\}$

the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $v_i \sim v$ for every $v \in V_i$. Notice that the corona product $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$.

Observe that $G \odot H$ is connected if and only if G is connected. Moreover, it is readily seen from the definition that this product is neither an associative nor a commutative operation. Figure 1.5 shows some examples of corona products and also underscores the fact that the corona product is not commutative.

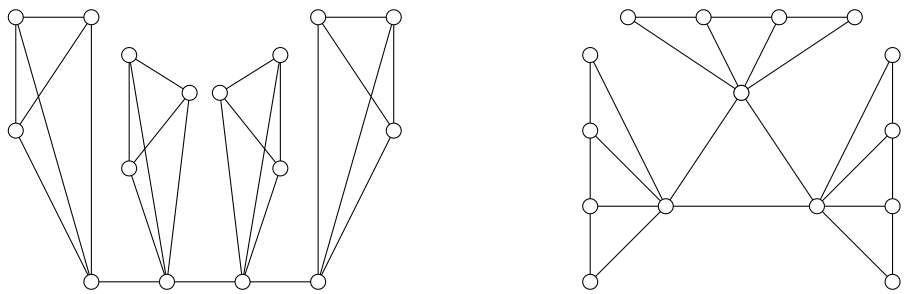

Figure 1.5: The corona product graphs $P_4 \odot C_3$ and $C_3 \odot P_4$.

The concept of corona product of two graphs was first introduced by Frucht and Harary [47]. This product is not too much popular and has not been widely investigated. One of the reasons could be that the corona product is a simple operation on two graphs and some mathematical properties could be directly consequences of its factors. Surprisingly, for the case of the k-metric dimension this is not the situation, which makes interesting its study for this non standard product. Moreover, there are a few remarkable studies on corona products, like for instance on some topological indices [97, 129], the chromatic number [48, 54, 133], the domination number [54], the toughness [19], and the metric dimension [8, 39, 41, 42, 54, 66, 83, 106, 132].

We also define an extended corona product between a graph G of order n and a family composed by n graphs. Let G be a graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family of graphs. The *corona product* graph $G \odot \mathcal{H}$ is defined as the graph obtained from G and $\mathcal{H}$ by taking one copy of G and joining by an edge each vertex of H_i with the ith vertex of G, [47]. In particular, if every $H_i \in \mathcal{H}$ holds that $H_i \cong H$, then we use the standard notation $G \odot H$ instead of $G \odot \mathcal{H}$. Note that $G \odot \mathcal{H}$ is also connected if and only if G is connected.

1.2.3 Cartesian product graphs

The *Cartesian product graph* $G \Box H$, of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, is the graph whose vertex set is $V(G \Box H) = V_1 \times V_2$ and any two distinct vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent in $G \Box H$ if and only if either:

(a) $x_1 = y_1$ and $x_2 \sim y_2$, or

(b) $x_1 \sim y_1$ and $x_2 = y_2$.

The Cartesian product is a straightforward and natural construction, and is in many respects the simplest graph product [57, 65]. Hypercubes, Hamming graphs and grid graphs are some particular cases of this product. The *Hamming graph* $H_{k,n}$ is the Cartesian product of k copies of the complete graph K_n, *i.e.*,

$$H_{k,n} = \underbrace{K_n \Box K_n \Box \ldots \Box K_n}_{k \text{ times}}$$

The *Hypercube* Q_n is defined as $H_{n,2}$. Moreover, the *grid graph* $P_k \Box P_n$ is the Cartesian product of the paths P_k and P_n, the *cylinder graph* $C_k \Box P_n$ is the Cartesian product of the cycle C_k and the path P_n, and the *torus graph* $C_k \Box C_n$ is the Cartesian product of the cycles C_k and C_n. Figure 1.6 shows two examples of Cartesian products.

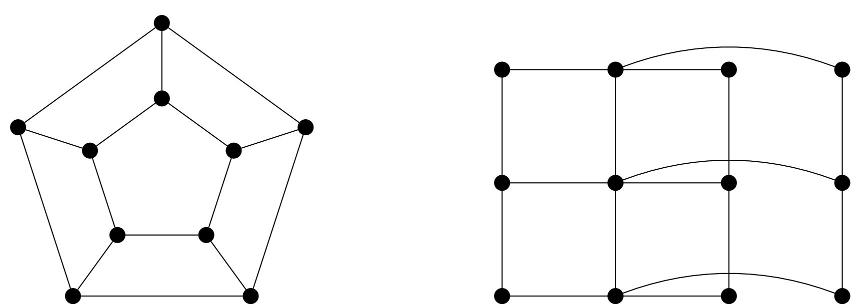

Figure 1.6: Two Cartesian product graphs: $C_5 \Box K_2$ and $K_{1,3} \Box P_3$.

This operation is commutative [57] in the sense that $G \Box H \cong H \Box G$, and is also associative, as the graphs $(F \Box G) \Box H$ and $F \Box (G \Box H)$ are naturally isomorphic. A Cartesian product of graphs is connected if and only if both of its factors are connected.

This product has been extensively investigated from various perspectives. For instance, the most popular open problem in the area of domination theory known as Vizing's conjecture [122]. Vizing suggested that the domination number of the Cartesian product of two graphs is at least as large as the product of domination numbers of its factors. Several researchers have worked on it, for instance, some partial results appears in [12, 57]. Moreover, Vizing [121] has investigated the independence number of Cartesian products. The chromatic number of this product has been completely studied in [110]. The connectivity and the hamiltonian properties of Cartesian products have been described in [123, 127] and [29], respectively. For more information on structure and properties of the Cartesian product of graphs we refer the reader to [57, 65].

1.2.4 Strong product graphs

The *strong product graph* $G \boxtimes H$ of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G \boxtimes H) = V_1 \times V_2$, where two distinct vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent in $G \boxtimes H$ if and only if one of the following holds.

- $x_1 = y_1$ and $x_2 \sim y_2$, or
- $x_1 \sim y_1$ and $x_2 = y_2$, or
- $x_1 \sim y_1$ and $x_2 \sim y_2$.

Other known names for the strong product are the *strong direct product* or the *symmetric composition*. Notice that $G \Box H$ and $G \times H$ are subgraphs of $G \boxtimes H$. Figure 1.7 shows two examples of strong products.

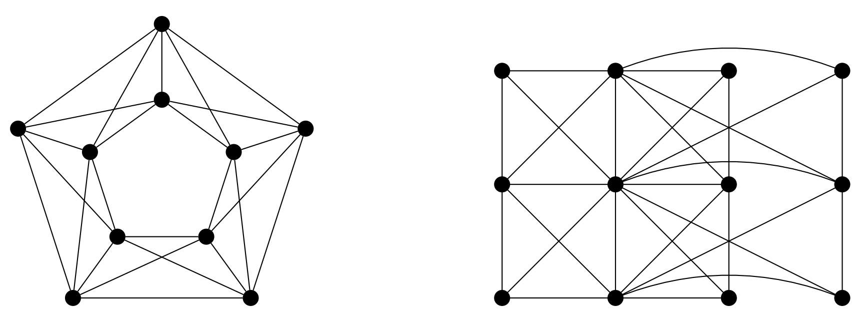

Figure 1.7: Two strong product graphs: $C_5 \boxtimes K_2$ and $K_{1,3} \boxtimes P_3$.

The commutativity of the strong product follows from the symmetry of the definition of adjacency and for associativity see [57, 65]. A strong product of graphs is connected if and only if every one of its factors is connected.

Remark 1.2. [57, 65] *Let G and H be two graphs. For every $u \in V(G)$ and $v \in V(H)$*

$$N_{G\boxtimes H}[(u,v)] = N_G[u] \times N_H[v].$$

As a direct consequence of the remark above the following result is obtained.

Corollary 1.3. *Let G and H be two graphs and let $u, u' \in V(G)$ and $v, v' \in V(H)$. The following assertions hold.*

(i) *If $(u', v') \in N_{G\boxtimes H}(u,v)$, then $u' \in N_G[u]$ and $v' \in N_G[v]$.*

(ii) *If $u' \in N_G(u)$ and $v' \in N_G(v)$, then $(u', v') \in N_{G\boxtimes H}(u,v)$.*

With the strong product is closely connected an important information theoretical parameter, which in general is very difficult to calculate - the Shannon capacity. The Shannon capacity of a graph G is defined as the limit of $\sqrt[k]{\alpha(G^k)}$ when n tends to infinity, and where $\alpha(G)$ denotes the independence number of the graph G and G^k is the strong product of G with itself k times. This problem has been attracted for several researchers and some partial results are presented in [5, 57].

Various properties of strong products have been also studied. The investigation encompasses, for instance, domination [57, 94], chromatic number [74, 120], connectivity [13, 124] and hamiltonian properties [36, 76]. For more information on the strong product we refer the reader to [57, 65].

Chapter 2

On (k, t)-dimensional graphs

Overview

This chapter is concerned with finding the largest integer k such that there exists a (k, t)-metric basis for a graph G. In this sense, we first give general results for any value of $t \geq 2$. Subsequently, we focus on those values of t that are at least the diameter of G and that are equal to two. Particularly, we study the values of k such that there exist k-metric bases for the lexicographic product and the corona product of graphs.

2.1 (k, t)-metric dimensional graphs

Throughout this chapter, unless otherwise stated, we will consider t as an integer greater than one.

It is clear that it is not possible to find a (k, t)-metric generator for a graph for every integer k. That is, given a graph G and the distance $d_{G,t}$, there exists an integer r such that G does not contain any (k, t)-metric generator for every $k > r$. According to that fact, we say that a graph G is (k, t)-*metric dimensional* if k is the largest integer such that there exists a (k, t)-metric basis of G. Notice that if G is a (k, t)-metric dimensional graph, then for each positive integer $r \leq k$, there exists at least one (r, t)-metric basis of G. Given a graph G and two different vertices $x, y \in V(G)$, we denote by $\mathcal{D}_{G,t}(x, y)$ the set of vertices that distinguish the pair x, y with regard to the metric $d_{G,t}$, *i.e.*,

$$\mathcal{D}_{G,t}(x, y) = \{z \in V : \; d_{G,t}(z, x) \neq d_{G,t}(z, y)\}.$$

We also define the set of nontrivial vertices that distinguish the pair x, y as $\mathcal{D}^*_{G,t}(x,y) = \mathcal{D}_{G,t}(x,y) - \{x,y\}$. Note that a set $S \subseteq V(G)$ is a (k,t)-metric generator for G if $|\mathcal{D}_{G,t}(x,y) \cap S| \geq k$ for every two different vertices $x, y \in V(G)$. It can also be noted that two different vertices $x, y \in V(G)$ belong to the same twin equivalence class of G if and only if $\mathcal{D}^*_{G,t}(x,y) = \emptyset$, or equivalently, if $\mathcal{D}_{G,t}(x,y) = \{x,y\}$.

Since for every different vertices $x, y \in V$ we have that $|\mathcal{D}_{G,t}(x,y)| \geq 2$, it follows that the whole vertex set V is a $(2,t)$-metric generator for G and, as a consequence, we deduce that every graph G is (k,t)-metric dimensional for some $k \geq 2$. On the other hand, for any graph G of order $n \geq 3$, there exists at least one vertex $v \in V$ and two vertices $x, y \in V$ such that $\{x,y\} \in N_G(v)$ or $d_{G,t}(x,v) = d_{G,t}(y,v) = t \geq 2$, so $v \notin \mathcal{D}_{G,t}(x,y)$ and, as a result, there is no n-metric dimensional graph of order $n \geq 3$. Comments above are recalled in the next remark.

Remark 2.1. *Let G be a (k,t)-metric dimensional graph of order $n \geq 2$ and let t be an integer greater than one. If $n \geq 3$, then $2 \leq k \leq n-1$. Moreover, G is (n,t)-metric dimensional if and only if $G \cong K_2$ or $G \cong N_2$.*

We define the following parameter

$$\mathcal{D}(G,t) = \min_{x,y \in V}\{|\mathcal{D}_{G,t}(x,y)|\}.$$

Theorem 2.2. *A graph G is (k,t)-metric dimensional if and only if $k = \mathcal{D}(G,t)$.*

Proof. (Necessity) If G is a (k,t)-metric dimensional graph, then for any (k,t)-metric basis B and any pair of different vertices $x, y \in V(G)$, we have $|B \cap \mathcal{D}_{G,t}(x,y)| \geq k$. Thus, $k \leq \mathcal{D}(G,t)$. Now we suppose that $k < \mathcal{D}(G,t)$. In such a case, for every $x', y' \in V(G)$ such that $|B \cap \mathcal{D}_{G,t}(x',y')| = k$, there exists $z_{x'y'} \in \mathcal{D}_{G,t}(x',y') - B$ such that $d_{G,t}(z_{x'y'}, x') \neq d_{G,t}(z_{x'y'}, y')$. Hence, the set

$$B \cup \left(\bigcup_{x',y' \in V(G):\ |B \cap \mathcal{D}_{G,t}(x',y')| = k} \{z_{x'y'}\} \right)$$

is a $(k+1,t)$-metric generator for G, which is a contradiction. Therefore, $k = \mathcal{D}(G,t)$.

(Sufficiency) Let $a,b \in V(G)$ such that $\min_{x,y\in V(G)} |\mathcal{D}_{G,t}(x,y)| = |\mathcal{D}_{G,t}(a,b)| = k$. Since no set $S \subseteq V(G)$ satisfies $|S \cap \mathcal{D}_{G,t}(a,b)| > k$ and $V(G)$ is a (k,t)-metric generator for G, we conclude that G is a (k,t)-metric dimensional graph. □

The characterization proved in Theorem 2.2 is a result on general graphs. We next particularize this for some specific classes of graphs or we bound its possible value in terms of other parameters of the graph. If two vertices u, v of G do not belong to the same twin equivalence class, then u or v, say u, has an adjacent vertex x that is not adjacent to v. Thus, $\{u, v, x\} \subseteq \mathcal{D}_{G,t}(u, v)$, and as a consequence, we deduce the following result.

Corollary 2.3. *A graph G is $(2,t)$-metric dimensional if and only if there are at least two vertices of G belonging to the same twin equivalence class.*

It is clear that P_2 and P_3 are $(2,t)$-metric dimensional. Now, a specific characterization for $(2,t)$-dimensional trees is obtained from Theorem 2.2 (or from Corollary 2.3).

Corollary 2.4. *A tree T of order $n \geq 4$ is $(2,t)$-metric dimensional if and only if T contains a support vertex which is adjacent to at least two leaves.*

An example of a $(2,t)$-metric dimensional tree is the star graph $K_{1,n-1}$, whose $(2,t)$-metric dimension is $\dim_{2,t}(K_{1,n-1}) = n-1$. On the other side, an example of a tree T which is not $(2,t)$-metric dimensional is drawn in Figure 2.1. By Corollary 2.4 and since $|\mathcal{D}_{T,t}(v_1, v_3)| = |\{v_1, v_3, v_5\}| = 3$, we have that T is $(3,t)$-metric dimensional. The set $V(T) - \{v_2\}$ is a $(3,2)$-metric basis and a $(3,3)$-metric basis of T, while $\{v_1, v_3, v_4, v_5\}$ is a $(3,4)$-metric basis, or equivalently, a 3-metric basis of T.

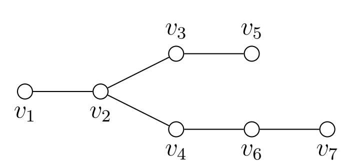

Figure 2.1: An example of $(3,t)$-metric dimensional tree T. The set $V(T) - \{v_2\}$ is a $(3,2)$-metric basis and a $(3,3)$-metric basis of T, while $\{v_1, v_3, v_4, v_5\}$ is a $(3,r)$-metric basis of T for any $r \geq 4$.

A *cut vertex* in a graph is a vertex whose removal increases the number of components of the graph and an *extreme vertex* is a vertex v such that the subgraph induced by $N[v]$ is isomorphic to a complete graph. Also, a *block* is a maximal biconnected subgraph[1] of the graph. Now, let $\mathfrak{F}$ be the family of sequences of connected graphs $G_1, G_2, \ldots, G_r$, $r \geq 2$, such that G_1 is a complete graph K_{n_1}, $n_1 \geq 2$, and G_i, $i \geq 2$, is obtained recursively from G_{i-1} by adding a complete graph K_{n_i}, $n_i \geq 2$, and identifying one vertex of G_{i-1} with one vertex of K_{n_i}.

From now on we say that a connected graph G is a *generalized tree*[2] if and only if there exists a sequence $\{G_1, G_2, \ldots, G_r\} \in \mathfrak{F}$ such that $G_r = G$ for some $r \geq 2$. The 1-metric dimension of these graphs was studied in [82]. Notice that in these generalized trees every vertex is either, a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that, if every K_{n_i} is isomorphic to K_2, then G_r is a tree, justifying the terminology used. With these concepts we give the following consequence of Theorem 2.2, which is a generalization of Corollary 2.4.

Corollary 2.5. *A generalized tree G is $(2,t)$-metric dimensional if and only if G contains at least two extreme vertices being adjacent to a common cut vertex.*

In particular we can state the following result on Cartesian product graphs.

Proposition 2.6. *Let G and H be two connected graphs of order $n \geq 2$ and $n' \geq 3$, respectively. Then $G \Box H$ is (k,t)-metric dimensional for some $k \geq 3$.*

Proof. Notice that for any vertex $(a,b) \in V(G \Box H)$, $N_{G \Box H}((a,b)) = (N_G(a) \times \{b\}) \cup (\{a\} \times N_H(b))$. Now, for any two distinct vertices $(a,b),(c,d) \in V(G \Box H)$ at least $a \neq c$ or $b \neq d$ and since H is a connected graph of order at least three, we have that $N_H(b) \neq \{d\}$ or $N_H(d) \neq \{b\}$. Thus, we obtain that $N_{G \Box H}((a,b)) \neq N_{G \Box H}((c,d))$. Therefore, the twin equivalence classes of $G \Box H$ are singletons and, by Remark 2.1 and Corollary 2.3, $G \Box H$ is (k,t)-metric dimensional for some $k \geq 3$. $\Box$

Now, according to Remark 2.1 we have that every graph of order $n \geq 2$, different from K_2 and N_2, is (k,t)-metric dimensional for some $k \leq n-1$. Next we characterize those graphs being $(n-1,t)$-metric dimensional. To this end, we first show two previous results.

[1] A biconnected graph is a connected graph having no cut vertices.

[2] In some works these graphs are called block graphs.

Proposition 2.7. *Let P_n be a path of order $n \geq 3$. If $n \leq t+2$, then P_n is $(n-1,t)$-metric dimensional. Otherwise, P_n is (k,t)-metric dimensional for some $k \leq n-2$.*

Proof. Since $n \geq 3$, by Remark 2.1, P_n is (k,t)-metric dimensional for some $k \in \{2,\ldots,n-1\}$. We now consider two cases:

(1) $n \leq t+2$. For any pair of different vertices $u, v \in V(P_n)$ there exists at most one vertex $w \in V(P_n)$ such that w does not distinguish u and v. Therefore, by Theorem 2.2, P_n is $(n-1,t)$-metric dimensional.

(2) $n > t+2$. Let $u, v \in V(P_n)$ be the leaves. Since $t \geq 2$, it follows $n > 4$. Let u', v' be the vertices adjacent to u and v, respectively. Since $n > t+2$, $d_{P_n,t}(u,v) = d_{P_n,t}(u,v') = t$ and $d_{P_n,t}(u',v) = d_{P_n,t}(u',v') = t$. So, $\mathcal{D}_{P_n,t}(v,v') \cap \{u,u'\} = \emptyset$, which means that $|\mathcal{D}_{P_n,t}(v,v')| \leq n-2$. Therefore, by Theorem 2.2, the graph P_n is (k,t)-metric dimensional for some $k \leq n-2$.

□

Proposition 2.8. *Let C_n be a cycle graph of order n. If $n \leq 2t+1$ and it is odd, then C_n is $(n-1,t)$-metric dimensional. Otherwise, C_n is (k,t)-metric dimensional for some $k \leq n-2$.*

Proof. Since $n \geq 3$, by Remark 2.1, C_n is (k,t)-metric dimensional for some $k \in \{2,\ldots,n-1\}$. We now consider three cases:

(1) $n \leq 2t+1$ and it is odd. For any pair of different vertices $u, v \in V(C_n)$ there exist only one vertex $w \in V(C_n)$ such that w does not distinguish u and v. Therefore, by Theorem 2.2, C_n is $(n-1,t)$-metric dimensional.

(2) n is even. In this case, C_n is 2-antipodal. For any pair of vertices $u, v \in V(C_n)$, such that $d_{C_n}(u,v) = 2l$, we can take a vertex x such that $d_{C_n}(u,x) = d_{C_n}(v,x) = l$. So, $\mathcal{D}_{C_n,t}(u,v) \cap \{x,y\} = \emptyset$, where y is antipodal to x. Therefore, by Theorem 2.2, the graph C_n is (k,t)-metric dimensional for some $k \leq n-2$.

(3) $n > 2t+1$ and it is odd. Let $u, v \in V(C_n)$ be two adjacent vertices. Let x, y be the antipodal vertices of u. Since $n > 2t+1$, we deduce that $d_{C_n}(u,x) = d_{C_n}(u,y) \geq t+1$, which gives $d_{C_n,t}(u,x) = d_{C_n,t}(u,y) = t$. Without loss of generality, we assume that $d_{C_n}(v,x) = d_{C_n}(v,y) + 1$.

Since $d_{C_n}(v,x) \geq t+1$, we deduce that $d_{C_n,t}(v,x) = d_{C_n,t}(v,y) = t$. So, $\mathcal{D}_{C_n,t}(x,y) \cap \{u,v\} = \emptyset$. Therefore, by Theorem 2.2, the graph C_n is (k,t)-metric dimensional for some $k \leq n-2$.

□

Once presented the two propositions above, we are now ready to present the characterization of $(n-1,t)$-metric dimensional graphs.

Theorem 2.9. *Let G be a graph order $n \geq 3$. The graph G is $(n-1,t)$-metric dimensional if and only if G is a path and $n \leq t+2$, or G is an odd cycle and $n \leq 2t+1$, or $G \cong K_1 \cup K_2$, or $G \cong N_3$.*

Proof. Since $n \geq 3$, by Remark 2.1, G is (k,t)-metric dimensional for some $k \in \{2,\ldots,n-1\}$. If G is a path of order $n \leq t+2$, then by Proposition 2.7 we have that G is $(n-1,t)$-metric dimensional. If G is a cycle of odd order $n \leq 2t+1$, then by Proposition 2.8 it follows that G is $(n-1,t)$-metric dimensional. If $G \cong K_1 \cup K_2$ or $G \cong N_3$, then it is straightforward to see that G is $(2,t)$-metric dimensional.

On the other side, let G be a $(n-1,t)$-metric dimensional graph. Hence, for every pair of different vertices $x,y \in V(G)$ there exists at most one vertex which does not distinguish x,y. Suppose $\Delta(G) > 2$ and let $v \in V(G)$ such that $\{u_1,u_2,u_3\} \subseteq N(v)$. Figure 2.2 shows all the possibilities for the links between these four vertices. Figures 2.2 (a), 2.2 (b) and 2.2 (d) show that v,u_1 do not distinguish u_2,u_3. Figure 2.2 (c) shows that u_1,u_2 do not distinguish v,u_3. Thus, from the situations above we deduce that there is a pair of different vertices which is not distinguished by at least two other different vertices. Thus, G is not a $(n-1,t)$-metric dimensional graph, which is a contradiction. So $\Delta(G) \leq 2$. If G is connected, then we have that G is either a path or a cycle, and by Propositions 2.7 and 2.8, we deduce that G is a path of order $n \leq t+2$, or G is an odd cycle of order $n \leq 2t+1$. If G is not connected, then each connected component is either a path, or a cycle or an isolated vertex. If one of the connected components C has order at least three, then there exists a vertex v of degree two. Let $N(v) = \{x,y\}$. The vertex v and any vertex belonging to a connected component different from C does not distinguish x,y, which is a contradiction. Thus, each connected component C of G satisfies $\Delta(C) \leq 1$. Suppose that there exists one connected component C of order two. In this case, if there exists other connected component of order two, or there exist at least two other connected component, then the pair of vertices belonging to C is not distinguished by at least two vertices, which is a contradiction. So in this case $G \cong K_1 \cup K_2$. Suppose that all connected components are isolated vertices. If there exist at least four connected components, then any pair of vertices of G is not distinguished by at least two vertices, which is a contradiction. Therefore, in this case $G \cong N_3$, and we conclude the proof. □

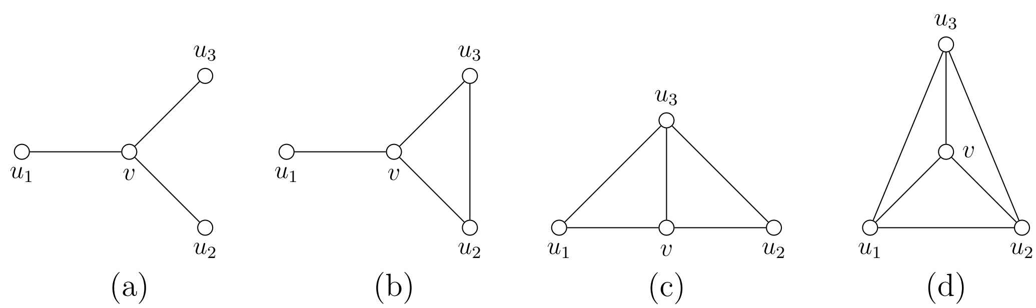

Figure 2.2: Possible cases for a vertex v with three adjacent vertices u_1, u_2, u_3.

2.2 On some families of k-metric dimensional graphs

From now on, given a graph G and two vertices $x, y \in V(G)$, for $t \geq D(G)$ we say that G is *k-metric dimensional* instead of (k,t)-metric dimensional. The concept of k-metric dimensional graph was introduced by Estrada-Moreno et al. in [34, 35]. In this section, we use the notation $\mathcal{D}(G)$, $\mathcal{D}_G(x,y)$ and $\mathcal{D}^*_G(x,y)$ instead of $\mathcal{D}(G,t)$, $\mathcal{D}_{G,t}(x,y)$ and $\mathcal{D}^*_{G,t}(x,y)$, respectively.

2.2.1 Bounding the value k for k-metric dimensional graphs

In order to continue presenting our results, we need to introduce some definitions. A vertex of degree at least three in a graph G will be called a *major vertex* of G. Any end vertex u of G is said to be a *terminal vertex* of a major vertex v of G if $d_G(u,v) < d_G(u,w)$ for every other major vertex w of G. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\mathcal{M}(G)$ be the set of exterior major vertices of G having terminal degree greater than one.

Given $w \in \mathcal{M}(G)$ and a terminal vertex u_j of w, we denote by $P(u_j, w)$ the shortest path that starts at u_j and ends at w. Let $l(u_j, w)$ be the length of $P(u_j, w)$. Now, given $w \in \mathcal{M}(G)$ and two terminal vertices u_j, u_r of w we denote by $P(u_j, w, u_r)$ the shortest path from u_j to u_r containing w, and by $\varsigma(u_j, u_r)$ the length of $P(u_j, w, u_r)$. Notice that, by definition of exterior major vertex, $P(u_j, w, u_r)$ is obtained by concatenating the paths $P(u_j, w)$ and $P(u_r, w)$, where w is the only vertex of degree greater than two lying on these paths. Finally, given $w \in \mathcal{M}(G)$ and the set of terminal vertices $U = \{u_1, u_2, \ldots, u_k\}$ of w, for $j \neq r$ we define $\varsigma(w) = \min\limits_{u_j, u_r \in U}\{\varsigma(u_j, u_r)\}$ and $l(w) = \min\limits_{u_j \in U}\{l(u_j, w)\}$.

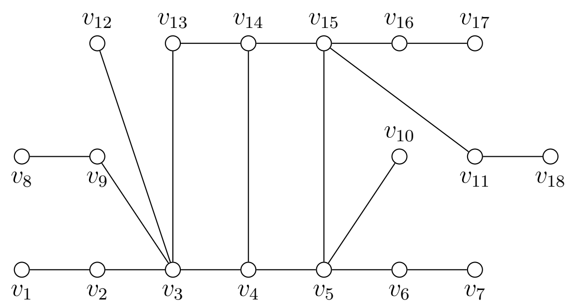

Figure 2.3: A graph G where $\varsigma(G) = 3$.

From the local parameters above we define the following global parameter

$$\varsigma(G) = \min_{w \in \mathcal{M}(G)}\{\varsigma(w)\}.$$

An example which helps to understand the notation above is given in Figure 2.3. In such a case we have $\mathcal{M}(G) = \{v_3, v_5, v_{15}\}$ and, for instance, $\{v_1, v_8, v_{12}\}$ are terminal vertices of v_3. So, v_3 has terminal degree three ($\mathrm{ter}(v_3) = 3$) and it follows that

$$\begin{aligned} l(v_3) &= \min\{l(v_{12}, v_3), l(v_8, v_3), l(v_1, v_3)\} \\ &= \min\{1, 2, 2\} = 1, \end{aligned}$$

and

$$\begin{aligned} \varsigma(v_3) &= \min\{\varsigma(v_{12}, v_1), \varsigma(v_{12}, v_8), \varsigma(v_8, v_1)\} \\ &= \min\{3, 3, 4\} = 3. \end{aligned}$$

Similarly, it is possible to observe that $\mathrm{ter}(v_5) = 2$, $l(v_5) = 1$, $\varsigma(v_5) = 3$, $\mathrm{ter}(v_{15}) = 2$, $l(v_{15}) = 2$ and $\varsigma(v_{15}) = 4$. Therefore, $\varsigma(G) = 3$.

According to this notation we present the following result.

Theorem 2.10. *Let G be a connected graph such that $\mathcal{M}(G) \neq \emptyset$. If G is k-metric dimensional, then $k \leq \varsigma(G)$.*

Proof. We claim that there exists at least one pair of different vertices $x, y \in V(G)$ such that $|\mathcal{D}_G(x,y)| = \varsigma(G)$. To see this, let $w \in \mathcal{M}(G)$ and let u_1, u_2 be two terminal vertices of w such that $\varsigma(G) = \varsigma(w) = \varsigma(u_1, u_2)$. Let u_1' and u_2' be the vertices adjacent to w in the shortest paths $P(u_1, w)$ and $P(u_2, w)$, respectively. Notice that it could happen $u_1' = u_1$ or $u_2' = u_2$. Since every vertex $v \notin V(P(u_1, w, u_2)) - \{w\}$ satisfies that $d_G(u_1', v) = d_G(u_2', v)$, and the only distinctive vertices of u_1', u_2' are those ones belonging to $P(u_1', u_1)$ and $P(u_2', u_2)$, we have that $|\mathcal{D}_G(u_1', u_2')| = \varsigma(G)$. Therefore, by Theorem 2.2, if G is k-metric dimensional, then $k \leq \varsigma(G)$. □

The upper bound of Theorem 2.10 is tight. For instance, it is achieved for every tree different from a path as it is further proved in Subsection 2.2.2, where the k-metric dimension of trees is studied.

Theorem 2.11. *Let G be a graph of order n different from a complete graph. If G is k-metric dimensional, then $k \leq n - \omega(G) + 1$.*

Proof. Let S be an $\omega(G)$-clique. Since G is not complete, there exists a vertex $v \notin S$ such that $N_S(v) \subsetneq S$. Let $u \in S$ with $v \not\sim u$. If $N_S(v) = S - \{u\}$, then $d(u,x) = d(v,x) = 1$ for every $x \in S - \{u\}$. Thus, $|\mathcal{D}_G(u,v)| \leq n - \omega(G) + 1$. On the other hand, if $N_S(v) \neq S - \{u\}$, then there exists $u' \in S - \{u\}$ such that $u' \not\sim v$. Thus, $d(u,v) = d(u',v) = 2$ and for every $x \in S - \{u, u'\}$, $d(u,x) = d(u',x) = 1$. So, $|\mathcal{D}_G(u,u')| \leq n - \omega(G) + 1$. Therefore, Theorem 2.2 leads to $k \leq n - \omega(G) + 1$. □

Examples where the previous bound is achieved are those connected graphs G of order n and clique number $\omega(G) = n - 1$. In such a case, $n - \omega(G) + 1 = 2$. Notice that in this case there exists at least two twin vertices. Hence, by Corollary 2.3 these graphs are 2-metric dimensional.

Theorem 2.12. *Let G be a graph of minimum degree $\delta(G) \geq 2$, maximum degree $\Delta(G) \geq 3$ and girth $\mathsf{g}(G) \geq 4$. If G is k-metric dimensional, then*

$$k \leq n - 1 - (\Delta(G) - 2) \sum_{i=0}^{\left\lfloor \frac{\mathsf{g}(G)}{2} \right\rfloor - 2} (\delta(G) - 1)^i.$$

Proof. Let $v \in V$ be a vertex of maximum degree in G. Since $\Delta(G) \geq 3$ and $\mathtt{g}(G) \geq 4$, there are at least three different vertices adjacent to v and $N(v)$ is an independent set[3]. Given $u_1, u_2 \in N(v)$ and $i \in \{0, \ldots, \left\lfloor \frac{\mathtt{g}(G)}{2} \right\rfloor - 2\}$ we define the following sets.

$$A_0 = N(v) - \{u_1, u_2\}.$$
$$A_1 = \bigcup_{x \in A_0} N(x) - \{v\}.$$
$$A_2 = \bigcup_{x \in A_1} N(x) - A_0.$$
$$\ldots$$
$$A_{\left\lfloor \frac{\mathtt{g}(G)}{2} \right\rfloor - 2} = \bigcup_{x \in A_{\left\lfloor \frac{\mathtt{g}(G)}{2} \right\rfloor - 3}} N(x) - A_{\left\lfloor \frac{\mathtt{g}(G)}{2} \right\rfloor - 4}.$$

Now, let $A = \{v\} \cup \left(\bigcup_{i=0}^{\left\lfloor \frac{\mathtt{g}(G)}{2} \right\rfloor - 2} A_i \right)$. Since $\delta(G) \geq 2$, we have that $|A| \geq 1 + (\Delta(G) - 2) \sum_{i=0}^{\left\lfloor \frac{\mathtt{g}(G)}{2} \right\rfloor - 2} (\delta(G) - 1)^i$. Also, notice that for every vertex $x \in A$, $d(u_1, x) = d(u_2, x)$. Thus, u_1, u_2 can be only distinguished by themselves and at most $n - |A| - 2$ other vertices. Therefore, $|\mathcal{D}_G(u_1, u_2)| \leq n - |A|$ and the result follows by Theorem 2.2. $\square$

The bound of Theorem 2.12 is sharp. For instance, it is attained for the graph in Figure 2.4. Since in this case $n = 8$, $\delta(G) = 2$, $\Delta(G) = 3$ and $\mathtt{g}(G) = 5$, we have that $k \leq n - 1 - (\Delta(G) - 2) \sum_{i=0}^{\left\lfloor \frac{\mathtt{g}(G)}{2} \right\rfloor - 2} (\delta(G) - 1)^i = 6$. Table 2.1 shows every pair of different vertices of this graph and their corresponding nontrivial distinctive vertices. Notice that by Theorem 2.2 the graph is 6-metric dimensional.

2.2.2 On k-metric dimensional trees

By Remark 2.1 we know that a path of order 2 is 2-metric dimensional, and by Theorem 2.9, any path of order $n \geq 3$ is $n-1$-metric dimensional. Thus, in this subsection we only considerer trees different from paths.

[3]An independent set or stable set is a set of vertices in a graph, no two of which are adjacent.

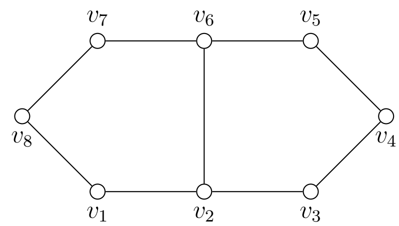

Figure 2.4: A graph that satisfies the equality in the upper bound of Theorem 2.12.

To study the k-metric dimensional trees different from paths, we need the terminology and notation already described in Subsection 2.2.1 and also the following one. Given an exterior major vertex v in a tree T and the set of its terminal vertices $v_1, \ldots, v_\alpha$, the subgraph induced by the set $\bigcup_{i=1}^{\alpha} V(P(v, v_i))$ is called a *branch* of T at v (a v-branch for short).

Theorem 2.13. *If T is a k-metric dimensional tree different from a path, then $k = \varsigma(T)$.*

Proof. Since T is not a path, $\mathcal{M}(T) \neq \emptyset$. Let $w \in \mathcal{M}(T)$ and let u_1, u_2 be two terminal vertices of w such that $\varsigma(T) = \varsigma(w) = \varsigma(u_1, u_2)$. Notice that, for instance, the two neighbours of w belonging to the paths $P(w, u_1)$ and $P(w, u_2)$, say u_1' and u_2' satisfy $|\mathcal{D}_T(u_1', u_2')| = \varsigma(T)$.

It only remains to prove that for every $x, y \in V(T)$ it holds that $|\mathcal{D}_T(x, y)| \geq \varsigma(T)$. Let $w \in \mathcal{M}(T)$ and let $T_w = (V_w, E_w)$ be the w-branch. Also we consider the set of vertices $V' = V(T) - \bigcup_{w \in \mathcal{M}(T)} V_w$. Note that $|V_w| \geq \varsigma(T) + 1$ for every $w \in \mathcal{M}(T)$. With this fact in mind, we consider three cases.

Case 1: $x \in V_w$ and $y \in V_{w'}$ for some $w, w' \in \mathcal{M}(T)$, $w \neq w'$. In this case x, y are distinguished by w or by w'. Now, if w distinguishes the pair x, y, then at most one element of V_w does not distinguish x, y (see Figure 2.5). So, x and y are distinguished by at least $|V_w| - 1$ vertices of T or by at least $|V_{w'}| - 1$ vertices of T.

Case 2: $x \in V'$ or $y \in V'$. Thus, $V' \neq \emptyset$ and, as a consequence, $|\mathcal{M}(T)| \geq 2$. Hence, we have one of the following situations.

- There exist two vertices $w, w' \in \mathcal{M}(T)$, $w \neq w'$, such that the shortest path from x to w and the shortest path from y to w' have empty intersection, or

x,y	$\mathcal{D}^*_G(x,y)$	x,y	$\mathcal{D}^*_G(x,y)$
v_1,v_3	$\{v_4,v_5,v_7,v_8\}$	v_1,v_2	$\{v_3,v_4,v_5,v_6,v_8\}$
v_1,v_5	$\{v_2,v_4,v_6,v_8\}$	v_1,v_4	$\{v_2,v_3,v_5,v_7,v_8\}$
v_1,v_6	$\{v_4,v_5,v_7,v_8\}$	v_2,v_3	$\{v_1,v_4,v_6,v_7,v_8\}$
v_1,v_7	$\{v_2,v_3,v_5,v_6\}$	v_2,v_4	$\{v_1,v_5,v_6,v_7,v_8\}$
v_1,v_8	$\{v_2,v_3,v_4,v_7\}$	v_2,v_8	$\{v_3,v_4,v_5,v_6,v_7\}$
v_2,v_5	$\{v_1,v_3,v_4,v_8\}$	v_3,v_8	$\{v_1,v_2,v_4,v_5,v_7\}$
v_2,v_6	$\{v_1,v_3,v_5,v_7\}$	v_4,v_6	$\{v_1,v_2,v_3,v_7,v_8\}$
v_2,v_7	$\{v_1,v_3,v_4,v_8\}$	v_4,v_7	$\{v_1,v_3,v_5,v_6,v_8\}$
v_3,v_4	$\{v_1,v_2,v_5,v_8\}$	v_5,v_6	$\{v_1,v_2,v_4,v_7,v_8\}$
v_3,v_5	$\{v_1,v_2,v_6,v_7\}$	v_5,v_8	$\{v_1,v_3,v_4,v_6,v_7\}$
v_3,v_6	$\{v_4,v_5,v_7,v_8\}$	v_6,v_7	$\{v_2,v_3,v_4,v_5,v_8\}$
v_3,v_7	$\{v_2,v_4,v_6,v_8\}$	v_6,v_8	$\{v_1,v_2,v_3,v_4,v_5\}$
v_4,v_5	$\{v_3,v_6,v_7,v_8\}$		
v_4,v_8	$\{v_1,v_3,v_5,v_7\}$		
v_5,v_7	$\{v_1,v_3,v_4,v_8\}$		
v_7,v_8	$\{v_1,v_4,v_5,v_6\}$		

Table 2.1: Pairs of vertices of the graph in Figure 2.4 and their nontrivial distinctive vertices.

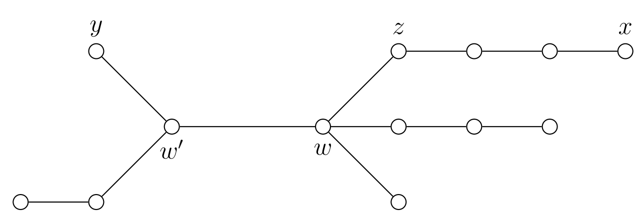

Figure 2.5: In this example, w distinguishes the pair x,y, and z is the only vertex in V_w that does not distinguish x,y.

- for every vertex $w'' \in \mathcal{M}(T)$, it follows that either y belongs to the shortest path from x to w'' or x belongs to the shortest path from y to w''.

In the first case, x,y are distinguished by vertices in V_w or by vertices in $V_{w'}$ and in the second one, x,y are distinguished by vertices in $V_{w''}$.

Case 3: $x,y \in V_w$ for some $w \in \mathcal{M}(T)$. If $x,y \in V(P(u_l,w))$ for some $l \in \{1,\dots,\mathrm{ter}(w)\}$, then there exists at most one vertex of $V(P(u_l,w))$

which does not distinguish x, y. Since $\mathrm{ter}(w) \geq 2$, the vertex w has a terminal vertex u_q with $q \neq l$. So, x, y are distinguished by at least $|V(P(u_l, w, u_q))| - 1$ vertices, and since $|V(P(u_l, w, u_q))| \geq \varsigma(T) + 1$, we are done. If $x \in V(P(u_l, w)$ and $y \in V(P(u_q, w)$ for some $l, q \in \{1, \ldots, \mathrm{ter}(w)\}$, $l \neq q$, then there exists at most one vertex of $V(P(u_l, w, u_q))$ which does not distinguish x, y. Since $|V(P(u_l, w, u_q))| \geq \varsigma(T) + 1$, the result follows.

Therefore, $\varsigma(T) = \mathcal{D}(T)$ and by Theorem 2.2 the proof is completed. $\square$

2.3 On some families of k-adjacency dimensional graph

From now on, given a graph G and two vertices $x, y \in V(G)$, we say that G is *k-adjacency dimensional* instead of $(k, 2)$-metric dimensional. The concept of k-adjacency dimensional graphs was introduced by Estrada-Moreno et al. in [33]. In this section, we use the notation $\mathcal{C}(G)$, $\mathcal{C}_G(x, y)$ and $\mathcal{C}^*_G(x, y)$ instead of $\mathcal{D}(G, 2)$, $\mathcal{D}_{G,2}(x, y)$ and $\mathcal{D}^*_{G,2}(x, y)$, respectively. Note that $\mathcal{C}(G) = \min_{x,y \in V(G)} \{|N_G(x) \triangledown N_G(y) \cup \{x, y\}|\}$.

As we shall see in Theorem 4.2, given a graph G, the problem of finding the value of k such that G is k-adjacency dimensional is easy to solve. Even so, we would point out some useful particular cases. For instance, by Corollary 2.3 we deduce that complete graph K_n and complete bipartite graph $K_{r,s}$ are 2-adjacency dimensional.

If $u, v \in V(G)$ are adjacent vertices of degree two and they are not twin vertices, then $|\mathcal{C}_G(u, v)| = 4$. Thus, for any integer $n \geq 5$, C_n is 4-adjacency dimensional and we can state the following more general remark.

Remark 2.14. *Let G be a twins free graph of minimum degree two. If G has two adjacent vertices of degree two, then G is 4-adjacency dimensional.*

For any hypercube Q_r, $r \geq 2$, we have $|\mathcal{C}_{Q_r}(u, v)| = 2r$ if $u \sim v$, $|\mathcal{C}_{Q_r}(u, v)| = 2r - 2$ if $d_{Q_r}(u, v) = 2$ and $|\mathcal{C}_{Q_r}(u, v)| = 2r + 2$ if $d_{Q_r}(u, v) \geq 3$. Hence, $\mathcal{C}(Q_r) = 2r - 2$.

Remark 2.15. *For any integer $r \geq 2$ the hypercube Q_r is $(2r-2)$-adjacency dimensional.*

It is straightforward to see that for any graph G of girth $\mathsf{g}(G) \geq 5$ and minimum degree $\delta(G) \geq 2$, $\mathcal{C}(G) \geq 2\delta(G)$. Hence, the following remark is immediate.

Remark 2.16. *Let G be a k-adjacency dimensional graph. If $\mathrm{g}(G) \geq 5$ and $\delta(G) \geq 2$, then $k \geq 2\delta(G)$.*

If there is an end vertex u in G whose support vertex v has degree two, then $|\mathcal{C}_G(u,v)| = |N_G[v]| = 3$. Hence, we deduce the following result.

Remark 2.17. *Let G be a twins free graph. If there exists an end vertex whose support vertex has degree two, then G is 3-adjacency dimensional.*

The case of trees is summarized in the following remark an we need the concept of exterior major vertex already presented in Subsection 2.2.1.

Remark 2.18. *Let T be a k-adjacency dimensional tree of order $n \geq 3$. Then $k \in \{2,3\}$ and $k = 2$ if and only if there are two leaves sharing a common support vertex.*

Proof. By Remark 2.3 we conclude that $k = 2$ if and only if there are two leaves sharing a common support vertex. Also, if T is a path different from P_3, then by Remark 2.17 we have that $k = 3$.

If T is not a path, then there exists at least one exterior major vertex u of terminal degree greater than one. Then, either u is the support vertex of all its terminal vertices, in which case Remark 2.3 leads to $k = 2$, or u has at least one terminal vertex whose support vertex has degree two, in which case Remark 2.17 leads to $k = 3$ if there are no leaves of T sharing a common support vertex. $\square$

Since $|\mathcal{C}_G(x,y)| \leq \delta(x) + \delta(y) + 2$, for all $x, y \in V(G)$, the following remark immediately follows.

Remark 2.19. *If G is a k-adjacency dimensional graph, then*

$$k \leq \min_{x,y \in V(G)} \{\delta(x) + \delta(y)\} + 2.$$

This bound is achieved, for instance, for any graph $G \cong C_n \odot K_1$. Also, a trivial example is the case of graphs having two isolated vertices, which are 2-adjacency dimensional.

Since any k-adjacency generator is a (k,t)-metric generator, and for any graph G of diameter at most two the distances $d_{G,t}$ and $d_{G,2}$ are equivalent, the following result is straightforward.

Remark 2.20. *If a graph G is k-adjacency dimensional and (k',t)-metric dimensional, then $k \leq k'$. Moreover, if $D(G) \leq 2$, then $k' = k$.*

2.4 k-metric dimensional product graphs

We explained earlier that we study the k-metric dimension of the lexicographic product and the corona product of graphs. Thus, we need to determine the values of k such that there exist k-metric bases for these products.

2.4.1 Lexicographic product graphs

We now analyse when the lexicographic product is k-metric dimensional. We first study the particular case of join graphs due its importance and its peculiarities.

Join graphs

Throughout this section we also use the concept of $\mathcal{C}(H)$ for a graph H already defined at the beginning of Section 2.3.

Proposition 2.21. *Let H be a graph of order $n' \geq 2$ and maximum degree $\Delta(H)$. The graph $K_1 + H$ is k-metric dimensional if and only if $k = \min\{\mathcal{C}(H), n' - \Delta(H) + 1\}$.*

Proof. If $x, y \in V(H)$, then $\mathcal{D}_{K_1+H}(x,y) = \mathcal{C}_H(x,y)$. Also, if $x \notin V(H)$ then $\mathcal{D}_{K_1+H}(x,y) = \{x\} \cup (V(H) - N_H(y))$. Therefore, by Theorem 2.2, the result follows. □

We next point out some consequences of Proposition 2.21.

