IX JORNADAS DE MATEMÁTICA DISCRETA Y ALGORÍTMICA

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Grupo de Matemática Discreta
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Presentación

Las IX Jornadas de Matemática Discreta y Algorítmica se celebraron en Tarragona del 7 al 9 de Julio de 2014 y fueron organizadas por los miembros del grupo de Matemática Discreta del Departamento de Ingeniería Informática y Matemática de la Universitat Rovira i Virgili. Este es un congreso que se ha celebrado de forma bianual desde que se hiciera por primera vez en Barcelona en el año 1998. Posteriores ediciones del mismo han sido celebradas en Palma de Mallorca (2000), Sevilla (2002), Cercedilla (2004), Soria (2006), Lleida (2008), Castro Urdiales (2010) y Almería (2012).

Este documento contiene los trabajos presentados en este congreso celebrado en Tarragona, los cuales son una amplia muestra del potencial de investigación de gran parte de los grupos españoles que han participado en las ediciones realizadas hasta hoy. En conjunto, estas actas contienen 58 trabajos, de los cuales 32 fueron presentados como comunicaciones orales y los restantes 26 en formato de póster.

Es merecido notar que la popularidad y difusión del congreso sigue en aumento, lo cual se puede verificar dada la notable presencia de investigadores de todos los grupos de investigación en temas de Matemática Discreta, así como la representación de investigadores extranjeros, como es el caso de México, Francia, Canadá, Australia y Portugal.

A pesar del poco patrocinio obtenido, debido a las dificultades económicas que afrontan actualmente las instituciones españolas, es necesario mencionar a aquellos que han aportado su granito de arena al desarrollo de este evento. Así, agradecemos a la Sociedad Catalana de Matemáticas, la Universitat Rovira i Virgili y la Real Sociedad Matemática Española por su apoyo económico e institucional. Agradecemos además la participación de los conferenciantes invitados, los profesores Sergi Elizalde (Dartmouth College, Estados Unidos), Günter Rote (Freie Universität Berlin, Alemania) y Oriol Serra (Universitat Politècnica de Catalunya, España). En especial agradecemos a los miembros del Grupo de Investigación en Matemática Discreta por todo el trabajo realizado. Y finalmente, como no puede ser de otra manera, nuestro más grato agradecimiento a todos los participantes en las Jornadas, los cuales hacen posible estos momentos de intercambio de ideas y resultados.

A todos, muchas gracias.
Atentamente,

El Comité Organizador

Índice general

An Interlacing approach for bounding the sum of Laplacian eigenvalues of graphs *

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Abstract. We apply eigenvalue interlacing techniques for obtaining lower and upper bounds for the sums of Laplacian eigenvalues of graphs, and characterize equality. This leads to generalizations of, and variations on theorems by Grone, and Grone & Merris. As a consequence we obtain inequalities involving bounds for some well-known parameters of a graph, such as edge-connectivity, and the isoperimetric number.

Key words: Laplacian matrix, eigenvalue interlacing, edge-connectivity, isoperimetric number

1 Eigenvalue interlacing

Throughout this paper, $G = (V, E)$ is a finite simple graph with $n = |V|$ vertices. Recall that the Laplacian matrix of G is $\mathcal{L} = \Delta - \boldsymbol{A}$ where Δ is the diagonal matrix of the vertex degrees and $\boldsymbol{A}$ is the adjacency matrix of G. Let us also recall the following basic result about interlacing (see [5], [2], or [1]).

Theorem 1. *Let $\boldsymbol{A}$ be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. For some $m < n$, let $\mathcal{S}$ be a real $n \times m$ matrix with orthonormal columns, $\mathcal{S}^\top \mathcal{S} = \boldsymbol{I}$, and consider the matrix $\boldsymbol{B} = \mathcal{S}^\top \boldsymbol{A} \mathcal{S}$, with eigenvalues $\mu_1 \geq \cdots \geq \mu_m$. Then,*

(*a*) *the eigenvalues of $\boldsymbol{B}$ interlace those of $\boldsymbol{A}$, that is,*

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \qquad i = 1, \ldots, m, \tag{1}$$

(*b*) *if the interlacing is tight, that is, for some $0 \leq k \leq m$, $\lambda_i = \mu_i$, $i = 1, \ldots, k$, and $\mu_i = \lambda_{n-m+i}$, $i = k+1, \ldots, m$, then $\mathcal{S}\boldsymbol{B} = \boldsymbol{A}\mathcal{S}$.*

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Two interesting particular cases are obtained by choosing appropriately the matrix $\mathcal{S}$. If $\mathcal{S} = [\boldsymbol{I} \quad \Omega]^\top$, then $\boldsymbol{B}$ is a principal submatrix of $\boldsymbol{A}$. If $\mathcal{P} = \{U_1, \ldots, U_m\}$ is a partition of $\{1, \ldots, n\}$ we can take for $\boldsymbol{B}$ the so-called *quotient matrix* of $\boldsymbol{A}$ with respect to $\mathcal{P}$.

The first case gives useful conditions for an induced subgraph G' of a graph G, because the adjacency matrix of G' is a principal submatrix of the adjacency matrix of G. However, the Laplacian matrix $\mathcal{L}'$ of G' is in general not a principal submatrix of the Laplacian matrix $\mathcal{L}$ of G. But $\mathcal{L}' + \Delta'$ is a principal submatrix of $\mathcal{L}$ for some nonnegative diagonal matrix Δ'. Therefore the left hand inequalities in (1) still hold for the Laplacian eigenvalues, because adding the positive semi-definite matrix Δ' decreases no eigenvalue.

In the case that $\boldsymbol{B}$ is a quotient matrix of $\boldsymbol{A}$ with respect to $\mathcal{P}$, the element b_{ij} of $\boldsymbol{B}$ is the average row sum of the block $\boldsymbol{A}_{i,j}$ of $\boldsymbol{A}$ with rows and columns indexed by U_i and U_j, respectively. If $\mathcal{P}$ has characteristic matrix $\boldsymbol{C}$ (that is, the columns of $\boldsymbol{C}$ are the characteristic vectors of $U_1, \ldots, U_m$) then we take $\mathcal{S} = \boldsymbol{C}\Delta^{-1/2}$, where $\Delta = \mathrm{diag}(|U_1|, \ldots, |U_m|) = \boldsymbol{C}^\top \boldsymbol{C}$. In this case, the quotient matrix $\boldsymbol{B}$ is in general not equal to $\mathcal{S}^\top \boldsymbol{A}\mathcal{S}$, but $\boldsymbol{B} = \Delta^{-1/2}\mathcal{S}^\top \boldsymbol{A}\mathcal{S}\Delta^{1/2}$, and thus $\boldsymbol{B}$ is similar to (and therefore has the same spectrum as) $\mathcal{S}^\top \boldsymbol{A}\mathcal{S}$. If the interlacing is tight, then (b) of Theorem 1 reflects that $\mathcal{P}$ is an *equitable* (or *regular*) partition of $\boldsymbol{A}$, that is, each block of the partition has constant row and column sums. In case $\boldsymbol{A}$ is the adjacency matrix of a graph G, equitability of $\mathcal{P}$ implies that the bipartite induced subgraph $G[U_i, U_j]$ is biregular for each $i \neq j$, and that the induced subgraph $G[U_i]$ is regular for each $i \in \{1, \ldots, m\}$. In case of tight interlacing for the quotient matrix of the Laplacian matrix of G, the first condition also holds, but the induced subgraphs $G[U_i]$ are not necessarily regular (in this case we speak about an *almost equitable*, or *almost regular* partition of G).

If a symmetric matrix $\boldsymbol{A}$ has