Corollary 2.22. *Let H be a nontrivial graph. If H is k-metric dimensional and $K_1 + H$ is k'-metric dimensional, then $k' \leq k$.*

Proof. By Proposition 2.21 we have that if $K_1 + H$ is a k'-metric dimensional graph, then $k' \leq \mathcal{C}(H)$. Since, for any $x, y \in V(H)$ we have $\mathcal{C}_H(x,y) \subseteq \mathcal{D}_H(x,y)$, we deduce that if H is k-metric dimensional, then $\mathcal{C}(H) \leq k$ and, as a consequence, $k' \leq k$. □

Corollary 2.23. *For any connected graph H of order $n' \geq 2$, the graph $K_1 + H$ is 2-metric dimensional if and only if $\Delta(H) = n' - 1$ or H has twins vertices.*

Notice that the previous corollary may also be derived from Corollary 2.3.

Corollary 2.24. *Let H be a connected graph of order $n' \geq 4$ and maximum degree $\Delta(H) = n' - 2$. If H does not contain twin vertices, then $K_1 + H$ is 3-metric dimensional.*

Proof. Since H does not contain twin vertices, for every $x, y \in V(H)$ there exists $z \in \mathcal{C}_H(x, y) - \{x, y\}$. Thus, $\mathcal{C}(H) \geq 3$. Now, since $n' - \Delta(H) + 1 = 3$, by Proposition 2.21 we can deduce the result. □

The *wheel graph* $W_{1,n}$ is the join graph $K_1 + C_n$ and the *fan graph* $F_{1,n}$ is the join graph $K_1 + P_n$.

Corollary 2.25. *For any $n \geq 4$, the fan graph $F_{1,n}$ is 3-metric dimensional, and for any $n \geq 5$, the wheel graph $W_{1,n}$ is 4-metric dimensional.*

We now show a property on the $(n' - \Delta(H) + 1)$-metric bases of $K_1 + H$.

Proposition 2.26. *Let H be a nontrivial graph of order n'. If $K_1 + H$ is $(n' - \Delta(H) + 1)$-metric dimensional, then the vertex of K_1 belongs to every $(n' - \Delta(H) + 1)$-metric basis of $K_1 + H$.*

Proof. Let v be the vertex of K_1. Notice that for every $x \in V(H)$, we have

$$\mathcal{D}_{K_1+H}(x, v) = (V(H) - N_H(x)) \cup \{v\}.$$

For every $x \in V(H)$ such that $N_H(x) = \Delta(H)$ we have that $n' - \Delta(H) + 1 = |\,(V(H) - N(x)) \cup \{v\}| = |\mathcal{D}_{K_1+H}(x, v)|$. Thus, for any $(n' - \Delta(H) + 1)$-metric basis B we have $\mathcal{D}_{K_1+H}(x, v) \subseteq B$ and, since $v \in \mathcal{D}_{K_1+H}(x, v)$, we conclude that $v \in B$. □

By Proposition 2.26 we deduce that if the vertex of K_1 does not belong to any k-metric basis of $K_1 + H$, then $K_1 + H$ is not $(n' - \Delta(H) + 1)$-metric dimensional. Thus, by Proposition 2.21 we obtain the following result.

Proposition 2.27. *Let H be a nontrivial graph. If the vertex of K_1 does not belong to any k-metric basis of $K_1 + H$, then $K_1 + H$ is $\mathcal{C}(H)$-metric dimensional.*

Proposition 2.28. *Let G and H be two graphs of order $n \geq 2$ and $n' \geq 2$, respectively. The graph $G + H$ is k-metric dimensional if and only if $k = \min\{\mathcal{C}(G), \mathcal{C}(H), n - \Delta(G) + n' - \Delta(H)\}$.*

Proof. If $x, y \in V(G)$, then $\mathcal{D}_{G+H}(x,y) = \mathcal{C}_G(x,y)$. Analogously, if $x, y \in V(H)$, then $\mathcal{D}_{G+H}(x,y) = \mathcal{C}_H(x,y)$. Also, if $x \in V(G)$ and $y \in V(H)$, then $\mathcal{D}_{G+H}(x,y) = (V(G) - N_G(x)) \cup (V(H) - N_H(y))$. Therefore, by Theorem 2.2, the result follows. □

General lexicographic product graphs

Twin vertices play a highly significant role into studying the k-metric dimension of graphs, as we shall observe through our exposition. In this sense, we need to use a formal terminology already set forth above in Section 1.1. Now, for any graph G of order n, a family composed by n nontrivial graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$ and $u_i \in V(G)$, we define in $G \circ \mathcal{H}$ the following local parameter:

$$\mathcal{T}(u_i,\mathcal{H}) = \begin{cases} |V(H_i)|, & \text{if } u_i \in S(G),\\ \\ \displaystyle\min_{u_j,u_l \in FT(u_i)} \{\delta(H_j) + \delta(H_l) + 2\}, & \text{if } u_i \in FT(G),\\ \\ \displaystyle\min_{u_j,u_l \in TT(u_i)} \{|V(H_j)| - \Delta(H_j) + |V(H_l)| - \Delta(H_l)\}, & \text{if } u_i \in TT(G). \end{cases}$$

Moreover, we define a global parameter from the local parameter defined above,

$$\mathcal{T}(G \circ \mathcal{H}) = \min_{u_i \in V(G)} \{\mathcal{T}(u_i, \mathcal{H})\}.$$

We also define

$$\mathcal{C}(\mathcal{H}) = \min_{H_i \in \mathcal{H}} \{\mathcal{C}(H_i)\}.$$

With all the tools presented till this point, we are now prepare to give our first result regarding the value k for which a lexicographic product graph is k-metric dimensional.

Theorem 2.29. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family of n nontrivial graphs. The graph $G \circ \mathcal{H}$ is k-metric dimensional if and only if $k = \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}$.*

Proof. By Theorem 2.2, it is only necessary to prove that $\mathcal{D}(G \circ \mathcal{H}) = \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}$. Hence, let $(u_i, v), (u_j, w) \in V(G \circ \mathcal{H})$ be two different vertices. We analyse two cases.

Case 1. $i = j$. In this case $v \neq w$. By Remark 1.1 (i) and (ii), it follows that $\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_i, w)) = \{u_i\} \times \mathcal{C}_{H_i}(v, w)$. Thus,

$$R_1 = \min_{(u_i,v),(u_i,w)\in V(G\circ\mathcal{H})} \{|\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_i, w))|\} = \min_{H_i\in\mathcal{H}} \{\mathcal{C}(H_i)\} = \mathcal{C}(\mathcal{H}).$$

Case 2. $i \neq j$. If u_i, u_j are not twins, then $\mathcal{D}^*_G(u_i, u_j) \neq \emptyset$. So, for every $u_l \in \mathcal{D}^*_G(u_i, u_j)$ it follows $V(H_l) \subsetneq \mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_j, w))$ or equivalently $|V(H_l)| < |\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_j, w))|$. Thus,

$$\begin{aligned} R_2 &= \min_{(u_i,v),(u_j,w)\in V(G\circ\mathcal{H})} \{|\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_j, w)))|\} \\ &> \min_{H_l\in\mathcal{H}} \{|V(H_l)|\} \\ &\geq \min_{H_l\in\mathcal{H}} \{|\mathcal{C}(H_l)|\} = \mathcal{C}(\mathcal{H}). \end{aligned}$$

Notice that R_2 is strictly greater than R_1. So, the minimum between them is R_1.

Now, we assume that u_i, u_j are twins, so $\mathcal{D}^*(u_i, u_j) = \emptyset$. Hence we consider two possibilities for u_i, u_j in the next statements, where the conclusions are consequences of Remark 1.1 (i) and (ii).

Subcase 2.1: If $u_i \sim u_j$, then $|\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_j, w))| = |(V(H_i) - N_{H_i}(v)) \cup (V(H_j) - N_{H_j}(w))|$. So, it follows that

$$\begin{aligned} R_3 &= \min_{(u_i,v),(u_j,w)\in V(G\circ\mathcal{H})} \{|\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_j, w))|\} \\ &= \min\{|(V(H_i) - N_{H_i}(v)) \cup (V(H_j) - N_{H_j}(w))|\} \\ &= \min\{|V(H_i)| - \Delta(H_i) + |V(H_j)| - \Delta(H_j)\} \\ &= \min_{u_l\in V(G)} \{\mathcal{T}(u_l, \mathcal{H})\} \\ &= \mathcal{T}(G \circ \mathcal{H}). \end{aligned}$$

Subcase 2.2: If $d_G(u_i, u_j) = 2$, then $|\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_j, w))| = |N_{H_i}[v] \cup$

$N_{H_j}[w]|$. Similarly, we obtain that

$$\begin{aligned}
R_4 &= \min_{(u_i,v),(u_j,w)\in V(G\circ\mathcal{H})}\{|\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,w))|\} \\
&= \min\{|N_{H_i}[v]\cup N_{H_j}[w]|\} \\
&= \min\{\delta(H_i)+\delta(H_j)+2\} \\
&= \min_{u_l\in V(G)}\{\mathcal{T}(u_l,\mathcal{H})\} \\
&= \mathcal{T}(G\circ\mathcal{H}).
\end{aligned}$$

As a conclusion of all the statements above, it is obtained that

$$\begin{aligned}
\mathcal{D}(G\circ\mathcal{H}) &= \min_{(u_i,v),(u_j,w)\in V(G\circ\mathcal{H})}\{|\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,w))|\} \\
&= \min\left\{\min_{i=j}\{|\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,w))|\},\min_{j\neq i}\{|\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,w))|\}\right\} \\
&= \min\{R_1,R_2,R_3,R_4\} \\
&= \min\{R_1,R_3,R_4\} \\
&= \min\{\mathcal{C}(\mathcal{H}),\mathcal{T}(G\circ\mathcal{H})\}.
\end{aligned}$$

Therefore the proof is completed. □

We next emphasize some particular cases of Theorem 2.29 when the lexicographic product graphs have some specific structure which are related with the existence or not of twin vertices in the graph G.

Corollary 2.30. *Let G be a connected twins free graph of order $n\geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs. Then $G\circ\mathcal{H}$ is $\mathcal{C}(\mathcal{H})$-metric dimensional.*

Corollary 2.31. *Let G be a connected nontrivial graph and let H be a graph of order $n'\geq 2$.*

(i) *If G is twins free, then the graph $G\circ H$ is k-metric dimensional if and only if $k=\mathcal{C}(H)$.*

(ii) *If G contains at least one false twin and one true twin, then the graph $G\circ H$ is k-metric dimensional if and only if $k=\min\{2\delta(H)+2,2(n'-\Delta(H)),\mathcal{C}(H)\}$.*

(iii) *If G is true twins free and contains at least one false twin, then the graph $G\circ H$ is k-metric dimensional if and only if $k=\min\{2\delta(H)+2,\mathcal{C}(H)\}$.*

(iv) *If G is false twins free and contains at least one true twin, then the graph $G \circ H$ is k-metric dimensional if and only if $k = \min\{2(n' - \Delta(H)), \mathcal{C}(H)\}$.*

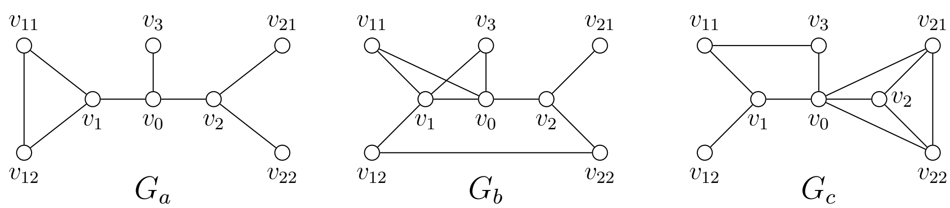

Figure 2.6: The graph G_x, $x \in \{a, b, c\}$, satisfies the conditions of Corollary 2.31 (ii), (iii) and (iv) respectively.

As some instances of graphs G that satisfy the conditions of the corollary above we next construct some examples. In Figure 2.6, the vertices v_{11} and v_{12} of the graph G_a are true twins, as well as v_{21} and v_{22} are false twins. So, G_a contains two false twins and two true twins and satisfies the premise of Corollary 2.31 (ii), and as a consequence, for any graph H of order $n' \geq 2$, we have that $G_a \circ H$ is k-metric dimensional for $k = \min\{2\delta(H) + 2, 2(n' - \Delta(H)), \mathcal{C}(H)\}$. Similarly, G_b is a true twins free graph and it has two false twin vertices, v_{11} and v_3. Thus, $G_b \circ H$ is k-metric dimensional for $k = \min\{2\delta(H) + 2, \mathcal{C}(H)\}$. Finally, the graph G_c is false twins free and it has two true twin vertices, v_{21} and v_{22}, and consequently, $G_c \circ H$ is k-metric dimensional for $k = \min\{2(n' - \Delta(H)), \mathcal{C}(H)\}$.

We also point out the particular case $k = 2$ of Theorem 2.29.

Corollary 2.32. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs. The graph $G \circ \mathcal{H}$ is 2-metric dimensional if and only if at least one of the following statements holds,*

(i) *there exists $H_i \in \mathcal{H}$ which has twins or,*

(ii) *there exist two true twin vertices $u_i, u_j \in V(G)$ such that $\Delta(H_i) = n_i - 1$ and $\Delta(H_j) = n_j - 1$.*

(iii) *there exist two false twin vertices $u_i, u_j \in V(G)$ such that H_i and H_j contain at least an isolated vertex.*

2.4.2 Corona product graphs

If there exists a graph $H_i \in \mathcal{H}$ such that H_i has twin vertices, then for any graph G, the corona graph $G \odot \mathcal{H}$ has twin vertices. Notice also that any two vertices of G are not twins in $G \odot \mathcal{H}$. Therefore, according to Corollary 2.3 we deduce the following result.

Remark 2.33. *For any connected graph G of order n and any family $\mathcal{H}$ composed by n connected nontrivial graphs, the corona graph $G \odot \mathcal{H}$ is 2-metric dimensional if and only if there exists a 2-metric dimensional graph $H_i \in \mathcal{H}$.*

Corollary 2.34. *Let G be a connected graph. Then,*

(i) *For $n \geq 2$, the graph $G \odot K_n$ is 2-metric dimensional.*

(ii) *The graphs $G \odot P_3$ and $G \odot C_4$ are 2-metric dimensional.*

Theorem 2.35. *Let G be a connected nontrivial graph of order n and let $\mathcal{H}$ be a family composed by n nontrivial graphs. Then, $G \odot \mathcal{H}$ is k-metric dimensional if and only if $k = \mathcal{C}(\mathcal{H})$.*

Proof. We claim that $\mathcal{C}(\mathcal{H}) = \min\limits_{x,y \in V(G\odot\mathcal{H})}\{|\mathcal{D}_{G\odot\mathcal{H}}(x,y)|\}$. Notice that, for every $u,v \in V(H_i)$, we have that $|\mathcal{C}_{H_i}(u,v)| \leq |V(H_i)|$. Let x,y be two different vertices of $G \odot \mathcal{H}$. We consider the following cases.

Case 1. If $x \in V(H_i)$ and $y \in V(H_j)$, $i \neq j$, then we obtain that $\mathcal{D}_{G\odot\mathcal{H}}(x,y) = \bigcup\limits_{v_l \in \mathcal{D}_G(v_i,v_j)} (V(H_l) \cup \{v_l\})$.

Case 2. If $x,y \in V(G)$, then we assume that $x = v_i$ and $y = v_j$. So, it follows that $\mathcal{D}_{G\odot\mathcal{H}}(x,y) = \bigcup\limits_{v_l \in \mathcal{D}_G(v_i,v_j)} (V(H_l) \cup \{v_l\})$.

Case 3. If $x \in V(H_i)$ and $y \in V(G)$, then $y = v_j$ for some $j \in \{1,\ldots,n\}$ and we consider the following. If $j = i$, then $\mathcal{D}_{G\odot\mathcal{H}}(x,y) = V(G \odot \mathcal{H}) - N_{H_i}(x)$. Now, if $j \neq i$, then we have $\mathcal{D}_{G\odot\mathcal{H}}(x,y) \supseteq V(H_j)$.

Case 4. If $x,y \in V(H_i)$, then $\mathcal{D}_{G\odot\mathcal{H}}(x,y) = \mathcal{C}_{H_i}(x,y)$.

Now, notice that from Cases 1, 2 and 3, $|\mathcal{D}_{G\odot\mathcal{H}}(x,y)| \geq \min\limits_{H_i \in \mathcal{H}}\{|V(H_i)|\} \geq \min\limits_{H_i \in \mathcal{H}}\{\mathcal{C}(H_i)\} = \mathcal{C}(\mathcal{H})$. Also, in Case 4, for every $x,y \in V(H_i)$ we have that

$|\mathcal{D}_{G\odot\mathcal{H}}(x,y)| = |\mathcal{C}_{H_i}(x,y)| \geq \min_{H_j\in\mathcal{H}}\{\mathcal{C}(H_j)\} = \mathcal{C}(\mathcal{H})$. Thus,

$$\mathcal{C}(\mathcal{H}) \leq \min_{x,y\in V(G\odot\mathcal{H})}\{|\mathcal{D}_{G\odot\mathcal{H}}(x,y)|\}.$$

On the other hand, we consider the following.

$$\begin{aligned}
\min_{x,y\in V(G\odot\mathcal{H})}\{|\mathcal{D}_{G\odot\mathcal{H}}(x,y)|\} &\leq \min_{x,y\in V(G\odot\mathcal{H})-V(G)}\{|\mathcal{D}_{G\odot\mathcal{H}}(x,y)|\}\\
&\leq \min_{H_i\in\mathcal{H}}\{\min_{x,y\in V(H_i)}\{|\mathcal{D}_{G\odot\mathcal{H}}(x,y)|\}\}\\
&= \min_{H_i\in\mathcal{H}}\{\min_{x,y\in V(H_i)}\{|\mathcal{C}_{H_i}(x,y)|\}\}\\
&= \min_{H_i\in\mathcal{H}}\{\mathcal{C}(H_i)\}\\
&= \mathcal{C}(\mathcal{H}).
\end{aligned}$$

Therefore $\mathcal{C}(\mathcal{H}) = \min_{x,y\in V(G\odot\mathcal{H})}\{|\mathcal{D}_{G\odot\mathcal{H}}(x,y)|\}$ and, by Theorem 2.2, we conclude the proof. □

Notice that if every $H_i \in \mathcal{H}$ satisfies that $H_i \cong H$, then $\mathcal{C}(\mathcal{H}) = \mathcal{C}(H)$. Thus, the following result follows from Theorem 2.35.

Corollary 2.36. *Let G and H be two connected nontrivial graphs. Then $G \odot H$ is k-metric dimensional if and only if $k = \mathcal{C}(H)$.*

According to Theorem 2.35, if the corona graph $G \odot \mathcal{H}$ is k-metric dimensional, then the value of k is independent from the connected nontrivial graph G. Moreover, for any $x, y \in V(H_i)$ it holds $\mathcal{D}_{H_i}(x,y) \supseteq \mathcal{C}_{H_i}(x,y)$. Therefore, by Theorems 2.2 and 2.35 we deduce the following result.

Proposition 2.37. *Let $G\odot\mathcal{H}$ be a k-metric dimensional graph such that G is a connected nontrivial graph and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ is a family composed by nontrivial graphs, where H_i is k_i-metric dimensional for $i \in \{1, \ldots, n\}$. Then the following assertions hold:*

(i) $k \leq \min_{i\in\{1,\ldots,n\}}\{k_i\}$.

(ii) $k = k_j$ *if and only if* $\min_{i\in\{1,\ldots,n\}}\{\mathcal{C}(H_i)\} = \min_{x,y\in V(H_j)}\{|\mathcal{D}_{H_j}(x,y)|\}$.

(iii) *If* $k = k_j$, *then* $\mathcal{C}(H_j) = \min_{x,y\in V(H_j)}\{|\mathcal{D}_{H_j}(x,y)|\}$.

If a graph H has diameter $D(H) \leq 2$, then for every $x, y \in V(H)$ it holds $\mathcal{D}_H(x,y) = \mathcal{C}_H(x,y)$. Thus, the following result is deduced.

Corollary 2.38. *Let $G \odot \mathcal{H}$ be a k-metric dimensional graph where G is a connected nontrivial graph and $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ is a family composed by graphs such that H_i is k_i-metric dimensional and $D(H_i) \leq 2$, for every $i \in \{1, \ldots, n\}$. Then $k = \min_{i \in \{1,\ldots,n\}} \{k_i\}$.*

If $\mathsf{g}(H) \geq 5$, then for every $x, y \in V(H)$ we have that either $|N_H(x) \cap N_H(y)| = 1$ or $|N_H(x) \cap N_H(y)| = 0$. Hence, as a consequence of Theorem 2.35, the next result follows.

Corollary 2.39. *Let G be a connected nontrivial graph of order n and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family composed by δ-regular graphs where $\mathsf{g}(H_i) \geq 5$, for every $i \in \{1, \ldots, n\}$. Then $G \odot \mathcal{H}$ is a 2δ-metric dimensional graph.*

We would point out the following particular case of Corollary 2.39.

Remark 2.40. *Let G be a connected nontrivial graph. Then, for $n \geq 5$, the graph $G \odot C_n$ is 4-metric dimensional.*

If $x \in V(H)$ is an end vertex and $y \in V(H)$ is a support vertex of degree two which is adjacent to x, then $|\mathcal{C}_H(x,y)| = 3$. Thus, from Corollary 2.3 and Theorem 2.35 we deduce the following result.

Proposition 2.41. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs such that no graph belonging to $\mathcal{H}$ has twin vertices. If there exists $H \in \mathcal{H}$, having an end vertex whose support vertex has degree two, then $G \odot \mathcal{H}$ is a 3-metric dimensional graph.*

An interesting particular case of the result above is when the family $\mathcal{H}$ contains a path P_r of order $r \geq 4$ and no graph belonging to $\mathcal{H}$ has twin vertices. In such a case $G \odot \mathcal{H}$ is a 3-metric dimensional graph.

Chapter 3

On the (k,t)-metric dimension of graphs

Overview

The current chapter is concerned with finding formulae and bounds for the (k,t)-metric dimension of some graphs. We also describe some classes of graphs where these bounds are achieved. Despite we give some results for the (k,t)-metric dimension for any $t \geq 2$, we emphasize in the particular cases of the k-metric dimension and the k-adjacency dimension of a graph G, *i.e.*, the cases when $t \geq D(G)$ and $t = 2$, respectively. We also find exact values of the k-metric dimension of some families of lexicographic product graphs and Corona product graphs, or general lower and upper bounds, and express these in terms of invariants of the factor graphs.

3.1 On the (k,t)-metric dimension of graphs

As in the previous chapter, throughout this chapter, unless otherwise stated, we will consider t as an integer greater than one.

In this section we study the problem of computing or bounding the (k,t)-metric dimension of several families of graphs. The following result is direct consequence of the fact that any (k,t)-metric generator for G is also a $(k,t+1)$-metric generator for G.

Remark 3.1 (Monotony of the (k,t)-metric dimension with respect to t)**.** *Let G be a (k',t)-metric dimensional graph. Then for any $k \in \{1,\ldots,k'\}$*

and any integers r, s such that $2 \leq r < s$,

$$\dim_{k,r}(G) \geq \dim_{k,s}(G).$$

Moreover, if $r \geq D(G)$, then $\dim_{k,r}(G) = \dim_{k,s}(G)$.

Theorem 3.2 (Monotony of the (k,t)-metric dimension with respect to k)**.** *Let G be a (k,t)-metric dimensional graph and let r, s be two integers. If $1 \leq r < s \leq k$, then $\dim_{r,t}(G) < \dim_{s,t}(G)$.*

Proof. Let B be a (k,t)-metric basis of G and let $x \in B$. Since all pairs of different vertices in $V(G)$ are distinguished by at least k vertices of B, we have that $B - \{x\}$ is a $(k-1,t)$-metric generator for G and, as a consequence, $\dim_{k-1,t}(G) \leq |B - \{x\}| < |B| = \dim_{k,t}(G)$. Proceeding analogously, we obtain that $\dim_{k-1,t}(G) > \dim_{k-2,t}(G)$ and, by a finite repetition of the process we obtain the result. □

Corollary 3.3. *Let G be a (k,t)-metric dimensional graph of order $n \geq 2$.*

(i) *For any $r \in \{2, \ldots, k\}$, $\dim_{r,t}(G) \geq \dim_{r-1,t}(G) + 1$.*

(ii) *For any $r \in \{1, \ldots, k\}$, $\dim_{r,t}(G) \geq \dim_{1,t}(G) + (r-1)$.*

(iii) *For any $r \in \{1, \ldots, k\}$, $\dim_{r,t}(G) \leq n - (k-r)$.*

Let $\mathcal{D}_{k,t}(G)$ be the set obtained as the union of the sets $\mathcal{D}_{G,t}(x,y)$ that distinguish a pair of different vertices x, y whenever $|\mathcal{D}_{G,t}(x,y)| = k$, *i.e.*,

$$\mathcal{D}_{k,t}(G) = \bigcup_{|\mathcal{D}_{G,t}(x,y)|=k} \mathcal{D}_{G,t}(x,y).$$

Remark 3.4. *If G is a (k,t)-metric dimensional graph, then for any (k,t)-metric basis B we have $\mathcal{D}_{k,t}(G) \subseteq B$, and as a consequence, $\dim_{k,t}(G) \geq |\mathcal{D}_{k,t}(G)|$.*

Proof. Since every pair of different vertices x, y is distinguished only by the elements of $\mathcal{D}_{G,t}(x,y)$, if $|\mathcal{D}_{G,t}(u,v)| = k$ for some u, v of G, then for any (k,t)-metric basis B we have $\mathcal{D}_{G,t}(u,v) \subseteq B$, and as a consequence, $\mathcal{D}_{k,t}(G) \subseteq B$. Therefore, the result follows. □

The bound given in Remark 3.4 is tight. For instance, for $t \geq D(G)$ we will show in Proposition 3.23 that there exists a family of trees attaining this bound for every k. Other examples for any positive integer $t \geq 2$ can be derived from the following result.

Theorem 3.5. *Let G be a (k,t)-metric dimensional graph of order $n \geq 2$. Then $\dim_{k,t}(G) = n$ if and only if $V(G) = \mathcal{D}_{k,t}(G)$.*

Proof. Suppose that $V(G) = \mathcal{D}_{k,t}(G)$. Now, since every (k,t)-metric dimensional graph G satisfies that $\dim_{k,t}(G) \leq n$, by Remark 3.4 we obtain that $\dim_{k,t}(G) = n$.

On the other hand, let $\dim_{k,t}(G) = n$. Note that for every $a, b \in V(G)$, we have $|\mathcal{D}_{G,t}(a,b)| \geq k$. If there exists at least one vertex $x \in V(G)$ such that $x \notin \mathcal{D}_{k,t}(G)$, then for every $a, b \in V(G)$, we have $|\mathcal{D}_{G,t}(a,b) - \{x\}| \geq k$ and, as a consequence, $V(G) - \{x\}$ is a (k,t)-metric generator for G, which is a contradiction. Therefore, $V(G) = \mathcal{D}_{k,t}(G)$. $\square$

Corollary 3.6. *Let G be a graph of order $n \geq 2$. Then $\dim_{2,t}(G) = n$ if and only if every vertex of G belongs to a non-singleton twin equivalence class.*

We will show other examples of graphs that satisfy Theorem 3.5 for $k \geq 3$. Let $W_{1,n}$ be the wheel graph and $F_{1,n} = K_1 + P_n$ be the fan graph. Since $V(F_{1,4}) = \mathcal{D}_{3,t}(F_{1,4})$ and $V(W_{1,5}) = \mathcal{D}_{4,t}(W_{1,5})$, by Theorem 3.5 we have that $\dim_{3,t}(F_{1,4}) = 5$ and $\dim_{4,t}(W_{1,5}) = 6$.

Given two nontrivial graphs G and H, it holds that any pair of twin vertices $x, y \in V(G)$ or $x, y \in V(H)$ are also twin vertices in $G + H$. As a direct consequence of Corollary 3.6, the next result holds.

Remark 3.7. *Let G and H be two nontrivial graphs of order n_1 and n_2, respectively. If all the vertices of G and H are twin vertices, then $G + H$ is $(2,t)$-metric dimensional and*

$$\dim_{2,t}(G + H) = n_1 + n_2.$$

Note that in Remark 3.7, the graphs G and H could be non-connected. Moreover, G and H could be nontrivial empty graphs. For instance, $N_r + N_s$, where N_r, N_s, $r, s > 1$, is the complete bipartite graph $K_{r,s}$ which satisfies that $\dim_{2,t}(K_{r,s}) = r + s$.

In general, we can state the following result.

Remark 3.8. *Let G be a connected graph, and let $U_1, U_2, \ldots, U_r$ be the non-singleton twin equivalence classes of G. Then*

$$\dim_{2,t}(G) \geq \sum_{i=1}^{r} |U_i|.$$

Proof. Since for two different vertices $x, y \in V(G)$ we have that $\mathcal{D}_{2,t}(x,y) = \{x,y\}$ if and only if there exists a twin equivalence class U_i such that $x, y \in U_i$, we deduce

$$\mathcal{D}_{2,t}(G) = \bigcup_{i=1}^{r} U_i.$$

Therefore, by Remark 3.4 we conclude the proof. □

Notice that the previous result leads to Corollary 3.6, so this bound is tight. Now, we consider the connected graph G of order $r+t$ obtained from an empty graph N_r of order $r \geq 2$ and a path P_t of order $t \geq 2$ by connecting every vertex of N_r to a given leaf of P_t. In this case, there are t singleton classes and one false twin equivalence class, say U_1, of cardinality r. By the previous result we have $\dim_{2,t}(G) \geq |U_1| = r$ and, since U_1 is a $(2,t)$-metric generator for G, we conclude that $\dim_{2,t}(G) = r$.

In particular we can state the following result on the strong product graphs.

Theorem 3.9. *Let G and H be two nontrivial connected graphs of order n and n', respectively. Let $U_1, U_2, \ldots, U_r$ be the true twin equivalence classes of G. Then*

$$\dim_{2,t}(G \boxtimes H) \geq n' \sum_{i=1}^{r} |U_i|.$$

Moreover, if every vertex of G is a true twin, then

$$\dim_{2,t}(G \boxtimes H) = nn'.$$

Proof. For any two vertices $a, c \in U_i$ and $b \in V(H)$,

$$\begin{aligned} N_{G\boxtimes H}[(a,b)] &= N_G[a] \times N_H[b] \\ &= N_G[c] \times N_H[b] \\ &= N_{G\boxtimes H}[(c,b)]. \end{aligned}$$

Thus, (a,b) and (c,b) are true twin vertices. Hence,

$$\mathcal{D}_{2,t}(G \boxtimes H) \supseteq \bigcup_{i=1}^{r} U_i \times V(H).$$

Therefore, by Remark 3.4 we conclude $\dim_{2,t}(G \boxtimes H) \geq n' \sum_{i=1}^{r} |U_i|$.

Finally, if every vertex of G is a true twin, then $\bigcup_{i=1}^{r} U_i = V(G)$ and, as a consequence, we obtain $\dim_{2,t}(G \boxtimes H) = nn'$. □

It was shown in [21] that $\dim(G) = 1$ if and only if $G \cong P_n$. The following result is a generalization.

Theorem 3.10. *Let G be a nontrivial graph. Then $\dim_{1,t}(G) = 1$ if and only if $G \cong K_1 \cup P_r$ or $G \cong P_{r+1}$ for some $r \in \{1,\dots,t\}$.*

Proof. Note that one of the vertices of $K_1 \cup K_1$ and one of the vertices of P_2 form a corresponding $(1,t)$-metric basis in the corresponding graph. For every $r \in \{2,\dots,t\}$ one of the leaves of P_r and one of the leaves of P_{r+1} are $(1,t)$-metric bases of $K_1 \cup P_r$ and P_{r+1}, respectively. Thus, for any $r \in \{1,\dots,t\}$ we have that if $G \cong K_1 \cup P_r$ or $G \cong P_{r+1}$, then $\dim_{1,t}(G) = 1$.

Now, suppose that $\dim_{1,t}(G) = 1$. We first show that $\Delta(G) \le 2$. To this end, suppose that there exists a vertex u of $\deg(u) \ge 3$. Thus, for every $v \in V$ there exist two vertices $x, y \in N_G(u)$ such that $d_{G,t}(v,x) = d_{G,t}(v,y)$. Hence, $\dim_{1,t}(G) \ge 2$, which is a contradiction, and as a consequence, $\Delta(G) \le 2$. Note that no vertex belonging to any $(1,t)$-metric basis has degree two, since this vertex does not distinguish a pair of its neighbours. As a consequence of this facts, we deduce that each connected component of G is a path or an isolated vertex. If G has at least three connected components, then no vertex belonging to a connected component distinguishes two vertices belonging to other two connected components. Hence, G has at most two connected components. If G has only one connected component, then G is a path of order at most $t+1$. Note that if G is a path of order at least $t+2$, then for any leaf v of the path there exist two vertices x, y such that $d_{G,t}(v,x) = d_{G,t}(v,y) = t$, which is a contradiction. Now, consider that G has two connected components. If these connected components have order greater than one, then no vertex belonging to a connected component distinguish two vertices belonging to the other connected component. So, if G has two connected components, then one of them is an isolated vertex and the other is a path of order at most t. Note that if $G \cong K_1 \cup P_r$ for some $r > t$, then for any leaf v of P_r there exists a vertex $x \in V(P_r)$ such that $d_{G,t}(v,x) = d_{G,t}(v,y) = t$, where y is the isolated vertex, which is a contradiction. □

Proposition 3.11. *Let k,t be two integers such that $k \geq 3$ and $t \geq k-1$. For any path P_r of order $r \geq k+1$,*

$$\dim_{k,t}(P_r) \geq k+1.$$

Moreover, if $k+1 \leq r \leq 2t-k+3$, then

$$\dim_{k,t}(P_r) = k+1.$$

Proof. Let $V(P_r) = \{v_1, v_2, \ldots, v_r\}$ such that $v_i \sim v_{i+1}$ for every $i \in \{1, \ldots, r-1\}$, and let S be a (k,t)-metric basis of P_r. Since $|S| \geq k \geq 3$, we deduce that $S \cap (V(P_r) - \{v_1, v_r\}) \neq \emptyset$. For any vertex $w \in S \cap (V(P_r) - \{v_1, v_r\})$ there exist at least two vertices $u, v \in V(P_r)$ such that $d_{P_r,t}(w,u) = d_{P_r,t}(w,v)$. Hence, $|S| = \dim_{k,t}(P_r) \geq k+1$.

Let $S' = \left\{v_{\lceil \frac{r}{2} \rceil - \lfloor \frac{k}{2} \rfloor}, v_{\lceil \frac{r}{2} \rceil - \lfloor \frac{k}{2} \rfloor + 1}, \ldots, v_{\lceil \frac{r}{2} \rceil + \lceil \frac{k}{2} \rceil}\right\}$ for some $r \geq k+1$. Note that $|S'| = k+1$. If $r \leq 2t-k+3$, then for any pair of different vertices $u, v \in V(P_r)$ there exists at most one vertex $w \in S'$ such that $d_{P_r,t}(w,u) = d_{P_r,t}(w,v)$. Thus, for every pair of different vertices $x, y \in V(P_r)$, there exists at least k vertices of S' such that they distinguish x, y. So S' is a (k,t)-metric generator for P_r. Therefore, $\dim_k(P_r, t) \leq |S'| = k+1$ and, consequently, the result follows. □

We now consider the limit case of the trivial bound $\dim_{k,t}(G) \geq k$.

Proposition 3.12. *If G is a nontrivial graph, then $\dim_{k,t}(G) = k$ if and only if $k \in \{1,2\}$ and $G \cong K_1 \cup P_r$ or $G \cong P_{r+1}$ for all $r \in \{1, \ldots, t\}$.*

Proof. The case $k=1$ was studied in Theorem 3.10. On the other hand, note that all the vertices of $K_1 \cup K_1$ and all the vertices of P_2 form a $(2,t)$-metric basis of $K_1 \cup K_1$ and P_2, respectively. For every $r \in \{2, \ldots, t\}$ the leaves of P_r and the leaves of P_{r+1} form a $(2,t)$-metric basis of $K_1 \cup P_r$ and P_{r+1}, respectively. Thus, for any $r \in \{1, \ldots, t\}$ we have that if $G \cong K_1 \cup P_r$ or $G \cong P_{r+1}$, then $\dim_{2,t}(G) = 2$.

Now, suppose that $\dim_{k,t}(G) = k$ for some $k \geq 2$. By Corollary 3.3 (ii) we have that $k = \dim_{k,t}(G) \geq \dim_{1,t}(G) + k - 1$, and as a consequence, $\dim_{1,t}(G) = 1$. Hence, by Theorem 3.10 it follows that $G \cong K_1 \cup P_r$ or $G \cong P_{r+1}$ for any $r \in \{1, \ldots, t\}$ and we are done for $k \in \{1,2\}$.

From now on we assume that $k \geq 3$. If $G \cong P_{r+1}$, then by Corollary 2.3 we deduce that $r \geq 3$. As a consequence of this fact and by Proposition

3.11 we have that $\dim_{k,t}(G) = k+1$, which is a contradiction. Since the isolated vertex of $K_1 \cup P_r$ does not distinguish any pair of different vertices of P_r, we obtain that $\dim_{k,t}(K_1 \cup P_r) \geq \dim_{k,t}(P_r)$. By Corollary 2.3, if $G \cong K_1 \cup P_r$, then $r \geq 4$. According to this fact and by Proposition 3.11, we conclude that $k+1 = \dim_{k,t}(P_r) \leq \dim_{k,t}(K_1 \cup P_r)$, which is a contradiction again. Therefore, if $k \geq 3$, then for any nontrivial graph G we deduce that $\dim_{k,t}(G) \geq k+1$. $\square$

The following result allows to extend the results on the (k,t)-metric dimension of lexicographic product graphs $G \circ \mathcal{H}$ to results on the $(k,2)$-metric dimension of $G \circ \mathcal{H}$, and vice versa.

Theorem 3.13. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed by nontrivial graphs. A set $A \subseteq V(G \circ \mathcal{H})$ is a (k,t)-metric generator for $G \circ \mathcal{H}$ if and only if A is a $(k,2)$-metric generator for $G \circ \mathcal{H}$, and as a consequence,*

$$\dim_{k,t}(G \circ \mathcal{H}) = \dim_{k,2}(G \circ \mathcal{H}).$$

Proof. By definition, any $(k,2)$-metric generator for a graph is also a (k,r)-metric generator for $r \geq 3$. Considering that any $(k, D(G))$-metric generator for a graph G is also a (k,t)-metric generator for $t > D(G)$, we only need to prove that any $(k, D(G \circ \mathcal{H}))$-metric generator for $G \circ \mathcal{H}$ is also a $(k,2)$-metric generator. For simplicity, we will use the terminology of k-metric generators and k-adjacency generators. Let $V(G) = \{u_1, \ldots, u_n\}$, let S be a k-metric generator for $G \circ \mathcal{H}$, and let $S_i = S \cap (\{u_i\} \times V(H_i))$ for every $u_i \in V(G)$. We differentiate the following four cases for two different vertices (u_i, v), $(u_j, w) \in V(G \circ \mathcal{H})$.

Case 1. $i = j$. In this case $v \neq w$. By Remark 1.1, no vertex from S_l, $l \neq i$, distinguishes (u_i, v) and (u_i, w). So it holds that $|\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_i,w)) \cap S_i| \geq k$. Since for any vertex $(u_i, x) \in S_i$ we have that $d_{G\circ\mathcal{H}}((u_i,x),(u_i,v)) = d_{G\circ\mathcal{H},2}((u_i,x),(u_i,v))$ and $d_{G\circ\mathcal{H}}((u,x),(u_i,w)) = d_{G\circ\mathcal{H},2}((u_i,x),(u_i,w))$, we conclude that $k \leq |\mathcal{C}_{G\circ\mathcal{H}}((u_i,v),(u_i,w)) \cap S_i| = |\mathcal{C}_{G\circ\mathcal{H}}((u_i,v),(u_i,w)) \cap S|$.

Case 2. $i \neq j$ and u_i, u_j are true twins. By Remark 1.1, no vertex from S_l, $l \notin \{i,j\}$, distinguishes (u_i, v) and (u_j, w). So $|\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,w)) \cap (S_i \cup S_j)| \geq k$. Since for any vertex $(u,x) \in S_i \cup S_j$ we have that $d_{G\circ\mathcal{H}}((u,x),(u_i,v)) =$

$d_{G\circ\mathcal{H},2}((u,x),(u_i,v))$ and $d_{G\circ\mathcal{H}}((u,x),(u_j,w)) = d_{G\circ\mathcal{H},2}((u,x),(u_j,w))$, we conclude that $k \le |\mathcal{C}_{G\circ\mathcal{H}}((u_i,v),(u_j,w)) \cap (S_i \cup S_j)| = |\mathcal{C}_{G\circ\mathcal{H}}((u_i,v),(u_j,w)) \cap S|$.

Case 3. $i \neq j$ and u_i, u_j are false twins. Analogous to the previous case.

Case 4. $i \neq j$ and u_i, u_j are not twins. Hence, there exists $u_l \in V(G)-\{u_i,u_j\}$ such that $d_{G,2}(u_l,u_i) \neq d_{G,2}(u_l,u_j)$. Hence, for any vertex $(u_l,x) \in S_l$ we have that

$$d_{G\circ\mathcal{H},2}((u_l,x),(u_i,v)) = d_{G,2}((u_l,u_i) \neq d_{G,2}((u_l,u_j) = d_{G\circ\mathcal{H},2}((u_l,x),(u_j,w)).$$

According to Case 1, we have that $|S_l| \ge k$. Therefore, we conclude that $k \le |\mathcal{C}_{G\circ\mathcal{H}}((u_i,v),(u_j,w)) \cap S_l| \le |\mathcal{C}_{G\circ\mathcal{H}}((u_i,v),(u_j,w)) \cap S|$.

In conclusion, S is a k-adjacency generator for $G \circ \mathcal{H}$. The proof is complete. □

3.1.1 Large families of graphs having a common (k,t)-metric generator

Let B be a (k,t)-metric basis of a graph $G=(V,E)$, and let $D(G,t) = \min\{D(G),t\}$. For any $r \in \{0,1,\ldots,D(G,t)\}$ we define the set

$$\mathbf{B}_r(B) = \bigcup_{x\in B}\{y \in V:\ d_{G,t}(x,y) \le r\}.$$

In particular, $\mathbf{B}_0(B) = B$ and $\mathbf{B}_1(B) = \displaystyle\bigcup_{x\in B} N_G[x]$. Moreover, since B is a (k,t)-metric basis of G, $|\mathbf{B}_{D(G,t)-1}(B)| \ge |V|-1$.

Let $G=(V,E)$ be a connected graph that is not complete. Given a (k,t)-metric basis B of G we say that a graph $G'=(V,E')$ belongs to the family $\mathcal{G}_B(G)$ if and only if $N_{G'}(v) = N_G(v)$, for every $v \in \mathbf{B}_{D(G,t)-2}(B)$. In particular, if $t=2$ and G is not a complete graph, then $G'=(V,E')$ belongs to the family $\mathcal{G}_B(G)$ if and only if $N_{G'}(x) = N_G(x)$, for every $x \in B$. Moreover, if G is a complete graph, we define $\mathcal{G}_B(G) = \{G\}$. By the definition of $\mathcal{G}_B(G)$, we deduce the following remark.

Remark 3.14. *Let B be a (k,t)-metric basis of a connected graph G, and let $G' \in \mathcal{G}_B(G)$. Then for any $b \in B$ and $v \in \mathbf{B}_{D(G,t)-1}(B)$, $d_{G,t}(b,v) = d_{G',t}(b,v)$. Moreover, $\langle \mathbf{B}_{D(G,t)-2}(B)\rangle \cong \langle \mathbf{B}_{D(G',t)-2}(B)\rangle$.*

Notice that if $\mathbf{B}_{D(G,t)-2}(B) \subsetneq V$, then any graph $G' \in \mathcal{G}_B(G)$ is isomorphic to a graph $G^* = (V, E^*)$ whose edge set E^* can be partitioned into two sets E_1^*, E_2^*, where E_1^* consists of all edges of G having at least one vertex in $\mathbf{B}_{D(G)-2}(B)$ and E_2^* is a subset of edges of a complete graph whose vertex set is $V - \mathbf{B}_{D(G,t)-2}(B)$. Hence, if $l = \binom{|V(G) - \mathbf{B}_{D(G,t)-2}(B)|}{2}$, then $\mathcal{G}_B(G)$ contains 2^l different graphs, where some of them could be isomorphic.

Theorem 3.15. *Any (k,t)-metric basis B of a graph G is a (k,t)-metric generator for any graph $G' \in \mathcal{G}_B(G)$, and as a consequence,*

$$\dim_{k,t}(G') \leq \dim_{k,t}(G).$$

Proof. Assume that B is a (k,t)-metric basis of a graph $G = (V, E)$, and $G' \in \mathcal{G}_B(G)$. We shall show that B is a (k,t)-metric generator for G'. To this end, we take two different vertices $u, v \in V$. Since B is a (k,t)-metric basis of G, there exists $B_{uv} \subseteq B$ such that $|B_{uv}| \geq k$ and for every $x \in B_{uv}$ we have that $d_{G,t}(x,u) \neq d_{G,t}(x,v)$. Now, consider the following two cases for u, v.

(1) $u, v \in \mathbf{B}_{D(G,t)-1}(B)$. In this case, since for every $x \in B_{uv}$ we have that $d_{G,t}(x,u) \neq d_{G,t}(x,v)$, Remark 3.14 leads to $d_{G',t}(x,u) \neq d_{G',t}(x,v)$ for every $x \in B_{uv}$.

(2) $u \in \mathbf{B}_{D(G,t)-1}(B)$ and $v \notin \mathbf{B}_{D(G,t)-1}(B)$. By definition of $\mathbf{B}_{D(G,t)-1}(B)$, we deduce that $d_{G',t}(x,u) \leq D(G,t) - 1$ for every $x \in B_{uv}$. Since $v \notin \mathbf{B}_{D(G,t)-1}(B)$, we have that $d_{G',t}(x,v) = D(G,t)$ for every $x \in B_{uv}$. So, $d_{G',t}(x,u) \leq D(G,t) - 1 < D(G,t) = d_{G',t}(x,v)$ for every $x \in B_{uv}$.

Notice that since B is a (k,t)-metric basis of G, the case $u, v \notin \mathbf{B}_{D(G,t)-1}(B)$ is not possible. According to the two cases above, B is a (k,t)-metric generator for G'. Therefore, $\dim_{k,t}(G') \leq |B| = \dim_{k,t}(G)$. □

By Proposition 3.12 we have that if G is a nontrivial graph, then $\dim_{k,t}(G) = k$ if and only if $k \in \{1,2\}$ and $G \cong K_1 \cup P_r$ or $G \cong P_{r+1}$ for every $r \in \{1, \ldots, t\}$. Thus, for any graph G of order at least $t+2$, $\dim_{k,t}(G) \geq k+1$. Therefore, the next corollary is a direct consequence of Theorem 3.15.

Corollary 3.16. *Let B be a (k,t)-metric basis of a graph G of order $n \geq t+2$ and let $G' \in \mathcal{G}_B(G)$. If $\dim_{k,t}(G) = k+1$, then $\dim_{k,t}(G') = k+1$.*

Figure 3.1 shows some graphs belonging to the family $\mathcal{G}_B(G)$ having a common $(2,2)$-metric generator $B = \{v_2, v_3, v_4, v_5\}$. Moreover, as we shall see in Theorem 3.27, B is also a common $(2,2)$-metric basis for all graphs belonging to $\mathcal{G}_B(G)$. In this case, the family $\mathcal{G}_B(G)$ contains $2^{10} = 1024$ different graphs, where some of them could be isomorphic.

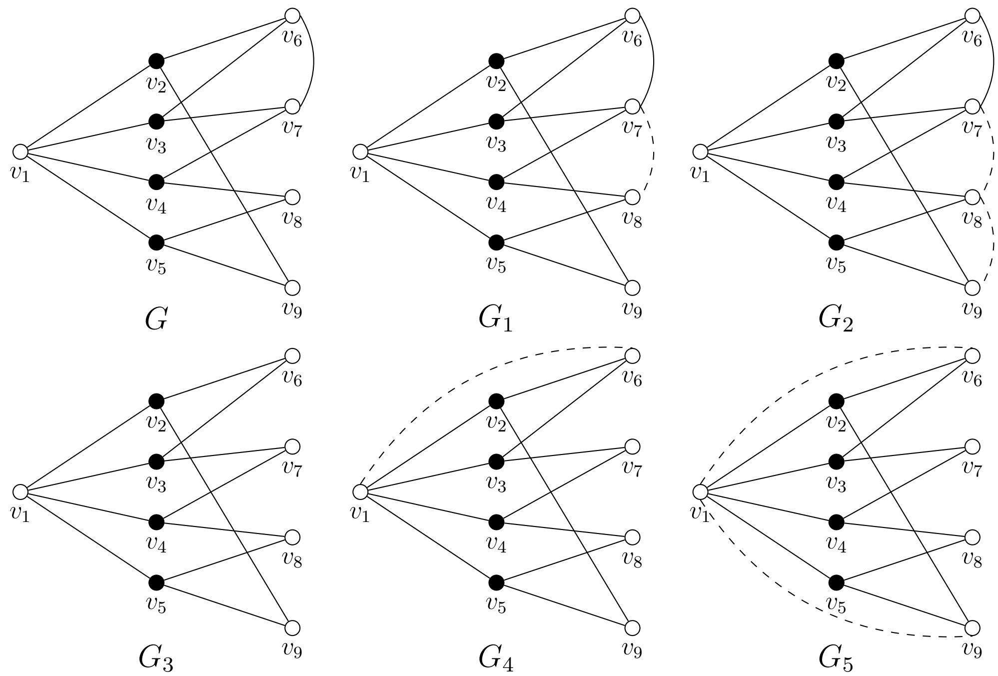

Figure 3.1: $B = \{v_2, v_3, v_4, v_5\}$ is a $(2,2)$-metric basis of G and $\{G, G_1, G_2, G_4, G_5\} \subset \mathcal{G}_B(G)$.

3.2 On the k-metric dimension of graphs

In this section we present some results that allow to compute the k-metric dimension of several families of graphs. We also give some tight bounds on the k-metric dimension of a graph.

We now present a lower bound for the k-metric dimension of a k'-metric dimensional graph G with $k' \geq k$. To this end, we require the use of the following function for any exterior major vertex $w \in V(G)$ having terminal degree greater than one, *i.e.*, $w \in \mathcal{M}(G)$. Notice that this function uses the concepts already defined in Subsection 2.2.1. Given an integer $r \leq k'$,

$$I_r(w) = \begin{cases} (\mathrm{ter}(w)-1)\,(r-l(w))+l(w), & \text{if } l(w) \leq \lfloor \frac{r}{2} \rfloor, \\ \\ (\mathrm{ter}(w)-1)\,\lceil \frac{r}{2} \rceil + \lfloor \frac{r}{2} \rfloor, & \text{otherwise.} \end{cases}$$

In Figure 2.3 we give an example of a graph G, which helps to clarify the notation above. Since every graph is at least 2-metric dimensional, we can consider the integer $r = 2$ and we have the following.

- Since $l(v_3) = 1 \leq \lfloor \frac{r}{2} \rfloor$, it follows that $I_r(v_3) = (\mathrm{ter}(v_3)-1)\,(r-l(v_3)) + l(v_3) = (3-1)(2-1)+1 = 3$.
- Since $l(v_5) = 1 \leq \lfloor \frac{r}{2} \rfloor$, it follows that $I_r(v_5) = (\mathrm{ter}(v_5)-1)\,(r-l(v_5)) + l(v_5) = (2-1)(2-1)+1 = 2$.
- Since $l(v_{15}) = 2 > \lfloor \frac{r}{2} \rfloor$, it follows that $I_r(v_{15}) = (\mathrm{ter}(v_{15})-1)\,\lceil \frac{r}{2} \rceil + \lfloor \frac{r}{2} \rfloor = (2-1)\,\lceil \frac{2}{2} \rceil + \lfloor \frac{2}{2} \rfloor = 2$.

Therefore, according to the result below, $\dim_2(G) \geq 3+2+2 = 7$.

Theorem 3.17. *If G is a k-metric dimensional graph such that $|\mathcal{M}(G)| \geq 1$, then for every $r \in \{1,\ldots,k\}$,*

$$\dim_r(G) \geq \sum_{w\in\mathcal{M}(G)} I_r(w).$$

Proof. Let S be an r-metric basis of G. Let $w \in \mathcal{M}(G)$ and let u_i, u_s be two different terminal vertices of w. Let u'_i, u'_s be the vertices adjacent to w in the paths $P(u_i,w)$ and $P(u_s,w)$, respectively. Notice that $\mathcal{D}_G(u'_i,u'_s) = V\,(P(u_i,w,u_s)) - \{w\}$ and, as a consequence, it follows that $|S \cap (V\,(P(u_i,w,u_s)) - \{w\})| \geq r$. Now, if $\mathrm{ter}(w) = 2$, then we have

$$|S \cap (V\,(P(u_i,w,u_s)) - \{w\})| \geq r = I_r(w).$$

Now, we assume $\mathrm{ter}(w) > 2$. Let W be the set of terminal vertices of w, and let u'_j be the vertex adjacent to w in the path $P(u_j,w)$ for every $u_j \in W$. Let $U(w) = \bigcup_{u_j\in W} V\,(P(u_j,w)) - \{w\}$ and let $x = \min_{u_j\in W}\{|S \cap V\,(P(u_j,w))|\}$. Since S is an r-metric generator of minimum cardinality (it is an r-metric basis of G), it is satisfied that $0 \leq x \leq \min\{l(w), \lfloor \frac{r}{2} \rfloor\}$. Let u_α be a terminal vertex such that $|S \cap (V\,(P(u_\alpha,w)) - \{w\})| = x$. Since for every terminal vertex $u_\beta \in W - \{u_\alpha\}$ we have that $|S \cap \mathcal{D}_G(u'_\beta, u'_\alpha)| \ge r$, it follows that $|S \cap (V(P(u_\beta, w)) - \{w\})| \ge r - x$. Thus,

$$\begin{aligned}|S \cap U(w)| &= \sum_{\beta=1,\beta\neq\alpha}^{\operatorname{ter}(w)} |S \cap (V(P(u_\beta, w)) - \{w\})| + \\ &\quad + |S \cap (V(P(u_\alpha, w)) - \{w\})| \\ &\ge (\operatorname{ter}(w) - 1)(r - x) + x.\end{aligned}$$

Now, if $x = 0$, then $|S \cap U(w)| \ge (\operatorname{ter}(w) - 1)\, r > I_r(w)$. On the contrary, if $x > 0$, then the function $f(x) = (\operatorname{ter}(w) - 1)(r - x) + x$ is decreasing with respect to x. So, the minimum value of f is achieved in the highest possible value of x. Thus, $|S \cap U(w)| \ge I_r(w)$. Since $\bigcap_{w \in \mathcal{M}(G)} U(w) = \emptyset$, it follows that

$$\dim_r(G) \ge \sum_{w \in \mathcal{M}(G)} |S \cap U(w)| \ge \sum_{w \in \mathcal{M}(G)} I_r(w).$$

□

Now, in order to give some consequences of the bound above we shall use some notation defined in Subsection 2.2.1 to introduce the following parameter.

$$\mu(G) = \sum_{v \in \mathcal{M}(G)} \operatorname{ter}(v).$$

Notice that for $k = 1$ Theorem 3.17 leads to the bound on the metric dimension of a graph, established by Chartrand *et al.* in [21]. In such a case, $I_1(w) = \operatorname{ter}(w) - 1$ for all $w \in \mathcal{M}(G)$ and thus,

$$\dim_1(G) \ge \sum_{w \in \mathcal{M}(G)} (\operatorname{ter}(w) - 1) = \mu(G) - |\mathcal{M}(G)|.$$

Next we give the particular cases of Theorem 3.17 for $r = 2$ and $r = 3$.

Corollary 3.18. *If G is a connected graph, then*

$$\dim_2(G) \ge \mu(G).$$

Proof. If $\mathcal{M}(G) = \emptyset$, then $\mu(G) = 0$ and the result is direct. Suppose that $\mathcal{M}(G) \neq \emptyset$. Since $I_2(w) = \operatorname{ter}(w)$ for all $w \in \mathcal{M}(G)$, we deduce that

$$\dim_2(G) \ge \sum_{w \in \mathcal{M}(G)} \operatorname{ter}(w) = \mu(G).$$

□

Corollary 3.19. *If G is k-metric dimensional for some $k \geq 3$, then*

$$\dim_3(G) \geq 2\mu(G) - |\mathcal{M}(G)|.$$

Proof. If $\mathcal{M}(G) = \emptyset$, then the result is direct. Suppose that $\mathcal{M}(G) \neq \emptyset$. Since $I_3(w) = 2\,\mathrm{ter}(w) - 1$ for all $w \in \mathcal{M}(G)$, we obtain that

$$\dim_3(G) \geq \sum_{w \in \mathcal{M}(G)} (2\,\mathrm{ter}(w) - 1) = 2\mu(G) - |\mathcal{M}(G)|.$$

□

In the next subsection we give some results concerning trees which show that the bounds proved in Theorem 3.17 and Corollaries 3.18 and 3.19 are tight. Specifically those results are Theorem 3.20 and Corollaries 3.21 and 3.22, respectively.

3.2.1 On the k-metric dimension of trees

Since any path is a particular case of a tree and its behaviour with respect to the k-metric dimension is relatively different, here we analyse them in a first instance. In Proposition 3.12 we noticed that for $k \in \{1, 2\}$ the k-metric dimension of a path $P_n (n \geqslant 2)$ is k. On the other hand, by Proposition 3.11 we deduce that for any integer $k \geq 3$ and any path graph P_n of order $n \geq k+1$, we have that $\dim_k(P_n) = k+1$.

We now continue with a formula for the r-metric dimension of any k-metric dimensional tree different from a path which, among other usefulness, shows that Theorem 3.17 is tight. In this proof we use the concept of branch already defined in Subsection 2.2.2.

Theorem 3.20. *If T is a tree which is not a path, then for any $r \in \{1, \ldots, \varsigma(T)\}$,*

$$\dim_r(T) = \sum_{w \in \mathcal{M}(T)} I_r(w).$$

Proof. Since T is not a path, T contains at least one vertex belonging to $\mathcal{M}(T)$. Let $w \in \mathcal{M}(T)$ and let $T_w = (V_w, E_w)$ be the w-branch. Also we consider the set $V' = V(T) - \bigcup_{w \in \mathcal{M}(T)} V_w$. For every $w \in \mathcal{M}(T)$, we suppose u_1 is a terminal vertex of w such that $l(u_1, w) = l(w)$. Let $U(w) = \{u_1, u_2, \ldots, u_s\}$ be the set of terminal vertices of w. Now, for every $u_j \in$

$U(w)$, let the path $P(u_j,w)=u_ju_j^1u_j^2\dots u_j^{l(u_j,w)-1}w$ and we consider the set $S(u_j,w)\subset V\left(P(u_j,w)\right)-\{w\}$ given by:

$$S(u_1,w)=\begin{cases}\left\{u_1,u_1^1,\dots,u_1^{l(w)-1}\right\}, & \text{if } l(w)\le\lfloor\frac{r}{2}\rfloor\\ \\ \left\{u_1,u_1^1,\dots,u_1^{\lfloor\frac{r}{2}\rfloor-1}\right\}, & \text{if } l(w)>\lfloor\frac{r}{2}\rfloor.\end{cases}$$

and for $j\ne 1$,

$$S(u_j,w)=\begin{cases}\left\{u_j,u_j^1,\dots,u_j^{r-l(w)-1}\right\}, & \text{if } l(w)\le\lfloor\frac{r}{2}\rfloor,\\ \\ \left\{u_j,u_j^1,\dots,u_j^{\lceil\frac{r}{2}\rceil-1}\right\}, & \text{if } l(w)>\lfloor\frac{r}{2}\rfloor.\end{cases}$$

According to this we have,

$$|S(u_j,w)|=\begin{cases}l(w), & \text{if } l(w)\le\lfloor\frac{r}{2}\rfloor \text{ and } u_j=u_1,\\ r-l(w), & \text{if } l(w)\le\lfloor\frac{r}{2}\rfloor \text{ and } u_j\ne u_1,\\ \lfloor\frac{r}{2}\rfloor, & \text{if } l(w)>\lfloor\frac{r}{2}\rfloor \text{ and } u_j=u_1,\\ \lceil\frac{r}{2}\rceil, & \text{if } l(w)>\lfloor\frac{r}{2}\rfloor \text{ and } u_j\ne u_1.\end{cases}$$

Let $S(w)=\displaystyle\bigcup_{u_j\in U(w)}S(u_j,w)$ and $S=\displaystyle\bigcup_{w\in\mathcal{M}(T)}S(w)$. Since for every $w\in\mathcal{M}(T)$ it follows that $\displaystyle\bigcap_{u_j\in U(w)}S(u_j,w)=\emptyset$ and $\displaystyle\bigcap_{w\in\mathcal{M}(T)}S(w)=\emptyset$, we obtain that $|S|=\displaystyle\sum_{w\in\mathcal{M}(T)}I_r(w)$.

Also notice that for every $w\in\mathcal{M}(T)$, such that $\mathrm{ter}(w)=2$ we have $|S(w)|=r$ and, if $\mathrm{ter}(w)>2$, then we have $|S(w)|\ge r+1$. We claim that S is an r-metric generator for T. Let u,v be two distinct vertices of T. We consider the following cases.

Case 1: $u,v\in V_w$ for some $w\in\mathcal{M}(T)$. We have the following subcases.

Subcase 1.1: $u,v\in V(P(u_j,w))$ for some $j\in\{1,\dots,\mathrm{ter}(w)\}$. Hence there exists at most one vertex of $S(w)\cap V(P(u_j,w))$ which does not distinguish u,v. If $\mathrm{ter}(w)=2$, then there exists at least one more exterior major vertex $w'\in\mathcal{M}(T)-\{w\}$. So, the elements of $S(w')$ distinguish u,v. Since $|S(w')|\ge r$, we deduce that at least r elements of S distinguish u,v. On the other hand, if $\mathrm{ter}(w)>2$, then since $|S(w)|\ge r+1$, we obtain that at least r elements of $S(w)$ distinguish u,v.

Subcase 1.2: $u \in V(P(u_j, w))$ and $v \in V(P(u_l, w))$ for some $j, l \in \{1, \ldots, \mathrm{ter}(w)\}$, $j \neq l$. According to the construction of the set $S(w)$, there exists at most one vertex of $(S(w) \cap (V(P(u_j, w, u_l)))$ which does not distinguish u, v.

Now, if $\mathrm{ter}(w) = 2$, then there exists $w' \in \mathcal{M}(T) - \{w\}$. If $d_T(u, w) = d_T(v, w)$, then the r elements of $S(w)$ distinguish u, v and, if $d_T(u, w) \neq d_T(v, w)$, then the elements of $S(w')$ distinguish u, v.

On the other hand, if $\mathrm{ter}(w) > 2$, then since $|S(w)| \geq r + 1$, we deduce that at least r elements of $S(w)$ distinguish u, v.

Case 2: $u \in V_w$ and $v \in V_{w'}$, for some $w, w' \in \mathcal{M}(T)$ with $w \neq w'$. In this case, either the vertices in $S(w)$ or the vertices in $S(w')$ distinguish u, v. Since $|S(w)| \geq r$ and $|S(w')| \geq r$ we have that u, v are distinguished by at least r elements of S.

Case 3: $u \in V'$ or $v \in V'$. Without loss of generality we assume $u \in V'$. Since $V' \neq \emptyset$, we have that there exist at least two different vertices in $\mathcal{M}(T)$. Hence, we have either one of the following situations.

- There exist two vertices $w, w' \in \mathcal{M}(T)$, $w \neq w'$, such that the shortest path from u to w and the shortest path from v to w' have empty intersection, or
- for every vertex $w'' \in \mathcal{M}(T)$, it follows that either v belongs to every shortest path from u to w'' or u belongs to every shortest path from v to w''.

Notice that in both situations, since $|S(w)| \geq r$, for every $w \in \mathcal{M}(T)$, we have that u, v are distinguished by at least r elements of S. In the first case, u and v are distinguished by the elements of $S(w)$ or by the elements of $S(w')$ and, in the second one, u and v are distinguished by the elements of $S(w'')$.

Therefore, S is an r-metric generator for T and, by Theorem 3.17, the proof is complete. $\square$

In the case $r = 1$, the formula of Theorem 3.20 leads to

$$\dim_1(T) = \mu(T) - |\mathcal{M}(T)|,$$

which is the result obtained in [21]. Other interesting particular cases are the following ones for $r = 2$ and $r = 3$, respectively. That is, by Theorem 3.20 we have the next results.

Corollary 3.21. *If T is a tree different from a path, then*

$$\dim_2(T) = \mu(T).$$

Corollary 3.22. *If T is a tree different from a path with $\varsigma(T) \geq 3$, then*

$$\dim_3(T) = 2\mu(T) - |\mathcal{M}(T)|.$$

As mentioned before, the two corollaries above show that the bounds given in Corollaries 3.18 and 3.19 are achieved.

We finish this subsection with a formula for the k-metric dimension of a k-metric dimensional tree with some specific structure. Given a graph G, we define $\mathcal{D}_k(G)$ as $\mathcal{D}_{k,t}(G)$ for $t \geq D(G)$. With this notation in mind, we show that the inequality $\dim_k(T) \geq |\mathcal{D}_k(T)|$, given in Remark 3.4, can be reached.

Proposition 3.23. *Let T be a tree different from a path and let $k \geq 2$ be an integer. If $\operatorname{ter}(w) = 2$ and $\varsigma(w) = k$ for every $w \in \mathcal{M}(T)$, then $\dim_k(T) = |\mathcal{D}_k(T)|$.*

Proof. Since every vertex $w \in \mathcal{M}(T)$ satisfies that $\operatorname{ter}(w) = 2$ and $\varsigma(w) = k$, we have that $\varsigma(T) = k$. Thus, by Theorem 2.13, T is k-metric dimensional tree. Since $I_k(w) = k$ for every $w \in \mathcal{M}(T)$, by Theorem 3.20 we have that $\dim_k(T) = k|\mathcal{M}(T)|$. Let u_r, u_s be the terminal vertices of w. As we have shown in the proof of Theorem 2.13, for every pair $x, y \in V(T)$ such that $x \notin V(P(u_r, w, u_s)) - \{w\}$ or $y \notin V(P(u_r, w, u_s)) - \{w\}$, it follows that x, y are distinguished by at least $k+1$ vertices of T and so $|\mathcal{D}^*_T(x, y)| > k-2$. Hence, if $|\mathcal{D}^*_T(x, y)| = k-2$, then $x, y \in V(P(u_r, w, u_s)) - \{w\}$ for some $w \in \mathcal{M}(T)$. If $d_T(x, w) \neq d_T(y, w)$, then x, y are distinguished by more than k vertices (those vertices not in $V(P(u_r, w, u_s)) - \{w\}$). Thus, if $|\mathcal{D}^*_T(x, y)| = k-2$, then $d_T(x, w) = d_T(y, w)$ and, as a consequence, $\mathcal{D}^*_T(x, y) = V(P(u_r, w, u_s)) - \{x, y, w\}$. Considering that $|V(P(u_r, w, u_s)) - \{w\}| = k$ and at the same time that $\bigcap_{w \in \mathcal{M}(T)} V(P(u_r, w, u_s)) = \emptyset$, we deduce $|\mathcal{D}_k(T)| = k|\mathcal{M}(T)|$. Therefore, $\dim_k(T) = |\mathcal{D}_k(T)|$. □

Figure 3.2 shows an example of a 3-metric dimensional tree. In this case $\mathcal{M}(T) = \{w, w'\}$, $\operatorname{ter}(w) = \operatorname{ter}(w') = 2$ and $\varsigma(w) = \varsigma(w') = 3$. Then Proposition 3.23 leads to $\dim_3(T) = |\mathcal{D}_3(T)| = |\{u_1, u_2, u_3, u'_1, u'_2, u'_3\}| = 6$.

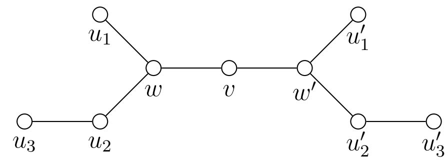

Figure 3.2: A 3-metric dimensional tree T for which $\dim_3(T) = |\mathcal{D}_3(T)| = 6$.

3.3 On the k-adjacency dimension of graphs

In this section we present some results that allow us to compute the k-adjacency dimension of several families of graphs. We also give some tight bounds on the k-adjacency dimension of a graph. For the graph G shown in Figure 3.3 we have $\dim_1(G) = 8 < 9 = \operatorname{adim}_1(G)$, $\dim_2(G) = 12 < 14 = \operatorname{adim}_2(G)$ and $\dim_3(G) = 20 = \operatorname{adim}_3(G)$. Note that the only 3-adjacency basis of G, and at the same time the only 3-metric basis, is $V(G) - \{0, 6, 12, 18\}$.

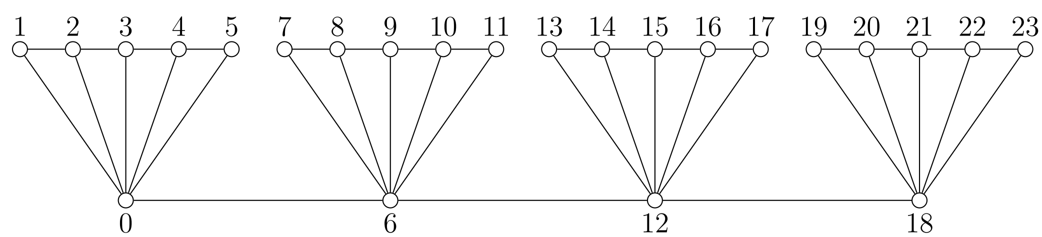

Figure 3.3: The set $\{2, 4, 6, 8, 10, 14, 16, 20, 21\}$ is a 1-adjacency basis of G, while the set $\{2l + 1 : \ l \in \{0, \ldots, 11\}\} \cup \{6, 12\}$ is a 2-adjacency basis and $V(G) - \{0, 6, 12, 18\}$ is a 3-adjacency basis.

In the same way, for the Petersen graph G we have that $\operatorname{adim}_6(G) = \operatorname{adim}_5(G) + 1 = \operatorname{adim}_4(G) + 2 = \operatorname{adim}_3(G) + 3 = 10$ and $\operatorname{adim}_2(G) = \operatorname{adim}_1(G) + 1 = 4$.

Since $\mathcal{C}_G(x, y) = \mathcal{C}_{\overline{G}}(x, y)$ for all $x, y \in V(G)$, we deduce the following result, which was previously observed for $k = 1$ by Jannesari and Omoomi in [67].

Remark 3.24. *For any nontrivial graph G and $k \in \{1, 2, \ldots, \mathcal{C}(G)\}$,*

$$\operatorname{adim}_k(G) = \operatorname{adim}_k(\overline{G}).$$

Moreover, A is a k-adjacency generator for G if and only if A is a k-adjacency generator for $\overline{G}$.

According to the Proposition 3.12, it is interesting to study the graphs where $\mathrm{adim}_k(G) = k+1$. To begin with, we state the following remark.

Remark 3.25. *If G is a graph of order $n \geq 7$, then* $\mathrm{adim}_1(G) \geq 3$.

Proof. Suppose, for purposes of contradiction, that $\mathrm{adim}_1(G) \leq 2$. By Proposition 3.12 we deduce that $\mathrm{adim}_1(G) = 2$. Let $B = \{u, v\}$ be an adjacency basis of G. Then for any $w \in V(G) - B$ the distance vector $(d_{G,2}(u,w), d_{G,2}(v,w))$ must belong to $\{(1,1),(1,2),(2,1),(2,2)\}$. Since $|V(G) - B| \geq 5$, by Dirichlet's box principle at least two elements of $V(G) - B$ have the same distance vector, which is a contradiction. Therefore, $\mathrm{adim}_1(G) \geq 3$. □

By Corollary 3.3 (ii) and Remark 3.25 we obtain the following result.

Theorem 3.26. *For any graph G of order $n \geq 7$ and $k \in \{1, \dots, \mathcal{C}(G)\}$,*

$$\mathrm{adim}_k(G) \geq k+2.$$

Our next result immediately follows from Theorems 3.15 and 3.26.

Theorem 3.27. *Let B be a k-adjacency basis of a graph G of order $n \geq 7$ and let $G' \in \mathcal{G}_B(G)$. If* $\mathrm{adim}_k(G) = k+2$, *then* $\mathrm{adim}_k(G') = k+2$.

An example of an application of the result above is shown in Figure 3.1, where $\mathrm{adim}_2(G') = 4$ for all $G' \in \mathcal{G}_B(G)$. In this case, as we mentioned above, $\mathcal{G}_B(G)$ contains $2^{10} = 1024$ different graphs.

From Remark 3.25 and Theorem 3.26, we only need to consider graphs of order $n \in \{3,4,5,6\}$ to determine those satisfying $\mathrm{adim}_k(G) = k+1$. If $n = 3$, then by Proposition 3.12 we conclude that $\mathrm{adim}_1(G) = 2$ or $\mathrm{adim}_2(G) = 3$ if and only if $G \in \{K_3, N_3\}$. For $k \in \{1,2\}$ and $n \in \{4,5,6\}$ the graphs satisfying $\mathrm{adim}_k(G) = k+1$ can be determined by a simple calculation. Here we just show some of these graphs in Figure 3.4. Finally, the cases $\mathrm{adim}_3(G) = 4$ and $\mathrm{adim}_5(G) = 5$ are studied in the following two remarks.

Remark 3.28. *A graph G of order greater than or equal to four satisfies* $\mathrm{adim}_3(G) = 4$ *if and only if $G \in \{P_4, C_5\}$.*

Proof. If $G \in \{P_4, C_5\}$, then it is straightforward to check that $\mathrm{adim}_3(G) = 4$. Assume that $B = \{v_1, \dots, v_4\}$ is a 3-adjacency basis of G. Since for any pair of vertices $v_i, v_j \in B$, there exists $v_l \in B \cap \mathcal{C}^*(v_i, v_j)$, by inspection we

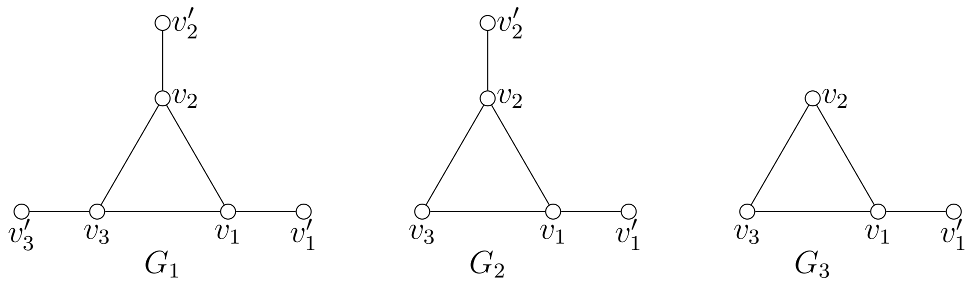

Figure 3.4: Any graph belonging to the families $\mathcal{G}_B(G_1)$, $\mathcal{G}_B(G_2)$ or $\{K_1 \cup K_3, G_3\}$, where $B = \{v_1, v_2, v_3\}$, satisfies $\mathrm{adim}_2(G) = 3$. The reader is referred to Subsection 3.1.1 for the construction of the families $\mathcal{G}_B(G_i)$.

can check that $\langle B \rangle \cong P_4$. We assume that $v_i \sim v_{i+1}$ for $i \in \{1, 2, 3\}$. If $V(G) - B = \emptyset$, then $G \cong P_4$. Suppose that there exists $v \in V(G) - B$. If $v \sim v_2$, then the fact that $|B \cap \mathcal{C}^*(v, v_1)| \geq 2$ leads to $v \sim v_3$ and $v \sim v_4$. Since $|B \cap \mathcal{C}^*(v, v_4)| \geq 2$ and $v \sim v_3$, it follows that $v \sim v_1$. Thus, v is connected to any vertex in B, which leads to $|B \cap \mathcal{C}^*(v, v_2)| = |\{v_4\}| = 1$, contradicting the fact that B is a 3-adjacency basis of G. Analogously if $v \sim v_3$, then we arrive to a similar contradiction. Thus, $v \sim v_1$ or $v \sim v_4$. If $v \sim v_1$ and $v \not\sim v_4$, then $|B \cap \mathcal{C}^*(v, v_2)| = |\{v_3\}| = 1$, contradicting the fact that B is a 3-adjacency basis of G. Now, if $v \sim v_1$ and $v \sim v_4$, then $G \cong C_5$. If $|V(G)| \geq 6$, then there exist $u, v \in V(G) - B$. Since $|B \cap \mathcal{C}(u, v)| \geq 3$, then either $|B \cap N(u)| \geq 2$ or $|B \cap N(v)| \geq 2$. Suppose that $|B \cap N(u)| \geq 2$. As discussed earlier, $B \cap N(u) = \{v_1, v_4\}$. Since $|B \cap \mathcal{C}(u, v)| \geq 3$, it follows that either $v \sim v_2$ or $v \sim v_3$, which, as we saw earlier, contradicts the fact that B is a 3-adjacency basis of G. □

By Corollary 3.3 (i) and Remark 3.28 we deduce that $\mathrm{adim}_4(G) \geq 6$ for any graph G of order at least five such that $G \not\cong C_5$. Since $\mathrm{adim}_4(C_5) = 5$, we obtain the following result.

Remark 3.29. *A graph G of order $n \geq 5$ satisfies that* $\mathrm{adim}_4(G) = 5$ *if and only if $G \cong C_5$.*

From Corollary 3.3 (i) and Remark 3.29, it follows that any 4-adjacency dimensional graph G of order six satisfies $\mathrm{adim}_4(G) = 6$, as the case of C_6.

3.4 On the k-metric dimension of product graphs

As we mentioned above, in this section we study the k-metric dimension of the lexicographic product graphs.

3.4.1 Lexicographic product graphs

In this subsection we focus into obtaining the (k,t)-metric dimension of the lexicographic product of graphs for $k \in \{1, \dots, \min\{\mathcal{T}(G\circ\mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$. Theorem3.13 allows us to give, without loss of generality, all results referred to the (k,t)-metric dimension of $G \circ \mathcal{H}$ in terms of $\mathrm{adim}_k(G \circ \mathcal{H})$ or $\dim_k(G \circ \mathcal{H})$ as we consider appropriate.

Join graphs

The following remark is a particular case of Corollary 3.6.

Remark 3.30. *Let H be a graph of order n. Then $\mathrm{adim}_2(K_1+H) = n+1$ if and only if $\Delta(H) = n-1$ and every vertex $v \in V(H)$ of degree $\delta(v) < n-1$ belongs to a non-singleton twin equivalence class.*

Proposition 3.31. *Let H be a graph of order $n \geq 2$ and $k \in \{1, \dots, \mathcal{C}(K_1 + H)\}$. Then*

$$\mathrm{adim}_k(K_1 + H) \geq \mathrm{adim}_k(H).$$

Proof. Let A be a k-adjacency basis of $K_1 + H$, $A_H = A \cap V(H)$ and let $x, y \in V(H)$ be two different vertices. Since $\mathcal{C}_{K_1+H}(x,y) = \mathcal{C}_H(x,y)$, it follows that $|A_H \cap \mathcal{C}_H(x,y)| = |A \cap \mathcal{C}_{K_1+H}(x,y)| \geq k$, and as a consequence, A_H is a k-adjacency generator for H. Therefore, $\mathrm{adim}_k(K_1 + H) = |A| \geq |A_H| \geq \mathrm{adim}_k(H)$. □

Theorem 3.32. *For any nontrivial graph H, the following assertions are equivalent:*

(i) *There exists a k-adjacency basis A of H such that $|A - N_H(y)| \geq k$, for all $y \in V(H)$.*

(ii) $\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H)$.

Proof. Let A be a k-adjacency basis of H such that $|A - N_H(y)| \geq k$, for all $y \in V(H)$. By Proposition 3.31 we have that $\mathrm{adim}_k(K_1 + H) \geq \mathrm{adim}_k(H)$. It remains to prove that $\mathrm{adim}_k(K_1 + H) \leq \mathrm{adim}_k(H)$. We will prove that A is a k-adjacency generator for $K_1 + H$. We differentiate two cases for two vertices $x, y \in V(K_1 + H)$. If $x, y \in V(H)$, then the fact that A is a k-adjacency basis of H leads to $k \leq |A \cap \mathcal{C}_H(x, y)| = |A \cap \mathcal{C}_{K_1+H}(x, y)|$. On the other hand, if x is the vertex of K_1 and $y \in V(H)$, then the fact that $\mathcal{C}_{K_1+H}(x, y) = \{x\} \cup (V(H) - N_H(y))$ and $|A - N_H(y)| \geq k$ leads to $|A \cap \mathcal{C}_{K_1+H}(x, y)| \geq k$. Therefore, A is a k-adjacency generator for $K_1 + H$, and as a consequence, $\mathrm{adim}_k(H) = |A| \geq \mathrm{adim}_k(K_1 + H)$.

On the other hand, let B be a k-adjacency basis of K_1+H such that $|B| = \mathrm{adim}_k(H)$ and let $B_H = B \cap V(H)$. Since for any $h_1, h_2 \in V(H)$ the vertex of K_1 does not belong to $\mathcal{C}_{K_1+H}(h_1, h_2)$, we conclude that B_H is a k-adjacency generator for H. Thus, $|B_H| = \mathrm{adim}_k(H)$ and, as a consequence, B_H is a k-adjacency basis of H. If there exists $h \in V(H)$ such that $|B_H - N_H(h)| < k$, then $|B \cap \mathcal{C}_{K_1+H}(v, h)| = |B_H - N_H(h)| < k$, which is a contradiction. Therefore, the result follows. □

Corollary 3.33. *Let H be a nontrivial graph such that* $\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H)$. *Then the vertex of K_1 does not belong to any k-adjacency basis A of $K_1 + H$.*

Proof. Let A be a k-adjacency basis of $K_1 + H$, let $x, y \in V(H)$ be two different vertices and let $A_H = A \cap V(H)$. Since $\mathcal{C}_{K_1+H}(x, y) = \mathcal{C}_H(x, y)$, it follows that $|A_H \cap \mathcal{C}_H(x, y)| = |A \cap \mathcal{C}_{K_1+H}(x, y)| \geq k$, and as a consequence, A_H is a k-adjacency generator for H. Thus, $\mathrm{adim}_k(H) = \mathrm{adim}_k(K_1 + H) = |A| \geq |A_H| \geq \mathrm{adim}_k(H)$, and as a consequence, $A_H = A$. Therefore, the vertex of K_1 does not belong to A. □

Our next result on graphs of diameter grater than or equal to six, is a consequence of Theorem 3.32.

Corollary 3.34. *For any graph H of diameter $D(H) \geq 6$ and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$,*

$$\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H).$$

Proof. Let S be a k-adjacency basis of H. We will show that $|S - N_H(x)| \geq k$, for all $x \in V(H)$. Suppose, for the purpose of contradiction, that there exists

$x \in V(H)$ such that $|S \cap (V(H) - N_H(x))| < k$. Let $F(x) = S \cap N_H[x]$. Notice that $|S| \geq k$ and hence $F(x) \neq \emptyset$.

From the assumptions above, if $V(H) = F(x) \cup \{x\}$, then $D(H) \leq 2$, which is a contradiction. If for every $y \in V(H) - (F(x) \cup \{x\})$ there exists $z \in F(x)$ such that $d_H(y,z) = 1$, then $d_H(v,v') \leq 4$ for all $v, v' \in V(H) - (F(x) \cup \{x\})$. Hence $D(H) \leq 4$, which is a contradiction. So, we assume that there exists a vertex $y' \in V(H) - (F(x) \cup \{x\})$ such that $d_H(y', z) > 1$, for every $z \in F(x)$, *i.e*, $N_H(y') \cap F(x) = \emptyset$. If $V(H) = F(x) \cup \{x, y'\}$, then by the connectivity of H we have $y' \sim x$ and, as consequence, $D(H) = 2$, which is also a contradiction. Hence, $V(H) - (F(x) \cup \{x, y'\}) \neq \emptyset$. Now, for any $w \in V(H) - (F(x) \cup \{x, y'\})$ we have that $|\mathcal{C}_H(y', w) \cap S| \geq k$ and, since $|S \cap (V(H) - N_H(x))| < k$ and $N_H(y') \cap F(x) = \emptyset$, we deduce that $N_H(w) \cap F(x) \neq \emptyset$. From this fact and the connectivity of H, we obtain that $d_H(y', w) \leq 5$. Hence $D(H) \leq 5$, which is also a contradiction. Therefore, if $D(H) \geq 6$, then for every $x \in V(H)$ we have that $|S \cap (V(H) - N_H(x))| \geq k$. Therefore, the result follows by Theorem 3.32. □

Corollary 3.35. *Let H be a graph of girth $\mathsf{g}(H) \geq 5$ and minimum degree $\delta(H) \geq 3$. Then for any $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$,*

$$\operatorname{adim}_k(K_1 + H) = \operatorname{adim}_k(H).$$

Proof. Let A be a k-adjacency basis of H and let $x \in V(H)$ and $y \in N_H(x)$. Since $\mathsf{g}(H) \geq 5$, for any $u, v \in N_H(y) - \{x\}$ we have that $\mathcal{C}_H(u,v) \cap N_H[x] = \emptyset$. Also, since $|\mathcal{C}_H(u,v) \cap A| \geq k$, we obtain that $|A - N_H(x)| \geq k$. Therefore, by Theorem 3.32 we conclude the proof. □

We now study the k-metric dimension of fan and wheel graphs. The case $k = 1$ for fan graphs was previously studied in [62] and for wheel graphs in [15].

Proposition 3.36. [15, 62]

(i) $$\operatorname{adim}_1(K_1 + P_n) = \begin{cases} 1, & \text{if } n = 1, \\ 2, & \text{if } n = 2,3,4,5, \\ 3, & \text{if } n = 6, \\ \left\lfloor \frac{2n+2}{5} \right\rfloor, & \text{otherwise.} \end{cases}$$

(ii) $$\operatorname{adim}_1(K_1 + C_n) = \begin{cases} 3, & \text{if } n = 3,6, \\ \left\lfloor \frac{2n+2}{5} \right\rfloor, & \text{otherwise.} \end{cases}$$

By Corollary 2.25, we know that the fan graphs $F_{1,n}$, $n \geq 4$, are 3-metric dimensional, so $\dim_k(F_{1,n})$ makes sense for $k \in \{1,2,3\}$. Thus, it only remains for us analyse the case $k \in \{2,3\}$. To this end, we will use the following notation. Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of the path P_n, and let $F_{1,n} = \langle u \rangle + P_n$. We assume that $u_i \sim u_{i+1}$ for each $i \in \{1, \ldots, n-1\}$.

We first present some useful lemmas.

Lemma 3.37. *Let $k \in \{2,3\}$ and let $n \geq 6$ be an integer. For any k-metric basis S of $F_{1,n}$ it holds $|S \cap V(P_n)| \geq 2k$.*

Proof. Notice that $\mathcal{D}_{F_{1,n}}(u_1,u_2) = \{u_1,u_2,u_3\}$ and $\mathcal{D}_{F_{1,n}}(u_{n-1},u_n) = \{u_{n-2}, u_{n-1}, u_n\}$. Since S is a k-metric basis of $F_{1,n}$, we have $|S \cap \mathcal{D}_{F_{1,n}}(u_1,u_2)| \geq k$ and $|S \cap \mathcal{D}_{F_{1,n}}(u_{n-1},u_n)| \geq k$. As $n \geq 6$, it holds $\mathcal{D}_{F_{1,n}}(u_1,u_2) \cap \mathcal{D}_{F_{1,n}}(u_{n-1},u_n) = \emptyset$. Therefore, $|S \cap V(P_n)| \geq 2k$. □

Lemma 3.38. *Let H be a nontrivial graph, let $K_1 + H$ be a k'-metric dimensional graph, and let $k \in \{1, \ldots, k'\}$. If for every k-metric basis S of $K_1 + H$ we have that $|S \cap V(H)| \geq k + \Delta(H)$, then the vertex of K_1 does not belong to any k-metric basis of $K_1 + H$.*

Proof. Let v be the vertex of K_1, and let S be a k-metric basis of $K_1 + H$. We will show that $S' = S - \{v\}$ is a k-metric generator for $K_1 + H$.

On the one hand, for every $x \in V(H)$ we have $|S' \cap \mathcal{D}_{K_1+H}(x,v)| = |S' \cap (V(H) - N_H(x))| \geq k$, as $|S' \cap V(H)| = |S \cap V(H)| \geq k + \Delta(H)$.

On the other hand, for any $x, y \in V(H)$ we have $|S' \cap \mathcal{D}_{K_1+H}(x,y)| = |S \cap \mathcal{D}_{K_1+H}(x,y)| \geq k$, as $v \notin \mathcal{D}_{K_1+H}(x,y)$.

Therefore, S' is a k-metric generator for $K_1 + H$ and, by the minimality of S, the set S' is a k-metric basis of $K_1 + H$. □

By performing some simple calculations, we observe that $\dim_2(F_{1,2}) = 3$, $\dim_2(F_{1,3}) = 4$, $\dim_2(F_{1,4}) = \dim_2(F_{1,5}) = 4$, and $\dim_3(F_{1,4}) = \dim_3(F_{1,5}) = 5$. The remaining values of $\dim_k(F_{1,n})$ are obtained in our next proposition.

Proposition 3.39. *For any integer $n \geq 6$,*

(i) $\dim_2(F_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil$.

(ii) $\dim_3(F_{1,n}) = n - \left\lfloor \frac{n-4}{5} \right\rfloor$

Proof.

(i) We shall prove that $A = \{u_i \in V(P_n) : i \equiv 1\ (2)\} \cup \{u_n\}$ is a 2-metric generator for $F_{1,n}$. Let x, y be two different vertices of $F_{1,n} = \langle u \rangle + P_n$.

If $x = u$, then $d_{F_{1,n}}(x, u_i) = 1$ for every $u_i \in V(P_n)$. Since $|A| \geq 4$ and there exist at most two vertices $u_j, u_l \in V(P_n)$ such that $d_{F_{1,n}}(y, u_j) = d_{F_{1,n}}(y, u_l) = 1$, we have $|\mathcal{D}_{F_{1,n}}(u, y) \cap A| \geq 2$.

Let us now assume that $x, y \in V(P_n)$. If $x, y \in A$, then they are distinguished by themselves and, if $x, y \notin A$, then there exist at least two vertices $u_i, u_j \in A$ such that $u_i, u_j \in N(x) \bigtriangledown N(y) \subset \mathcal{D}_{F_{1,n}}(x, y)$. Finally, if $x \in A$ and $y \notin A$, then there exists a vertex $u_l \in A - \{x\}$ such that $u_l \in N(y) - N(x)$. Therefore, A is a 2-metric generator for $F_{1,n}$ and, as a consequence, $\dim_2(F_{1,n}) \leq |A| = \left\lceil \frac{n+1}{2} \right\rceil$.

It remains to show that $\dim_2(F_{1,n}) \geq \left\lceil \frac{n+1}{2} \right\rceil$. With this aim, we take an arbitrary k-metric basis A' of $F_{1,n}$. Since $n \geq 6$, by Lemmas 3.37 and 3.38, $u \notin A'$. Notice that $\mathcal{D}_{F_{1,n}}(u_1, u_2) = \{u_1, u_2, u_3\}$ and $\mathcal{D}_{F_{1,n}}(u_{n-1}, u_n) = \{u_{n-2}, u_{n-1}, u_n\}$. Thus, $|A' \cap \{u_1, u_2, u_3\}| \geq 2$ and $|A' \cap \{u_{n-2}, u_{n-1}, u_n\}| \geq 2$. So, for $n = 6$, then $|A'| \geq 4$ and we are done. From now on, we consider $n \geq 7$. Let $M(P_n) = V(P_n) - \{u_1, u_2, u_3, u_{n-2}, u_{n-1}, u_n\}$. Assume for purposes of contradiction that $|A' \cap M(P_n)| \leq \left\lfloor \frac{n-6}{2} \right\rfloor - 1$. We consider the following three subcases.

(1) $n - 6 = 4p$ or $n - 6 = 4p + 1$ for some positive integer p. Let $Q_i = \{u_{4i}, u_{4i+1}, u_{4i+2}, u_{4i+3}\}$, $1 \leq i \leq p$. Notice that every $Q_i \subset M(P_n)$. Since $|A' \cap M(P_n)| < \left\lfloor \frac{n-6}{2} \right\rfloor = 2p$, there exists at least a set $Q_j = \{u_{4j}, u_{4j+1}, u_{4j+2}, u_{4j+3}\}$ such that $|Q_j \cap A'| \leq 1$. Since $\mathcal{D}_{F_{1,n}}(u_{4j+1}, u_{4j+2}) = \{u_{4j}, u_{4j+1}, u_{4j+2}, u_{4j+3}\}$, we deduce that u_{4j+1}, u_{4j+2} are distinguished by at most one vertex of A', which is a contradiction.

(2) $n - 6 = 4p + 2$ for some positive integer p. As above, let $Q_i = \{u_{4i}, u_{4i+1}, u_{4i+2}, u_{4i+3}\}$, $1 \leq i \leq p$. Notice that $M(P_n) = (\bigcup_{i=1}^{p} Q_i) \cup \{u_{4(p+1)}, u_{4(p+1)+1}\}$. If there exists at least one Q_i such that $|Q_i \cap A'| \leq 1$, then we have a contradiction as in the case above. Thus, $|Q_i \cap A'| \geq 2$ for all $1 \leq i \leq p$ and we

have

$$\begin{aligned} 2p &= \left\lfloor \frac{n-6}{2} \right\rfloor - 1 \\ &\geq |A' \cap M(P_n)| \\ &= \sum_{i=1}^{p} |Q_i \cap A'| + |A' \cap \{u_{4(p+1)}, u_{4(p+1)+1}\}| \\ &\geq 2p. \end{aligned}$$

As a consequence, it follows $|Q_j \cap A'| = 2$ for every $j \in \{1, \ldots, p\}$ and $A' \cap \{u_{4(p+1)}, u_{4(p+1)+1}\} = \emptyset$. Now, if $u_{4p+2}, u_{4p+3} \in A'$, then $u_{4p}, u_{4p+1} \notin A'$. Thus, u_{4p+1}, u_{4p+3} are distinguished only by u_{4p+3}, which is a contradiction. Conversely, if $u_{4p+2} \notin A'$ or $u_{4p+3} \notin A'$, then $|A' \cap \{u_{4p+2}, u_{4p+3}, u_{4(p+1)}, u_{4(p+1)+1}| \leq 1$ and, like in the previous case, we obtain that $u_{4p+3}, u_{4(p+1)}$ are distinguished by at most one vertex, which is also a contradiction.
(3) If $n - 6 = 4p + 3$, then we obtain a contradiction by proceeding analogously to Case 2 ($n - 6 = 4p + 2$).

Thus, $|A' \cap M(P_n)| \geq \left\lfloor \frac{n-6}{2} \right\rfloor$ and we obtain that $\dim_2(F_{1,n}) = |A'| = |A' \cap M(P_n)| + |A' \cap \mathcal{D}_{F_{1,n}}(u_1, u_2)| + |A' \cap \mathcal{D}_{F_{1,n}}(u_{n-1}, u_n)| \geq \left\lfloor \frac{n-6}{2} \right\rfloor + 4 = \left\lceil \frac{n+1}{2} \right\rceil$. Therefore, (i) follows.

(ii) Let $S = V(P_n) - \{u_i \in V(P_n) : i \equiv 0 \ (5) \ \wedge \ 1 \leq i \leq n-4\}$. Notice that $|S| = n - \left\lfloor \frac{n-4}{5} \right\rfloor$. We claim that S is a 3-metric generator for $F_{1,n}$. Let x, y be two different vertices of $F_{1,n}$.

If $x = u$, then $d_{F_{1,n}}(x, u_i) = 1$ for every $u_i \in V(P_n)$. Also, there exist at most two vertices $u_j, u_l \in V(P_n)$ such that $d_{F_{1,n}}(y, u_j) = d_{F_{1,n}}(y, u_l) = 1$. Since $|S| \geq 6$, the vertices x, y are distinguished by at least three vertices of S.

Now suppose $x, y \in V(P_n)$. According to the construction of S, there exist at least three different vertices $u_{i_1}, u_{i_2}, u_{i_3} \in S$ such that $d_{F_{1,n}}(x, u_{i_j}) \neq d_{F_{1,n}}(y, u_{i_j})$, with $j \in \{1, 2, 3\}$ (notice that x or y could be equal to some u_{i_j}, $j \in \{1, 2, 3\}$).

Thus, S is a 3-metric generator for $F_{1,n}$ and, as a result, $\dim_3(F_{1,n}) \leq |S| = n - \left\lfloor \frac{n-4}{5} \right\rfloor$.

It remains to show that $\dim_3(F_{1,n}) \geq n - \left\lfloor \frac{n-4}{5} \right\rfloor$. Now, let S' be a 3-metric basis of $F_{1,n}$. Since $n \geq 6$, by Lemmas 3.37 and 3.38, $u \notin S'$. Also, notice that two adjacent vertices u_i, u_{i+1} are distinguished by themselves and at least one neighbour u_{i-1} or u_{i+2}. So, at least three of them belong to S'. Now, if there exist three consecutive vertices $u_{i-1}, u_i, u_{i+1} \in S'$ such that $u_{i-2}, u_{i+2} \notin S'$, then the vertices u_{i-1}, u_{i+1} are not distinguished by at least three vertices of S', which is a contradiction. Thus, if two vertices $u_i, u_j \notin S'$, then $i - j \equiv 0 \ (5)$ and, as a consequence, per each five consecutive vertices of $V(P_n)$, at least four of them are in S', or equivalently, at most one does not belong to S'. Moreover, notice that $\mathcal{D}_{F_{1,n}}(u_1,u_2) = \{u_1,u_2,u_3\}$, $\mathcal{D}_{F_{1,n}}(u_1,u_3) = \{u_1,u_3,u_4\}$, $\mathcal{D}_{F_{1,n}}(u_{n-1},u_n) = \{u_{n-2},u_{n-1},u_n\}$, and $\mathcal{D}_{F_{1,n}}(u_{n-2},u_n) = \{u_{n-3},u_{n-2},u_n\}$. By Remark 3.4, $\{u_1,u_2,u_3,u_4,u_{n-3},u_{n-2},u_{n-1},u_n\} \subset S'$. Hence, $|\overline{S'}| \le \left\lfloor \frac{n-4}{5} \right\rfloor + 1$. Finally, we have that $\dim_3(F_{1,n}) = |S'| = n + 1 - |\overline{S'}| \ge n - \left\lfloor \frac{n-4}{5} \right\rfloor$. Therefore, $\dim_3(F_{1,n}) = n - \left\lfloor \frac{n-4}{5} \right\rfloor$. □

Let $V(C_n) = \{u_0, u_2, \ldots, u_{n-1}\}$ be the vertex set of the cycle C_n in $W_{1,n} = K_1 + C_n$, and let u be the central vertex of the wheel graph. From now on, all the operations with the subscripts of $u_i \in V(C_n)$ will be taken modulo n.

Since $W_{1,3}$ and $W_{1,4}$ have twin vertices, they are 2-metric dimensional graphs. Also, by Corollary 2.25 we know that the wheel graphs $W_{1,n}$, $n \ge 5$, are 4-metric dimensional, *i.e*, $\dim_k(W_{1,n})$ makes sense for $k \in \{1,2,3,4\}$. We now study $\dim_k(W_{1,n})$ for $k \in \{2,3,4\}$. To this end, we first give some useful results.

Lemma 3.40. *Let C_r be a cycle graph of order $r \ge 7$, and let $k \in \{2,3,4\}$. For any k-metric basis S of $W_{1,r}$ we have that $|S \cap V(C_r)| \ge k+2$.*

Proof. Let $V(C_r) = \{u_0, u_2, \ldots, u_{r-1}\}$ be the vertex set of the cycle C_r. The subscripts of $u_i \in V(C_r)$ will be taken modulo r. Notice that $\mathcal{D}_{W_{1,r}}(u_i, u_{i+1}) = \{u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$.

We first consider the case $r \ge 8$. Since $\mathcal{D}_{W_{1,r}}(u_i,u_{i+1}) \cap \mathcal{D}_{W_{1,r}}(u_{i+4},u_{i+5}) = \emptyset$, $|\mathcal{D}_{W_{1,r}}(u_i,u_{i+1}) \cap (S \cap V(C_r))| \ge k$ and $|\mathcal{D}_{W_{1,r}}(u_{i+4},u_{i+5}) \cap (S \cap V(C_r))| \ge k$, we deduce that $|S \cap V(C_r)| \ge 2k$. Thus, for $k \ge 2$ we have that $|S \cap V(C_r)| \ge k+2$.

We now consider the case $r = 7$. Since $\mathcal{D}_{W_{1,r}}(u_i,u_{i+1}) \cap \mathcal{D}_{W_{1,r}}(u_{i+4},u_{i+5}) = \{u_{i+6}\}$, in this case we have $|S \cap V(C_r)| \ge 2k-1$. So, for $k \in \{3,4\}$ it holds $|S| \ge k+2$. Now we take $k = 2$. Suppose that $|S \cap V(C_r)| = 3$. If $S \cap V(C_r)$ is composed by non-consecutive vertices, say $S \cap V(C_r) = \{u_i, u_{i+2}, u_{i+4}\}$, then $|\mathcal{D}_{W_{1,r}}(u_{i+4},u_{i+5}) \cap (S \cap V(C_r))| = 1$, which is a contradiction. If there are two consecutive vertices in $S \cap V(C_r)$, say $u_i, u_{i+1} \in S \cap V(C_r)$, then

$|\mathcal{D}_{W_{1,r}}(u_{i+3}, u_{i+4}) \cap (S \cap V(C_r))| \leq 1$, which is a contradiction. Hence, $|S \cap V(C_r)| \geq 4$ and, as a consequence, for $k=2$ we have that $|S \cap V(C_r)| \geq k+2$. □

By Lemmas 3.38 and 3.40 we deduce the next result.

Proposition 3.41. *Let C_r be a cycle graph of order $r \geq 7$, and let $k \in \{2,3,4\}$. Then the vertex of K_1 does not belong to any k-metric basis of $W_{1,r}$.*

Lemma 3.42. *Let H be a nontrivial graph, and let $K_1 + H$ be a k'-metric dimensional graph. Let $k \in \{1, \ldots, k'\}$ and $S \subseteq V(H)$. If for every $x, y \in V(H)$, $|S \cap \mathcal{D}_{K_1+H}(x,y)| \geq k$ and $|S| \geq k + \Delta(H)$, then S is a k-metric generator for $K_1 + H$.*

Proof. Let v be the vertex of K_1. Since for every $x, y \in V(H)$ we have that $|S \cap \mathcal{D}_{K_1+H}(x,y)| \geq k$, in order to prove that S is a k-metric generator for $K_1 + H$, it is enough proving that for every $x \in V(H)$ the condition $|\mathcal{D}_{K_1+H}(x,v) \cap S| \geq k$ is satisfied. Notice that for every $x \in V(H)$ we have that $\mathcal{D}_{K_1+H}(x,v) = (V(H) - N_H(x)) \cup \{v\}$. Since $|S| \geq k + \Delta(H)$, for every $x \in V(H)$ there exist k vertices $y \in S \cap (V(H) - N_H(x))$. Thus, for every $x \in V(H)$ it holds that $|\mathcal{D}_{K_1+H}(x,v) \cap S| \geq k$. Therefore, S is a k-metric generator for $K_1 + H$. □

By performing some simple calculations, we have that $\dim_2(W_{1,3}) = \dim_2(W_{1,4}) = \dim_2(W_{1,5}) = \dim_2(W_{1,6}) = 4$, $\dim_3(W_{1,5}) = \dim_3(W_{1,6}) = 5$, and $\dim_4(W_{1,5}) = \dim_4(W_{1,6}) = 6$. Next we present a formula for the k-metric dimension of wheel graphs for $n \geq 7$ and $k \in \{2,3,4\}$.

Proposition 3.43. *For any $n \geq 7$,*

(i) $\dim_2(W_{1,n}) = \left\lceil \frac{n}{2} \right\rceil$.

(ii) $\dim_3(W_{1,n}) = n - \left\lfloor \frac{n}{5} \right\rfloor$.

(iii) $\dim_4(W_{1,n}) = n$.

Proof. Since $n \geq 7$, by Proposition 3.41, the central vertex of $W_{1,n}$ does not belong to any k-metric basis of $W_{1,n}$. Thus, any k-metric basis of $W_{1,n}$ is a subset of $V(C_n)$. Let $S_k \subset V(C_n)$, $k \in \{2,3,4\}$, be a set of vertices of $W_{1,n}$ such that $|S_2| < \left\lceil \frac{n}{2} \right\rceil$, $|S_3| < n - \left\lfloor \frac{n}{5} \right\rfloor$, and $|S_4| < n$. We claim that

S_k is not a k-metric generator for $W_{1,n}$ with $k \in \{2,3,4\}$. Consider each S_k independently:

$k = 2$. Since $|S_2| < \left\lceil \frac{n}{2} \right\rceil$, there exist four consecutive vertices u_i, u_{i+1}, u_{i+2}, u_{i+3} such that at most one of them belongs to S_2. Thus, $|\mathcal{D}_{W_{1,n}}(u_{i+1}, u_{i+2}) \cap S_2| \leq 1$.

$k = 3$. Since $|S_3| < n - \left\lfloor \frac{n}{5} \right\rfloor$, there exist five consecutive vertices u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4} such that at most three of them belong to S_3. Thus, there exist four consecutive vertices $u_j, u_{j+1}, u_{j+2}, u_{j+3} \in \{u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\}$ such that at most two of them belong to S_3, with the exception of two cases. Hence, $|\mathcal{D}_{W_{1,n}}(u_{j+1}, u_{j+2}) \cap S_3| \leq 2$. The two exceptional cases are when either $u_{i+1}, u_{i+2}, u_{i+3} \in S_3$ or $u_i, u_{i+2}, u_{i+4} \in S_3$. In both cases, $|\mathcal{D}_{W_{1,n}}(u_{i+1}, u_{i+3}) \cap S_3| = 2$.

$k = 4$. Since $|S_4| < n$, there exist four consecutive vertices $u_i, u_{i+1}, u_{i+2}, u_{i+3}$ such that at most three of them belong to S_4. Thus, $|\mathcal{D}_{W_{1,n}}(u_{i+1}, u_{i+2}) \cap S_4| \leq 3$.

Therefore, as we claimed, S_k is not a k-metric generator for $W_{1,n}$, with $k \in \{2,3,4\}$, and so $\dim_2(W_{1,n}) \geq \left\lceil \frac{n}{2} \right\rceil$, $\dim_3(W_{1,n}) \geq n - \left\lfloor \frac{n}{5} \right\rfloor$ and $\dim_4(W_{1,n}) \geq n$.

Since $n \geq 7$, by Proposition 3.41, the central vertex of $W_{1,n}$ does not belong to any k-metric basis of $W_{1,n}$. Thus, $V(C_n)$ is a 4-metric generator for $W_{1,n}$ and, as a result, $\dim_4(W_{1,n}) = n$. It remains to show that $\dim_2(W_{1,n}) \leq \left\lceil \frac{n}{2} \right\rceil$ and $\dim_3(W_{1,n}) \leq n - \left\lfloor \frac{n}{5} \right\rfloor$. With this aim, let $A_k \subset V(C_n)$, $k \in \{2,3\}$, be a set of vertices such that u_i belongs to A_2 or A_3 if and only if i is odd or $i \not\equiv 0 \ (5)$, respectively. Notice that $|A_2| = \left\lceil \frac{n}{2} \right\rceil$ and $|A_3| = n - \left\lfloor \frac{n}{5} \right\rfloor$. We shall show that for every $u_i, u_j \in V(C_n)$, $i \neq j$, it holds $|\mathcal{D}_{W_{1,n}}(u_i, u_j) \cap A_k| \geq k$ and hence, by Lemmas 3.40 and 3.42, we have that A_k is a k-metric generator for $W_{1,n}$. Consider each A_k separately:

$k = 2$. If $u_i, u_j \in A_2$, then the result is straightforward. If $u_i \in A_2$ and $u_j \notin A_2$, then $\{u_i, u_k\} \subseteq A_2 \cap \mathcal{D}_{W_{1,n}}(u_i, u_j)$, for some $u_k \in N(u_j) - N[u_i]$. Also, if $u_i, u_j \notin A_2$, then $\{u_k, u_l\} \subseteq A_2 \cap \mathcal{D}_{W_{1,n}}(u_i, u_j)$, where $u_k, u_l \in N(u_i) \triangledown N(u_j)$.

$k = 3$. If $u_i, u_j \in A_3$, then $\{u_i, u_j, u_k\} \subseteq A_3 \cap \mathcal{D}_{W_{1,n}}(u_i, u_j)$, where $u_k \in$

$A_3 \cap (N[u_i]\nabla N[u_j])$. If $u_i \in A_3$ and $u_j \notin A_3$, then $\{u_i, u_k, u_l\} \subseteq A_3 \cap \mathcal{D}_{W_{1,n}}(u_i, u_j)$, where $u_k, u_l \in A_3 \cap (N[u_j]\nabla N[u_i])$. Finally, if $u_i, u_j \notin A_3$, then $\{u_k, u_l, u_m\} \subseteq A_3 \cap \mathcal{D}_{W_{1,n}}(u_i, u_j)$, where $u_k, u_l, u_m \in N(u_i) \cup N(u_j)$.

Therefore, A_k is a k-metric generator for $W_{1,n}$, with $k \in \{2,3\}$ and, as a consequence, the result follows. □

Proposition 3.44. [67] *For any integer $n \geq 4$,*

$$\mathrm{adim}_1(P_n) = \mathrm{adim}_1(C_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Notice that by Propositions 3.36 and 3.44, for any $n \geq 4$, $n \neq 6$, we have that

$$\mathrm{adim}_1(P_n) = \mathrm{adim}_1(K_1 + P_n) = \mathrm{adim}_1(C_n) = \mathrm{adim}_1(K_1 + C_n).$$

We now show the relationship between the k-adjacency dimension of fan (wheel) graphs and path (cycle) graphs. By Theorem 2.2 we have that any path graph of order at least four is 3-adjacency dimensional and any cycle graph of order at least five is 4-adjacency dimensional. From Propositions 3.31, 3.39 and 3.43 we will derive closed formulae for the k-adjacency dimension of paths (for $k \in \{2,3\}$) and cycles (for $k \in \{2,3,4\}$).

Proposition 3.45. *For any integer $n \geq 4$,*

$$\mathrm{adim}_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil \text{ and } \mathrm{adim}_3(P_n) = n - \left\lfloor \frac{n-4}{5} \right\rfloor.$$

Proof. Let $k \in \{2,3\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} for every $i \in \{1, \ldots, n-1\}$.

We first consider the case $n \geq 7$. Since $\mathcal{C}_{P_n}(v_1, v_2) = \{v_1, v_2, v_3\}$ and $\mathcal{C}_{P_n}(v_{n-1}, v_n) = \{v_{n-2}, v_{n-1}, v_n\}$, we deduce that for any k-adjacency basis A of P_n and any $y \in V(T)$, $|A - N_{P_n}(y)| \geq k$. Hence, Theorem 3.32 leads to $\mathrm{adim}_k(K_1 + P_n) = \mathrm{adim}_k(P_n)$. Therefore, by Proposition 3.39 we deduce the result for $n \geq 7$.

Now, for $n = 6$, since $\mathcal{C}_{P_6}(v_1, v_2) = \{v_1, v_2, v_3\}$ and $\mathcal{C}_{P_6}(v_5, v_6) = \{v_4, v_5, v_6\}$, we deduce that $\mathrm{adim}_2(P_6) \geq 4$ and $\mathrm{adim}_3(P_6) = 6$. In addition, $\{v_1, v_3, v_4, v_6\}$ is a 2-adjacency generator for P_6 and so $\mathrm{adim}_2(P_6) = 4$.

From now on, let $n \in \{4,5\}$. By Proposition 3.31 we have $\dim_k(K_1 + P_n) \geq \mathrm{adim}_k(P_n)$. It remains to prove that $\mathrm{adim}_k(K_1 + P_n) \leq \mathrm{adim}_k(P_n)$.

If $n = 4$ or $n = 5$, then by Proposition 3.12, $\operatorname{adim}_2(P_n) \geq 3$. Note that $\{v_1, v_2, v_4\}$ and $\{v_1, v_3, v_5\}$ are 2-adjacency generators for P_4 and P_5, respectively. Thus, $\operatorname{adim}_2(P_4) = \operatorname{adim}_2(P_5) = 3$. Let A be a 3-adjacency basis of P_n, where $n \in \{4, 5\}$. Since $\mathcal{C}_{P_n}(v_1, v_2) = \{v_1, v_2, v_3\}$ and $\mathcal{C}_{P_n}(v_{n-1}, v_n) = \{v_{n-2}, v_{n-1}, v_n\}$, we have that $(A \cap \mathcal{C}_{P_n}(v_1, v_2)) \cup (A \cap \mathcal{C}_{P_n}(v_{n-1}, v_n)) = V(P_n)$, and as consequence, $A = V(P_n)$. Therefore, $\operatorname{adim}_3(P_4) = 4$ and $\operatorname{adim}_3(P_5) = 5$ and, as a consequence, the result follows. $\square$

Proposition 3.46. *For any integer $n \geq 5$,*

$$\operatorname{adim}_2(C_n) = \left\lceil \frac{n}{2} \right\rceil, \ \operatorname{adim}_3(C_n) = n - \left\lfloor \frac{n}{5} \right\rfloor \textit{ and } \operatorname{adim}_4(C_n) = n.$$

Proof. Let $k \in \{2, 3, 4\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} and the subscripts are taken modulo n.

We first consider the case $n \geq 7$. Since $\mathcal{C}_{C_n}(v_{i+3}, v_{i+4}) = \{v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$, we deduce that for any k-adjacency basis A of C_n, $|A - N_{C_n}(v_i)| \geq k$. Hence, Theorem 3.32 leads to $\operatorname{adim}_k(K_1 + C_n) = \operatorname{adim}_k(C_n)$. Therefore, by Proposition 3.43 we deduce the result for $n \geq 7$.

From now on, let $n \in \{5, 6\}$. By Proposition 3.31 we have $\dim_k(K_1 + G) \geq \operatorname{adim}_k(G)$. It remains to prove that $\operatorname{adim}_k(K_1 + H) \leq \operatorname{adim}_k(H)$.

By Theorem 3.2, we deduce that $2 = \operatorname{adim}_1(C_5) < \operatorname{adim}_2(C_5) < \operatorname{adim}_3(C_5) < \operatorname{adim}_4(C_5) \leq 5$. Hence, $\operatorname{adim}_2(C_5) = 3$, $\operatorname{adim}_3(C_5) = 4$ and $\operatorname{adim}_4(C_5) = 5$. Therefore, for $n = 5$ the result follows.

By Theorem 3.2, $\operatorname{adim}_2(C_6) > \operatorname{adim}_1(C_6) = 2$ and, since $\{v_1, v_3, v_5\}$ is a 2-adjacency generator for C_6, we obtain that $\operatorname{adim}_2(C_6) = 3$. Now, let A_4 be a 4-adjacency basis of C_6. If $|A_4| \leq 5$, then there exists at least one vertex which does not belong to A_4, say v_1. Then, $|\mathcal{C}_{C_n}(v_1, v_2) \cap A_4| \leq 3$, which is a contradiction. Thus, $\operatorname{adim}_4(C_6) = |A_4| = 6$. Let $A_3^1 = \{v_1, v_2, v_3, v_4\}$, $A_3^2 = \{v_1, v_2, v_3, v_5\}$ and $A_3^3 = \{v_1, v_2, v_4, v_5\}$. Note that any manner of selecting four different vertices from C_6 is equivalent to some of these A_3^1, A_3^2, A_3^3. Since $|\mathcal{C}_{C_n}(v_5, v_6) \cap A_3^1| = |\{v_1, v_4\}| = 2 < 3$, $|\mathcal{C}_{C_n}(v_4, v_6) \cap A_3^2| = |\{v_1, v_3\}| = 2 < 3$ and $|\mathcal{C}_{C_n}(v_1, v_2) \cap A_3^3| = |\{v_1, v_2\}| = 2 < 3$, we deduce that $\operatorname{adim}_3(C_6) \geq 5 > |A_3^1| = |A_3^2| = |A_3^3| = 4$. By Theorem 3.2, $5 \leq \operatorname{adim}_3(C_6) < \operatorname{adim}_4(C_6) \leq 6$. Thus, $\operatorname{adim}_3(C_6) = 5$ and, as a consequence, the result follows. $\square$

By Propositions 3.45 and 3.46, $\operatorname{adim}_3(P_n) = n$ for $n \in \{4, \ldots, 8\}$ and $\operatorname{adim}_4(C_n) = n$ for $n \geq 5$. These are examples of graphs satisfying conditions of Theorem 3.5.

By Propositions 3.36, 3.39, 3.43, 3.44, 3.45 and 3.46 we observe that for any $k \in \{1,2,3\}$ and $n \geq 7$, $\operatorname{adim}_k(K_1 + P_n) = \operatorname{adim}_k(P_n)$ and for any $k \in \{1,2,3,4\}$, $\operatorname{adim}_k(K_1 + C_n) = \operatorname{adim}_k(C_n)$. The next result is devoted to characterize the trees where $\operatorname{adim}_k(K_1 + T) = \operatorname{adim}_k(T)$. To this end, we recall that the *eccentricity* of a vertex v in a connected graph G is the maximum distance between v and any other vertex u of G.

Proposition 3.47. *Let T be a tree. The following statements hold.*

(a) $\operatorname{adim}_1(K_1+T) = \operatorname{adim}_1(T)$ *if and only if* $T \notin \mathcal{F}_1 = \{P_2, P_3, P_6, K_{1,n}, T'\}$, *where $n \geq 3$ and T' is obtained from $P_5 \cup \{K_1\}$ by joining by an edge the vertex of K_1 to the central vertex of P_5.*

(b) $\operatorname{adim}_2(K_1+T) = \operatorname{adim}_2(T)$ *if and only if* $T \notin \mathcal{F}_2 = \{P_r, K_{1,n}, T'\}$, *where $r \in \{2,\ldots,5\}$, $n \geq 3$ and T' is a graph obtained from $K_{1,n} \cup K_2$ by joining by an edge one leaf of $K_{1,n}$ to one leaf of K_2.*

(c) $\operatorname{adim}_3(K_1+T) = \operatorname{adim}_3(T)$ *if and only if* $T \notin \mathcal{F}_3 = \{P_4, P_5\}$.

Proof. For any $k \in \{1,2,3\}$ and $T \in \mathcal{F}_k$, a simple inspection shows that $\operatorname{adim}_k(K_1 + T) \neq \operatorname{adim}_k(T)$. From now on, assume that $T \notin \mathcal{F}_k$, for $k \in \{1,2,3\}$, and let $\operatorname{Ext}(T)$ be the number of exterior major vertices of T. We differentiate the following three cases.

Case 1. $T = P_n$. The result is a direct consequence of combining Propositions 3.36 and 3.44 for $k = 1$ and Propositions 3.39 and 3.45 for $k > 1$.

In the following cases we shall show that there exists a k-adjacency basis A of T such that $|A - N_T(v)| \geq k$, for all $v \in V(T)$. Therefore, the result follows by Theorem 3.32.

Case 2. $\operatorname{Ext}(T) = 1$. Let u be the only exterior major vertex of T.

We first take $k = 1$. Since any two vertices adjacent to u must be distinguished by at least one vertex, we have that all paths from u to its terminal vertices, except at most one, contain at least one vertex in A. Thus, $|A - N_T(y)| \geq 1$, for all $y \in V(T) - \{u\}$. Now we shall show that $|A - N_T(u)| \geq 1$. If $u \in A$ or $A \not\subseteq N_T(u)$, then we are done, so we suppose that for any adjacency basis A of T, $u \notin A$ and $A \subseteq N_T(u)$. If there exists a leaf v such that $d_T(u,v) \geq 4$, then the support v' of v satisfies $\mathcal{C}_T(v,v') \cap A = \emptyset$, which is a contradiction. Hence, the eccentricity of u satisfies $2 \leq \epsilon(u) \leq 3$. If w is a leaf of T such that $d_T(u,w) = \epsilon(u)$, then the vertex $u' \in N_T(u)$ belonging to the path from u to w must belong to A and, as a consequence $A' = (A - \{u'\}) \cup \{w\}$ is an adjacency basis of T, which is a contradiction.

We now take $k = 2$. Let A be a 2-adjacency basis of T. Since any two vertices adjacent to u must be distinguished by at least two vertices in A, either all paths joining u to its terminal vertices contain at least one vertex of A or all but one contain at least two vertices of A. Thus, any vertex $y \in V(T) - \{u\}$ and any 2-adjacency basis A of T satisfy that $|A - N_T(y)| \geq 2$.

If there exist two vertices $v, v' \in V(T)$ such that $d_T(u,v) \geq 3$ and $d_T(u,v') \geq 3$, then $|A - N_T(u)| \geq 2$, as $|A \cap \mathcal{C}_2(v,v')| \geq 2$. On the other hand, if there exists only one leaf v such that $d_T(u,v) \geq 3$ and another leaf w such that $d_T(u,w) = 2$, we have that in order to distinguish v and its support as well as w and its support, $|A \cap N_T[v]| \geq 1$ and $|A \cap \{u,w\}| \geq 1$ and, as a result, $|A - N_T(u)| \geq 2$. Now, since $T \notin \mathcal{F}_2$ it remains to consider the case where u has eccentricity two. Let v, w be two leaves such that $d_T(u,v) = d_T(u,w) = 2$. If $|N_T(u)| = 3$, then the set A composed by u and its three terminal vertices is a 2-adjacency basis of T such that $|A - N_T(u)| \geq 2$. Assume that $|N_T(u)| \geq 4$. In order to distinguish v and its support vertex v', as well as w and its support vertex w', any 2-adjacency basis A of T must contain at least two vertices of $\{u,v,v'\}$ and at least two vertices of $\{u,w,w'\}$. If $u \notin A$, then $v, w \in A$, and as a consequence, $|A - N_T(u)| \geq 2$. Assume that $u \in A$. In this case, if $A - N_T[u] \neq \emptyset$, then $|A - N_T(u)| \geq 2$. Otherwise, $A \subseteq N_T[u]$ and $\{u, v', w'\} \subset A$ and, as a consequence, $A' = (A - \{v'\}) \cup \{v\}$ is a 2-adjacency basis of T and $|A' - N_T(u)| \geq 2$.

Finally, suppose that there exists exactly one leaf v such that $d_T(u,v) = 2$. Let v' be the support vertex of v. In this case, $V(T) - \{v'\}$ is a 2-adjacency basis A of T such that $|A - N_T(u)| \geq 2$.

We now take $k = 3$. In this case, there exist two leaves v, w such that $d_T(u,v) \geq 2$ and $d_T(u,w) \geq 2$. Since v and its support vertex v' must be distinguished by at least three vertices, they must belong to any 3-adjacency basis. Analogously, w and its support vertex w' must belong to any 3-adjacency basis. In general, any leaf that is not adjacent to u and its support vertex belong to any 3-adjacency basis of T. Moreover, there exists at most one terminal vertex x adjacent to u. If x exists, it must be distinguished from any vertex belonging to $N_T(u) - \{x\}$ by at least three vertices. Thus, they must belong to any 3-adjacency basis. Any vertex y different from u and any 3-adjacency basis A of T satisfy $v, v' \in A - N_T(y)$ or $w, w' \in A - N_T(y)$. If $v, v' \in A - N_T(y)$ and $w, w' \in A - N_T(y)$, then $|A - N_T(y)| \geq 3$. Otherwise, assuming without loss of generality that $v, v' \in A - N_T(y)$, there exists a terminal vertex z different from w such that $y \not\sim z$. Thus, again $|A - N_T(y)| \geq 3$. If $d_T(u, v) = 2$, then v, v' are distinguished only by u, v, v', so u must belong to any 3-adjacency basis of T. Thus, for any 3-adjacency basis A of T we have that $u, v, w \in A - N_T(u)$, and as a consequence, $|A - N_T(u)| \geq 3$. Finally, if $d_T(u, v) > 2$ and $d_T(u, w) > 2$, then $v, v', w, w' \in A - N_T(u)$. Hence $|A - N_T(u)| \geq 3$.

Case 3. $\mathrm{Ext}(T) \geq 2$. In this case, there are at least two exterior major vertices u, v of T having terminal degree at least two. Let u_1, u_2 be two terminal vertices of u and v_1, v_2 be two terminal vertices of v. Let u_1' and u_2' be the vertices adjacent to u in the paths $u - u_1$ and $u - u_2$, respectively. Likewise, let v_1' and v_2' be the vertices adjacent to v in the paths $v - v_1$ and $v - v_2$, respectively. Notice that it is possible that $u_1 = u_1'$, $u_2 = u_2'$, $v_1 = v_1'$ or $v_2 = v_2'$. Note also that $\mathcal{C}(u_1', u_2') = (N_T[u_1'] \cup N_T[u_2']) - \{u\}$ and $\mathcal{C}(v_1', v_2') = (N_T[v_1'] \cup N_T[v_2']) - \{v\}$. Since for any k-adjacency basis A of T it holds that $|\mathcal{C}(u_1', u_2') \cap A| \geq k$ and $|\mathcal{C}(v_1', v_2') \cap A| \geq k$, and for any vertex $w \in V(T)$ we have that $(A - N_T(w)) \cap \mathcal{C}(u_1', u_2') = \emptyset$ or $(A - N_T(w)) \cap \mathcal{C}(v_1', v_2') = \emptyset$, we conclude that $|A - N_T(w)| \geq k$. □

By Remark 3.24 we know that $\mathrm{adim}_k(H) = \mathrm{adim}_k(\overline{H})$ for any nontrivial graph H and $k \in \{1, 2, \ldots, \mathcal{C}(H)\}$. Thus, by Corollaries 3.33, 3.34, 3.35 and Propositions 3.36, 3.39, 3.43, 3.44, 3.45, 3.46 and 3.47 we deduce the following results.

Proposition 3.48. *Let H be a nontrivial graph such that* $\mathrm{adim}_k(K_1 \cup \overline{H}) = \mathrm{adim}_k(H)$. *Then the vertex of K_1 does not belong to any k-adjacency basis of $K_1 \cup \overline{H}$.*

Proposition 3.49. *If H is a graph of diameter $D(H) \geq 6$, or H has girth $g(H) \geq 5$ and minimum degree $\delta(H) \geq 3$, then* $\mathrm{adim}_k(K_1 \cup \overline{H}) = \mathrm{adim}_k(H) = \mathrm{adim}_k(\overline{H})$ *for any $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$.*

Proposition 3.50. *Let T be a tree. The following statements hold.*

(a) $\operatorname{adim}_1(K_1 \cup \overline{T}) = \operatorname{adim}_1(T) = \operatorname{adim}_1(\overline{T})$ *if and only if* $T \notin \mathcal{F}_1 = \{P_2, P_3, P_6, K_{1,n}, T'\}$*, where* $n \geq 3$ *and* T' *is obtained from* $P_5 \cup K_1$ *by joining by an edge the vertex of* K_1 *to the central vertex of* P_5*.*

(b) $\operatorname{adim}_2(K_1 \cup \overline{T}) = \operatorname{adim}_2(T) = \operatorname{adim}_2(\overline{T})$ *if and only if* $T \notin \mathcal{F}_2 = \{P_r, K_{1,n}, T'\}$*, where* $r \in \{2, \ldots, 5\}$*,* $n \geq 3$ *and* T' *is a graph obtained from* $K_{1,n} \cup K_2$ *by joining by an edge one leaf of* $K_{1,n}$ *to one leaf of* K_2*.*

(c) $\operatorname{adim}_3(K_1 \cup \overline{T}) = \operatorname{adim}_3(T) = \operatorname{adim}_3(\overline{T})$ *if and only if* $T \notin \mathcal{F}_3 = \{P_4, P_5\}$*.*

Moreover, if T *is a path* P_n *of order* n*, then*

(i) $\operatorname{adim}_1(K_1 \cup \overline{P_n}) = \operatorname{adim}_1(P_n) = \operatorname{adim}_1(\overline{P_n}) = \left\lfloor \frac{2n+2}{5} \right\rfloor$ *for* $n \geq 7$*.*

(ii) $\operatorname{adim}_2(K_1 \cup \overline{P_n}) = \operatorname{adim}_2(P_n) = \operatorname{adim}_2(\overline{P_n}) = \left\lceil \frac{n+1}{2} \right\rceil$ *for* $n \geq 6$*.*

(iii) $\operatorname{adim}_3(K_1 \cup \overline{P_n}) = \operatorname{adim}_3(P_n) = \operatorname{adim}_3(\overline{P_n}) = n - \left\lfloor \frac{n-4}{5} \right\rfloor$ *for* $n \geq 6$*.*

Proposition 3.51. *For any cycle* C_n *of order* $n \geq 7$*, the following statements hold.*

(i) $\operatorname{adim}_1(K_1 \cup \overline{C_n}) = \operatorname{adim}_1(C_n) = \operatorname{adim}_1(\overline{C_n}) = \left\lfloor \frac{2n+2}{5} \right\rfloor$*.*

(ii) $\operatorname{adim}_2(K_1 \cup \overline{C_n}) = \operatorname{adim}_2(C_n) = \operatorname{adim}_2(\overline{C_n}) = \left\lceil \frac{n}{2} \right\rceil$*.*

(iii) $\operatorname{adim}_3(K_1 \cup \overline{C_n}) = \operatorname{adim}_3(C_n) = \operatorname{adim}_3(\overline{C_n}) = n - \left\lfloor \frac{n}{5} \right\rfloor$*.*

(iv) $\operatorname{adim}_4(K_1 \cup \overline{C_n}) = \operatorname{adim}_4(C_n) = \operatorname{adim}_4(\overline{C_n}) = n$*.*

From now on, we shall study some cases where $\operatorname{adim}_k(K_1+H) > \operatorname{adim}_k(H)$. First of all, notice that by Corollary 3.34, if H is a connected graph and $\operatorname{adim}_k(K_1 + H) \geq \operatorname{adim}_k(H) + 1$, then $D(H) \leq 5$ and, by Corollary 3.35, if H has minimum degree $\delta(H) \geq 3$, then it has girth $\mathsf{g}(H) \leq 4$. We would point out the following consequence of Theorem 3.32.

Corollary 3.52. *If* $\operatorname{adim}_k(K_1 + H) \geq \operatorname{adim}_k(H) + 1$*, then either* H *is connected or* H *has exactly two connected components, one of which is an isolated vertex.*

Proof. Let A be a k-adjacency basis of H. We differentiate three cases for H.

Case 1. There are two connected components H_1 and H_2 of H such that $|V(H_1)| \geq 2$ and $|V(H_2)| \geq 2$. As for any $i \in \{1, 2\}$ and $u, v \in V(H_i)$,

$|C_H(u,v) \cap A| = |C_{H_i}(u,v) \cap A| \geq k$ we deduce that $|A \cap V(H_1)| \geq k$ and $|A \cap V(H_2)| \geq k$. Hence, if $x \in V(H_1)$, then $|A - N_H(x)| \geq |A \cap V(H_2)| \geq k$ and if $x \in V(H) - V(H_1)$, then $|A - N_H(x)| \geq |A \cap V(H_1)| \geq k$. Thus, by Theorem 3.32, $\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H)$.

Case 2. There is a connected component H_1 of H such that $|V(H_1)| \geq 2$ and there are two isolated vertices $u, v \in V(H)$. From $C_H(u,v) = \{u,v\}$ we conclude that $k \leq 2$ and $|\{u,v\} \cap A| \geq k$. Moreover, for any $x, y \in V(H_1)$, $x \neq y$, we have that $|C_H(x,y) \cap A| = |C_{H_1}(u,v) \cap A| \geq k$ and so $|A \cap V(H_1)| \geq k$. Hence, if $x \in V(H_1)$, then $|A - N_H(x)| \geq |\{u,v\} \cap A| \geq k$ and if $x \in V(H) - V(H_1)$, then $|A - N_H(x)| \geq |A \cap V(H_1)| \geq k$. Thus, by Theorem 3.32, $\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H)$.

Case 3. $H \cong N_n$, for $n \geq 2$. In this case $k \in \{1,2\}$, $\mathrm{adim}_1(K_1 + N_n) = \mathrm{adim}_1(N_n) = n - 1$ and $\mathrm{adim}_2(K_1 + N_n) = \mathrm{adim}_2(N_n) = n$.

Therefore, according to the three cases above, the result follows. $\square$

By Proposition 3.31 and Theorem 3.32, $\mathrm{adim}_k(K_1 + H) \geq \mathrm{adim}_k(H) + 1$ if and only if for any k-adjacency basis A of H, there exists $h \in V(H)$ such that $|A - N_H(h)| < k$. Consider, for instance, the graph G shown in Figure 3.1. The only 2-adjacency basis of G is $B = \{v_2, v_3, v_4, v_5\}$ and $|B - N_G(v_1)| = 0$, so $\mathrm{adim}_2(K_1 + G) \geq \mathrm{adim}_2(G) + 1 = 5$. It is easy to check that $A = \{v_1, v_6, v_7, v_8, v_9\}$ is a 2-adjacency generator for $K_1 + G$, and so $\mathrm{adim}_2(K_1 + G) = \mathrm{adim}_2(G) + 1 = 5$. We emphasize that neither $B \cup \{v_1\}$ nor $B \cup \{x\}$ are 2-adjacency bases of $\langle x \rangle + G$.

Proposition 3.53. *Let H be a graph of order $n \geq 2$ and let $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$. If for any k-adjacency basis A of H, there exists $h \in V(H)$ such that $|A - N_H(h)| = k - 1$ and $|A - N_H(h')| \geq k - 1$, for all $h' \in V(H)$, then*

$$\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H) + 1.$$

Proof. If for any k-adjacency basis A of H, there exists $h \in V(H)$ such that $|A - N_H(h)| = k - 1$, then by Theorem 3.32, $\mathrm{adim}_k(K_1 + H) \geq \mathrm{adim}_k(H) + 1$.

Now, let A be a k-adjacency basis of H and let v be the vertex of K_1. Since $|A - N_H(h')| \geq k - 1$, for all $h' \in V(H)$, the set $A \cup \{v\}$, is a k-adjacency generator for $K_1 + H$ and, as a consequence, $\mathrm{adim}_k(K_1 + H) \leq |A \cup \{v\}| = \mathrm{adim}_k(H) + 1$. $\square$

The graph H shown in Figure 3.5 has six 3-adjacency bases. For instance, one of them is $B = \{1, 2, 3, 4, 5, 8, 9\}$ and the remaining ones can be found

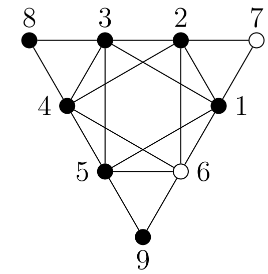

Figure 3.5: The set $B = \{1, 2, 3, 4, 5, 8, 9\}$ is a 3-adjacency basis of this graph.

by symmetry. Notice that for any 3-adjacency basis, say A, there are two vertices i, j such that $|A - N_H(i)| = 2$, $|A - N_H(j)| = 2$ and $|A - N_H(l)| \geq 3$, for all $l \neq i, j$. In particular, for the basis B we have $i = 3$ and $j = 4$. Therefore, Proposition 3.53 leads to $\mathrm{adim}_3(K_1 + H) = \mathrm{adim}_3(H) + 1 = 8$.

By Theorem 3.32 and Proposition 3.53 we deduce the following result previously obtained in [67].

Proposition 3.54. [67] *Let H be graph of order $n \geq 2$. If for any adjacency basis A of H, there exists $h \in V(H) - A$ such that $A \subseteq N_H(h)$, then*

$$\mathrm{adim}_1(K_1 + H) = \mathrm{adim}_1(H) + 1,$$

otherwise,

$$\mathrm{adim}_1(K_1 + H) = \mathrm{adim}_1(H).$$

Theorem 3.55. *For any nontrivial graph H,*

$$\mathrm{adim}_2(K_1 + H) \leq \mathrm{adim}_2(H) + 2.$$

Proof. Let A be a 2-adjacency basis of H and let u be the vertex of K_1. Notice that there exists at most one vertex $x \in V(H)$ such that $A \subseteq N_H(x)$. Now, if $|A - N_H(v)| \geq 1$ for all $v \in V(H)$, then we define $X = A \cup \{u\}$ and, if there exists $x \in V(H)$ such that $A \subseteq N_H(x)$, then we define $X = A \cup \{x, u\}$. We claim that X is a 2-adjacency generator for $K_1 + H$. To show this, we first note that for any $y \in V(H)$ we have that $|\mathcal{C}_{K_1+H}(u, y) \cap X| = |((A - N_H(y)) \cup \{u\}) \cap X| \geq 2$. Moreover, for any $a, b \in V(H)$ we have that $\mathcal{C}_{K_1+H}(a, b) = \mathcal{C}_H(a, b)$. Therefore, X is a 2-adjacency generator for $K_1 + H$ and, as a consequence, $\mathrm{adim}_2(K_1 + H) \leq \mathrm{adim}_2(H) + 2$. $\square$

We would point out that if for any 2-adjacency basis A of a graph H, there exists a vertex x such that $A \subseteq N_H(x)$, then not necessarily $\mathrm{adim}_2(K_1+$

$H) = \mathrm{adim}_2(H) + 2$. To see this, consider the graph G shown in Figure 3.1, where $\{v_2, v_3, v_4, v_5\}$ is the only 2-adjacency basis of G and $\{v_2, v_3, v_4, v_5\} \subseteq N_H(v_1)$. However, $\{v_1, v_6, v_7, v_8, v_9\}$ is a 2-adjacency basis of $K_1 + G$ and so $\mathrm{adim}_2(K_1 + H) = \mathrm{adim}(H) + 1$. Now, we prove some results showing that the inequality given in Theorem 3.55 is tight.

Theorem 3.56. *Let H be a nontrivial graph. If there exists a vertex x of degree $\delta(x) = |V(H)| - 1$ not belonging to any 2-adjacency basis of H, then*

$$\mathrm{adim}_2(K_1 + H) = \mathrm{adim}_2(H) + 2.$$

Proof. Let u be the vertex of K_1 and let $x \in V(H)$ be a vertex of degree $\delta(x) = |V(H)| - 1$ not belonging to any 2-adjacency basis of H. In such a case, $\mathcal{C}_{K_1+H}(x, u) = \{x, u\}$ and, as a result, both x and u must belong to any 2-adjacency basis X of $K_1 + H$. Since $X - \{u\}$ is a 2-adjacency generator for H and $x \in X - \{u\}$ we conclude that $|X - \{u\}| \geq \mathrm{adim}_2(H) + 1$ and so $\mathrm{adim}_2(K_1 + H) = |X| \geq \mathrm{adim}_2(H) + 2$. By Theorem 3.55 we conclude the proof. $\square$

Examples of graphs satisfying the premises of Theorem 3.56 are the fan graphs $F_{1,n}$ and the wheel graphs $W_{1,n}$ for $n \geq 7$. For these graphs we have $\mathrm{adim}_2(K_1 + F_{1,n}) = \mathrm{adim}_2(F_{1,n}) + 2$ and $\mathrm{adim}_2(K_1 + W_{1,n}) = \mathrm{adim}_2(W_{1,n}) + 2$.

Theorem 3.57. *Let H be a graph having an isolated vertex v and a vertex u of degree $\delta(x) = |V(H)| - 2$. If for any 2-adjacency basis B of H, neither u nor v belongs to B, then*

$$\mathrm{adim}_2(K_1 + H) = \mathrm{adim}_2(H) + 2.$$

Proof. Let u be the vertex of K_1. Since $\mathcal{C}_{K_1+H}(x, u) = \{x, u, v\}$, at least two vertices of $\{x, u, v\}$ must belong to any 2-adjacency basis X of $K_1 + H$. Then we have that $x \in X - \{u\}$ or $v \in X - \{u\}$. Since $X - \{u\}$ is a 2-adjacency generator for H, we conclude that if $|X \cap \{x, v\}| = 1$, then $\mathrm{adim}_2(K_1 + H) > |X - \{u\}| \geq \mathrm{adim}_2(H) + 1$, whereas if $|X \cap \{x, v\}| = 2$, then $\mathrm{adim}_2(K_1 + H) \geq |X - \{u\}| \geq \mathrm{adim}_2(H) + 2$. Hence, $\mathrm{adim}_2(K_1 + H) = |X| \geq \mathrm{adim}_2(H) + 2$. By Theorem 3.55 we conclude the proof. $\square$

For instance, we take a family of graphs $\mathcal{G} = \{G_1, G_2, \ldots\}$ such that for any $G_i \in \mathcal{G}$, every vertex in $V(G_i)$ belongs to a non-singleton true twin equivalence class. Then $X = \bigcup_{G_i \in \mathcal{G}} V(G_i)$ is the only 2-adjacency basis of $H = K_1 \cup (K_1 + \bigcup_{G_i \in \mathcal{G}} G_i)$. Therefore, $\mathrm{adim}_2(K_1 + H) = \mathrm{adim}_2(H) + 2$.

Proposition 3.58. *Let H be graph and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$. If there exists a vertex $x \in V(H)$ and a k-adjacency basis A of H such that $A \subseteq N_H(x)$, then*

$$\mathrm{adim}_k(K_1 + H) \leq \mathrm{adim}_k(H) + k.$$

Proof. Let u be the vertex of K_1 and assume that there exists a vertex $v_1 \in V(H)$ and a k-adjacency basis A of H such that $A \subseteq N_H(v_1)$. Since $k \leq |V(H)| - \Delta(H) + 1$, we have that $|V(H) - N_H(v_1)| \geq k - 1$. With this fact in mind, we shall show that $X = A \cup \{u\} \cup A'$ is a k-adjacency generator for $K_1 + H$, where $A' = \emptyset$ if $k = 1$ and $A' = \{v_1, v_2, \ldots, v_{k-1}\} \subset V(H) - N_H(v_1)$ if $k \geq 2$. To this end we only need to check that $|\mathcal{C}_{K_1+H}(u, v) \cap X| \geq k$, for all $v \in V(H)$. On one hand, $|\mathcal{C}_{K_1+H}(u, v_1) \cap X| = |\{u\} \cup A'| = k$. On the other hand, since $A \subseteq N_H(v_1)$, for any $v \in V(H) - \{v_1\}$ we have that $|A - N_H(v)| \geq k$ and, as a consequence, $|\mathcal{C}_{K_1+H}(u, v) \cap X| \geq k$. Therefore, X is a k-adjacency generator for $K_1 + H$ and, as a result, $\mathrm{adim}_k(K_1 + H) \leq |X| = \mathrm{adim}_k(H) + k$. □

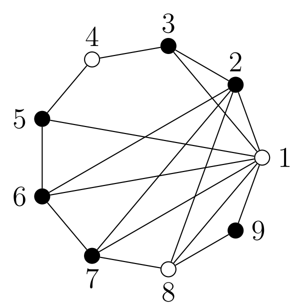

Figure 3.6: The set $A = \{2, 3, 5, 6, 7, 9\}$ is the only 3-adjacency basis of H and $A \subset N_H(1)$.

The bound above is tight. It is achieved, for instance, for the graph shown in Figure 3.6. In this case $\mathrm{adim}_3(K_1 + H) = \mathrm{adim}_3(H) + 3 = 9$. The set $\{2, 3, 5, 6, 7, 9\}$ is the only 3-adjacency basis of H, whereas $\langle u \rangle + H$ has four 3-adjacency bases, *i.e.,* $\{1, 2, 3, 4, 5, 6, 7, 8, u\}$, $\{1, 2, 3, 4, 5, 6, 7, 9, u\}$ $\{1, 2, 3, 4, 5, 7, 8, 9, u\}$ and $\{1, 2, 3, 4, 6, 7, 8, 9, u\}$.

Conjecture 3.59. *Let H be graph of order $n \geq 2$ and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$. Then*

$$\mathrm{adim}_k(K_1 + H) \leq \mathrm{adim}_k(H) + k.$$

We have shown that Conjecture 3.59 is true for any graph H and $k \in \{1, 2\}$, and for any H and k satisfying the premises of Proposition 3.58.

Moreover, in order to assess the potential validity of Conjecture 3.59, we explored the entire set of graphs of order $n \leq 11$ and minimum degree two by means of an exhaustive search algorithm. This search yielded no graph H such that $\operatorname{adim}_k(K_1+H) > \operatorname{adim}_k(H)+k$, $k \in \{3,4\}$, a fact that empirically supports our conjecture.

Two different vertices u, v of $G+H$ belong to the same twin equivalence class if and only if at least one of the following three statements hold.

(a) $u, v \in V(G)$ and u, v belong to the same twin equivalence class of G.

(b) $u, v \in V(H)$ and u, v belong to the same twin equivalence class of H.

(c) $u \in V(G)$, $v \in V(H)$, $N_G[u] = V(G)$ and $N_H[v] = V(H)$.

The following two remarks are direct consequence of Corollary 3.6.

Remark 3.60. *Let G and H be two graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. Then* $\operatorname{adim}_2(G+H) = n_1 + n_2$ *if and only if one of the two following statements hold.*

(a) *Every vertex of G belongs to a non-singleton twin equivalence class of G and every vertex of H belongs to a non-singleton twin equivalence class of H.*

(b) $\Delta(G) = n_1 - 1$, $\Delta(H) = n_2 - 1$, *every vertex $u \in V(G)$ of degree $\delta(u) < n_1 - 1$ belongs to a non-singleton twin equivalence class of G and every vertex $v \in V(H)$ of degree $\delta(v) < n_2 - 1$ belongs to a non-singleton twin equivalence class of H.*

Theorem 3.61. *Let G and H be two nontrivial graphs. Then the following assertions hold:*

(i) *For any $k \in \{1, \ldots, \mathcal{C}(G+H)\}$,*

$$\operatorname{adim}_k(G+H) \geq \operatorname{adim}_k(G) + \operatorname{adim}_k(H).$$

(ii) *For any $k \in \{1, \ldots, \min\{\mathcal{C}(H), \mathcal{C}(K_1+G)\}\}$*

$$\operatorname{adim}_k(G+H) \leq \operatorname{adim}_k(K_1+G) + \operatorname{adim}_k(H).$$

Proof. First we proceed to deduce the lower bound. Let A be a k-adjacency basis of $G+H$, $A_G = A \cap V(G)$, $A_H = A \cap V(H)$ and let $x, y \in V(G)$ be two different vertices. Notice that $A_G \neq \emptyset$ and $A_H \neq \emptyset$, as $n_1 \geq 2$ and $n_2 \geq 2$. Now, since $\mathcal{C}_{G+H}(x,y) = \mathcal{C}_G(x,y)$, it follows that $|A_G \cap \mathcal{C}_G(x,y)| = |A \cap \mathcal{C}_{G+H}(x,y)| \geq k$, and as a consequence, A_G is a k-adjacency generator for G. By analogy we deduce that A_H is a k-adjacency generator for H. Therefore, $\mathrm{adim}_k(G+H) = |A| = |A_G| + |A_H| \geq \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$.

To obtain the upper bound, first we suppose that there exists a k-adjacency basis U of $K_1 + G$ such that the vertex of K_1 does not belong to U. We claim that for any 2-adjacency basis B of H the set $X = U \cup B$ is a k-adjacency generator for $G+H$. To see this we take two different vertices $a, b \in V(G+H)$. If $a, b \in V(G)$, then $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{K_1+G}(a,b) \cap U| \geq k$. If $a, b \in V(H)$, then $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_H(a,b) \cap B| \geq k$. Now, assume that $a \in V(G)$ and $b \in V(H)$. Since U is a k-adjacency generator for $\langle b \rangle + G$, we have that $|\mathcal{C}_{\langle b \rangle+G}(a,b) \cap U| \geq k$. Hence, $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{\langle b \rangle+G}(a,b) \cap U| \geq k$. Therefore, X is a k-adjacency generator for $G+H$ and, as a consequence, $\mathrm{adim}_k(G+H) \leq |X| = |U| + |B| = \mathrm{adim}_k(K_1+G) + \mathrm{adim}_k(H)$.

Suppose from now on that the vertex u of K_1 belongs to any k-adjacency basis U of $K_1 + G$. We differentiate two cases:

Case 1. For any k-adjacency basis B of H, there exists a vertex x such that $B \subseteq N_H(x)$. We claim that $X = U' \cup (B \cup \{x\})$ is a k-adjacency generator for $G+H$, where $U' = U - \{u\}$. To see this we take two different vertices $a, b \in V(G+H)$. Notice that since B is k-adjacency basis of H, there exists exactly one vertex $x \in V(H)$ such that $B \subseteq N_H(x)$ and for any $y \in V(H) - \{x\}$ it holds $|B - N_H(y)| \geq k$. If $a, b \in V(G)$, then $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{K_1+G}(a,b) \cap U'| = |\mathcal{C}_{K_1+G}(a,b) \cap U| \geq k$. If $a, b \in V(H)$, then $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_H(a,b) \cap (B \cup \{x\})| \geq k$. Now, assume that $a \in V(G)$ and $b \in V(H)$. Since $U' \cup \{b\}$ is a k-adjacency basis of $\langle b \rangle + G$, we have that $|\mathcal{C}_{\langle b \rangle+G}(a,b) \cap U'| \geq k-1$. Furthermore, $|\mathcal{C}_{\langle a \rangle+H}(a,b) \cap (B \cup \{x\})| \geq 1$. Hence, $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{\langle b \rangle+G}(a,b) \cap U'| + |\mathcal{C}_{\langle a \rangle+H}(a,b) \cap (B \cup \{x\})| \geq k$. Therefore, X is a k-adjacency generator for $G+H$ and, as a consequence, $\mathrm{adim}_k(G+H) \leq |X| = |U'| + |B \cup \{x\}| = (\mathrm{adim}_k(K_1+G) - 1) + (\mathrm{adim}_k(H) + 1) = \mathrm{adim}_k(K_1+G) + \mathrm{adim}_k(H)$.

Case 2. There exists a k-adjacency basis B' of H such that $|B' - N_H(h')| \geq 1$, for all $h' \in V(H)$. We take $X = U' \cup B'$ and we proceed as above to show that X is a k-adjacency generator for $G+H$. As above, for $a, b \in$

$V(G)$ or $a, b \in V(H)$ we deduce that $|\mathcal{C}_{G+H}(a,b) \cap X| \geq k$. Now, for $a \in V(G)$ and $b \in V(H)$ we have $|\mathcal{C}_{\langle b\rangle+G}(a,b) \cap U'| \geq k-1$ and $|\mathcal{C}_{\langle a\rangle+H}(a,b) \cap B'| \geq 1$. Hence, $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{\langle b\rangle+G}(a,b) \cap U'| + |\mathcal{C}_{\langle a\rangle+H}(a,b) \cap B| \geq k$ and, as a consequence, $\mathrm{adim}_k(G+H) \leq |X| = |U'| + |B'| = (\mathrm{adim}_k(K_1+G) - 1) + \mathrm{adim}_k(H) \leq \mathrm{adim}_k(K_1+G) + \mathrm{adim}_k(H)$. □

By Proposition 3.54 and Theorem 3.61 we obtain the following result.

Proposition 3.62. *Let G and H be two nontrivial graphs. If for any adjacency basis A of G, there exists $g \in V(G)$ such that $A \subseteq N_G(g)$ and for any adjacency basis B of H, there exists $h \in V(H)$ such that $B \subseteq N_H(h)$, then*

$$\mathrm{adim}_1(G+H) = \mathrm{adim}_1(G) + \mathrm{adim}_1(H) + 1$$

Otherwise,

$$\mathrm{adim}_1(G+H) = \mathrm{adim}_1(G) + \mathrm{adim}_1(H).$$

Corollary 3.63. *Let G and H be two nontrivial graphs and $k \in \{1,\dots,\mathcal{C}(G+H)\}$. If $\mathrm{adim}_k(K_1+G) = \mathrm{adim}_k(G)$, then*

$$\mathrm{adim}_k(G+H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H).$$

In the previous subsection we showed that there are several classes of graphs where $\mathrm{adim}_k(K_1+G) = \mathrm{adim}_k(G)$. This is the case, for instance, of graphs of diameter $D(G) \geq 6$, or $G \in \{P_n, C_n\}$, $n \geq 7$, or graphs of girth $\mathsf{g}(G) \geq 5$ and minimum degree $\delta(G) \geq 3$. Hence, for any of these graphs, any nontrivial graph H, and any $k \in \{1,\dots,\min\{\mathcal{C}(H), \mathcal{C}(K_1+G)\}\}$ we have that $\mathrm{adim}_k(G+H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$.

Theorem 3.64. *Let G and H be two nontrivial graphs. Then the following assertions are equivalent:*

(i) *There exists a k-adjacency basis A_G of G and a k-adjacency basis A_H of H such that $|(A_G - N_G(x)) \cup (A_H - N_H(y))| \geq k$, for all $x \in V(G)$ and $y \in V(H)$.*

(ii) $\mathrm{adim}_k(G+H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$.

Proof. Let A_G be a k-adjacency basis of G and and let A_H be a k-adjacency basis of H such that $|(A_G - N_G(x)) \cup (A_H - N_H(y))| \geq k$, for all $x \in V(G)$ and $y \in V(H)$. By Theorem 3.61, $\mathrm{adim}_k(G+H) \geq \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$. It remains to prove that $\mathrm{adim}_k(G+H) \le \mathrm{adim}_k(G)+\mathrm{adim}_k(H)$. We will prove that $A = A_G \cup A_H$ is a k-adjacency generator for $G+H$. We differentiate three cases for two vertices $x, y \in V(G+H)$. If $x, y \in V(G)$, then the fact that A_G is a k-adjacency basis of G leads to $k \le |A_G \cap \mathcal{C}_G(x,y)| = |A \cap \mathcal{C}_{G+H}(x,y)|$. Analogously we deduce the case $x, y \in V(H)$. If $x \in V(G)$ and $y \in V(H)$, then the fact that $\mathcal{C}_{G+H}(x,y) = (V(G) - N_G(x)) \cup (V(H) - N_H(y))$ and $|(A_G - N_G(x)) \cup (A_H - N_H(y))| \ge k$ leads to $|A \cap \mathcal{C}_{G+H}(x,y)| \ge k$. Therefore, A is a k-adjacency generator for $G+H$, as a consequence, $|A| = |A_G|+|A_H| = \mathrm{adim}_k(G) + \mathrm{adim}_k(H) \ge \mathrm{adim}_k(G+H)$.

On the other hand, let B be a k-adjacency basis of $G+H$ such that $|B| = \mathrm{adim}_k(G)+\mathrm{adim}_k(H)$ and let $B_G = B \cap V(G)$ and $B_H = B \cap V(H)$. Since for any $g_1, g_2 \in V(G)$ and $h \in V(H)$, $h \notin \mathcal{C}_{G+H}(g_1, g_2)$, we conclude that B_G is a k-adjacency generator for G and, by analogy, B_H is a k-adjacency generator for H. Thus, $|B_G| \le \mathrm{adim}_k(G)$, $|B_H| \le \mathrm{adim}_k(H)$ and $|B_G| + |B_H| = |B| = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$. Hence, $|B_G| = \mathrm{adim}_k(G)$, $|B_H| = \mathrm{adim}_k(H)$ and, as a consequence, B_G and B_H are k-adjacency bases of G and H, respectively. If there exists $g \in V(G)$ and $h \in V(H)$ such that $|(B_G - N_G(g)) \cup (B_H - N_H(h))| < k$, then $|B \cap \mathcal{C}_{G+H}(g,h)| = |(B_G - N_G(g)) \cup (B_H - N_H(h))| < k$, which is a contradiction. Therefore, the result follows. □

We would point out the following particular cases of the previous result[1].

Corollary 3.65. *Let C_n be a cycle graph of order $n \ge 5$ and $P_{n'}$ a path graph of order $n' \ge 4$. If $G \in \{K_t + C_n, N_t + C_n\}$, then*

$$\mathrm{adim}_1(G) = \left\lfloor \frac{2n+2}{5} \right\rfloor + t - 1 \text{ and } \mathrm{adim}_2(G) = \left\lceil \frac{n}{2} \right\rceil + t.$$

If $G \in \{K_t + P_{n'}, N_t + P_{n'}\}$, then

$$\mathrm{adim}_1(G) = \left\lfloor \frac{2n'+2}{5} \right\rfloor + t - 1 \text{ and } \mathrm{adim}_2(G) = \left\lceil \frac{n'+1}{2} \right\rceil + t.$$

Proof. Let $G_1 \in \{K_t, N_t\}$ and $G_2 \in \{P_n, C_n\}$. By Propositions 3.45 and 3.46 we deduce that $\mathrm{adim}_2(G_2) - \Delta(G_2) \ge 1$. On the other hand, for any 2-adjacency basis A of G_1 and $x \in V(G_1)$ we have $|B - N_{G_1}(y)| \in \{1, t\}$. Therefore, by Theorem 3.64 we obtain the result for $G = G_1 + G_2$. □

[1]Notice that for $n \ge 7$ and $n' \ge 6$, this result can be derived from Corollary 3.63.

Corollary 3.66. *Let G be a graph of order $n \geq 7$ and maximum degree $\Delta(G) \leq 3$. Then for any integer $r \geq 2$ and $H \in \{K_r, N_r\}$,*

$$\mathrm{adim}_2(G+H) = \mathrm{adim}_2(G) + r.$$

Proof. By Theorem 3.26 we deduce that $\mathrm{adim}_2(G) \geq 4$, so for any 2-adjacency basis A of G and $x \in V(G)$ we have $|A - N_G(x)| \geq 1$. Moreover, for any 2-adjacency basis B of H and $y \in V(H)$ we have $|B - N_H(y)| \in \{1, r\}$. Therefore, by Theorem 3.64 we obtain the result. □

Corollary 3.67. *Let G and H be two graphs of order at least seven such that G is k_1-adjacency dimensional and H is k_2-adjacency dimensional. For any integer k such that $\Delta(G) + \Delta(H) - 4 \leq k \leq \min\{k_1, k_2\}$,*

$$\mathrm{adim}_k(G+H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H).$$

Proof. By Theorem 3.26, for any positive integer $k \leq \min\{k_1, k_2\}$, we have $\mathrm{adim}_k(G) \geq k+2$ and $\mathrm{adim}_k(H) \geq k+2$. Thus, if $k \geq \Delta(G) + \Delta(H) - 4$, then $(\mathrm{adim}_k(G) - \Delta(G)) + (\mathrm{adim}_k(H) - \Delta(H)) \geq k$. Therefore, by Theorem 3.64 we conclude the proof. □

As a particular case of the result above we derive the following remark.

Remark 3.68. *Let G and H be two 3-regular graphs of order at least seven. Then*

$$\mathrm{adim}_2(G+H) = \mathrm{adim}_2(G) + \mathrm{adim}_2(H).$$

General lexicographic product graphs

Note that a trivial upper bound on the k-metric dimension of $G \circ \mathcal{H}$ is $|V(G \circ \mathcal{H})|$, which is tight at least for $k = 2$. To see this, we can firstly consult the notation given by Section 1.1 and we can refer to Corollary 3.6, which states that the 2-metric dimension of a graph G is equal to its order if and only if G has no singleton twin equivalence classes. Considering this fact, we can conclude the next result.

Remark 3.69. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed by nontrivial graphs. Then $\dim_2(G \circ \mathcal{H}) = |V(G \circ \mathcal{H})|$ if and only if the following statements hold.*

(i) *For every $u_i \in S(G)$, the graph $H_i \in \mathcal{H}$ has no singleton twin equivalence classes.*

(ii) *For every $u_i \in TT(G)$, either the graph $H_i \in \mathcal{H}$ has no singleton twin equivalence classes or H_i has exactly one singleton twin equivalence class $\{v_i\}$, where $\delta(v_i) = n_i - 1$, and there exists $u_j \in TT(u_i)$ such that $H_j \in \mathcal{H}$ has a vertex v_j of degree $\delta(v_j) = n_j - 1$.*

(iii) *For every $u_i \in FT(G)$, either the graph $H_i \in \mathcal{H}$ has no singleton twin equivalence classes or H_i has exactly one singleton twin equivalence class $\{v_i\}$, where $\delta(v_i) = 0$, and there exists $u_j \in FT(u_i)$ such that $H_j \in \mathcal{H}$ has a vertex v_j of degree $\delta(v_j) = 0$.*

On the other hand, now we give a lower bound for $\dim_k(G \circ \mathcal{H})$, in terms of $\operatorname{adim}_k(H_i)$ for every $H_i \in \mathcal{H}$, which is also tight.

Theorem 3.70. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed by nontrivial graphs. For any $k \in \{1, \ldots, \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$, any k-metric basis B of $G \circ \mathcal{H}$, and any $u_i \in V(G)$, $B_i = \{v : (u_i, v) \in B\}$ is a k-adjacency generator for H_i and, as a consequence, $|B_i| \geq \operatorname{adim}_k(H_i)$. Moreover,*

$$\dim_k(G \circ \mathcal{H}) \geq \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

Proof. Let B be a k-metric basis for $G \circ \mathcal{H}$ and let $B_i = \{v : (u_i, v) \in B\}$. By Remark 1.1, we deduce that for any $(u_i, v), (u_i, v') \in \{u_i\} \times V(H_i)$, $v \neq v'$, it holds that $|\mathcal{D}_{G \circ \mathcal{H}}((u_i, v), (u_i, v')) \cap (\{u_i\} \times B_i)| \geq k$. Also, by Remark 1.1 again, $\mathcal{D}_{G \circ \mathcal{H}}((u_i, v), (u_i, v')) = \{u_i\} \times \mathcal{C}_{H_i}(v, v')$, and as a consequence, $|B_i \cap \mathcal{C}_{H_i}(v, v')| \geq k$. Thus, B_i is a k-adjacency generator for H_i and we obtain that $|B_i| \geq \operatorname{adim}_k(H_i)$. Therefore, $\dim_k(G \circ \mathcal{H}) = |B| = \sum_{i=1}^{n} |B_i| \geq \sum_{i=1}^{n} \operatorname{adim}_k(H_i)$. □

Later on, in Theorem 3.72, we show the tightness of the result above. To this end we need some extra notation. Given a graph G with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and a family of graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$, we define the following properties on the triplet $(G, \mathcal{H}, k)$. We must remark that in order to simplify the notations and statements of our exposition, even though the graphs G and H has no any relationship between them, the next properties are stated in such a way that seems there exists some connection.

Property $\mathcal{P}_1$: G is true twins free, otherwise for any $u_i \in TT(G)$, where $TT(u_i) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$, there exist i_r k-adjacency bases $A^t_{i_1}, A^t_{i_2}, \ldots, A^t_{i_r}$

of $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$, respectively, such that for every $j,l \in \{1,\ldots,r\}$, $j \neq l$, and every $x \in V(H_{i_j})$ and $y \in V(H_{i_l})$ it follows,

$$|(A^t_{i_j} - N_{H_{i_j}}(x)) \cup (A^t_{i_l} - N_{H_{i_l}}(y))| \geq k.$$

Notice that Property $\mathcal{P}_1$ ensures that for any $u_i, u_j \in TT(G)$, $i \neq j$, there exist two k-adjacency bases A^t_i, A^t_j of H_i, H_j, respectively, such that vertices belonging to $\{u_i\} \times H_i$ are distinguished from vertices belonging to $\{u_j\} \times H_j$ by at least k vertices of $(\{u_i\} \times A^t_i) \cup (\{u_j\} \times A^t_j)$.

An example which helps to clarify the property above is, for instance, the triplet $(K_3, \mathcal{H}, 2)$, where $V(K_3) = \{u_1, u_2, u_3\}$ and $\mathcal{H} = \{C^1_5, C^2_5, C^3_5\}$. Figure 3.7 shows the family of graphs $\mathcal{H}$. In this case $TT(u_1) = \{u_1, u_2, u_3\} = TT(K_3)$, since there is only one true twin equivalence class. If we take as 2-adjacency bases $A_{1_1} = \{v^1_1, v^1_3, v^1_4\}$, $A_{1_2} = \{v^2_1, v^2_3, v^2_4\}$ and $A_{1_3} = \{v^3_1, v^3_3, v^3_4\}$ of C^1_5, C^2_5 and C^3_5, respectively, then $(K_3, C_5, 2)$ satisfies Property $\mathcal{P}_1$. For instance, if $x = v^1_2$ and $y = v^3_5$, then $|(A_{1_1} - N_{C^1_5}(v^1_2)) \cup (A_{1_3} - N_{C^3_5}(v^3_5))| = |(A_{1_1} - \{v^1_1, v^1_3\}) \cup (A_{1_3} - \{v^3_1, v^3_4\})| = |\{v^1_4\} \cup \{v^3_3\}| = 2 \geq 2$.

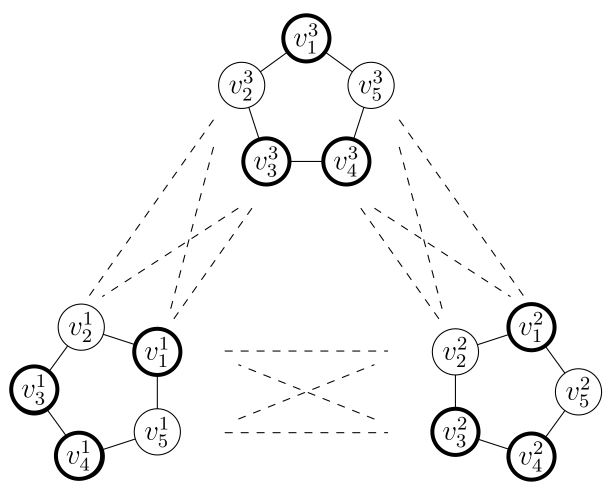

Figure 3.7: Sketch of lexicographic product $K_3 \circ C_5$, where the dashed line between two cycles C_5 means that each vertex of a cycle is connected to all vertices of the other cycle. The vertices represented by thick lines form a 2-adjacency basis of each copy of C_5.

Property $\mathcal{P}_2$: G is false twins free, otherwise for any $u_i \in FT(G)$, where $FT(u_i) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$, there exist i_r k-adjacency bases $A^f_{i_1}, A^f_{i_2}, \ldots, A^f_{i_r}$ of $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$, respectively, such that for every $j,l \in \{1,\ldots,r\}$, $j \neq l$, and every $x \in V(H_{i_j})$ and $y \in V(H_{i_l})$ it follows,

$$|(A^f_{i_j} \cap N_{H_{i_j}}[x]) \cup (A^f_{i_l} \cap N_{H_{i_l}}[y])| \geq k.$$

Notice that Property $\mathcal{P}_2$ ensures that for any $u_i, u_j \in FT(G)$, $i \neq j$, there exist two k-adjacency bases A^f_i, A^f_j of H_i, H_j, respectively, such that vertices belonging to $\{u_i\} \times H_i$ are distinguished from vertices belonging to $\{u_j\} \times H_j$ by at least k vertices of $\left(\{u_i\} \times A^f_i\right) \cup \left(\{u_j\} \times A^f_j\right)$.

Further on we will see a triplet $(G, \mathcal{H}, k)$ that satisfy Properties $\mathcal{P}_1$ and $\mathcal{P}_2$ at the same time, when $\mathcal{H}$ is a family of paths of order greater than three and/or cycles of order greater than four, G is any nontrivial connected graph and $k \in \{2, 3\}$.

To continue our exposition we need some extra notation. Given a family of graphs $\mathcal{H} = \{H_1, \ldots, H_n\}$, we define $\overline{\mathcal{H}}$ as the family of complement graphs of each $H_i \in \mathcal{H}$, *i.e.*, $\overline{\mathcal{H}} = \{\overline{H}_1, \ldots, \overline{H}_n\}$. We now see how Properties $\mathcal{P}_1$ and $\mathcal{P}_2$ behave for the case of the triplet $(G, \overline{\mathcal{H}}, k)$. To this end, we need to define two other properties on the triplet $(G, \mathcal{H}, k)$.

Property $\mathcal{P}_3$: G is true twins free, otherwise for any $u_i \in TT(G)$, where $TT(u_i) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$, there exist i_r k-adjacency bases $A^t_{i_1}, A^t_{i_2}, \ldots, A^t_{i_r}$ of $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$, respectively, such that for every $j, l \in \{1, \ldots, r\}$, $j \neq l$, and every $x \in V(H_{i_j})$ and $y \in V(H_{i_l})$, it follows

$$|(A^t_{i_j} \cap N_{H_{i_j}}[x]) \cup (A^t_{i_l} \cap N_{H_{i_l}}[y])| \geq k.$$

Property $\mathcal{P}_4$: G is false twins free, otherwise for any $u_i \in FT(G)$, where $FT(u_i) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$, there exist i_r k-adjacency bases $A^f_{i_1}, A^f_{i_2}, \ldots, A^f_{i_r}$ of $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$, respectively, such that for every $j, l \in \{1, \ldots, r\}$, $j \neq l$, and every $x \in V(H_{i_j})$ and $y \in V(H_{i_l})$ it follows,

$$|(A^f_{i_j} - N_{H_{i_j}}(x)) \cup (A^f_{i_l} - N_{H_{i_l}}(y))| \geq k.$$

Next claim relates all the properties above while using them in $(G, \mathcal{H}, k)$ or $(G, \overline{\mathcal{H}}, k)$.

Claim 3.71. *Let G be a graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of n graphs. Then,*

(i) *the triplet $(G, \overline{\mathcal{H}}, k)$ satisfies Property $\mathcal{P}_1$ if and only if $(G, \mathcal{H}, k)$ satisfies Property $\mathcal{P}_3$,*

(ii) *the triplet* $(G, \overline{\mathcal{H}}, k)$ *satisfies Property* $\mathcal{P}_2$ *if and only if* $(G, \mathcal{H}, k)$ *satisfies Property* $\mathcal{P}_4$.

Proof. For any graph H, any $A \subseteq V(H)$, and any $v \in V(H)$ we have that $A - N_{\overline{H}}(v) = A \cap N_H[v]$. Thus, for any $u_i \in TT(G)$ the set of k-adjacency bases $\{A^t_{i_1}, A^t_{i_2}, \ldots, A^t_{i_r}\}$ which makes that the triplet $(G, \overline{\mathcal{H}}, k)$ satisfies Property $\mathcal{P}_1$, also makes that $(G, \mathcal{H}, k)$ satisfies Property $\mathcal{P}_3$ and vice versa. Therefore (i) follows. The item (ii) follows similarly from the same fact that $A \cap N_{\overline{H}}[v] = A - N_H(v)$. □

In this point we are able to give one of the main results of this work and its powerful consequences.

Theorem 3.72. *Let G be a connected graph of order $n \geq 2$, let $\mathcal{H}$ be a family composed by n nontrivial graphs and let $k \in \{1, \ldots, \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$.*

(i) *The triplet* $(G, \mathcal{H}, k)$ *satisfies Properties* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *if and only if*

$$\dim_k(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

(ii) *The triplet* $(G, \mathcal{H}, k)$ *satisfies Properties* $\mathcal{P}_3$ *and* $\mathcal{P}_4$ *if and only if*

$$\dim_k(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

Proof.
(i) (Necessity) We assume that the triplet $(G, \mathcal{H}, k)$ satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_2$. If $u_i \in TT(G)$ or $u_i \in FT(G)$, then we take a k-adjacency basis A^t_i or A^f_i of H_i as defined in Property $\mathcal{P}_1$ or Property $\mathcal{P}_2$, respectively. Also, if $u_i \in V(G)$ is not a twin vertex, then we take any k-adjacency basis A_i of H_i.

We claim that

$$B = \left(\bigcup_{u_i \in TT(G)} \{u_i\} \times A^t_i \right) \cup \left(\bigcup_{u_i \in FT(G)} \{u_i\} \times A^f_i \right) \cup \left(\bigcup_{u_i \in S(G)} \{u_i\} \times A_i \right).$$

is a k-metric generator for $G \circ \mathcal{H}$.

We differentiate the following four cases for two different vertices (u_i, v), $(u_j, w) \in V(G \circ \mathcal{H})$.

Case 1. $i = j$. In this case $v \neq w$. We have three possibilities for the vertex u_i

- $u_i \in TT(G)$, in which case $B \cap (\{u_i\} \times V(H_i)) = \{u_i\} \times A_i^t$,
- $u_i \in FT(G)$, in which case $B \cap (\{u_i\} \times V(H_i)) = \{u_i\} \times A_i^f$,
- $u_i \in S(G)$, in which case $B \cap (\{u_i\} \times V(H_i)) = \{u_i\} \times A_i$.

Since A_i^t, A_i^f and A_i are k-adjacency bases of H_i, we obtain that $|\mathcal{C}_{H_i}(v,w) \cap A_i^t| \geq k$, $|\mathcal{C}_{H_i}(v,w) \cap A_i^f| \geq k$ and $|\mathcal{C}_{H_i}(v,w) \cap A_i| \geq k$. In any case, as $\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_i,w)) = \{u_i\} \times \mathcal{C}_{H_i}(v,w)$, we conclude that $|B \cap \mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_i,w))| \geq k$.

Case 2. $i \neq j$ and u_i, u_j are true twins. So $\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,w)) = (V(H_i) - N_{H_i}(v)) \cup (V(H_j) - N_{H_j}(w))$. Since $(G,\mathcal{H},k)$ satisfies Property $\mathcal{P}_1$, there exist at least k elements of $(\{u_i\} \times A_i^t) \cup (\{u_j\} \times A_j^t) \subseteq B$ distinguishing $(u_i,v),(u_j,w)$.

Case 3. $i \neq j$ and u_i, u_j are false twins. Thus $\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,w)) = N_{H_i}[v] \cup N_{H_j}[w]$. Since $(G,\mathcal{H},k)$ satisfies Property $\mathcal{P}_2$, there exist at least k elements of $(\{u_i\} \times A_i^f) \cup (\{u_j\} \times A_j^f) \subseteq B$ distinguishing $(u_i,v),(u_j,w)$.

Case 4. $i \neq j$ and u_i, u_j are not twins. Hence, there exists $u_l \in \mathcal{D}_G^*(u_i,u_j)$. So, the set $B \cap (\{u_l\} \times V(H_l))$ is either $\{u_l\} \times A_l^t$ or $\{u_l\} \times A_l^f$ or $\{u_l\} \times A_l$. Hence, (u_i,v) and (u_j,w) are distinguished by, at least k elements of B.

Therefore, B is a k-metric generator for $G\circ\mathcal{H}$, and consequently, $\dim_k(G\circ\mathcal{H}) \leq |B| = \sum_{i=1}^n \operatorname{adim}_k(H_i)$. By Theorem 3.70, we conclude that $\dim_k(G\circ\mathcal{H}) = \sum_{i=1}^n \operatorname{adim}_k(H_i)$.

(Sufficiency) Assume that $\dim_k(G\circ\mathcal{H}) = \sum_{i=1}^n \operatorname{adim}_k(H_i)$. Let B be a k-metric basis of $G\circ\mathcal{H}$ and let $B_i = \{v : (u_i,v) \in B\}$ for every $u_i \in V(G)$. By Theorem 3.70, we have that $|B_i| \geq \operatorname{adim}_k(H)$. According to this fact and since $\sum_{i=1}^n \operatorname{adim}_k(H_i) = \dim_k(G\circ\mathcal{H}) = |B| = \sum_{i=1}^n |B_i| \geq \sum_{i=1}^n \operatorname{adim}_k(H_i)$, we deduce that $|B_i| = \operatorname{adim}_k(H_i)$. So B_i is a k-adjacency basis of H_i.

We first consider that G is not true twins free, *i.e.*, there exist two true twin vertices u_i and u_j. Let $v \in V(H_i)$ and $v' \in V(H_j)$ such that $H_i, H_j \in \mathcal{H}$. In such case we obtain that $\mathcal{D}_{G\circ\mathcal{H}}((u_i,v),(u_j,v')) = (\{u_i\} \times (V(H) - N_H(v))) \cup (\{u_j\} \times (V(H) - N_H(v')))$. Suppose, for purposes of contradiction, that for each k-adjacency basis A_i of H_i and each k-adjacency basis A_j of H_j there exist two vertices $x \in V(H_i)$ and $x' \in V(H_j)$ such that $|(A_i - N_{H_i}(x)) \cup$

$(A_j - N_{H_j}(x'))| < k$. Since B_i, B_j are k-adjacency bases of H_i and H_j, respectively, there exist two vertices $w \in V(H_i)$ and $w' \in V(H_j)$ such that $|(B_i - N_{H_i}(w)) \cup (B_j - N_{H_j}(w'))| < k$. Hence, $|B \cap \mathcal{D}_{G\circ\mathcal{H}}((u_i, w), (u_j, w'))| = |\,(\{u_i\} \times (B_i - N_H(w))) \cup (\{u_j\} \times (B_j - N_H(w')))\,| < k$, which is a contradiction. Therefore, B_i and B_j satisfy the condition $|(\{u_i\} \times (B_i - N_H(v))) \cup (\{u_j\} \times (B_j - N_H(v')))| \geq k$ for every $v \in V(H_i)$ and every $v' \in V(H_j)$, and as a consequence, the triplet $(G, \mathcal{H}, k)$ satisfies Property $\mathcal{P}_1$. If G is true twins free, then the triplet $(G, \mathcal{H}, k)$ directly satisfies Property $\mathcal{P}_1$.

We now consider that G is not false twins free, *i.e.*, there exist two false twin vertices u_i and u_j. Let $v \in V(H_i)$ and $v' \in V(H_j)$ such that $H_i, H_j \in \mathcal{H}$. In this case, $\mathcal{D}_{G\circ\mathcal{H}}((u_i, v), (u_j, v')) = (\{u_i\} \times N_{H_i}[v]) \cup \big(\{u_j\} \times N_{H_j}[v']\big)$. Suppose, for purposes of contradiction, that for each k-adjacency basis A_i of H_i and each k-adjacency basis A_j of H_j, there exist two vertices $y \in V(H_i)$ and $y' \in V(H_j)$ such that $|(A_i \cap N_{H_i}[y]) \cup (A_j \cap N_{H_j}[y'])| < k$. Since B_i, B_j are k-adjacency bases of H_i an H_j, respectively, there exist two vertices $z \in V(H_i)$ and $z' \in V(H_j)$ such that $|(B_i \cap N_{H_i}[z]) \cup (B_j \cap N_{H_j}[z'])| < k$. Hence, $|B \cap \mathcal{D}_{G\circ\mathcal{H}}((u_i, z), (u_j, z'))| = |\,(\{u_i\} \times (B_i \cap N_H[z])) \cup (\{u_j\} \times (B_j \cap N_H[z']))\,| < k$, which is a contradiction. Therefore, B_i and B_j satisfy $|\,(\{u_i\} \times (B_i \cap N_H[z])) \cup (\{u_j\} \times (B_j \cap N_H[z']))\,| \geq k$ for every $v \in V(H_i)$ and every $v' \in V(H_j)$, and as a consequence, the triplet $(G, \mathcal{H}, k)$ satisfies Property $\mathcal{P}_2$. If G is false twins free, then the triplet $(G, \mathcal{H}, k)$ satisfies Property $\mathcal{P}_2$.

(ii) Proceeding analogously to the proof of (i) and considering that $\mathrm{adim}_k(\overline{H}_i) = \mathrm{adim}_k(H_i)$ for every $H_i \in \mathcal{H}$ and also considering Claim 3.71, we conclude this proof. □

The previous theorem is a generalization for $k \in \{1, \dots, \mathcal{C}(\mathcal{H})\}$ of a result obtained by Jannesari and Omoomi [67] for the 1-metric dimension of $G \circ H$, *i.e*, for $\dim_1(G \circ \mathcal{H})$ when graphs belonging to $\mathcal{H}$ are isomorphic to the same graph H.

Assume now that the k-adjacency dimension of every graph of a given family $\mathcal{H}'$ is known. Hence, as a measure of the reach of Theorem 3.72, the following consequences are deduced. Notice that we can then compute, not only the k-metric dimension of $G \circ \mathcal{H}'$, but also that of $G \circ \overline{\mathcal{H}}'$, for a huge quantity of graphs G. If G is a connected graph of order $n \geq 2$ and $\mathcal{H}$ is a family composed by n nontrivial graphs, then Theorem 3.72 gives us the conditions for which the problem of computing the k-metric dimension of

$G \circ \mathcal{H}$ and $G \circ \overline{\mathcal{H}}$ is reduced to computing the k-adjacency dimension of the graphs $H_i \in \mathcal{H}$.

Corollary 3.73. *Let G be a connected graph of order $n \geq 2$, let $\mathcal{H}$ be a family composed by n nontrivial graphs and let $k \in \{1, \ldots, \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$. Then the following statements hold.*

(i) *If G is twins free, then*

$$\dim_k(G \circ \mathcal{H}) = \dim_k(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

(ii) *If G is false twins free and $(G, \mathcal{H}, k)$ holds Property $\mathcal{P}_1$,*

$$\dim_k(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

(iii) *If G is true twins free and $(G, \mathcal{H}, k)$ holds Property $\mathcal{P}_2$,*

$$\dim_k(G \circ \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

(iv) *If G is false twins free and $(G, \mathcal{H}, k)$ holds Property $\mathcal{P}_3$,*

$$\dim_k(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

(v) *If G is true twins free and $(G, \mathcal{H}, k)$ holds Property $\mathcal{P}_4$,*

$$\dim_k(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

A natural question is now the following one. Can we realize triplets $(G, \mathcal{H}, k)$ satisfying Properties $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ or $\mathcal{P}_4$? To proceed in this direction, we first need to present some useful lemmas which allow us to describe some realizations of the triplet $(G, \mathcal{H}, k)$ in concordance with Properties $\mathcal{P}_1$ and $\mathcal{P}_2$.

Lemma 3.74. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs. If $\operatorname{adim}_k(H) - \Delta(H) \geq \lceil \frac{k}{2} \rceil$ for every $H \in \mathcal{H}$ and $k \in \{1, \ldots, \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$, then $(G, \mathcal{H}, k)$ satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_4$.*

Proof. Let A_i, A_j, $i \neq j$, be two k-adjacency bases of $H_i, H_j \in \mathcal{H}$, respectively. Since $\operatorname{adim}_k(H_i) - \Delta(H_i) \geq \lceil \frac{k}{2} \rceil$ and $\operatorname{adim}_k(H_j) - \Delta(H_j) \geq \lceil \frac{k}{2} \rceil$, it follows that for every $v \in V(H_i)$ and $w \in V(H_j)$, $|A_i - N_{H_i}(v)| \geq \lceil \frac{k}{2} \rceil$ and $|A_j - N_{H_j}(w)| \geq \lceil \frac{k}{2} \rceil$, which implies that $(G, \mathcal{H}, k)$ satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_4$. □

Lemma 3.75. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n graphs without isolated vertices. If $\Delta(H) - 1 \leq \lfloor \frac{k}{2} \rfloor$ for every $H \in \mathcal{H}$ and $k \in \{1, \ldots, \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$, then $(G, \mathcal{H}, k)$ satisfies Properties $\mathcal{P}_2$ and $\mathcal{P}_3$.*

Proof. Let $H \in \mathcal{H}$, let $v \in V(H)$, and let A be k-adjacency basis of H. Since $N_H(v) \neq \emptyset$, for every $w \in N_H(v)$ we have that $|N_H(w) - \{v\}| \leq \Delta(H) - 1 \leq \lfloor \frac{k}{2} \rfloor$. Now, as $|A \cap \mathcal{C}(v, w)| \geq k$, we obtain $|A \cap N_H[v]| \geq \lceil \frac{k}{2} \rceil$. Thus, for every $H_l, H_j \in \mathcal{H}$, $l \neq j$, and every $v \in V(H_l)$, $w \in V(H_j)$ it follows that $|(A_l \cap N_{H_l}[v]) \cup (A_j \cap N_{H_j}[w])| \geq k$, where A_l, A_j are k-adjacency bases of H_l, H_j, respectively. Therefore, $(G, \mathcal{H}, k)$ satisfies Properties $\mathcal{P}_2$ and $\mathcal{P}_3$. □

According to Proposition 3.45 and 3.46, and the lemmas above, we can notice now that, for instance, any triplet $(G, \mathcal{H}, k)$, where G is any connected graph, $\mathcal{H}$ is formed by paths of order greater than three and/or cycles of order greater than four, and $k \in \{2, 3\}$ (or if $\mathcal{H}$ is only formed by cycles, then also happens for $k = 4$), satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_2$, and Properties $\mathcal{P}_3$ and $\mathcal{P}_4$. In this sense, by Theorem 3.72, the previous lemmas and Propositions 3.45 and 3.46, we give a closed formulae for the lexicographic product of any connected graph G and this family $\mathcal{H}$ of graphs.

Theorem 3.76. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{P_{q_1}, \ldots, P_{q_r}, C_{q_{r+1}}, \ldots, C_{q_n}\}$. If $q_i \geq 4$ for $1 \leq i \leq r$ and $q_i \geq 5$ for $r + 1 \leq i \leq n$, then*

(i) $$\dim_2(G \circ \mathcal{H}) = \dim_2(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{r} \left\lceil \frac{q_i + 1}{2} \right\rceil + \sum_{i=r+1}^{n} \left\lceil \frac{q_i}{2} \right\rceil$$

(ii) $$\dim_3(G \circ \mathcal{H}) = \dim_3(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{r} \left(q_i - \left\lfloor \frac{q_i - 4}{5} \right\rfloor \right) + \sum_{i=r+1}^{n} \left(q_i - \left\lfloor \frac{q_i}{5} \right\rfloor \right).$$

Moreover, if $\mathcal{H} = \{C_{q_1}, \ldots, C_{q_n}\}$ and $q_i \geq 5$, then $\dim_4(G \circ \mathcal{H}) = \dim_4(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{n} q_i$.

Note that for any connected graph G of order n and any family $\mathcal{H}$ composed by n graphs we have that $\dim_4(G \circ \mathcal{H}) = \dim_4(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{n} q_i = |V(G \circ \mathcal{H})|$, and these are two other examples where the trivial upper bound is reached.

On the other hand, Theorem 3.76 does not consider the case $k = 1$. Note that from Proposition 3.44 and Lemma 3.74, we deduce that any triplet $(G, \mathcal{H}, 1)$, where G is any connected graph, and $\mathcal{H}$ is formed by paths of order at least seven and/or cycles of order at least seven, satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_4$. However, for $k = 1$ paths and cycles do not satisfy the condition of Lemma 3.75. The following lemma provide us a family $\mathcal{H}$ composed by paths and cycles where it makes sure that $(G, \mathcal{H}, 1)$ satisfies Properties $\mathcal{P}_2$ and $\mathcal{P}_3$.

Lemma 3.77. *Let P_n and C_n be a path and a cycle graph of order $n \geq 7$. If $n \mod 5 \in \{1, 3\}$, then no adjacency basis of P_n or C_n is a dominating set. Otherwise, there exist adjacency bases of P_n and C_n that are dominating sets.*

Proof. In C_n, consider the path $v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4}$, where the subscripts are taken modulo n, and an adjacency basis B. If $v_i, v_{i+2} \in B$ and $v_{i+1} \notin B$, then $\{v_{i+1}\}$ is said to be a 1-gap of B. Likewise, if $v_i, v_{i+3} \in B$ and $v_{i+1}, v_{i+2} \notin B$, then $\{v_{i+1}, v_{i+2}\}$ is said to be a 2-gap of B and if $v_i, v_{i+4} \in B$ and $v_{i+1}, v_{i+2}, v_{i+3} \notin B$, then $\{v_{i+1}, v_{i+2}, v_{i+3}\}$ is said to be a 3-gap of B. Since B is an adjacency basis of C_n, it has no gaps of size 4 or larger and it has at most one 3-gap. Moreover, every 2- or 3-gap must be neighboured by two 1-gaps and the number of gaps of either size is at most $\mathrm{adim}_1(C_n)$. We now differentiate the following cases for C_n:

1. $n = 5k$, $k \geq 2$. In this case, $\mathrm{adim}_1(C_n) = 2k$ and $n - \mathrm{adim}_1(C_n) = 3k$. Since any 2-gap must be neighboured by two 1-gaps, any adjacency basis has at most k 2-gaps. Any set B having exactly k 2-gaps and exactly k 1-gaps is an adjacency basis of C_n, as $|B| \geq 2k = \mathrm{adim}_1(C_n)$ and $|(N_{C_n}(x) \cap B) \triangledown (N_{C_n}(y) \cap B)| \geq 1$ for any pair of different vertices $x, y \in V(C_n) - B$. Since the number of vertices of $V(C_n) - B$ belonging to a 1- or 2-gap is $3k = n - |B|$, we deduce that B has no 3-gaps, *i.e.* it is a dominating set.

2. $n = 5k + 1$, $k \geq 2$. In this case, $\mathrm{adim}_1(C_n) = 2k$ and $n - \mathrm{adim}_1(C_n) = 3k + 1$. As in the previous case, any adjacency basis B has at most k

2-gaps. Now, assume that B has no 3-gaps. Then $|V(C_n) - B| = 3k < 3k+1 = n - |B|$, which is a contradiction. Thus, any B has a 3-gap, *i.e.* it is not dominating.

3. $n = 5k+2$, $k \geq 1$. In this case, $\mathrm{adim}_1(C_n) = 2k+1$ and $n - \mathrm{adim}_1(C_n) = 3k+1$. As in the previous cases, any adjacency basis has at most k 2-gaps. Moreover, any set B having exactly k 2-gaps and exactly $k+1$ 1-gaps is an adjacency basis of C_n, and the number of vertices of $V(C_n) - B$ belonging to a 1- or 2-gap is $3k+1 = n - |B|$, so B has no 3-gaps, *i.e.* it is a dominating set.

4. $n = 5k+3$, $k \geq 1$. In this case, $\mathrm{adim}_1(C_n) = 2k+1$ and $n - \mathrm{adim}_1(C_n) = 3k+2$. As in the previous cases, any adjacency basis B has at most k 2-gaps. Now assume that B has no 3-gaps. Then $|V(C_n) - B| = 3k+1 < 3k+2 = n - |B|$, which is a contradiction. Thus, any B has a 3-gap, *i.e.* it is not dominating.

5. $n = 5k+4$, $k \geq 1$. In this case, $\mathrm{adim}_1(C_n) = 2k+2$ and $n - \mathrm{adim}_1(C_n) = 3k+2$. Assume that some adjacency basis B has $k+1$ 2-gaps. Then, B would have at least $k+1$ 1-gaps, making $|V(C_n) - B| \geq 3k+3$, which is a contradiction. So, any adjacency basis has at most k 2-gaps. As in cases 1 and 3 the previous cases, any set B having exactly k 2-gaps and exactly $k+2$ 1-gaps is an adjacency basis of C_n, and the number of vertices of $V(C_n) - B$ belonging to a 1- or 2-gap is $3k+2 = n - |B|$, so B has no 3-gaps, *i.e.* it is a dominating set.

As a consequence of all the cases above, the results follows for C_n.

Consider now the path P_n, where $n \mod 5 \in \{0,2,4\}$, and let C'_n be the cycle obtained from P_n by joining its leaves v_1 and v_n by an edge. Let B be an adjacency basis of C'_n which is also a dominating set and satisfies $v_1, v_n \notin B$ (at least one such B exists). We have that every $u \in B$ and every $v \in V(P_n) - B$ satisfy $d_{C'_n,2}(u,v) = d_{P_n,2}(u,v)$, so B is also an adjacency basis and a dominating set of P_n.

To conclude, consider the path P_n, $n \mod 5 \in \{1,3\}$, and let C'_n be the cycle obtained from P_n by joining its leaves v_1 and v_n by an edge. Consider $V = V(P_n) = V(C_n)$, and let B be an adjacency basis of P_n. If $v_1, v_n \in B$ or $v_1, v_n \notin B$, then every vertex $v \in V - B$ has the same adjacency representation in C'_n with respect to B as in P_n, so B is an adjacency basis of

C_n. Moreover, some vertex $w \in V-B$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so B is not a dominating set of P_n. We now treat the case where $v_1 \in B$ and $v_n \notin B$. If $v_{n-1} \notin B$ then B is not a dominating set of P_n. If $v_{n-1} \in B$ and $v_2 \notin B$, we have that $d_{C'_n,2}(v_2, v_{n-1}) = d_{P_n,2}(v_2, v_{n-1}) = 2 \neq 1 = d_{P_n,2}(v_n, v_{n-1}) = d_{C'_n,2}(v_n, v_{n-1})$, whereas for any other pair of different vertices $x, y \in V-B$ there exists $z \in B$ such that $d_{C'_n,2}(x,z) = d_{P_n,2}(x,z) \neq d_{P_n,2}(y,z) = d_{C'_n,2}(y,z)$, so B is an adjacency basis of C'_n where $\{v_n\}$ is a 1-gap. In consequence, some vertex $w \in (V-B)-\{v_n\}$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so B is not a dominating set of P_n. Finally, if $v_2, v_{n-1} \in B$, then for any pair of different vertices $x, y \in V-B$ there exists $z \in B - \{v_1\}$ such that $d_{C'_n,2}(x,z) = d_{P_n,2}(x,z) \neq d_{P_n,2}(y,z) = d_{C'_n,2}(y,z)$, so B is an adjacency basis of C'_n where $\{v_n\}$ is a 1-gap. As in the previous case, some vertex $w \in (V-B)-\{v_n\}$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so B is not a dominating set of P_n. The proof is complete. $\square$

According to Lemma 3.77, we deduce that any triplet $(G, \mathcal{H}, 1)$ satisfies Properties $\mathcal{P}_2$ and $\mathcal{P}_3$, whenever G is any connected graph, and $\mathcal{H}$ is formed by paths of order at least seven and/or cycles of order at least seven, where at most one of these orders n_i holds that $n_i \mod 5 \in \{1,3\}$. Therefore, by Theorem 3.72 and Proposition 3.44 we can conclude the following result.

Theorem 3.78. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{P_{q_1}, \ldots, P_{q_r}, C_{q_{r+1}}, \ldots, C_{q_n}\}$. If $q_i \geq 7$ and there exists at most one q_j such that $q_j \mod 5 \in \{1,3\}$, then*

$$\dim_1(G \circ \mathcal{H}) = \dim_1(G \circ \overline{\mathcal{H}}) = \sum_{i=1}^{n} \left\lfloor \frac{2q_i+2}{5} \right\rfloor.$$

To finish this subsection, we continue now with some examples of classes of graphs achieving the equality in the bound of Theorem 3.70. To this end, we need the following lemma, in order to give another possible triplet satisfying Properties $\mathcal{P}_1$ and $\mathcal{P}_4$.

Lemma 3.79. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family of graphs. If every $H \in \mathcal{H}$ has diameter $D(H) \geq 6$, or has girth $\mathtt{g}(H) \geq 5$ and minimum degree $\delta(H) \geq 3$, then for $k \in \{1, \ldots, \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$ the triplet $(G, \mathcal{H}, k)$ satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_4$.*

Proof. By Corollaries 3.34 and 3.35, $\mathrm{adim}_k(H) = \dim_k(K_1 + H)$ for every $H \in \mathcal{H}$. By Theorem 3.32 there exists a k-adjacency basis A of H such that $|A - N_H(v)| \geq k$ for all $v \in V(H)$. Thus, we deduce that for any G and $k \in \{1, \dots, \mathcal{C}(G)\}$, the triplet $(G, \mathcal{H}, k)$ satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_4$. □

Finishing this subsection, as we mention before, now we are able to give a result in which we describe some other classes of graphs achieving the bound of Theorem 3.70. That is, by Corollary 3.73 (ii) and (v), and Lemma 3.79 we obtain the following.

Theorem 3.80. *Let G be a connected false twins free graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \dots, H_n\}$ be a family composed by n graphs such that every $H \in \mathcal{H}$ has diameter $D(H) \geq 6$, or has girth $\mathsf{g}(H) \geq 5$ and minimum degree $\delta(H) \geq 3$. Then for any $k \in \{1, \dots, \min\{\mathcal{T}(G \circ \mathcal{H}), \mathcal{C}(\mathcal{H})\}\}$, $\dim_k(G \circ \mathcal{H}) = \displaystyle\sum_{i=1}^{n} \mathrm{adim}_k(H_i)$. Moreover, if G is a connected true twins free graph of order $n \geq 2$, then $\dim_k(G \circ \overline{\mathcal{H}}) = \displaystyle\sum_{i=1}^{n} \mathrm{adim}_k(H_i)$.*

The particular case of the 2-metric dimension of $G \circ H$

As it can be inferred from the seen so far, the closed formulae for the value of $\dim_2(G \circ H)$ depends on the 2-adjacency dimension of H. Clearly, from Theorem 3.70 we deduce that for any connected graph G of order $n \geq 2$ and any nontrivial graph H, it follows $\dim_2(G \circ H) \geq n \cdot \mathrm{adim}_2(H)$, which leads to the next result.

Corollary 3.81. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. Then, there exists a non-negative integer $f(G, H)$ such that*

$$\dim_2(G \circ H) = n \cdot \mathrm{adim}_2(H) + f(G, H).$$

From now on, our goal is to determine the value of $f(G, H)$. To begin with, we rephrase the properties given above in order to facilitate their comprehension in this particular case. Since in this particular case $k = 2$, we only define the properties for a pair of graphs (G, H).

Property $\mathcal{P}_1$: G is true twins free, otherwise there exists a 2-adjacency basis A of H such that $A \not\subseteq N_H(v)$, for all $v \in V(H)$.

Property $\mathcal{P}_2$: G is false twins free, otherwise there exists a 2-adjacency basis A of H which is a dominating set of H.

To give the first case for a possible value of $f(G,H)$, we will state two particular cases of two more general already known results (see Theorem 3.72 and Corollary 3.73).

Theorem 3.82. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. Then $\dim_2(G \circ H) = n \cdot \mathrm{adim}_2(H)$ if and only if the pair (G,H) satisfies Properties $\mathcal{P}_1$ and $\mathcal{P}_2$.*

Corollary 3.83. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. Then the following statements hold.*

(i) *If G is a twins free graph, then*

$$\dim_2(G \circ H) = \dim_2(G \circ \overline{H}) = n \cdot \mathrm{adim}_2(H).$$

(ii) *If G is a false twins free graph and the pair (G,H) satisfies Property $\mathcal{P}_1$, then*

$$\dim_2(G \circ H) = n \cdot \mathrm{adim}_2(H).$$

(iii) *If G is true twins free graph and the pair (G,H) satisfies Property $\mathcal{P}_2$, then*

$$\dim_2(G \circ H) = n \cdot \mathrm{adim}_2(H).$$

Theorem 3.82 leads to the case when $f(G,H) = 0$. We next define the following four properties for a graph H which will be used to give the remaining possible values of $f(G,H)$. We must remark that the proofs of the forthcoming results showing the values of $f(G,H)$ are very similar with respect to the structure and the technique used. However, the complementary use of the next four properties (and its negations) makes necessary the almost complete development of each proof. Those analogous cases will be avoided.

Property $\mathcal{P}_3$. For each 2-adjacency basis A of H there exists a vertex v such that $A \subseteq N_H(v)$.

Property $\mathcal{P}_4$. For each 2-adjacency basis A of H there exists a vertex v such that $A \cap N_H[v] = \emptyset$.

Property $\mathcal{P}_5$. For each 2-adjacency generator S for H of cardinality $\mathrm{adim}_2(H)+1$ there exists a vertex v such that $|S - N_H(v)| \le 1$.

Property $\mathcal{P}_6$. For each 2-adjacency generator S for H of cardinality $\mathrm{adim}_2(H)+1$ there exists a vertex v such that $|S \cap N_H[v]| \le 1$.

Since the 2-adjacency generators of a graph H are simultaneously 2-adjacency generators of its complement $\overline{H}$, we deduce the following remark.

Remark 3.84. *Let G be a connected nontrivial graph such that $\overline{G}$ is connected and let H be a nontrivial graph. The following assertion hold.*

(i) *The pair (G,H) satisfies Property $\mathcal{P}_1$ if and only if the pair $(\overline{G},\overline{H})$ satisfies Property $\mathcal{P}_2$.*

(ii) *The graph H satisfies Property $\mathcal{P}_3$ if and only if $\overline{H}$ satisfies Property $\mathcal{P}_4$.*

(iii) *The graph H satisfies Property $\mathcal{P}_5$ if and only if $\overline{H}$ satisfies Property $\mathcal{P}_6$.*

Our first result regarding the existence of a possibly non-zero value for $f(G,H)$ is given in the next theorem.

Theorem 3.85. *Let G be a connected graph of order $n \ge 2$ and let H be a nontrivial graph. If the pair (G,H) satisfies $\mathcal{P}_2$ and H satisfies Properties $\mathcal{P}_3$ and $\mathcal{P}_5$, then*

$$\dim_2(G \circ H) = n \cdot \mathrm{adim}_2(H) + |TT(G)|.$$

Proof. Let B be a 2-metric basis of $G \circ H$ and, for any $u_l \in V(G)$, let $B_l = \{v : (u_l, v) \in B\}$. Suppose that there exists $u_i \in TT(G)$. In this case there exists $u_j \in TT(u_i) - \{u_i\}$. We claim that $|B_i \cup B_j| \ge 2 \cdot \mathrm{adim}_2(H) + 2$. Suppose, for purposes of contradiction, that

$$|B_i \cup B_j| \le 2 \cdot \mathrm{adim}_2(H) + 1. \qquad (3.1)$$

By Theorem 3.70 we know that B_i and B_j are 2-adjacency generators for H and so from (3.1) we have that at least one of them is a 2-adjacency basis of

H, say B_i. By Property $\mathcal{P}_3$, there exists $w \in V(H)$ such that $B_i \subseteq N_H(w)$. If B_j is a 2-adjacency basis of H, then there exists $w' \in V(H)$ such that $B_j \subseteq N_H(w')$. Thus, $|B \cap \mathcal{D}_{G\circ H}((u_i,w),(u_j,w'))| = |\,(\{u_i\} \times (B_i - N_H(w))) \cup (\{u_j\} \times (B_j - N_H(w')))\,| = 0$, which is a contradiction. If B_j is not a 2-adjacency basis of H, then $|B_j| = \operatorname{adim}_2(H) + 1$. By Property $\mathcal{P}_5$, there exists a vertex $w'' \in V(H)$ such that $|B_j - N_H(w'')| \leq 1$. Thus, $|B \cap \mathcal{D}_{G\circ H}((u_i,w),(u_j,w''))| = |\,(\{u_i\} \times (B_i - N_H(w))) \cup (\{u_j\} \times (B_j - N_H(w'')))\,| \leq 1$, which is a contradiction again. As a consequence $|B_i \cup B_j| \geq 2 \cdot \operatorname{adim}_2(H) + 2$. Therefore, any two vertices u_i, u_j in the same true twin equivalence class of G satisfy $|B_i \cup B_j| \geq 2 \cdot \operatorname{adim}_2(H) + 2$, which leads to

$$\left|\bigcup_{u_j \in TT(u_i)} B_j\right| \geq |TT(u_i)|(\operatorname{adim}_2(H) + 1). \tag{3.2}$$

Finally, since the intersection of any two true twin equivalence classes of a graph is empty, by Theorem 3.70 and (3.2) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \operatorname{adim}_2(H) + |TT(G)|$.

On the other hand, we consider a 2-adjacency basis A of H satisfying the following. If G has at least a false twin equivalence class, then A is taken such that every $z \in V(H)$ satisfies $A \cap N_H[z] \neq \emptyset$, which is possible by Property $\mathcal{P}_2$. Otherwise, we take an arbitrary 2-adjacency basis A of H. Let v_c be the vertex of H such that $A \subseteq N_H(v_c)$, which exists by Property $\mathcal{P}_3$. We shall show that $B' = \bigcup_{u_i \in V(G)} (\{u_i\} \times A) \cup \bigcup_{u_i \in TT(G)} \{(u_i, v_c)\}$ is a 2-metric generator for $G \circ H$. Note that $|B'| = n \cdot \operatorname{adim}_2(H) + |TT(G)|$. We analyse the following four cases in order to prove that any two different vertices $(u_i, v), (u_j, v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. In this case, $v \neq v'$. As A is a 2-adjacency basis of H, we obtain that $|A \cap \mathcal{C}_H(v,v')| \geq 2$ and since $\mathcal{D}_{G\circ H}((u_i,v),(u_i,v')) = \{u_i\} \times \mathcal{C}_H(v,v')$, we conclude that $|B' \cap \mathcal{D}_{G\circ H}((u_i,v),(u_i,v'))| \geq |(\{u_i\} \times A) \cap \mathcal{D}_{G\circ H}((u_i,v),(u_i,v'))| \geq 2$.

Case 2: $i \neq j$ and u_i, u_j are true twins. In this case we have that, $\mathcal{D}_{G\circ H}((u_i,v),(u_j,v')) = (\{u_i\} \times (V(H) - N_H(v))) \cup (\{u_j\} \times (V(H) - N_H(v')))$. If $v = v_c$, then $|B' \cap (\{u_i\} \times (V(H) - N_H(v)))| = |(\{u_i\} \times (A \cup \{v_c\})) \cap (\{u_i\} \times (V(H) - N_H(v)))| = |\{(u_i, v_c)\}| = 1$. Now, if $v \neq v_c$, then due to the fact that A is a 2-adjacency basis of H and $A \subseteq N_H(v_c)$, it follows that $2 \le |A \cap \mathcal{C}_H(v, v_c)| = |A - N_H(v))| \le |B' \cap (\{u_i\} \times (V(H) - N_H(v)))|$. In any case, $|B' \cap (\{u_i\} \times (V(H) - N_H(v)))| \ge 1$. Analogously, we deduce that $|B' \cap (\{u_j\} \times (V(H) - N_H(v')))| \ge 1$. Therefore, $|B' \cap \mathcal{D}_{G\circ H}((u_i, v), (u_j, v'))| \ge 2$.

Case 3: $i \ne j$ and u_i, u_j are false twins. In this case, $\mathcal{D}_{G\circ H}((u_i, v), (u_j, v')) = (\{u_i\} \times N_H[v]) \cup (\{u_j\} \times N_H[v'])$. Since $A \cap N_H[w] \ne \emptyset$ for all $w \in V(H)$, we deduce that $|B' \cap \mathcal{D}_{G\circ H}((u_i, v), (u_j, v'))| = |(\{u_i\} \times (A \cap N_H[v])) \cup (\{u_j\} \times (A \cap N_H[v']))| \ge 2$.

Case 4: $i \ne j$ and u_i, u_j are not twins. Clearly, there exists a vertex $u_l \in \mathcal{D}^*_G(u_i, u_j)$, and as a consequence, the elements of $B' \cap (\{u_l\} \times V(H))$ distinguish the vertices (u_i, v) and (u_j, v'). Since $|B' \cap (\{u_l\} \times V(H))| \ge |A| \ge 2$, we have that $|B' \cap \mathcal{D}_{G\circ H}((u_i, v), (u_i, v'))| \ge 2$.

Therefore, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G \circ H) \le |B'| = n \cdot \mathrm{adim}_2(H) + |TT(G)|$, which completes the proof. □

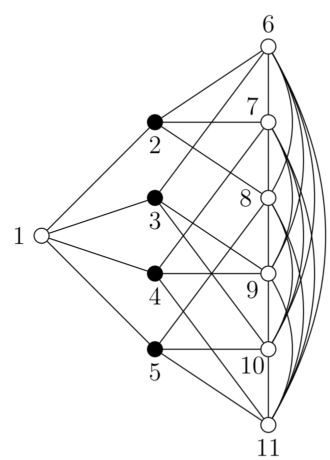

Figure 3.8: The set of black vertices $\{2, 3, 4, 5\}$ is the only 2-adjacency basis of this graph.

The premises of Theorem 3.85 are satisfied for the graph H shown in Figure 3.8 and any nontrivial connected graph G. Notice that the set of black vertices $N_H(1) = \{2, 3, 4, 5\}$ is the only 2-adjacency basis of H and so H satisfies Property $\mathcal{P}_3$. It can be checked that for every 2-adjacency generator

S for H of cardinality $|S| = \mathrm{adim}_2(H) + 1$ it follows that $N_H(1) \subset S$. Thus, Property $\mathcal{P}_5$ is satisfied. Since the 2-adjacency basis of H is a dominating set, we conclude that for any connected nontrivial graph G, the pair (G, H) satisfies Property $\mathcal{P}_2$. Note that this is an example where H does not have any dominating vertex.

If H is a graph having exactly one dominating vertex which does not belong to any 2-adjacency basis, then H satisfies Properties $\mathcal{P}_3$ and $\mathcal{P}_5$.

Corollaries 3.33, 3.34, 3.35 and Proposition 3.47 allow us to state the following consequences of Theorem 3.85.

Proposition 3.86. *Let G be a connected false twins free graph of order $n \geq 2$ and let H be a graph having at least one of the following properties.*

(i) *$H = K_1 + H'$, where H' is a graph of diameter $D(H') \geq 6$.*

(ii) *$H = K_1 + H'$, where $g(H') \geq 5$ and $\delta(H') \geq 3$.*

(iii) *$H = K_1 + H'$, such that H' is a tree that does not belong to $\{P_r, K_{1,n}, T'\}$, where $r \in \{2, \ldots, 5\}$, $n \geq 3$ and T' is a graph obtained from $K_{1,n} \cup K_2$ by joining by an edge one leaf of $K_{1,n}$ to one leaf of K_2.*

Then, $\dim_2(G \circ H) = n \cdot \mathrm{adim}_2(H') + |TT(G)|$.

We now show several others consequences of Theorem 3.85 for some particular families of graphs.

Proposition 3.87. *Let G be connected graph of order $n \geq 2$.*

(i) *If H is a fan graph $K_1 + P_{n'}$ such that $n' \geq 6$, then* $\dim_2(G \circ H) = n\lceil \frac{n'+1}{2} \rceil + |TT(G)|$.

(ii) *If H is a wheel graph $K_1 + C_{n'}$ such that $n' \geq 7$, then* $\dim_2(G \circ H) = n\lceil \frac{n'}{2} \rceil + |TT(G)|$.

(iii) *If H is a star graph $K_{1,n'}$ such that $n' \geq 2$, then* $\dim_2(G \circ H) = nn' + |TT(G)|$.

Proof. (i) By Corollary 3.33 and Proposition 3.47, the vertex of K_1 does not belong to any 2-adjacency basis of $K_1 + P_{n'}$. Hence, H satisfies Properties $\mathcal{P}_3$ and $\mathcal{P}_5$. It only remains to prove that the pair (G, H) satisfies Property $\mathcal{P}_2$.

Let v_0 be the vertex of K_1, let A be a 2-adjacency basis of $K_1+P_{n'}$ and let $V(P_{n'})=\{v_1,\ldots,v_{n'}\}$ where $v_j\sim v_{j+1}$ for $1\le j\le n'-1$. If $v_{j-1},v_j,v_{j+1}\notin A$, then $|A\cap\mathcal{C}_{K_1+P_{n'}}(v_{j-1},v_j)|\le 1$ and $|A\cap\mathcal{C}_{K_1+P_{n'}}(v_j,v_{j+1})|\le 1$, which is a contradiction. Thus, for every $j\in\{2,\ldots,n'-1\}$, it follows that $|A\cap N_{K_1+P_{n'}}[v_j]|\ge 1$. If $v_1,v_2\notin A$ or $v_{n'-1},v_{n'}\notin A$, then $|A\cap\mathcal{C}_{K_1+P_{n'}}(v_1,v_2)|\le 1$ or $|A\cap\mathcal{C}_{K_1+P_{n'}}(v_{n'-1},v_{n'})|\le 1$, respectively, which is a contradiction. So, $|A\cap N_{K_1+P_{n'}}[v_1]|\ge 1$ and $|A\cap N_{K_1+P_{n'}}[v_{n'}]|\ge 1$. Now, since $|A|\ge 2$, we deduce that $|A\cap N_{K_1+P_{n'}}[v_0]|=|A\cap V(K_1+P_{n'})|\ge 2$. Thus, the pair (G,H) satisfies Property $\mathcal{P}_2$. Therefore, by Theorem 3.85 and Propositions 3.39 and 3.45, we conclude the proof.

(ii) Proceeding analogously to the previous case, but now using Propositions 3.43 and 3.46 instead of Propositions 3.39 and 3.45, the result follows.

(iii) Let $K_{1,n'}=K_1+N_{n'}$. Note that the vertices of $N_{n'}$ are twins in $K_{1,n'}$ and, as a consequence, $V(N_{n'})$ is the only 2-adjacency basis of $K_{1,n'}$. Thus, $K_{1,n'}$ satisfies Properties $\mathcal{P}_3$ and $\mathcal{P}_5$. Note that for every $v\in V(K_{1,n'})$ it follows that $|V(N_{n'})\cap N_{K_{1,n'}}[v]|\ge 1$. So, the pair (G,H) satisfies Property $\mathcal{P}_2$. Therefore, by Theorem 3.85 we conclude the proof. □

In order to present other possible values for $f(G,H)$, from now on, we will refer to $\neg\mathcal{P}_i$ as the negation of $\mathcal{P}_i$. Besides, we will say that $\varphi(G)$ and $\tau(G)$ are the number of false and true equivalence classes of a graph G, respectively. Last, we denote by $\nu(G)$ the number of non-singleton twin equivalence classes, *i.e.*, $\nu(G)=\varphi(G)+\tau(G)$.

Theorem 3.88. *Let G be a connected graph of order $n\ge 2$ and let H be a nontrivial graph. If the pair (G,H) satisfies Property $\mathcal{P}_2$ and H satisfies Properties $\mathcal{P}_3$ and $\neg\mathcal{P}_5$, then*

$$\dim_2(G\circ H)=n\cdot\mathrm{adim}_2(H)+|TT(G)|-\tau(G).$$

Proof. Let B be a 2-metric basis of $G\circ H$. For any $u_l\in V(G)$ we define $B_l=\{v:(u_l,v)\in B\}$. Suppose that there exists $u_i\in TT(G)$ and let $u_j\in TT(u_i)-\{u_i\}$. We claim that $|B_i\cup B_j|\ge 2\cdot\mathrm{adim}_2(H)+1$. Suppose, for purposes of contradiction, that $|B_i\cup B_j|=2\cdot\mathrm{adim}_2(H)$. Since by Theorem 3.70 we have that B_i and B_j are 2-adjacency generators for H, we deduce that B_i and B_j are 2-adjacency bases of H. By Property $\mathcal{P}_3$, there exist two vertices $w, w' \in V(H)$ such that $B_i \subseteq N_H(w)$ and $B_j \subseteq N_H(w')$. Thus, $|B \cap \mathcal{D}_{G\circ H}((u_i,w),(u_j,w'))| = |(\{u_i\} \times (B_i - N_H(w))) \cup (\{u_j\} \times (B_j - N_H(w')))| = 0$, which is a contradiction. Thus, $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 1$ and, as a consequence,

$$\left| \bigcup_{u_j \in TT(u_i)} B_j \right| \geq |TT(u_i)|(\mathrm{adim}_2(H) + 1) - 1. \tag{3.3}$$

Finally, since the intersection of any two true twin equivalence classes of a graph is empty, by Theorem 3.70 and (3.3) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \mathrm{adim}_2(H) + |TT(G)| - \tau(G)$.

We will now show that

$$B' = \bigcup_{u_i \in (V(G) - TT(G)) \cup M(G)} (\{u_i\} \times A) \cup \bigcup_{u_i \in TT(G) - M(G)} (\{u_i\} \times S)$$

is a 2-metric generator for $G \circ H$, where $M(G)$ is a set composed by exactly one vertex of each true twin equivalence class of G. Also, A and S are a 2-adjacency basis and a 2-adjacency generator for H, respectively, as described at next. If G has at least a false twin equivalence class, then we take A as a 2-adjacency basis of H such that every $z \in V(H)$ satisfies $A \cap N_H[z] \neq \emptyset$, which is possible by Property $\mathcal{P}_2$. Otherwise, we take A as an arbitrary 2-adjacency basis of H. We take S as a 2-adjacency generator for H of cardinality $\mathrm{adim}_2(H) + 1$ that satisfies $|S - N_H(v)| \geq 2$ for every $v \in V(H)$, which exists by Property $\neg\mathcal{P}_5$. Note that, $|B'| = n \cdot \mathrm{adim}_2(H) + |TT(G)| - \tau(G)$.

We differentiate the following four cases in order to prove that any two different vertices $(u_i, v), (u_j, v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. It is analogous to Case 1 of the proof of Theorem 3.85.

Case 2: $i \neq j$ and u_i, u_j are true twins. In this case, we have that $\mathcal{D}_{G\circ H}((u_i, v), (u_j, v')) = (\{u_i\} \times (V(H) - N_H(v))) \cup (\{u_j\} \times (V(H) - N_H(v')))$. Note that $u_i \notin M(G)$ or $u_j \notin M(G)$, say $u_j \notin M(G)$. Thus, $B' \cap (\{u_j\} \times V(H)) = \{u_j\} \times S$ and, since $|S - N_H(v')| \geq 2$, we deduce that $|B' \cap \mathcal{D}_{G\circ H}((u_i, v), (u_j, v'))| \geq |\{u_j\} \times (S - N_H(v'))| \geq 2$.

Case 3: $i \neq j$ and u_i, u_j are false twins. It is analogous to Case 3 of the proof of Theorem 3.85.

Case 4: $i \neq j$ and u_i, u_j are not twins. It is analogous to Case 4 of the proof of Theorem 3.85.

Therefore, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G \circ H) \leq |B'| = n \cdot \operatorname{adim}_2(H) + |TT(G)| - \tau(G)$, which completes the proof. $\square$

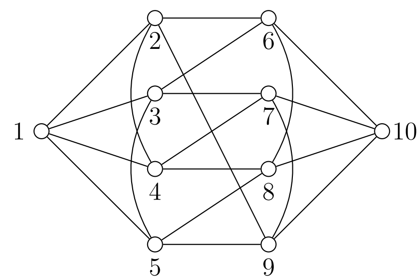

Figure 3.9: In the graph H, any 2-adjacency basis is given by $N_H(v)$ with $v \in V(H)$.

Consider the graph H shown in Figure 3.9. Notice that only the sets $N_H(v)$, with $v \in V(H)$ form 2-adjacency bases of H, and as a consequence, H satisfies Property $\mathcal{P}_3$. In addition, the 2-adjacency generator $S = \{1, 2, 3, 4, 8\}$ for H satisfies that $|S| = \operatorname{adim}_2(H) + 1$ and $|S - N_H(v)| \geq 2$ for every $v \in V(H)$. Thus, H satisfies Property $\neg\mathcal{P}_5$. Note that the value $\min\limits_{v \in V(H)} \{|S - N_H(v)|\} = 2$ is attained for vertices belonging to $\{1, 4, 5, 6\}$. Besides, the 2-adjacency basis $N_H(4) = \{1, 2, 7, 8\}$ of H is also a dominating set, and as a consequence, the pair (G, H) satisfies Property $\mathcal{P}_2$ for any connected nontrivial graph G. Therefore, by Theorem 3.88, we have that $\dim_2(G \circ H) = n \cdot \operatorname{adim}_2(H) + |TT(G)| - \tau(G) = 4n + |TT(G)| - \tau(G)$. For instance, if $G = K_n$, then we have $\dim_2(K_n \circ H) = 4n + |TT(K_n)| - \tau(K_n) = 4n + n - 1 = 5n - 1$.

Theorem 3.89. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. If the pair (G, H) satisfies Property $\mathcal{P}_1$ and H satisfies Properties $\mathcal{P}_4$ and $\mathcal{P}_6$, then*

$$\dim_2(G \circ H) = n \cdot \operatorname{adim}_2(H) + |FT(G)|.$$

Proof. Let B be a 2-metric basis of $G \circ H$ and let $B_i = \{v : (u_i, v) \in B\}$. Suppose that u_i and u_j are false twins in G. We claim that $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 2$. Suppose, for purposes of contradiction, that $|B_i \cup B_j| \leq 2 \cdot \mathrm{adim}_2(H) + 1$. According to Theorem 3.70, B_i and B_j are 2-adjacency generators for H. Hence, B_i or B_j, say B_i, is a 2-adjacency basis of H. By Property $\mathcal{P}_4$, there exists $w \in V(H)$ such that $B_i \cap N_H[w] = \emptyset$. If B_j is a 2-adjacency basis of H, then by Property $\mathcal{P}_4$, there exists $w' \in V(H)$ such that $B_j \cap N_H[w'] = \emptyset$. Thus, $|B \cap \mathcal{D}_{G\circ H}((u_i, w), (u_j, w'))| = |\,(\{u_i\} \times (B_i \cap N_H[w])) \cup (\{u_j\} \times (B_j \cap N_H[w']))\,| = 0$, which is a contradiction. If B_j is not a 2-adjacency basis of H, then $|B_j| = \mathrm{adim}_2(H) + 1$. Now, by Property $\mathcal{P}_6$, there exists a vertex $w'' \in V(H)$ such that $|B_j \cap N_H[w'']| \leq 1$. Thus, $|B \cap \mathcal{D}_{G\circ H}((u_i, w), (u_j, w''))| = |\,(\{u_i\} \times (B_i \cap N_H[w])) \cup (\{u_j\} \times (B_j \cap N_H[w'']))\,| \leq 1$, which is a contradiction again. So, as we have claimed, $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 2$. Hence, any two vertices u_i, u_j in the same false twin equivalence class of G satisfy $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 2$, and as a consequence, we obtain that

$$\left| \bigcup_{u_j \in FT(u_i)} B_j \right| \geq |FT(u_i)|(\mathrm{adim}_2(H) + 1). \tag{3.4}$$

Finally, since the intersection of any two false twin equivalence classes of a graph is empty, by Theorem 3.70 and (3.4) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \mathrm{adim}_2(H) + |FT(G)|$.

On the other hand, we consider a 2-adjacency basis A of H satisfying the following. If G has at least a true twin equivalence class, then we choose the 2-adjacency basis A, such that every $z \in V(H)$ satisfies $A - N_H(z) \neq \emptyset$, which is possible by Property $\mathcal{P}_1$. Otherwise, we choose A as an arbitrary 2-adjacency basis of H. Let v_c be the vertex of H such that $A \cap N_H[v_c] = \emptyset$, which exists by Property $\mathcal{P}_4$. We will show that $B' = \bigcup_{u_i \in V(G)} (\{u_i\} \times A) \cup \bigcup_{u_i \in FT(G)} \{(u_i, v_c)\}$ is a 2-metric generator for $G \circ H$. Note that $|B'| = n \cdot \mathrm{adim}_2(H) + |FT(G)|$. We analyse the following four cases in order to prove that any two different vertices $(u_i, v), (u_j, v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. It is analogous to Case 1 of the proof of Theorem 3.85.

Case 2: $i \neq j$ and u_i, u_j are true twins. In this case we have that $\mathcal{D}_{G\circ H}((u_i, v),$

$(u_j, v')) = (\{u_i\} \times (V(H) - N_H(v))) \cup (\{u_j\} \times (V(H) - N_H(v')))$. Since $A - N_H(z) \neq \emptyset$ for all $z \in V(H)$, we deduce that $|B' \cap \mathcal{D}_{G\circ H}((u_i, v), (u_j, v'))| = |(\{u_i\} \times (A - N_H(v))) \cup (\{u_j\} \times (A - N_H(v')))| \geq 2$.

Case 3: $i \neq j$ and u_i, u_j are false twins. In this case, $\mathcal{D}_{G\circ H}((u_i, v), (u_j, v')) = (\{u_i\} \times N_H[v]) \cup (\{u_j\} \times N_H[v'])$. If $v = v_c$, then $|B' \cap (\{u_i\} \times N_H[v])| = |(\{u_i\} \times (A \cup \{v_c\})) \cap (\{u_i\} \times N_H[v])| = |\{(u_i, v_c)\}| = 1$. Now, if $v \neq v_c$, then, due to the fact that A is a 2-adjacency basis of H and $A \cap N_H[v_c] = \emptyset$, it follows that $2 \leq |A \cap \mathcal{C}_H(v, v_c)| = |A \cap N_H[v]| \leq |B' \cap (\{u_i\} \times N_H[v])|$. In both cases, $|B' \cap (\{u_i\} \times N_H[v])| \geq 1$. Analogously, we deduce that $|B' \cap (\{u_j\} \times N_H[v'])| \geq 1$. Therefore, $|B' \cap \mathcal{D}_{G\circ H}((u_i, v), (u_j, v'))| \geq 2$.

Case 4: $i \neq j$ and u_i, u_j are not twins. It is analogous to Case 4 of the proof of Theorem 3.85.

Therefore, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G\circ H) \leq |B'| = n \cdot \operatorname{adim}_2(H) + |FT(G)|$, which completes the proof. □

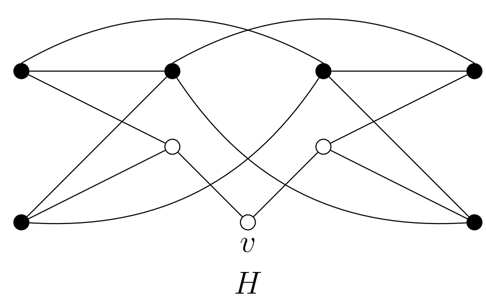

Figure 3.10: The set of black vertices A is the only 2-adjacency basis of the graph H.

The premises of Theorem 3.89 are satisfied for the graph H shown in Figure 3.10 and any connected nontrivial graph G. Note that the black vertices are twins and thus, they belong to any 2-adjacency basis of H. In addition, the set of black vertices A is a 2-adjacency generator for H and therefore, it is the only 2-adjacency basis of H. Since $A \cap N[v] = \emptyset$, the graph H satisfies Property $\mathcal{P}_4$. Since any 2-adjacency generator S for H must contain the set of twin vertices of H, we deduce that $A \subseteq S$. Thus, if $|S| = \operatorname{adim}_2(H) + 1 = |A| + 1$, then $|S - N_H(v)| \leq 1$, and as a consequence,

H satisfies Property $\mathcal{P}_6$. Since $A \not\subseteq N_H(v)$ for all $v \in V(H)$, we conclude that for any connected nontrivial graph G, the pair (G,H) satisfies Property $\mathcal{P}_1$. Note that this is an example where H does not have any isolated vertex.

Note that by Remark 3.84, any pair of graph (G,H) satisfies the conditions of Theorem 3.85 if and only if $(\overline{G},\overline{H})$ satisfies the conditions of Theorem 3.89. Thus, in the next two propositions we give some families of pairs of graphs satisfying the conditions of Theorem 3.89 from Propositions 3.86 and 3.87.

Proposition 3.90. *Let G be a connected true twins free graph of order $n \geq 2$ and let H be a graph having at least one of the following properties.*

(i) $H = K_1 \cup \overline{H'}$, *where H' is a graph of diameter $D(H') \geq 6$.*

(ii) $H = K_1 \cup \overline{H'}$, *where $g(H') \geq 5$ and $\delta(H') \geq 3$.*

(iii) $H = K_1 \cup \overline{H'}$ *such that H' is a tree that does not belong to $\{P_r, K_{1,n}, T'\}$, where $r \in \{2,\ldots,5\}$, $n \geq 3$ and T' is a graph obtained from $K_{1,n} \cup K_2$ by joining by an edge one leaf of $K_{1,n}$ to one leaf of K_2.*

Then $\dim_2(G \circ H) = n \cdot \operatorname{adim}_2(H') + |FT(G)|$.

Proposition 3.91. *Let G be a connected graph of order $n \geq 2$.*

(i) *If $H = K_1 \cup \overline{P}_{n'}$ such that $n' \geq 6$, then* $\dim_2(G \circ H) = n\lceil \frac{n'+1}{2} \rceil + |FT(G)|$.

(ii) *If $H = K_1 \cup \overline{C}_{n'}$ such that $n' \geq 7$, then* $\dim_2(G \circ H) = n\lceil \frac{n'}{2} \rceil + |FT(G)|$.

(iii) *If $H = K_1 \cup K_{n'}$ such that $n' \geq 2$, then* $\dim_2(G \circ H) = nn' + |FT(G)|$.

Theorem 3.92. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. If the pair (G,H) satisfies Property $\mathcal{P}_1$ and H satisfies Properties $\mathcal{P}_4$ and $\neg\mathcal{P}_6$, then*

$$\dim_2(G \circ H) = n \cdot \operatorname{adim}_2(H) + |FT(G)| - \varphi(G).$$

Proof. Let B be a 2-metric basis of $G \circ H$ and let $B_i = \{v : (u_i, v) \in B\}$. Suppose that u_i and u_j are false twins in G. We claim that $|B_i \cup B_j| \geq 2 \cdot \operatorname{adim}_2(H) + 1$. Suppose, for purposes of contradiction, that $|B_i \cup B_j| = 2 \cdot \operatorname{adim}_2(H)$. According to Theorem 3.70, B_i and B_j are 2-adjacency generators for H. Hence, B_i and B_j are 2-adjacency bases of H. By Property $\mathcal{P}_4$, there exist $w, w' \in V(H)$ such that $B_i \cap N_H[w] = \emptyset$ and $B_j \cap$

$N_H[w'] = \emptyset$. Thus, $|B \cap \mathcal{D}_{G\circ H}((u_i,w),(u_j,w'))| = |\,(\{u_i\} \times (B_i \cap N_H[w])) \cup (\{u_j\} \times (B_j \cap N_H[w']))\,| = 0$, which is a contradiction. So, as we have claimed before, $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H)+1$. Hence, any two vertices u_i, u_j in the same false twin equivalence class of G satisfy $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H)+1$, and as a consequence, we obtain that

$$\left|\bigcup_{u_j \in FT(u_i)} B_j\right| \geq |FT(u_i)|(\mathrm{adim}_2(H)+1)-1. \tag{3.5}$$

Finally, since the intersection of any two false twin equivalence classes of a graph is empty, by Theorem 3.70 and (3.5) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \mathrm{adim}_2(H) + |FT(G)| - \varphi(G)$.

We will show that

$$B' = \bigcup_{u_i \in (V(G)-FT(G))\cup M(G)} (\{u_i\} \times A) \cup \bigcup_{u_i \in FT(G)-M(G)} (\{u_i\} \times S)$$

is a 2-metric generator for $G \circ H$, where $M(G)$ is a set composed by exactly one vertex of each false twin equivalence class of G. Also A and S are a 2-adjacency basis and a 2-adjacency generator for H, respectively, chosen as follows. If G has at least a true twin equivalence class, then we take A as a 2-adjacency basis of H such that every $z \in V(H)$ satisfies $A \cap N_H[z] \neq \emptyset$, which is possible by Property $\mathcal{P}_1$. Otherwise, we take A as an arbitrary 2-adjacency basis of H. Also, we take S as a 2-adjacency generator for H of cardinality $\mathrm{adim}_2(H)+1$ that satisfies $|S \cap N_H[z']| \geq 2$ for every $z' \in V(H)$, which exists by Property $\neg\mathcal{P}_6$. Note that, $|B'| = n \cdot \mathrm{adim}_2(H) + |TT(G)| - \varphi(G)$.

We differentiate the following four cases in order to prove that any two different vertices $(u_i,v),(u_j,v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. It is analogous to Case 1 of the proof of Theorem 3.85.

Case 2: $i \neq j$ and u_i, u_j are true twins. It is analogous to Case 2 of the proof of Theorem 3.89.

Case 3: $i \neq j$ and u_i, u_j are false twins. In this case, $\mathcal{D}_{G\circ H}((u_i,v),(u_j,v')) = (\{u_i\} \times N_H[v]) \cup (\{u_j\} \times N_H[v'])$. Note that $u_i \notin M(G)$ or $u_j \notin M(G)$, say $u_j \notin M(G)$. Thus, $B' \cap (\{u_j\} \times V(H)) = \{u_j\} \times S$. By the definition of S, we have that it is a 2-adjacency generator for H such that $|S \cap N_H[v']| \geq 2$. Therefore, we deduce that $|B' \cap \mathcal{D}_{G\circ H}((u_i,v),(u_j,v'))| \geq |\{u_j\} \times (S \cap N_H[v'])| \geq 2$.

Case 4: $i \neq j$ and u_i, u_j are not twins. It is analogous to Case 4 of the proof of Theorem 3.85.

Therefore, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G \circ H) \leq |B'| = n \cdot \operatorname{adim}_2(H) + |FT(G)| - \varphi(G)$, which completes the proof. □

Consider the graph H shown in Figure 3.9. Note that by Remark 3.84, any pair of graphs (G,H) satisfies the conditions of Theorem 3.88 if and only if $(\overline{G}, \overline{H})$ satisfies the conditions of Theorem 3.92. Therefore, by Theorem 3.92, we have that $\dim_2(G \circ \overline{H}) = n \cdot \operatorname{adim}_2(H) + |FT(G)| - \varphi(G) = 4n + |FT(G)| - \varphi(G)$ for any connected graph G of order $n \geq 2$. For instance, if $G = K_{1,n-1}$, then we can compute $\dim_2(K_{1,n-1} \circ \overline{H}) = 4n + |FT(K_{1,n-1})| - \varphi(K_{1,n-1}) = 4n + (n-1) - 1 = 5n - 2$.

Theorem 3.93. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. If H satisfies Properties $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$, then*

$$\dim_2(G \circ H) = \dim_2(G \circ \overline{H}) = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)|.$$

Proof. Let B be a 2-metric basis of $G \circ H$ and let $B_i = \{v : (u_i, v) \in B\}$. If u_i and u_j are true twins in G, then by using an analogous procedure as in the first part of the proof of Theorem 3.85 we obtain that $|B_i \cup B_j| \geq 2 \cdot \operatorname{adim}_2(H) + 2$, which leads to

$$\left| \bigcup_{u_j \in TT(u_i)} B_j \right| \geq |TT(u_i)|(\operatorname{adim}_2(H) + 1). \tag{3.6}$$

Similarly, if u_i and u_j are false twins in G, then as in the first part of the proof of Theorem 3.89 we obtain that $|B_i \cup B_j| \geq 2 \cdot \operatorname{adim}_2(H) + 2$, which leads to

$$\left| \bigcup_{u_j \in FT(u_i)} B_j \right| \geq |FT(u_i)|(\operatorname{adim}_2(H) + 1). \tag{3.7}$$

Now, since the intersection of any two (true and false) twin equivalence classes of a graph is empty, by Theorem 3.70, (3.6) and (3.7) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)|$.

On the other hand, we take an arbitrary 2-adjacency basis A of H. Let v_c, v'_c be the two vertices of H such that $A \subseteq N_H(v_c)$ and $A \cap N_H[v'_c] = \emptyset$, which exist by Properties $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively. We will show that $B' = \bigcup_{u_i \in V(G)} (\{u_i\} \times A) \cup \bigcup_{u_i \in TT(G)} \{(u_i, v_c)\} \cup \bigcup_{u_i \in FT(G)} \{(u_i, v'_c)\}$ is a 2-metric generator for $G \circ H$. Note that $|B'| = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)|$. We analyse the following four cases in order to prove that any two different vertices $(u_i, v), (u_j, v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. It is analogous to Case 1 of the proof of Theorem 3.85.

Case 2: $i \neq j$ and u_i, u_j are true twins. It is analogous to Case 2 of the proof of Theorem 3.85.

Case 3: $i \neq j$ and u_i, u_j are false twins. We proceed analogously to Case 3 of the proof of Theorem 3.89, using v'_c instead of v_c.

Case 4: $i \neq j$ and u_i, u_j are not twins. It is analogous to Case 4 of the proof of Theorem 3.85.

Thus, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G \circ H) \leq |B'| = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)|$. Therefore $\dim_2(G \circ H) = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)|$. Finally, since each 2-adjacency basis of $\overline{H}$ is also a 2-adjacency basis of H, by Remark 3.84 we conclude that $\overline{H}$ satisfies the conditions of the theorem, which means that $\dim_2(G \circ \overline{H}) = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)|$ and the proof is complete. □

For instance, we take a family of graphs $\mathcal{H}$ such that for any $H \in \mathcal{H}$ we have that $\Delta(H) \geq 1$ and every vertex in $V(H)$ belongs to a non-singleton twin equivalence class. If $\bigcup_{H \in \mathcal{H}} V(H)$ is different from a complete graph, then the set $\bigcup_{H \in \mathcal{H}} V(H)$ belongs to any 2-adjacency generator of $H' = K_1 + (K_1 \cup \bigcup_{H \in \mathcal{H}} H)$. Moreover, $\bigcup_{H \in \mathcal{H}} V(H)$ is a 2-generator for H', and as a consequence, it is the only 2-adjacency basis. Let u be the universal vertex of H' and let $v \in V(H')$ be the vertex of degree 1. Note that $N_{H'}(u) \subseteq \bigcup_{H \in \mathcal{H}} V(H)$ and $N_{H'}[v] \cap \bigcup_{H \in \mathcal{H}} V(H) = \emptyset$. Thus, Properties $\mathcal{P}_3$ and $\mathcal{P}_4$ are satisfied. Since $\bigcup_{H \in \mathcal{H}} V(H)$ is subset of any 2-adjacency generator for H', Properties $\mathcal{P}_5$ and $\mathcal{P}_6$ are also satisfied. Therefore, if we take any connected graph G of order $n \geq 2$, then by Theorem 3.93 we have that $\dim_2(G \circ H') = n \cdot \mathrm{adim}_2(H') + |TT(G)| + |FT(G)| = n(|V(H')| - 2) + |TT(G)| + |FT(G)|$. In Figure 3.11 we show an example of a graph $H' = K_1 + (K_1 \cup C_4 \cup K_2)$ that satisfies the Properties $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$.

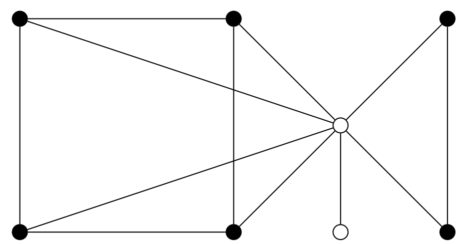

Figure 3.11: The graph $K_1 + (K_1 \cup C_4 \cup K_2)$ satisfies Properties $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$, and the set of black vertices is its only 2-adjacency basis.

Theorem 3.94. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. If H satisfies Properties $\mathcal{P}_3$, $\mathcal{P}_4$, $\neg\mathcal{P}_5$ and $\neg\mathcal{P}_6$, then*

$$\dim_2(G \circ H) = \dim_2(G \circ \overline{H}) = n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \nu(G).$$

Proof. Let B be a 2-metric basis of $G \circ H$ and let $B_i = \{v : (u_i, v) \in B\}$. If u_i and u_j are true twins in G, then as in the first part of the proof of Theorem 3.88 we obtain that $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 1$. Thus

$$\left| \bigcup_{u_j \in TT(u_i)} B_j \right| \geq |TT(u_i)|(\mathrm{adim}_2(H) + 1) - 1. \qquad (3.8)$$

Also, if u_i and u_j are false twins in G, then as in the first part of the proof of Theorem 3.92 we obtain that $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 1$, which leads to

$$\left| \bigcup_{u_j \in FT(u_i)} B_j \right| \geq |FT(u_i)|(\mathrm{adim}_2(H) + 1) - 1. \qquad (3.9)$$

Since the intersection of any two twin equivalence classes of a graph is empty, by Theorem 3.70, (3.8) and (3.9) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \mathrm{adim}_2(H) + |FT(G)| - \nu(G)$.

On the other hand, we take an arbitrary 2-adjacency basis A of H. Let v_c, v'_c be the two vertices of H such that $A \subseteq N_H(v_c)$ and $A \cap N_H[v'_c] = \emptyset$, which exist by Properties $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively. Also, we take two 2-adjacency generators S and S' for H of cardinality $\mathrm{adim}_2(H)+1$ that satisfy $|S - N_H(v)| \geq 2$ and $|S' \cap N_H[v]| \geq 2$ for every $v \in V(H)$, which exist by Properties $\neg\mathcal{P}_5$ and $\neg\mathcal{P}_6$, respectively. Last, we take two vertex sets $M(G)$ and $M'(G)$ such that $M(G)$ is composed by exactly one vertex of each true twin equivalence class of G and $M'(G)$ is composed by exactly one vertex of each false twin equivalence class of G.

We will show that

$$\begin{aligned} B' = & \bigcup_{u_i \in (V(G)-TT(G)-FT(G)) \cup M(G) \cup M'(G)} (\{u_i\} \times A) \cup \\ & \cup \bigcup_{u_i \in TT(G)-M(G)} (\{u_i\} \times S) \cup \\ & \cup \bigcup_{u_i \in FT(G)-M'(G)} (\{u_i\} \times S') \end{aligned}$$

is a 2-metric generator for $G \circ H$. Note that, $|B'| = n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \nu(G)$. We analyse the following four cases in order to prove that any two different vertices $(u_i, v), (u_j, v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. It is analogous to Case 1 of the proof of Theorem 3.85.

Case 2: $i \neq j$ and u_i, u_j are true twins. It is analogous to Case 2 of the proof of Theorem 3.88.

Case 3: $i \neq j$ and u_i, u_j are false twins. We proceed analogously to Case 3 of the proof of Theorem 3.92, using S' instead of S.

Case 4: $i \neq j$ and u_i, u_j are not twins. It is analogous to Case 4 of the proof of Theorem 3.85.

Therefore, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G \circ H) \leq |B'| = n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \nu(G)$. Finally, as in Theorem 3.93, we also have that $\dim_2(G \circ \overline{H}) = n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \nu(G)$, which completes the proof. □

The graph H shown in Figure 3.12 has only two 2-adjacency bases $A = \{2, 3, 4, 5\}$ and $A' = \{6, 7, 8, 9\}$. Note that $N_H(1) \subseteq A$ and $N_H[10] \cap A = \emptyset$,

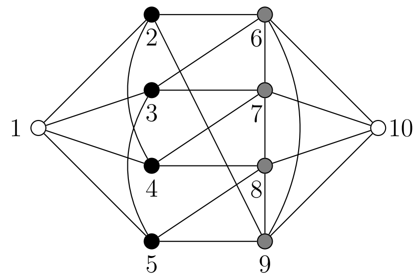

Figure 3.12: This graph satisfies the conditions of Theorem 3.94 and its only two adjacency bases are $\{2,3,4,5\}$ and $\{6,7,8,9\}$.

as well as, $N_H(10) \subseteq A'$ and $N_H[1] \cap A' = \emptyset$. Thus, H satisfies Properties $\mathcal{P}_3$ and $\mathcal{P}_4$. The set $S = \{1,2,3,8,10\}$ is a 2-adjacency generator for H such that $|S| = \operatorname{adim}(H)+1$, $|S - N_H(v)| \geq 2$ and $|S \cap N_H[v]| \geq 2$ for every $v \in V(H)$. Note that the value $\min_{v\in V(H)}\{|S - N_H(v)|\} = 2$ is attained for vertices belonging to $\{4,5,6,7,9\}$ and the value $\min_{v\in V(H)}\{|S \cap N_H[v]|\} = 2$ is attained for vertices belonging to $\{2,3,8,10\}$. Hence, H satisfies Properties $\neg\mathcal{P}_5$ and $\neg\mathcal{P}_6$, and as a consequence, it satisfies conditions of Theorem 3.94.

Theorem 3.95. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. If H satisfies Properties $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\neg\mathcal{P}_6$, then*

$$\dim_2(G \circ H) = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)| - \varphi(G).$$

Proof. Let B be a 2-metric basis of $G \circ H$ and let $B_i = \{v : (u_i, v) \in B\}$. If u_i and u_j are true twins in G, then as in the first part of the proof of Theorem 3.85 we obtain that $|B_i \cup B_j| \geq 2 \cdot \operatorname{adim}_2(H) + 2$, which leads to

$$\left|\bigcup_{u_j \in TT(u_i)} B_j\right| \geq |TT(u_i)|(\operatorname{adim}_2(H)+1). \tag{3.10}$$

Also, if u_i and u_j are false twins in G, then as in the first part of the proof of Theorem 3.92 we have $|B_i \cup B_j| \geq 2 \cdot \operatorname{adim}_2(H) + 1$, and as a consequence,

$$\left|\bigcup_{u_j \in FT(u_i)} B_j\right| \geq |FT(u_i)|(\operatorname{adim}_2(H)+1) - 1. \tag{3.11}$$

Since the intersection of any two twin equivalence classes of a graph is empty, by Theorem 3.70, (3.10) and (3.11) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \operatorname{adim}_2(H) + |FT(G)| - \varphi(G)$.

On the other hand, we take an arbitrary 2-adjacency basis A of H. Let v_c, v'_c be the two vertices of H such that $A \subseteq N_H(v_c)$ and $A \cap N_H[v'_c] = \emptyset$, which exist by Properties $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively. Also, we take a 2-adjacency generator S for H of cardinality $\mathrm{adim}_2(H)+1$ satisfying $|S \cap N_H(v)| \geq 2$ for every $v \in V(H)$, which exists by Property $\neg\mathcal{P}_6$. Last, we take a vertex set $M(G)$ composed by exactly one vertex of each false twin equivalence class of G.

We will show that

$$B' = \bigcup_{u_i \in (V(G)-FT(G)) \cup M(G)} (\{u_i\} \times A) \cup \bigcup_{u_i \in TT(G)} \{(u_i, v_c)\} \cup \bigcup_{u_i \in FT(G)-M(G)} (\{u_i\} \times S)$$

is a 2-metric generator for $G \circ H$. Note that, $|B'| = n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \varphi(G)$. We analyse the next four cases in order to prove that any two different vertices $(u_i, v), (u_j, v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. It is analogous to Case 1 of the proof of Theorem 3.85.

Case 2: $i \neq j$ and u_i, u_j are true twins. It is analogous to Case 2 of the proof of Theorem 3.85.

Case 3: $i \neq j$ and u_i, u_j are false twins. It is analogous to Case 3 of the proof of Theorem 3.92.

Case 4: $i \neq j$ and u_i, u_j are not twins. It is analogous to Case 4 of the proof of Theorem 3.85.

Therefore, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G \circ H) \leq |B'| = n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \varphi(G)$. $\square$

The graph H shown in Figure 3.13 has only one 2-adjacency basis given by $A = \{2,3,4,5\}$. Note that $N_H(1) \subseteq A$ and $N_H[10] \cap A = \emptyset$. Thus, H satisfies Properties $\mathcal{P}_3$ and $\mathcal{P}_4$. The set $S = \{2,3,6,9,10\}$ is a 2-adjacency generator for H such that $|S| = \mathrm{adim}(H)+1$ and $|S \cap N_H[v]| \geq 2$ for every $v \in V(H)$. Note that the value $\min_{v \in V(H)} \{|S \cap N_H[v]|\} = 2$ is attained for vertices belonging to $\{1,2,3,4,5\}$. Hence, H satisfies Property $\neg\mathcal{P}_6$. It can be checked that there exists at least one vertex $v \in V(H)$ such that $|S -$

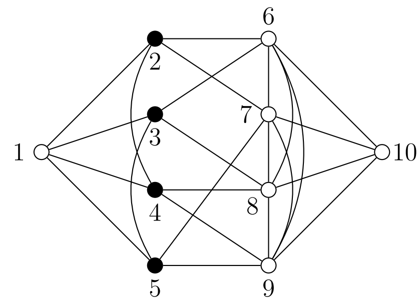

Figure 3.13: This graph satisfies Properties $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\neg\mathcal{P}_6$, and the set of black vertices $\{2,3,4,5\}$ is its only 2-adjacency basis.

$N_H(v)| = 1$ for each one of the eleven 2-adjacency generators for H having cardinality $\mathrm{adim}(H)+1$. So, H satisfies Property $\mathcal{P}_5$, and as a consequence, it satisfies conditions of Theorem 3.95.

Theorem 3.96. *Let G be a connected graph of order $n \geq 2$ and let H be a nontrivial graph. If H satisfies Properties $\mathcal{P}_3$, $\mathcal{P}_4$, $\neg\mathcal{P}_5$ and $\mathcal{P}_6$, then*

$$\dim_2(G \circ H) = n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \tau(G).$$

Proof. Let B be a 2-metric basis of $G \circ H$ and let $B_i = \{v : (u_i, v) \in B\}$. If u_i and u_j are true twins in G, then as in the first part of the proof of Theorem 3.88 we obtain that $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 1$. So we have

$$\left| \bigcup_{u_j \in TT(u_i)} B_j \right| \geq |TT(u_i)|(\mathrm{adim}_2(H) + 1) - 1. \tag{3.12}$$

Also, if u_i and u_j are false twins in G, then as in the first part of the proof of Theorem 3.89 we obtain that $|B_i \cup B_j| \geq 2 \cdot \mathrm{adim}_2(H) + 2$. Thus,

$$\left| \bigcup_{u_j \in FT(u_i)} B_j \right| \geq |FT(u_i)|(\mathrm{adim}_2(H) + 1). \tag{3.13}$$

Since the intersection of any two twin equivalence classes of a graph is empty, by Theorem 3.70, (3.12) and (3.13) we conclude that $\dim_2(G \circ H) = |B| = \sum_{i=1}^{n} |B_i| \geq n \cdot \mathrm{adim}_2(H) + |TT(G)| + |FT(G)| - \tau(G)$.

On the other hand, we take an arbitrary 2-adjacency basis A of H. Let v_c, v_c' be the two vertices of H such that $A \subseteq N_H(v_c)$ and $A \cap N_H[v_c'] = \emptyset$, which exist by Properties $\mathcal{P}_3$ and $\mathcal{P}_4$, respectively. Also, we take a 2-adjacency generator S for H of cardinality $\mathrm{adim}_2(H) + 1$ satisfying $|S -$

$N_H(v)| \geq 2$ for every $v \in V(H)$, which exists by Property $\neg\mathcal{P}_5$. Last, we take a vertex set $M(G)$ composed by exactly one vertex of each true twin equivalence class of G.

We will show that

$$B' = \bigcup_{u_i \in (V(G)-TT(G)) \cup M(G)} (\{u_i\} \times A) \cup \bigcup_{u_i \in TT(G)-M(G)} (\{u_i\} \times S) \cup \bigcup_{u_i \in FT(G)} \{(u_i, v_c')\}$$

is a 2-metric generator for $G \circ H$. Note that, $|B'| = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)| - \nu(G)$. We analyse the following four cases in order to prove that any two different vertices $(u_i, v), (u_j, v') \in V(G \circ H)$ are distinguished by at least two vertices in B'.

Case 1: $i = j$. It is analogous to Case 1 of the proof of Theorem 3.85.

Case 2: $i \neq j$ and u_i, u_j are true twins. It is analogous to Case 2 of the proof of Theorem 3.88.

Case 3: $i \neq j$ and u_i, u_j are false twins. We proceed analogously to Case 3 of the proof of Theorem 3.89, using v_c' instead of v_c.

Case 4: $i \neq j$ and u_i, u_j are not twins. It is analogous to Case 4 of the proof of Theorem 3.85.

Therefore, B' is a 2-metric generator for $G \circ H$ and, as a consequence, $\dim_2(G \circ H) \leq |B'| = n \cdot \operatorname{adim}_2(H) + |TT(G)| + |FT(G)| - \tau(G)$. $\square$

Note that by Remark 3.84, any graph H satisfies the conditions of Theorem 3.95 if and only if $\overline{H}$ satisfies the conditions of Theorem 3.96. The graph H shown in Figure 3.13 satisfies the conditions of Theorem 3.95. Therefore, $\overline{H}$ is an example of a graph satisfying the conditions of Theorem 3.96.

Notice that the assumptions of Theorems 3.82, 3.85, 3.88, 3.89, 3.92, 3.93, 3.94, 3.95 and 3.96 cover all the possible values for $f(G,H)$ while computing $\dim_2(G \circ H)$, where G is a connected nontrivial graph and H is a nontrivial graph. Therefore, it is possible to compute $\dim_2(G \circ H)$ in terms of the values of $\operatorname{adim}_2(H)$, $|TT(G)|$, $|FT(G)|$, $\tau(G)$, $\varphi(G)$ and $\nu(G)$.

3.4.2 Corona product graphs

In this subsection we compute or bound the k-metric dimension of corona product graphs. To do so, we present a generalization of a result obtained in [41, 42] for $k = 1$. At the same time, this result shows the strong relationship existing between the k-metric dimension of $G \odot \mathcal{H}$ and the k-adjacency dimension of the graphs belonging to $\mathcal{H}$. By Remark 3.24, this relationship can be extended to $G \odot \overline{\mathcal{H}}$.

Theorem 3.97. *If G is a connected graph of order $n \geq 2$ and $\mathcal{H} = \{H_1, \ldots, H_n\}$ is a family composed by n nontrivial graphs, then for $k \in \{1, \ldots, \mathcal{C}(\mathcal{H})\}$*

$$\dim_k(G \odot \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i) = \dim_k(G \odot \overline{\mathcal{H}}).$$

Proof. We will show that $\dim_k(G \odot \mathcal{H}) \leq \sum_{i=1}^{n} \operatorname{adim}_k(H_i)$. Let S_i be a k-adjacency basis of H_i. In order to show that $S = \bigcup_{i=1}^{n} S_i$ is a k-metric generator for $G \odot \mathcal{H}$, we analyse the following four cases for two different vertices $x, y \in V(G \odot \mathcal{H})$.

1. $x, y \in V(H_i)$. Since S_i is a k-adjacency basis of H_i, it follows that $|S_i \cap \mathcal{C}_{H_i}(x,y)| \geq k$. Since $\mathcal{C}_{H_i}(x,y) = \mathcal{D}_{G\odot\mathcal{H}}(x,y)$, we deduce that $|S \cap \mathcal{D}_{G\odot\mathcal{H}}(x,y)| \geq k$.

2. $x \in V(H_i)$ and $y \in V(H_j)$, $j \neq i$. For every $v \in S_j$, we have $d_{G\odot\mathcal{H}}(y,v) \leq 2 < 3 \leq d_{G\odot\mathcal{H}}(x,v)$. Since S_j is a k-adjacency basis of H_j, we deduce that $|S_j| \geq k$, and consequently, $|S \cap \mathcal{D}_{G\odot\mathcal{H}}(x,y)| \geq k$.

3. $x \in V(H_i)$ and $y \in V(G)$. If $y = u_i$, then for every $v \in S_j$ such that $j \neq i$, we have $d_{G\odot\mathcal{H}}(x,v) = d_{G\odot\mathcal{H}}(x,y) + d_{G\odot\mathcal{H}}(y,v) > d_{G\odot\mathcal{H}}(y,v)$. Now, if $y = u_j$ such that $j \neq i$, then for every $v \in S_j$ we have $d_{G\odot\mathcal{H}}(x,v) = d_{G\odot\mathcal{H}}(x,y) + d_{G\odot\mathcal{H}}(y,v) > d_{G\odot\mathcal{H}}(y,v)$. Since S_j is a k-adjacency basis of H_j, we deduce that $|S_j| \geq k$, and as a consequence, $|S \cap \mathcal{D}_{G\odot\mathcal{H}}(x,y)| \geq k$.

4. $x = u_i \in V(G)$ and $y = u_j \in V(G)$. For every $v \in S_j$, we have $d_{G\odot\mathcal{H}}(x,v) = d_{G\odot\mathcal{H}}(x,y) + d_{G\odot\mathcal{H}}(y,v) > d_{G\odot\mathcal{H}}(y,v)$. Since S_j is a k-adjacency basis of H_j, we deduce that $|S_j| \geq k$, and thus, $|S \cap \mathcal{D}_{G\odot\mathcal{H}}(x,y)| \geq k$.

Hence, S is a k-metric generator for $G \odot \mathcal{H}$ and, as a consequence,

$$\dim_k(G \odot \mathcal{H}) \leq \sum_{i=1}^{n} |S_i| = \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

It only remains to prove that $\dim_k(G \odot \mathcal{H}) \geq \sum_{i=1}^{n} \operatorname{adim}_k(H_i)$. To do this, let B be a k-metric basis of $G \odot \mathcal{H}$ and, let $B_i = B \cap V(H_i)$. We claim that B_i is a k-adjacency generator for H_i. To this end, consider two different vertices $x, y \in V(H_i)$. Since $\mathcal{D}_{G\odot\mathcal{H}}(x,y) \cap (V(G \odot \mathcal{H}) - V(H_i)) = \emptyset$, we deduce that $|\mathcal{D}_{G\odot\mathcal{H}}(x,y) \cap B_i| \geq k$. Since for every $u \in B_i$ we have that $d_{G\odot\mathcal{H}}(x,u) = d_{G\odot\mathcal{H},2}(x,u)$ and $d_{G\odot\mathcal{H}}(y,u) = d_{G\odot\mathcal{H},2}(y,u)$, we conclude that $\mathcal{C}_{H_i}(x,y) = \mathcal{D}_{G\odot\mathcal{H}}(x,y)$ and, as a consequence, $|\mathcal{C}_{H_i}(x,y) \cap B_i| \geq k$. So, B_i is a k-adjacency generator for H_i and, consequently, $|B_i| \geq \operatorname{adim}_k(H_i)$. Therefore,

$$\dim_k(G \odot \mathcal{H}) = |B| \geq \sum_{i=1}^{n} |B_i| \geq \sum_{i=1}^{n} \operatorname{adim}_k(H_i).$$

We have shown that $\dim_k(G \odot \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i)$ and by analogy we deduce that $\dim_k(G \odot \overline{\mathcal{H}}) = \sum_{i=1}^{n} \operatorname{adim}_k(\overline{H}_i)$. By Remark 3.24, we have that $\sum_{i=1}^{n} \operatorname{adim}_k(H_i) = \sum_{i=1}^{n} \operatorname{adim}_k(\overline{H}_i)$, and so, the result follows. $\square$

Note that the k-metric dimension of corona product graphs is not equivalent to its k-adjacency dimension as in the case of lexicographic product graphs. For the graph $G \cong P_4 \odot P_5$ shown in Figure 3.3 we have $\dim_1(G) = 8 < 9 = \operatorname{adim}_1(G)$, $\dim_2(G) = 12 < 14 = \operatorname{adim}_2(G)$ and $\dim_3(G) = 20 = \operatorname{adim}_3(G)$. The only 3-adjacency basis of G, and at the same time the only 3-metric basis, is $V(G) - \{0, 6, 12, 18\}$.

According to Theorem 3.97, $\dim_k(G \odot \mathcal{H}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i)$, and considering that $\operatorname{adim}_k(H_i) \geq k$, we have a lower bound of kn on $\dim_k(G \odot \mathcal{H})$. By Proposition 3.12 and Theorem 3.97, we can deduce when this lower bound is tight.

Proposition 3.98. *For any connected graph G of order $n \geq 2$ and any family $\mathcal{H}$ composed by n nontrivial graphs, $\dim_k(G \odot \mathcal{H}) = kn$ if and only if $k \in \{1, 2\}$ and for every $H \in \mathcal{H}$ we have that $H \in \{P_2, P_3, \overline{P}_2, \overline{P}_3\}$.*

Our next result is obtained as a consequence of Theorem 3.97 and the fact that $\dim_k(H) \leq \operatorname{adim}_k(H) \leq |V(H)|$ for any nontrivial graph H and $k \in \{1, \ldots, \mathcal{C}H)\}$.

Theorem 3.99. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs. Then for every $k \in \{1, \ldots, \mathcal{C}(\mathcal{H})\}$,*

$$\sum_{i=1}^{n} \dim_k(H_i) \leq \dim_k(G \odot \mathcal{H}) \leq \sum_{i=1}^{n} |V(H_i)|.$$

By Theorem 3.97 it follows when the lower bound of Theorem 3.99 is achieved.

Theorem 3.100. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs. Then $\dim_k(G \odot \mathcal{H}) = \sum_{i=1}^{n} \dim_k(H_i)$ if and only if $\dim_k(H_i) = \operatorname{adim}_k(H_i)$ for every $H_i \in \mathcal{H}$.*

We know that if a graph H has diameter $D(H) \leq 2$, then $d_{H,t}$ and $d_{H,2}$ are equivalents. So $\dim_k(H) = \operatorname{adim}_k(H)$. Thus, the following result is a particular case of previous theorem.

Theorem 3.101. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs such that every $H_i \in \mathcal{H}$ has $D(H_i) \leq 2$. Then for every $k \in \{1, \ldots, \mathcal{C}(\mathcal{H})\}$, $\dim_k(G \odot \mathcal{H}) = \sum_{i=1}^{n} \dim_k(H_i)$.*

By Theorem 3.13, we have that $\dim_k(G \circ \mathcal{H}) = \operatorname{adim}_k(G \circ \mathcal{H})$ for any connected graph G and any family $\mathcal{H}$ composed by nontrivial graphs. Therefore, by Theorem 3.100 we deduce the following result.

Theorem 3.102. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{G_1 \circ \mathcal{H}_1, G_2 \circ \mathcal{H}_2, \ldots, G_n \circ \mathcal{H}_n\}$, where G_i is a connected graph of order $n_i \geq 2$ and $\mathcal{H}_i$ is a family composed by n_i nontrivial graphs with $i \in \{1, 2, \ldots, n\}$. Then for every $k \in \{1, \ldots, \mathcal{C}(\mathcal{H})\}$, $\dim_k(G \odot \mathcal{H}) = \sum_{i=1}^{n} \dim_k(G_i \circ \mathcal{H}_i)$.*

By Theorems 3.76 and 3.102, we deduce the following two results.

Proposition 3.103. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{G_1 \circ \mathcal{H}_1, G_2 \circ \mathcal{H}_2, \ldots, G_n \circ \mathcal{H}_n\}$, where G_i is a connected graph of order $n_i \geq 2$ and $\mathcal{H}_i$ is a family composed by n_i paths with $i \in \{1, 2, \ldots, n\}$. If every path $P_{i,j} \in \mathcal{H}_i$ has order $q_{i,j} \geq 4$, then*

(i) $$\dim_2(G \odot \mathcal{H}) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left\lceil \frac{q_{i,j}+1}{2} \right\rceil.$$

(ii) $$\dim_3(G \odot \mathcal{H}) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left(q_{i,j} - \left\lfloor \frac{q_{i,j}-4}{5} \right\rfloor \right).$$

Note that Theorem 3.76 leads to the conclusion that in the previous proposition each $\mathcal{H}_i$, where $i \in \{1, 2, \ldots, n\}$, can also be formed by the complement of paths of order at least four.

Proposition 3.104. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{G_1 \circ \mathcal{H}_1, G_2 \circ \mathcal{H}_2, \ldots, G_n \circ \mathcal{H}_n\}$, where G_i is a connected graph of order $n_i \geq 2$ and $\mathcal{H}_i$ is a family composed by n_i cycles with $i \in \{1, 2, \ldots, n\}$. If every cycle $C_{i,j} \in \mathcal{H}_i$ has order $q_{i,j} \geq 5$, then*

(i) $\dim_2(G \odot \mathcal{H}) = \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n_i} \left\lceil \frac{q_{i,j}}{2} \right\rceil$.

(ii) $\dim_3(G \odot \mathcal{H}) = \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n_i} \left(q_{i,j} - \left\lfloor \frac{q_{i,j}}{5} \right\rfloor \right)$.

(iii) $\dim_4(G \odot \mathcal{H}) = \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n_i} q_{i,j}$.

By Theorem 3.76 we deduce that in the previous proposition each $\mathcal{H}_i$, where $i \in \{1, 2, \ldots, n\}$, can also be formed by the complement of cycles of order at least five.

By Theorems 3.78 and 3.102, we deduce the following result.

Proposition 3.105. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{G_1 \circ \mathcal{H}_1, G_2 \circ \mathcal{H}_2, \ldots, G_n \circ \mathcal{H}_n\}$, where G_i is a connected graph of order $n_i \geq 2$ and $\mathcal{H}_i$ is a family composed by n_i paths and/or cycles with $i \in \{1, 2, \ldots, n\}$. If every $H_{i,j} \in \mathcal{H}_i$ has order $q_{i,j} \geq 7$ and in each family $\mathcal{H}_i$ there exists at most one $q_{i,l}$ such that $q_{i,l} \mod 5 \in \{1, 3\}$, then*

$$\dim_1(G \odot \mathcal{H}) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left\lfloor \frac{2q_{i,j} + 2}{5} \right\rfloor$$

Our following result is a direct consequence of Theorems 3.5 and 3.97.

Proposition 3.106. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family composed by n nontrivial graphs. Then $\dim_k(G \odot \mathcal{H}) = \sum_{i=1}^{n} |V_i|$ if and only if $k = \mathcal{C}(\mathcal{H})$ and $\mathcal{D}_{k,2}(H_i) = V(H_i)$ for every graph $H_i \in \mathcal{H}$.*

The graphs P_4 and C_6 are two examples for a graph H satisfying the conditions of Proposition 3.106. Notice that $\mathcal{C}(P_4) = 3$ and $\operatorname{adim}_3(P_4) = 4$. Also, $\mathcal{C}(C_6) = 4$ and $\operatorname{adim}_4(C_6) = 6$. Therefore, for any nontrivial graph G of order n, $\dim_3(G \odot P_4) = 4n$ and $\dim_4(G \odot C_6) = 6n$.

From Corollary 3.6 we can deduce the particular case of Proposition 3.106 for $k = 2$.

Proposition 3.107. *Let G be a connected graph of order $n \geq 2$, and let $\mathcal{H}$ be a family composed by nontrivial graphs. Every vertex of $H_i \in \mathcal{H}$ belongs to a non-singleton twin equivalence class if and only if*

$$\dim_2(G \odot \mathcal{H}) = \sum_{i=1}^{n} |V_i|.$$

We must point out that Theorems 3.99 and 3.101 are generalizations of previous results established in [132] for the case $k = 1$.

Notice that there are values for $\dim_k(G \odot \mathcal{H})$ non-achieving the bounds given in Theorem 3.99. If there exists a graph $H_i \in \mathcal{H}$ such that $\dim_k(H_i) < \operatorname{adim}_k(H_i) < |V(H_i)|$, then

$$\sum_{i=1}^{n} \dim_k(H_i) < \dim_k(G \odot \mathcal{H}) < \sum_{i=1}^{n} |V_i|.$$

The results given in Proposition 3.109 show some examples. Note that $\dim_k(P_{n_i}) = k+1 < \operatorname{adim}_k(P_{n_i}) < n_i = |V(P_{n_i})|$ for $n_i \geq 9$ and $k \in \{1,2,3\}$.

In order to present our next result we introduce a new definition. Given a family of n graphs $\mathcal{H}$, we denote by $K_1 \diamond \mathcal{H}$ the family of graphs formed by the graphs $K_1 + H_i$ for every $H_i \in \mathcal{H}$, *i.e.*, $K_1 \diamond \mathcal{H} = \{K_1 + H_1, K_1 + H_2, \ldots, K_1 + H_n\}$.

Proposition 3.108. *Let G be a connected graph of order $n \geq 2$, and let $\mathcal{H}$ be a family composed by n nontrivial graphs. Then for any $k \in \{1, \ldots, \mathcal{C}(\mathcal{H})\}$,*

$$\dim_k(G \odot \mathcal{H}) = \dim_k(G \odot \overline{\mathcal{H}}) = \dim_k\left(G \odot (K_1 \diamond \mathcal{H})\right)$$

if each $H_i \in \mathcal{H}$ holds one of the following statements.

(i) *H_i has diameter $D(H_i) \geq 6$.*

(ii) *H_i has girth $\mathtt{g}(H) \geq 5$ and minimum degree $\delta(H) \geq 3$.*

(iii) *H_i is a cycle graph of order at least seven.*

(iv) *H_i is a tree T such that the following statements hold.*

   (a) *If $k = 1$, then $T \notin \mathcal{F}_1 = \{P_2, P_3, P_6, K_{1,n}, T'\}$, where $n \geq 3$ and T' is obtained from $P_5 \cup \{K_1\}$ by joining by an edge the vertex of K_1 to the central vertex of P_5.*

(b) *If $k=2$, then $T \notin \mathcal{F}_2 = \{P_r, K_{1,n}, T'\}$, where $r \in \{2,\ldots,5\}$, $n \geq 3$ and T' is a graph obtained from $K_{1,n} \cup K_2$ by joining by an edge one leaf of $K_{1,n}$ to one leaf of K_2.*

(c) *If $k=3$, then $T \notin \mathcal{F}_3 = \{P_4, P_5\}$.*

Proof. Since for every $H_i \in \mathcal{H}$, it follows $D(K_1+H_i)=2$, by Theorem 3.101, $\dim_k(G \odot (K_1 \diamond \mathcal{H})) = \sum_{i=1}^{n} \dim_k(K_1+H_i)$. Also, by Corollaries 3.33, 3.34, 3.35, Propositions 3.43, 3.44, 3.46 and 3.47, and Theorem 3.97, $\dim_k(G\odot\mathcal{H}) = \dim_k(G\odot\overline{\mathcal{H}}) = \sum_{i=1}^{n} \operatorname{adim}_k(H_i) = \sum_{i=1}^{n} \dim_k(K_1+H_i)$. So, the result follows. □

The next result shows a relationship between $\dim_k(G\odot\mathcal{H})$ and $\dim_k(G\odot(K_1\diamond\mathcal{H}))$ for a family $\mathcal{H}$ of paths of order greater than five and $k\in\{1,2,3\}$. We only consider $k\in\{1,2,3\}$, since for $n'\geq 6$ we have that $\mathcal{C}(P_{n'}) = \mathcal{C}(F_{1,n'}) = 3$, and as a consequence, by Theorem 2.35, $G\odot\mathcal{H}$ and $G\odot(K_1\diamond\mathcal{H})$ are 3-metric dimensional. Thus, by Theorem 3.101 and Propositions 3.36, 3.39 and 3.108, we obtain the following result.

Proposition 3.109. *Let G be a connected graph of order $n\geq 2$, and let $\mathcal{H}$ be a family of paths. If every path $P_i\in\mathcal{H}$ has order n_i, then the following statements hold.*

(i) *If $n_i \geq 7$ for $i\in\{1,\ldots,n\}$, then* $\dim_1(G\odot\mathcal{H}) = \dim_1(G\odot\overline{\mathcal{H}}) = \dim_1(G\odot(K_1\diamond\mathcal{H})) = \sum_{i=1}^{n}\left\lfloor \frac{2n_i+2}{5}\right\rfloor$.

(ii) *If $n_i \geq 6$ for $i\in\{1,\ldots,n\}$, then* $\dim_2(G\odot\mathcal{H}) = \dim_2(G\odot\overline{\mathcal{H}}) = \dim_2(G\odot(K_1\diamond\mathcal{H})) = \sum_{i=1}^{n}\left\lceil \frac{n_i+1}{2}\right\rceil$.

(iii) *If $n_i \geq 6$ for $i\in\{1,\ldots,n\}$, then* $\dim_3(G\odot\mathcal{H}) = \dim_3(G\odot\overline{\mathcal{H}}) = \dim_3(G\odot(K_1\diamond\mathcal{H})) = \sum_{i=1}^{n}\left(n_i - \left\lfloor \frac{n_i-4}{5}\right\rfloor\right)$.

Finally, we present a relationship between $\dim_k(G\odot\mathcal{H})$ and $\dim_k(G\odot(K_1\diamond\mathcal{H}))$ for a family $\mathcal{H}$ of cycles of order greater than six and $k\in\{1,2,3,4\}$. We only consider $k\in\{1,2,3,4\}$, since for $n'\geq 7$ we have that $\mathcal{C}(C_{n'}) = \mathcal{C}(W_{1,n'}) = 4$, as a consequence, by Corollary 2.36, $G\odot\mathcal{H}$ and $G\odot(K_1\diamond\mathcal{H})$ are 4-metric dimensional. Thus, by Theorem 3.101 and Propositions 3.36, 3.43 and 3.108, we obtain the following result.

Proposition 3.110. *Let G be a connected graph of order $n\geq 2$, and let $\mathcal{H}$ be a family of n cycles. If every cycle $C_i\in\mathcal{H}$ has order $n_i\geq 7$, then*

(i) $\dim_1(G \odot \mathcal{H}) = \dim_1(G \odot \overline{\mathcal{H}}) = \dim_1(G \odot (K_1 \diamond \mathcal{H})) = \sum_{i=1}^{n} \left\lfloor \frac{2n_i+2}{5} \right\rfloor$.

(ii) $\dim_2(G \odot \mathcal{H}) = \dim_2(G \odot \overline{\mathcal{H}}) = \dim_2(G \odot (K_1 \diamond \mathcal{H})) = \sum_{i=1}^{n} \left\lceil \frac{n_i}{2} \right\rceil$.

(iii) $\dim_3(G \odot \mathcal{H}) = \dim_3(G \odot \overline{\mathcal{H}}) = \dim_3(G \odot (K_1 \diamond \mathcal{H})) = \sum_{i=1}^{n} \left(n_i - \left\lfloor \frac{n_i}{5} \right\rfloor\right)$.

(iv) $\dim_4(G \odot \mathcal{H}) = \dim_4(G \odot \overline{\mathcal{H}}) = \dim_4(G \odot (K_1 \diamond \mathcal{H})) = \sum_{i=1}^{n} n_i$.

Chapter 4

Computability of the (k,t)-dimensional problem and the k-metric dimension problem

Overview

In this chapter we study the computability of some problems concerning the (k,t)-metric dimension of graphs. Namely, we propose an algorithm which can be solved in cubic time with regard to order of the graph, for finding the value of k such that a graph is (k,t)-dimensional. We devise others particular algorithms for computing the value of k such that the lexicographic product and the corona product are k-metric dimensional. Despite these algorithms can also be solved in cubic time, we reduce its constant factor. Moreover, we prove that the problem of computing the k-metric dimension of graphs is NP-hard. However, the problem of computing the k-metric dimension of trees is solved in linear time with respect to the order of trees. To this end, we give three algorithms, the first one is for computing the value of k such that a tree is k-metric dimensional, the second one is for finding the value of the k-metric dimension of a tree, and the last one is for finding a k-metric basis of a tree.

4.1 The (k,t)-metric dimensional graph problem

We now consider the problem of finding the integer k for which a given graph G of order n is (k,t)-metric dimensional for an integer $t \geq 2$. By Remark 2.1 we know that if G is a (k,t)-metric dimensional graph of order $n \geq 3$, then $2 \leq k \leq n-1$. Therefore, the above mentioned problem would be expressed in the following way.

(k,t)-DIMENSIONAL GRAPH PROBLEM INSTANCE: A connected graph G of order $n \geq 3$ and an integer $t \geq 2$. PROBLEM: Find the integer k, $2 \leq k \leq n-1$, such that G is (k,t)-metric dimensional.

Theorem 4.1. *Let G be a connected graph of order $n \geq 3$, and let $t \geq 2$ be an integer. The time complexity of computing the value k for which G is (k,t)-metric dimensional is $O(n^3)$.*

Proof. We assume that the graph G is represented by its adjacency matrix $\mathbf{A_G}$. We recall that $\mathbf{A_G}$ is a symmetric $(n \times n)$-matrix given by

$$\mathbf{A_G}(i,j) = \begin{cases} 1, & \text{if } u_i \sim u_j, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.2, the problem is reduced to finding the value of $\mathcal{D}(G,t)$. To this end, we can initially compute the distance matrix $\mathbf{D_G}$ from the matrix $\mathbf{A_G}$ by using the well-known Floyd-Warshall algorithm [109, 125], which has time complexity $O(n^3)$. The distance matrix $\mathbf{D_G}$ is symmetric of order $n \times n$ whose rows and columns are labelled by vertices, with entries between 0 and $n-1$ (or ∞ if G is not connected). Now observe that for every $x, y \in V(G)$ we have that $z \in \mathcal{D}_{G,t}(x,y)$ if and only if $\min\{\mathbf{D_G}(x,z), t\} \neq \min\{\mathbf{D_G}(x,z), t\}$.

Given the distance matrix DistM_G, computing how many vertices belong to $\mathcal{D}_{G,t}(x,y)$ for each of the $\binom{|V(G)|}{2}$ pairs $x, y \in V(G)$ can be checked in linear time. Therefore, the overall running time of such a process is bounded by the cubic time of the Floyd-Warshall algorithm. $\square$

4.1.1 The particular case of product graphs

In this section we analyse the (k,t)-DIMENSIONAL GRAPH PROBLEM for the particular case of join graphs, corona and lexicographic product graphs. In the study of these graphs the parameter $\mathcal{C}(G)$ is involved, as we have seen in Propositions 2.21 and 2.28, and Theorems 2.29 and 2.35. By Theorem 4.1 we learned that $\mathcal{C}(G)$ can be computed in $O(|V(G)|^3)$ time. However, we propose an algorithm for the computability of this parameter, where the value of the constant factor is reduced, despite the fact that the time complexity remains of the same order.

Theorem 4.2. *For any nontrivial graph G, the value of $\mathcal{C}(G)$ can be computed in $O(|V(G)|^3)$ time.*

Proof. We assume that the graph G is represented by its adjacency matrix $\mathbf{A_G}$. Now observe that for every $x,y \in V(G)$ we have that any $z \in V(G) - \{x,y\}$ belongs to $\mathcal{C}_G(x,y)$ if and only if $\mathbf{A_G}(x,z) \neq \mathbf{A_G}(y,z)$. Considering this, we can compute $|\mathcal{C}_G(x,y)|$ in linear time for each pair $x,y \in V(G)$. Therefore, the overall running time for determining $\mathcal{C}(G)$ is dominated by the cubic time of computing the value of $|\mathcal{C}_G(x,y)|$ for the $\binom{|V(G)|}{2}$ pairs of vertices x,y of G. $\square$

Considering that the minimum and the maximum degree of any graph of order n can be computed in $O(n^2)$ time, by Propositions 2.21 and 2.28, and Corollary 2.31, we deduce the following result.

Proposition 4.3. *The following assertions hold:*

(i) *For any graph H of order $n \geq 2$, the value of k for which $K_1 + H$ is k-metric dimensional can be computed in $O(n^3)$ time.*

(ii) *For any two graphs G and H of order $n \geq 2$ and $n' \geq 2$, respectively, the value of k for which $G+H$ is k-metric dimensional can be computed in $O\left(\max\{n^3, n'^3\}\right)$ time.*

(iii) *For any connected nontrivial graph G and any graph H of order $n' \geq 2$, the value of k for which $G \circ H$ is k-metric dimensional can be computed in $O(n'^3)$ time.*

The following result is direct consequence of Theorems 2.35 and 4.2.

Remark 4.4. *Let G be a connected nontrivial graph of order n and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed by nontrivial graphs. Then the value of k for which $G \odot \mathcal{H}$ is k-metric dimensional can be computed in $O(\sum_{i=1}^{n} |V(H_i)|^3)$ time.*

It was shown in Theorem 4.1 that the value of k for which a graph G is k-metric dimensional can be computed in cubic time with regard to the order of G. The value of k for which $G \circ \mathcal{H}$ is k-metric dimensional can compute in $O\left(\left(n + \sum_{i=1}^{n} n_i\right)^3\right)$ time. A natural question which raises now regards with the existence of an algorithm that could allow us to compute the value of k for which $G \circ \mathcal{H}$ is k-metric dimensional in a lower order. The next result solves precisely that fact, where the general complexity is slightly improved.

Proposition 4.5. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a family composed by nontrivial graphs. Then the value of k for which $G \circ \mathcal{H}$ is k-metric dimensional can be computed in*

$$O\left(\max\left\{n^3 + \sum_{u_i \in TT(G) \cup TF(G)} |V(H_i)|^2, \sum_{i=1}^{n} |V(H_i)|^3\right\}\right).$$

Proof. By Theorem 2.29 we learn that $k = \min\{\mathcal{T}(G \circ \mathcal{H}, \mathcal{C}(\mathcal{H}))$. To compute $\mathcal{T}(G \circ \mathcal{H})$, it is first necessary to obtain the twin equivalence classes of G. We assume that the graph G is represented by its adjacency matrix $\mathbf{A_G}$. Now, note that u_i, u_j are twins if and only if for every $u_r \in V(G) - \{u_i, u_j\}$, we have that $\mathbf{A_G}(i, r) = \mathbf{A_G}(j, r)$. Given two twin vertices u_i, u_j, if $\mathbf{A_G}(i, j) = 1$, then u_i, u_j are true twins, otherwise they are false twins. Note that determining if two vertices are twins can be checked in linear time. In the worst case, when all twin equivalence classes are singletons, it would be necessary to check that any two vertices are twins between them or not. Thus, we conclude that determining the twin equivalence classes of G can be computed in $O(n^3)$. Once determined the twin equivalence classes of G, we have the following three possibilities for each twin equivalence class U_G of G.

- If $U_G = \{u_i\}$, then we take the order n_i of H_i, as the representative value of this class.

- If U_G is a false twin equivalence class, then we take $\min_{u_j, u_l \in U_G} \{\delta(H_j) + \delta(H_l) + 2\}$ as the representative value of this class.

- If U_G is a true twin equivalence class, then we take $\min\limits_{u_j,u_l\in U_G}\{|V(H_j)| - \Delta(H_j) + |V(H_l)| - \Delta(H_l)\}$ as the representative value of this class.

We observe that $\mathcal{T}(G\circ\mathcal{H})$ is the minimum of the representative values of each twin equivalence class. The minimum and maximum degrees $\delta(H_i)$ and $\Delta(H_i)$ of the graphs H_i (of order n_i) can be computed in $O(n_i^2)$. So, computing the representative value of each non-singleton twin equivalence class U_G can be done in $O(\sum_{u_i\in U_G} n_i^2)$. Therefore, we can compute the value of $\mathcal{T}(G\circ\mathcal{H})$ in $O(n^3+\sum_{u_i\in TT(G)\cup TF(G)} n_i^2)$.

On the other hand, by Proposition 4.5 we have that $\mathcal{C}(\mathcal{H})$ can be computed in $O(\sum_{i=1}^{n} n_i^3)$, which completes the proof. □

4.2 The k-metric dimension problem

Since the problem of computing the value k' for which a given graph is k'-metric dimensional can be solved in polynomial time, we can study the problem of deciding whether the k-metric dimension, $k\le k'$, of G is less than or equal to r, for some $r\ge k+1$, *i.e.*, the following decision problem.

k-METRIC DIMENSION PROBLEM
INSTANCE: A k'-metric dimensional graph G of order $n\ge 3$ and integers k,r such that $1\le k\le k'$ and $k+1\le r\le n$.
QUESTION: Is $\dim_k(G)\le r$?

We next prove that the k-METRIC DIMENSION PROBLEM is NP-complete. We must remark that for $k=1$ the problem above was proved to be NP-complete by Khuller *et al.* [72], although a previous claim about it was first presented in [49]. Moreover, the NP-completeness of this problem (when $k=1$) restricted to the case of planar graphs was settled in [27]. As a kind of generalization of the technique used in [72] for $k=1$, we also use a reduction from 3-SAT in order to prove the NP-completeness of the k-METRIC DIMENSION PROBLEM. We recall 3-SAT decision problem.

SATISFIABILITY (3-SAT)
INSTANCE: Collection $Q = \{Q_1, \dots, Q_s\}$ of clauses on finite set U of variables such that $|Q_i| = 3$ for $i \in \{1, \dots, s\}$.
QUESTION: Is there a truth assignment for U that satisfies all the clauses in Q?

Our problem is clearly in NP, since verifying that a given subset $S \subseteq V(G)$ with $k+1 \leq |S| \leq r$ is a k-metric generator for a graph G, can be done in polynomial time by using some similar procedure like that described in the proof of Theorem 4.1. In order to present the reduction from 3-SAT, we need some terminology and notation. From now on, we assume $x_1, \dots, x_n$ are variables; $Q_1, \dots, Q_s$ are clauses; and $x_1, \overline{x_1}, x_2, \overline{x_2}, \dots, x_n, \overline{x_n}$ are literals, where x_i represents a positive literal of the variable while $\overline{x_i}$ represents a negative literal.

We consider an arbitrary input to 3-SAT, that is, a boolean formula $\mathcal{F}$ with n variables and s clauses. In this reduction, without loss of generality, we assume that the formula $\mathcal{F}$ has $n \geq 4$ variables. Let $X = \{x_1, x_2, \dots, x_n\}$ be the set of variables and let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_s\}$ be the set of clauses. Now we construct a graph G_F in the following way.

- For every $x_i \in X$, we take an even cycle C^i of order $4\left\lceil \frac{k}{2} \right\rceil + 2$ and we denote by F_i (the false node) and by T_i (the true node) two diametral vertices of C^i. Then we denote by $f_i^1, f_i^2, \dots, f_i^{2\left\lceil \frac{k}{2} \right\rceil}$ the half vertices of C^i closest to F_i and we denote by $t_i^1, t_i^2, \dots, t_i^{2\left\lceil \frac{k}{2} \right\rceil}$ the half vertices of C^i closest to T_i (see Figure 4.1).

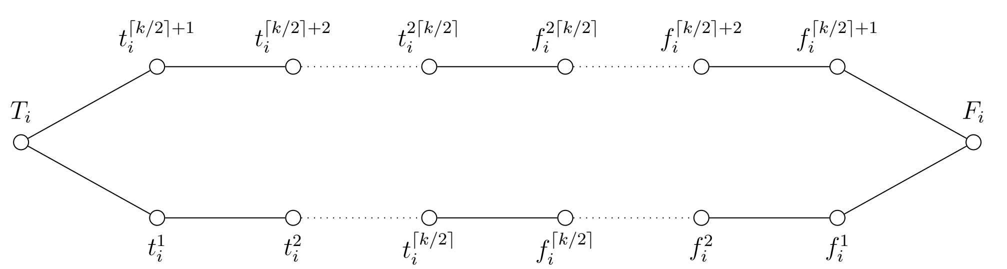

Figure 4.1: The cycle C^i associated to the variable x_i.

- For every clause $Q_j \in \mathcal{Q}$, we take a star graph $K_{1,4}$ with central vertex u_j and leaves $u_j^1, u_j^2, u_j^3, u_j^4$. If $k \geq 3$, then we subdivide the edge $u_j u_j^2$ until we obtain a shortest $u_j - u_j^2$ path of order $\left\lceil \frac{k}{2} \right\rceil + 1$, as well as, we subdivide the edge $u_j u_j^3$ until we obtain a shortest $u_j - u_j^3$ path of order $\left\lfloor \frac{k}{2} \right\rfloor + 1$ (see Figure 4.2). We denote by $P(u_j^2, u_j^3)$ the shortest $u_j^2 - u_j^3$ path of length k obtained after subdivision. The star graph remains unchanged for $k \in \{1, 2\}$.

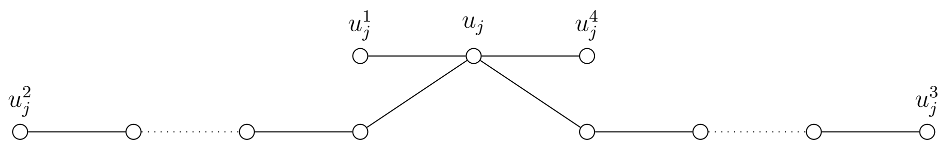

Figure 4.2: The subgraph associated to the clause Q_j.

- If a variable x_i occurs as a positive literal in a clause Q_j, then we add the edges $T_i u_j^1$, $F_i u_j^1$ and $F_i u_j^4$ (see Figure 4.3).
- If a variable x_i occurs as a negative literal in a clause Q_j, then we add the edges $T_i u_j^1$, $F_i u_j^1$ and $T_i u_j^4$ (see Figure 4.3).

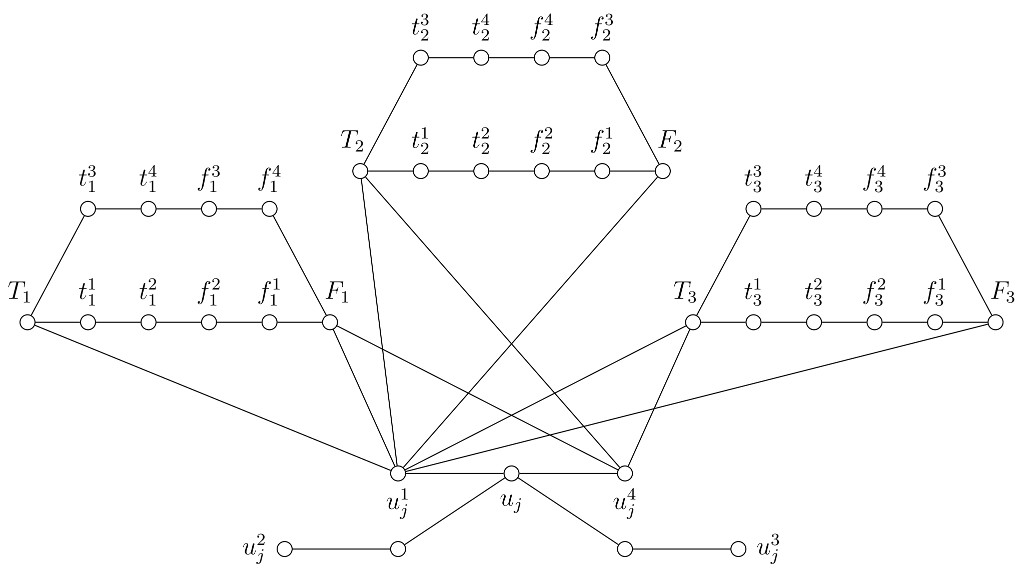

Figure 4.3: The subgraph associated to the clause $Q_j = (x_1 \vee \overline{x_2} \vee \overline{x_3})$ (taking $k = 4$).

- Finally, for every $l \in \{1, \ldots, n\}$ such that x_l and $\overline{x_l}$ do not occur in a clause Q_j we add the edges $T_l u_j^1$, $T_l u_j^4$, $F_l u_j^1$ and $F_l u_j^4$.

Notice that the graph G_F obtained from the procedure above has order $n\left(4\left\lceil\frac{k}{2}\right\rceil+2\right)+s(k+3)$. We also observe that given the formula $\mathcal{F}$, the graph G_F can be constructed in polynomial time. Next we prove that $\mathcal{F}$ is satisfiable if and only if $\dim_k(G_F)=k(n+s)$. To do so, we first notice some properties of G_F.

Remark 4.6. *Let $x_i \in X$. Then there exist two different vertices $a, b \in V(C^i)$ such that they are distinguished only by vertices of the cycle C^i and, as consequence, for any k-metric basis S of G_F, we have that $|S \cap V(C^i)| \geq k$.*

Proof. To observe that it is only necessary to take the two vertices of C^i adjacent to T_i or adjacent to F_i. □

Remark 4.7. *Let $Q_j \in \mathcal{Q}$. Then there exist two different vertices x, y in the shortest $u_j^2 - u_j^3$ path such that they are distinguished only by vertices of the itself shortest $u_j^2 - u_j^3$ path and, as consequence, for any k-metric basis S of G_F, we have that $|S \cap V(P(u_j^2, u_j^3))| \geq k$.*

Proof. To observe that it is only necessary to take the two vertices of $P(u_j^2, u_j^3)$ adjacent to u_j. □

Proposition 4.8. *Let $\mathcal{F}$ be an arbitrary input to 3-SAT problem. Then the graph G_F associated to $\mathcal{F}$ satisfies that $\dim_k(G_F) \geq k(n+s)$.*

Proof. As a consequence of Remarks 4.6 and 4.7 we obtain that for every variable $x_i \in X$ and for every clause $Q_j \in \mathcal{Q}$ the set of vertices of G_F associated to each variable or clause, contains at least k vertices of every k-metric basis for G_F. Thus, the result follows. □

Theorem 4.9. *k-METRIC DIMENSION PROBLEM is NP-complete.*

Proof. Let $\mathcal{F}$ be an arbitrary input to 3-SAT problem having more than three variables and let G_F be the graph associated to $\mathcal{F}$. We shall show that $\mathcal{F}$ is satisfiable if and only if $\dim_k(G_F)=k(n+s)$.

We first assume that $\mathcal{F}$ is satisfiable. From Proposition 4.8 we have that $\dim_k(G_F) \geq k(n+s)$. Now, based on a satisfying assignment of $\mathcal{F}$, we shall give a set S of vertices of G_F, of cardinality $|S|=k(n+s)$, which is k-metric generator.

Suppose we have a satisfying assignment for $\mathcal{F}$. For every clause $Q_j \in \mathcal{Q}$ we add to S all the vertices of the set $V(P(u_j^2, u_j^3)) - \{u_j\}$. For a variable

$x_i \in X$ we consider the following. If the value of x_i is true, then we add to S the vertices $t_i^1, t_i^2, \ldots, t_i^{2\lceil \frac{k}{2} \rceil}$. On the contrary, if the value of x_i is false, then we add to S the vertices $f_i^1, f_i^2, \ldots, f_i^{2\lceil \frac{k}{2} \rceil}$.

We shall show that S is a k-metric generator for G_F. Let a, b be two different vertices of G_F. We consider the following cases.

Case 1. $a, b \in V(C^i)$ for some $i \in \{1, \ldots, n\}$. Hence, there exists at most one vertex $y \in S \cap V(C^i)$ such that $d(a, y) = d(b, y)$. If $d(a, w) \neq d(b, w)$ for every vertex $w \in S \cap V(C^i)$, then since $|S \cap V(C^i)| = k$, we have that $|\mathcal{D}_G(a, b) \cap S| = k$. On the other hand, if there exist one vertex $y \in S \cap V(C^i)$ such that $d(a, y) = d(b, y)$, then $d(a, T_i) \neq d(b, T_i)$ and $d(a, F_i) \neq d(b, F_i)$. Thus, for every $w \in S - V(C^i)$ it follows that $d(a, w) \neq d(b, w)$ and a, b are distinguished by more than k vertices of S.

Case 2. $a, b \in V(P(u_j^2, u_j^3))$. Hence, there exists at most one vertex $y' \in S \cap V(P(u_j^2, u_j^3))$ such that $d(a, y') = d(b, y')$. But, in this case, $d(a, u_j) \neq d(b, u_j)$ and so for every $w \in S - V(P(u_j^2, u_j^3))$ it follows that $d(a, w) \neq d(b, w)$ and a, b are distinguished by at least k vertices of S.

Case 3. $a = u_j^1$ and $b = u_j^4$. Since the clause Q_j is satisfied, there exists $i \in \{1, \ldots, n\}$, *i.e*, a variable x_i occurring in the clause Q_j such that either

- $a \sim T_i$, $b \not\sim T_i$ and $S \cap V(C^i) = \left\{ t_i^1, t_i^2, \ldots, t_i^{2\lceil \frac{k}{2} \rceil} \right\}$, *i.e*, a variable x_i occurring as a positive literal in Q_j and has the value true in the assignment, or
- $a \sim F_i$, $b \not\sim F_i$ and $S \cap V(C^i) = \left\{ f_i^1, f_i^2, \ldots, f_i^{2\lceil \frac{k}{2} \rceil} \right\}$, *i.e*, a variable x_i occurring as a negative literal in Q_j and has the value false in the assignment.

Thus, in any case we have that for every $w \in S \cap V(C^i)$ it follows $d(a, w) < d(b, w)$ and a, b are distinguished by at least k vertices of S.

Case 4. $a \in V(C^i)$ and $b \in V(C^l)$ for some $i, l \in \{1, \ldots, n\}$, $i \neq l$. In this case, if there is a vertex $z \in S \cap V(C^i)$ such that $d(a, z) = d(b, z)$, then for every vertex $w \in S \cap V(C^l)$ it follows that $d(a, w) \neq d(b, w)$. So a, b are resolved by at least k vertices of S.

Case 5. $a \in V(C^i)$ and $b \in V(P(u_j^2, u_j^3))$. It is similar to the case above.

Case 6. $a \in \{u_j^1, u_j^4\}$ and $b \notin \{u_j^1, u_j^4\}$. If $b \in V(C^i)$, for some $i \in \{1, \ldots, n\}$, then all elements of $S \cap V(P(u_j^2, u_j^3)$ distinguish a, b. Now, let w be one of the two vertices adjacent to u_j in $P(u_j^2, u_j^3)$. If $b \in V(P(u_j^2, u_j^3)) - \{w\}$, then all elements of $S \cap V(C^i)$ distinguish a, b. On the other hand, since $n \geq 4$, if $b = w$, then there exists a variable x_l not occurring in the clause Q_j. Thus, the vertex a is adjacent to T_l and to F_l and, as a consequence, the vertices of $S \cap V(C^l)$ distinguish a, b.

As a consequence of the cases above, we have that S is a k-metric generator for G_F. Therefore, $\dim_k(G_F) = k(n+s)$.

Next we prove that, if $\dim_k(G_F) = k(n+s)$, then $\mathcal{F}$ is satisfiable. To this end, we show that there exists a k-metric basis S of G_F such that we can set an assignment of the variables, so that $\mathcal{F}$ is satisfiable. We take S in the same way as the k-metric generator for G_F described above. Since S is a k-metric generator for G_F of cardinality $k(n+s)$, it is also a k-metric basis. Note that for any cycle C_i either $S \cap V(C_i) = \left\{t_i^1, t_i^2, \ldots, t_i^{2\lceil \frac{k}{2} \rceil}\right\}$ or $S \cap V(C_i) = \left\{f_i^1, f_i^2, \ldots, f_i^{2\lceil \frac{k}{2} \rceil}\right\}$.

In this sense, we set an assignment of the variables as follows. Given a variable $x_i \in X$, if $S \cap \left\{t_i^1, t_i^2, \ldots, t_i^{2\lceil \frac{k}{2} \rceil}\right\} = \emptyset$, then we set x_i to be false. Otherwise we set x_i to be true. We claim that this assignment satisfies $\mathcal{F}$.

Consider any clause $Q_j \in \mathcal{Q}$ and let $x_{j_1}, x_{j_2}, x_{j_3}$ the variables occurring in Q_j. Recall that for each clause Q_h, we have that $S \cap V(P(u_h^2, u_h^3)) = V(P(u_h^2, u_h^3)) - \{u_h\}$. Besides no vertex of $V(C_l)$ associated to a variable x_l, $l \neq j_1, j_2, j_3$, nor any vertex of $S \cap V(P(u_h^2, u_h^3))$ associated to a clause Q_h, distinguishes the vertices u_j^1 and u_j^4. Thus u_j^1 and u_j^4 must be distinguished by at least k vertices belonging to $V(C_{j_1}) \cup V(C_{j_2}) \cup V(C_{j_3})$ associated to the variables $x_{j_1}, x_{j_2}, x_{j_3}$.

Now, according to the way in which we have added the edges between the vertices T_{j_1}, T_{j_2}, T_{j_3}, F_{j_1}, F_{j_2}, F_{j_3} and u_j^1, u_j^4, we have that u_j^1 and u_j^4 are distinguished by at least k vertices of S if and only if one of the following statements holds.

- There exists $l \in \{1,2,3\}$ for which the variable x_{j_l} occurs as a negative literal in the clause Q_j and $S \cap \left\{t_j^1, t_j^2, \ldots, t_j^{2\lceil \frac{k}{2} \rceil}\right\} = \emptyset$ (in such a case x_{j_l} is set to be false).
- There exists $l \in \{1,2,3\}$ for which the variable x_{j_l} occurs as a positive literal in the clause Q_j and $S \cap \left\{t_j^1, t_j^2, \ldots, t_j^{2\lceil \frac{k}{2} \rceil}\right\} \neq \emptyset$ (in such a case x_{j_l} is set to be true).

As a consequence of the two cases above, we have that if at least k vertices of S distinguish u_j^1, u_j^4, then the setting of x_{j_l}, $l \in \{1,2,3\}$, is such that it satisfies the clause Q_j. Therefore $\mathcal{F}$ is satisfiable. □

According to the theorem above we have the following result.

Corollary 4.10. *The problem of finding the k-metric dimension of graphs is NP-hard.*

4.3 The particular case of trees

We must first recall that for the particular case of trees, it is already known from [72] that the problem of computing its 1-metric dimension can be done in linear time. Moreover, it was recently proved in [27] that also for the case of outerplanar graphs, this problem can be solved in polynomial time. We next deal with the problem of computing the k-metric dimension of trees for $k \geq 2$.

In order to continue presenting our results, we need to use some definitions exposed at the beginning of subsection 2.2.1. An example of a tree T which helps to remember the notation of this subsection is given in Figure 4.4. In such a case we have that $\mathcal{M}(T) = \{6, 12, 26\}$, $\{1, 4\}$ is the set of terminal vertices of vertex 6, $\{9, 11\}$ is the set of terminal vertices of vertex 12 and $\{15, 20, 23\}$ is the set of terminal vertices of vertex 26. For instance, for the vertex 26 we have that $l(26) = \min\{l(15, 26), l(20, 26), l(23, 26)\} = \min\{5, 3, 3\} = 3$ and $\varsigma(26) = \min\{\varsigma(15, 20), \varsigma(15, 23), varsigma(20, 23)\} = \min\{8, 8, 6\} = 6$. Analogously, we deduce that $l(6) = 2$, $\varsigma(6) = 5$, $l(12) = 1$ and $\varsigma(12) = 3$. Therefore, we conclude that $\varsigma(T) = \min\{\varsigma(6), \varsigma(12), \varsigma(26)\} = \min\{5, 3, 6\} = 3$.

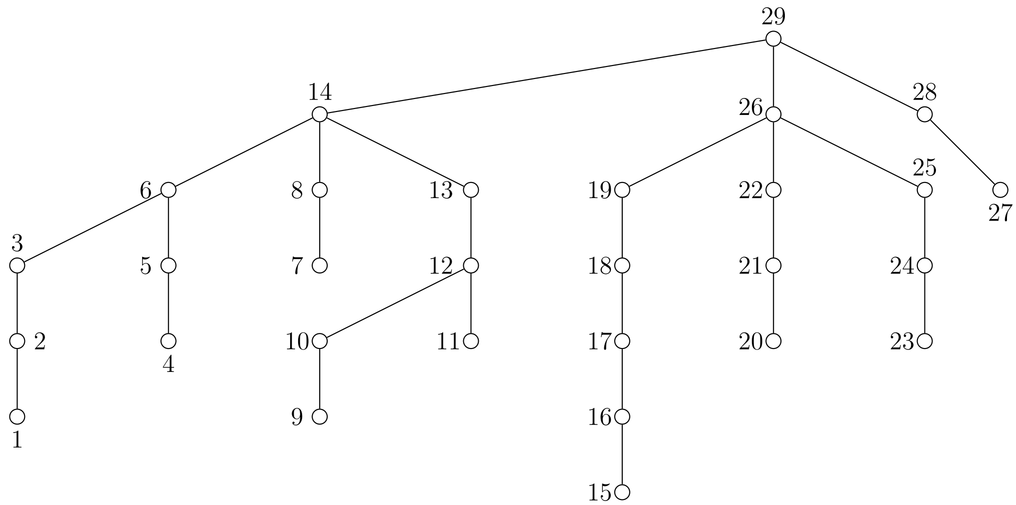

Figure 4.4: A tree T where $\varsigma(T) = 3$. Note that vertices are labeled through a post-order traversal.

4.3.1 On k-metric dimensional trees different from paths

In this subsection we focus on the problem of finding the integer k for which a tree is k-metric dimensional. By Theorem 2.9, we know that a graph G of order $n \geq 3$ is $(n-1)$-metric dimensional if and only if G is a path or G is an odd cycle. Thus, this result allows us to consider only those trees that are not paths. Theorem 2.13 is the base of the algorithm presented in this subsection.

Now we consider the problem of finding the integer k such that a tree T of order n is k-metric dimensional.

k-DIMENSIONAL TREE PROBLEM
INSTANCE: A tree T different from a path of order n
PROBLEM: Find the integer k, $2 \leq k \leq n-1$, such that T is k-metric dimensional

Algorithm 1:

Input: A tree T different from a path rooted in a major vertex v.

Output: The value k for which T is k-metric dimensional.

1. For any vertex $u \in V(T)$ visited by post-order traversal as shown in Figure 4.4, assign a pair (a_u, b_u) in the following way:

   (a) If u does not have any child (u is a leaf), then $a_u = 1$ and $b_u = \infty$.

(b) If u has only one child (u has degree 2), then $a_u = a_{u'} + 1$ and $b = b_{u'}$, where the pair $(a_{u'}, b_{u'})$ was assigned to the child vertex of u. Note that $a_{u'}$ can be ∞. Thus, in such case, $a_u = \infty$.

(c) If u has at least two children (u is a major vertex), then $a_u = \infty$ and $b_u = \min\{a_{u_1} + a_{u_2}, b_{min}\}$, where a_{u_1} and a_{u_2} are the two minimum values among all possible pairs (a_{u_i}, b_{u_i}) assigned to the children of u, and b_{min} is the minimum value among all the b_{u_i}'s.

2. The value k for which T is k-metric dimensional equals b_v (the second element of the pair assigned to the root v).

Figure 4.5 shows an example of a run of Algorithm 1 for the tree shown in Figure 4.4.

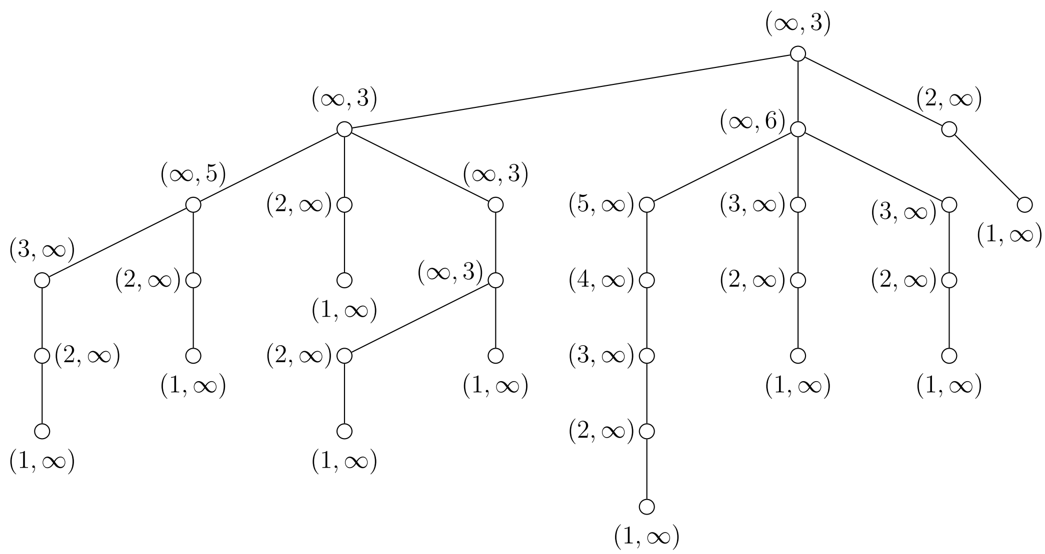

Figure 4.5: Algorithm 1 yields that this tree is 3-metric dimensional.

Remark 4.11. *Let T be a tree different from a path of order n.* Algorithm 1 *computes the integer k, $2 \leq k \leq n-1$, such that T is k-metric dimensional.*

Proof. Let v be the major vertex taken as the root of the tree T different from a path, and let (a_v, b_v) be the pair stored in v by Algorithm 1. We show that $b_v = \varsigma(T)$. Since v is a major vertex, it has at least three children. Let $t \geq 3$ be the number of children of v and let $S_1, \ldots, S_t$ be the subtrees whose roots are the children $v_1, \ldots, v_t$ of v, respectively. We differentiate two cases:

1. There exist at least two subtrees that are paths. In this case $v \in \mathcal{M}(T)$. Let $S_1, \ldots, S_{t'}$ be the subtrees that are paths, where $2 \leq t' \leq t$. In this case, after running Algorithm 1, each root v_i of S_i, $1 \leq i \leq t'$, stores the pair (a_{v_i}, ∞), where a_{v_i} is the number of vertices of S_i. Note that $\varsigma(v) = a_{v_1} + a_{v_2}$, where a_{v_1} and a_{v_2} are the two minimum values among all a_{v_i}'s belonging to the pairs (a_{v_i}, b_{v_i}) stored by the children of v such that $1 \leq i \leq t'$. If $t' = t$, then v is the only exterior major vertex of T, and Algorithm 1 stores in v the pair $(a_v, b_v) = (\infty, \varsigma(v)) = (\infty, \varsigma(T))$. Assume now that $t' < t$. Thus, there exists at least one subtree that is not a path. Let $S_{t'+1}, \ldots, S_t$ be the subtrees that are not paths. For each root v_i of S_i, $t'+1 \leq i \leq t$, if v_i is a major vertex, then we take the vertex $v'_i = v_i$. Otherwise, v'_i is the first descendant of v_i that is a major vertex. In this case, Algorithm 1 recursively stores in v'_i the pair $(\infty, b_{v'_i})$, where $b_{v'_i} = \min\limits_{v' \in \mathcal{M}(T) \cap V(S_i)} \{\varsigma(v')\}$. In both cases, $b_{v_i} = b_{v'_i}$, where (∞, b_{v_i}) is the pair stored in v_i by Algorithm 1. Therefore, by Algorithm 1, the root v stores the pair $(a_v, b_v) = (\infty, \min\{\varsigma(v), b_{min}\}) = (\infty, \varsigma(T))$, where $b_{min} = \min\limits_{t'+1 \leq i \leq t} \{b_i\}$.

2. There exists at most one subtree that is a path. In this case $v \notin \mathcal{M}(T)$. Let $S_1, \ldots, S'_t$ be the subtrees that are not paths, where $1 \leq t' \leq t$. For each root v_i of S_i, $1 \leq i \leq t'$, if v_i is a major vertex, then we take the vertex $v'_i = v_i$. Otherwise, v'_i is the first descendant of v_i that is a major vertex. In this case, Algorithm 1 recursively stores in v'_i the pair $(\infty, b_{v'_i})$, where $b_{v'_i} = \min\limits_{v' \in \mathcal{M}(T) \cap V(S_i)} \{\varsigma(v')\}$. In both cases, $b_{v_i} = b_{v'_i}$, where (∞, b_{v_i}) is the pair stored in v_i by Algorithm 1. Note in this case, at least one of two minimum values among all a_{v_i} of pairs (a_{v_i}, b_{v_i}) stored by the children of v is infinity. Therefore, by Algorithm 1, v stores the pair $(a_v, b_v) = (\infty, b_{min}) = (\infty, \varsigma(T))$, where $b_{min} = \min\limits_{1 \leq i \leq t'} \{b_{v_i}\}$.

In any case, $b_v = \varsigma(T)$, and by Theorem 2.13 the result follows. □

Corollary 4.12. *The positive integer k for which a tree different from a path is k-metric dimensional can be computed in linear time with respect to the order of the tree.*

4.3.2 On the k-metric bases and the k-metric dimension of trees different from paths

In this subsection we propose an algorithm to compute the k-metric dimension of a tree and another to determine a k-metric basis. By Proposition 3.11 we know that for any integer $k \geq 3$ and any path graph P_n of order $n \geq k+1$, $\dim_k(P_n) = k+1$. We observe that, for instance, if P_n is a path of order n and the two leaves of P_n belong to a set $S \subseteq V(P_n)$ of cardinality $k+1$, then S is a k-metric basis of P_n. Thus, we center our attention to those trees different from paths.

We recall a function for any exterior major vertex $w \in \mathcal{M}(T)$, shown in Section 3.2, that allows us to compute the k-metric dimension of any $k \leq \varsigma(T)$. Notice that this function uses the concepts already defined at the beginning of Subsection 2.2.1. Given an integer $k \leq \varsigma(T)$,

$$I_k(w) = \begin{cases} (ter(w)-1)\,(k-l(w)) + l(w), & \text{if } l(w) \leq \lfloor \frac{k}{2} \rfloor, \\ \\ (ter(w)-1)\,\lceil \frac{k}{2} \rceil + \lfloor \frac{k}{2} \rfloor, & \text{otherwise.} \end{cases}$$

Theorem 3.20 is the base of the two algorithms presented in this subsection. Now we consider the problem of computing the k-metric dimension of a tree T of order n, different from a path, for any $k \leq \varsigma(T)$.

k-METRIC DIMENSION TREE PROBLEM
INSTANCE: A tree T of order n
PROBLEM: Compute the k-metric dimension of T, for any $k \leq \varsigma(T)$

Algorithm 2:
Input: A tree T different from a path rooted in a major vertex v.
Output: The k-metric dimension of T for any $k \leq \varsigma(T)$.

1. For any vertex $u \in V(T)$ visited by post-order traversal as shown in Figure 4.4, assign a pair (a_u, b_u) in the following way:
   (a) If u does not have any child (u is a leaf), then $a_u = 1$ and $b_u = \infty$.
   (b) If u has only one child (u has degree 2), then $a_u = a_{u'} + 1$ and $b_u = b_{u'}$, where the pair $(a_{u'}, b_{u'})$ was assigned to the child vertex of u. Note that $a_{u'}$ can be ∞, in which case $a_u = \infty$.

(c) If u has at least two children (u is a major vertex), then $a_u = \infty$. Let a_{min} be the minimum value among all a_{u_i}'s in the pairs (a_{u_i}, b_{u_i}) assigned to the children of u, let c_u be the number of labels a_{u_i} different from ∞, and let s_u be the sum of all $b_{u_i} \neq \infty$. If $c_u \leq 1$, then $b_u = s_u$. If $c_u \geq 2$ and $a_{min} \leq \lfloor \frac{r}{2} \rfloor$, then $b_u = a_{min} + (c_u - 1)(r - a_{min}) + s_u$. If $c_u \geq 2$ and $a_{min} > \lfloor \frac{r}{2} \rfloor$, then $b_u = \left\lfloor \frac{r}{2} \right\rfloor + (c_u - 1) \left\lceil \frac{r}{2} \right\rceil + s_u$.

2. The k-metric dimension of T is b_v.

Figure 4.6 shows an example of a run of Algorithm 2 for computing the 3-metric dimension of the tree shown in Figure 4.4.

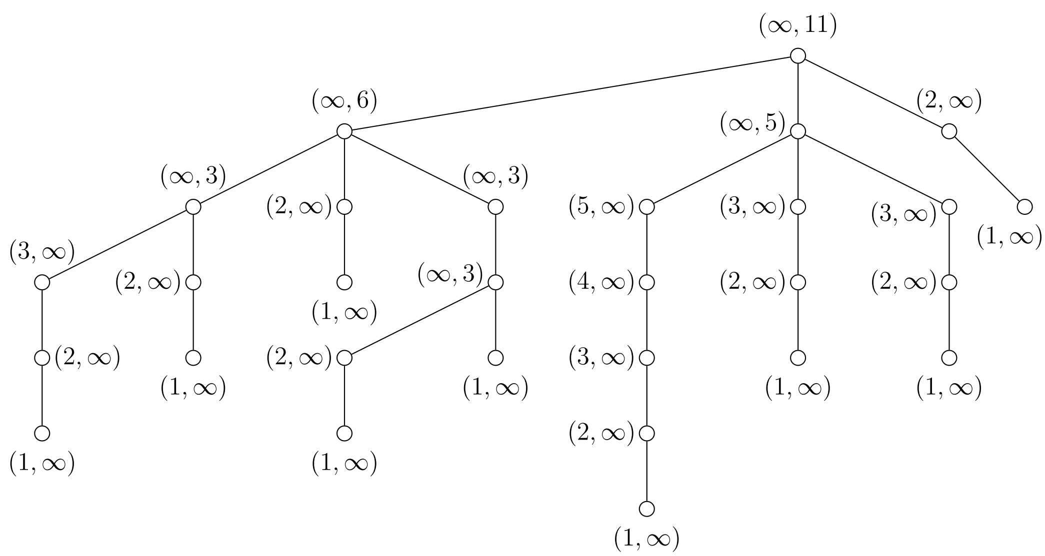

Figure 4.6: Algorithm 2 yields that 3-metric dimension of this tree is 11.

Remark 4.13. *Let T be a tree different from a path.* Algorithm 2 *computes the k-metric dimension of T for any $k \leq \varsigma(T)$.*

Proof. Let v be the major vertex taken as a root of the tree T different from a path and let (a, b) be the pair stored in v once Algorithm 2 has been executed. We shall show that $b_v = \sum_{v' \in \mathcal{M}(T)} I_k(v')$. Since v is a major vertex, it has at least three children. Let $t \geq 3$ be the number of children of v and let $S_1, \ldots, S_t$ be the subtrees whose roots are the children $v_1, \ldots, v_t$ of v, respectively. We differentiate two cases:

1. There exist at least two subtrees that are paths. In this case $v \in \mathcal{M}(T)$. Let $S_1, \ldots, S_{t'}$ be the subtrees that are paths, where $2 \le t' \le t$. Hence, once executed Algorithm 2, each root v_i of S_i, $1 \le i \le t'$, stores the pair (a_{v_i}, ∞), where a_{v_i} is the number of vertices of S'_i. Note that in this case $\mathrm{ter}(v) = c_v = t' \ge 2$ and $l(v) = a_{min}$. If $a_{min} \le \lfloor \frac{k}{2} \rfloor$, then $I_k(v) = a_{min} + (c_v - 1)(k - a_{min})$. Otherwise, $I_k(v) = \lfloor \frac{k}{2} \rfloor + (c_v - 1)\lceil \frac{k}{2} \rceil$. If $t' = t$, then v is the only exterior major vertex and $s_v = 0$. As a consequence, Algorithm 2 has assigned to v the pair $(a_v, b_v) = (\infty, I_k(v))$. Assume that $t' < t$. Thus, there exists at least one subtree that is not path. Let $S_{t'+1}, \ldots, S_t$ be the subtrees that are not paths. We consider the root v_i of S_i, $t' + 1 \le i \le t$. If v_i is a major vertex, then we take the vertex $v'_i = v_i$. Otherwise, v'_i is the first descendant of v_i that is a major vertex. In this case, Algorithm 2 recursively assigns to v'_i the pair $(\infty, b_{v'_i})$, where $b_{v'_i} = \displaystyle\sum_{v' \in \mathcal{M}(T) \cap V(S_i)} I_k(v')$. In both cases, $b_{v_i} = b_{v'_i}$, where (∞, b_{v_i}) is the pair assigned to v_i by Algorithm 2. Hence, $s_v = \displaystyle\sum_{v' \in \mathcal{M}(T) - \{v\}} I_k(v')$. Therefore, the execution of Algorithm 2 assigns to v the pair $(a_v, b_v) = \left(\infty, I_k(v) + \displaystyle\sum_{v' \in \mathcal{M}(T) - \{v\}} I_k(v')\right) = \left(\infty, \displaystyle\sum_{v' \in \mathcal{M}(T)} I_k(v')\right)$.

2. There exists at most one subtree that is a path. In this case $v \notin \mathcal{M}(T)$ and $c_v \le 1$. Let $S_1, \ldots, S'_t$ be the subtrees that are not paths, where $1 \le t' \le t$. For each root v_i of S_i, $1 \le i \le t'$, if v_i is a major vertex, then we take the vertex $v'_i = v_i$. Otherwise, v'_i is the first descendant of v_i that is a major vertex. Hence, Algorithm 2 recursively assigns to v'_i the pair $(\infty, b_{v'_i})$, where $b_{v'_i} = \displaystyle\sum_{v' \in \mathcal{M}(T) \cap V(S_i)} I_k(v')$. In both cases, $b_{v_i} = b_{v'_i}$, where (∞, b_{v_i}) is the pair stored in v_i by an execution of Algorithm 2. Hence, $s_v = \displaystyle\sum_{v' \in \mathcal{M}(T)} I_k(v')$. Note in this case, at most one of all the a_{v_i}'s belonging to the pairs (a_{v_i}, b_{v_i}) assigned to the children of v is different from infinity. As a consequence, $c_v \le 1$. Therefore, Algorithm 2 assigns to v the pair $(a_v, b_v) = \left(\infty, \displaystyle\sum_{v' \in \mathcal{M}(T)} I_k(v')\right)$.

In any case, $b_v = \displaystyle\sum_{v' \in \mathcal{M}(T)} I_k(v')$, and by Theorem 3.20 the result follows. □

Corollary 4.14. *The k-metric dimension of any tree T different from a path, for any $k \leq \varsigma(T)$, can be computed in linear time with respect to the order of T.*

Now we consider the problem of finding a k-metric basis of a tree different from a path for any $k \leq \varsigma(T)$. To this end, we present an algorithm quite similar to Algorithm 2, which is based on the k-metric basis of T proposed in the proof of Theorem 3.20.

k-METRIC BASIS TREE PROBLEM
INSTANCE: A tree T of order n different from a path
PROBLEM: Find a k-metric basis of T, for any $k \leq \varsigma(T)$

Algorithm 3:
Input: A tree T different from a path rooted in a major vertex v.
Output: A k-metric basis of T for any $k \leq \varsigma(T)$.

1. For any vertex $u \in V(T)$ visited by post-order traversal as shown in Figure 4.4, assign a pair (a_u, b_u) in the following way:

   (a) If u does not have any child (u is a leaf), then $a = \{u\}$ and $b = \emptyset$.

   (b) If u has only one child (u has degree 2), then $b_u = b_{u'}$, where the pair $(a_{u'}, b_{u'})$ was assigned to the child vertex of u. If $a_{u'} = \emptyset$, then $a_u = \emptyset$. If $a_{u'} \neq \emptyset$, then $a_u = a_{u'} \cup \{u\}$.

   (c) If u has at least two children (u is a major vertex), then $a_u = \emptyset$. Let a_{min} be a set of minimum cardinality among all a_{u_i} belonging to the pairs (a_{u_i}, b_{u_i}) assigned to the children of u, let c_u be the number of a_{u_i} which are different from an empty set, and let d_u be the union of all b_{u_i}. If $c_u \leq 1$, then $b_u = d_u$. If $c_u \geq 2$ and $|a_{min}| \leq \lfloor \frac{k}{2} \rfloor$, then we remove elements of each $a_{u_i} \neq a_{min}$ until its cardinality is $k - |a_{min}|$. If $c_u \geq 2$ and $|a_{min}| > \lfloor \frac{k}{2} \rfloor$, then we remove elements of each $a_{u_i} \neq a_{min}$ until its cardinality is $\lceil \frac{k}{2} \rceil$, and we remove elements of a_{min} until its cardinality is $\lfloor \frac{k}{2} \rfloor$. Then
   $$b_u = a_{min} \cup \left(\bigcup_{a_{u_i} \neq a_{min}} a_{u_i} \right) \cup d_u.$$

2. A k-metric basis of T is stored in b_v.

Remark 4.15. *Let T be a tree different from a path.* Algorithm 3 *finds a k-metric basis of T for any $k \leq \varsigma(T)$.*

Proof. Given an exterior major vertex $w \in \mathcal{M}(T)$ such that $u_1, u_2, \ldots, u_t$ are its terminal vertices and $l(w) = l(u_{min}, w)$, we define the vertex set $B_k(w)$ in the following way. If $l(v) \leq \lfloor \frac{k}{2} \rfloor$, then $|B_k(w) \cap (V(P(u_j, w)) - \{w\})| = k - l(v)$, for any $j \neq min$, and $V(P(u_{min}, w)) - \{w\} \subset B_k(w)$. Otherwise, $|B_k(w) \cap (V(P(u_j, w)) - \{w\})| = \lceil \frac{k}{2} \rceil$, for any $j \neq min$, and $|B_k(w) \cap (V(P(u_{min}, w)) - \{w\})| = \lfloor \frac{k}{2} \rfloor$. It was shown in [34], that $\bigcup_{w \in \mathcal{M}(T)} B_k(w)$ is a k-metric basis of T. Let v be the major vertex taken as a root of the tree T different from a path, and let (a_v, b_v) be the pair assigned to v once executed Algorithm 3. We show that the vertex set $b_v = \bigcup_{w \in \mathcal{M}(T)} B_k(w)$. Since v is a major vertex, it has at least three children. Let $t \geq 3$ be the number of children of v and let $S_1, \ldots, S_t$ be the subtrees whose roots are the children $v_1, \ldots, v_t$ of v, respectively. We differentiate two cases:

1. There exist at least two subtrees that are paths. In this case $v \in \mathcal{M}(T)$. Let $S_1, \ldots, S_{t'}$ be the subtrees that are paths, where $2 \leq t' \leq t$. Hence, Algorithm 3 assigns to each root v_i of S_i, $1 \leq i \leq t'$, the pair $(a_{v_i}, \emptyset)$, where $a_{v_i} = V(S_i)$. Note that in that situation $\mathrm{ter}(v) = c_v = t' \geq 2$ and $l(v) = |a_{min}|$. If $t' = t$, then v is the only exterior major vertex and $d_v = \emptyset$. As a consequence, Algorithm 3 assigns to v the pair $(\emptyset, B_k(v))$. Assume now that $t' < t$. Thus, there exists at least one subtree that is not a path. Let $S_{t'+1}, \ldots, S_t$ be the subtrees that are not paths. For each root v_i of S_i, $t'+1 \leq i \leq t$, if v_i is a major vertex, then we take the vertex $v_i' = v_i$. Otherwise, v_i' is the first descendant of v_i that is a major vertex. Hence, Algorithm 3 recursively stores in v_i' the pair $(\infty, b_{v_i'})$, where $b_{v_i'} = \displaystyle\bigcup_{v' \in \mathcal{M}(T) \cap V(S_i)} B_k(v')$. In both cases, $b_{v_i} = b_{v_i'}$, where (∞, b_{v_i}) is the pair stored in v_i by Algorithm 3. Hence, $d_v = \displaystyle\bigcup_{v' \in \mathcal{M}(T) - \{v\}} B_k(v')$. Therefore, Algorithm 3 assigns to v the pair
$$(a_v, b_v) = \left(\emptyset, B_k(v) \cup \bigcup_{v' \in \mathcal{M}(T) - \{v\}} B_k(v')\right) = \left(\emptyset, \bigcup_{v' \in \mathcal{M}(T)} B_k(v')\right).$$

2. There exists at most one subtree that is a path. In this case $v \notin \mathcal{M}(T)$ and $c_v \leq 1$. Let $S_1, \ldots, S_t'$ be the subtrees that are not paths, where $1 \leq t' \leq t$. For each root v_i of S_i, $1 \leq i \leq t'$, if v_i is a major vertex, then

we take the vertex $v_i' = v_i$. Otherwise, v_i' is the first descendant of v_i that is a major vertex. Hence, Algorithm 3 has recursively assigned to v_i' the pair $(\infty, b_{v_i'})$, where $b_{v_i'} = \bigcup_{v' \in \mathcal{M}(T) \cap V(S_i)} B_k(v')$. Again, $b_{v_i} = b_{v_i'}$, where (∞, b_{v_i}) is the pair stored in v_i by Algorithm 3. Thus, $d_v = \bigcup_{v' \in \mathcal{M}(T)} B_k(v')$. Note in such case, that at most one of all possible a_{v_i}'s belonging to the pairs (a_{v_i}, b_{v_i}) assigned to the children of v, is different from infinity. As a consequence, $c_v \leq 1$. Therefore, Algorithm 3 assigns to v the pair $(a_v, b_v) = \left(\emptyset, \bigcup_{v' \in \mathcal{M}(T)} B_k(v')\right)$.

In any case, $b_v = \bigcup_{v' \in \mathcal{M}(T)} B_k(v')$, and the result follows. □

Corollary 4.16. *A k-metric basis of any tree different from a path, for any $k \leq \varsigma(T)$, can be computed in linear time with respect to the order of T.*

We have proved that for any $k \geq 1$ the problem of determining the k-metric dimension of any tree can be solved in linear time, as it was done before for the case $k = 1$. It is known that the 1-metric dimension of the outerplanar graphs can be computed in polynomial time [27], and we conjecture that for $k \geq 2$ the problem of determining the k-metric dimension of any outerplanar graph can also be solved in polynomial time.

Conclusions

In this thesis we have studied the (k,t)-metric dimension of graphs. The central results of the thesis are focused on the k-metric dimension and the k-adjacency dimension as particular cases of the (k,t)-metric dimension of a graph G.

We were interested in finding the largest integer k for which there exist (k,t)-metric bases of a graph. To this end, we have introduced the concept of (k,t)-metric dimensional graph. We have analytically determined or bounded the value of k for some specific classes of graphs. Moreover, we have devised a cubic time algorithm for computing this value in the general case.

We have obtained closed formulae and tight bounds for the (k,t)-metric dimension of some graphs. For instance, we have described those graphs that, for some values of k, have (k,t)-metric dimension equal to k. We have characterized the paths where the (k,t)-metric dimension equals $k+1$. We have also shown how to construct large families of graphs having a (k,t)-metric basis of a graph as a common (k,t)-metric generator. On the other hand, we have bounded the value of the k-metric dimension in terms of distance-related parameters, pointing out some cases where these bounds are reached. In particular, we have given a formula for computing the k-metric dimension of any tree.

We have found a strong relationship between the k-metric dimension of some product graphs and the k-adjacency dimension of one of its factors. Therefore, we have also studied this parameter, in more detail. In particular, we have proved that the k-metric dimension of corona product graphs equals the sum of the k-adjacency dimensions of the second factors. For lexicographic product graphs, we have characterized the cases where this relation holds, deepening in the particular case of join graphs.

Finally, we have analysed the computability of the studied parameters. We have shown that the problem of finding the value k such that a graph is (k,t)-metric dimensional can be solved in cubic time with respect to the order of the graph. Moreover, we have proposed linear-time algorithms for computing the k-metric dimension and a k-metric basis of any tree. However, we have proved that computing the k-metric dimension of arbitrary graphs is NP-hard, so the problem is difficult in the general case.

Contributions of the thesis

The results presented in this work led to elaborate six papers, four of which has been published or accepted. Three of these papers have been accepted in ISI-JCR journals and the other was published in a peer-reviewed journal that, in the year 2013 ranked 53/251 (first quartile) in the category "Mathematics, Applied" of ISI-JCR, having Impact Factor 1.232. Furthermore, the two other paper has been submitted to ISI-JCR journals. Other results have been presented in international conferences or workshops.

Publications in ISI-JCR journals

- A. Estrada-Moreno, Y. Ramírez-Cruz, J.A. Rodríguez-Velázquez. On the adjacency dimension of graphs. *Applicable Analysis and Discrete Mathematics*, in press.
  `https://doi.org/10.2298/AADM151109022E`.

- A. Estrada-Moreno, I. G. Yero, J.A. Rodríguez-Velázquez. The k-metric dimension of corona product graphs. *Bulletin of the Malaysian Mathematical Sciences Society*, in press.
  `http://dx.doi.org/10.1007/s40840-015-0282-2`.

- A. Estrada-Moreno, I. G. Yero, J.A. Rodríguez-Velázquez. The k-metric dimension of the lexicographic product of graphs. *Discrete Mathematics*, in press.
  `http://dx.doi.org/10.1016/j.disc.2015.12.024`

Publications in a peer-reviewed journal (ISI-JCR, Q1, until 2014)

- A. Estrada-Moreno, J. A. Rodríguez-Velázquez, I. G. Yero. The k-metric dimension of a graph. *Applied Mathematics & Information Sciences*, 9(6), 2829–2840, 2015. `http://naturalspublishing.com/files/published/05a21265hsd7y2.pdf`.

Papers submitted to ISI-JCR journals

- Ismael G. Yero, Alejandro Estrada-Moreno, Juan A. Rodríguez-Velázquez. On the complexity of computing the k-metric dimension of graphs. Submitted to *Discrete Applied Mathematics* (2015).

- Alejandro Estrada-Moreno, Ismael G. Yero, Juan A. Rodríguez-Velázquez. Relationships between the 2-metric dimension and the 2-adjacency dimension in the lexicographic product of graphs. Submitted to *Graphs and Combinatorics* (2015).

Other Publications

- Alejandro Estrada-Moreno, Ismael G. Yero, Juan A. Rodríguez-Velázquez. k-metric resolvability in graphs. *Electronic Notes in Discrete Mathematics*, 46, 121–128, 2014. `http://dx.doi.org/10.1016/j.endm.2014.08.017`.

Participations in specialized conferences

- Ismael G. Yero, Alejandro Estrada-Moreno, Juan A. Rodríguez-Velázquez, On the complexity of computing the k-metric dimension of graphs, Algorithmic Graph Theory on the Adriatic Coast, Koper, Eslovenia (2015).

- Alejandro Estrada-Moreno, Ismael G. Yero, Juan A. Rodríguez-Velázquez, k-metric resolvability in graphs, Jornadas de Matemática Discreta y Algorítmica, Spain (2014).

- Ismael G. Yero, Alejandro Estrada-Moreno, Juan A. Rodríguez-Velázquez, The k-metric dimension of graphs, Seventh Cracow Conference on Graph Theory, Rytro, Poland (2014).

Workshops

1. Alejandro Estrada-Moreno, Ismael G. Yero, Juan A. Rodríguez-Velázquez, Yunior Ramírez-Cruz. The k-adjacency dimension of graphs in *2nd URV Doctoral Workshop in Computer Science and Mathematics*, pp 47–50, 2015, Llibres URV, ISBN-13: 978-84-8424-399-1.

2. Alejandro Estrada-Moreno, Ismael G. Yero, Juan A. Rodríguez-Velázquez. The k-metric dimension of a graph. *1st URV Doctoral Workshop in Computer Science and Mathematics*, pp 37–40, 2014, Llibres URV, ISBN-13: 978-84-8424-339-7.

Future works

- The (k,t)-metric dimension of a graph.

  The central results of the thesis are focused on the k-metric dimension and the k-adjacency dimension as particular cases of the (k,t)-metric dimension of a graph G for $t \geq D(G)$ and $t = 2$, respectively. However, it would be interesting to particularly study the (k,t)-metric dimension of a graph for others values of t. We have also shown a motivation for the (k,t)-metric dimension of a graph related to robot's navigation in network. A natural question would be if there is another interesting application of the (k,t)-metric dimension of a graph.

- The (k,t)-metric dimension of product graphs.

  We have proved that the (k,t)-metric dimension of lexicographic product graphs is the same for any $t \geq 2$. However, this does not happen in the case of corona product graphs. Therefore, it would be interesting to study the (k,t)-metric dimension of corona product graphs for different values of t. Moreover, is it possible to extend the study of the (k,t)-metric dimension to other graph product?

- Computability of the (k,t)-metric dimension.

  We have proved that the problem of determining the k-metric dimension of a graph is NP-Hard. A natural question would be if it is possible to extend this previous study to the problem of computing the (k,t)-metric dimension of any graph G for values of t less than the diameter of G.

- Computability of the (k,t)-metric dimension for the case of outerplanar graphs.

  Given that the 1-metric dimension of the outerplanar graphs can be computed in polynomial time [27], one could conjecture that for $k \geq 2$ the analogous problem can also be solved in polynomial time. Moreover, a natural question would be if it is possible to extend this previous study to the problem of computing the (k,t)-metric dimension of any graph G for values of t less than the diameter of G.

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Symbol Index

The symbols are arranged in the order of the first appearance in the work. Page numbers refer to definitions.

Index

2

B

C

D

E

F

G

H

J

K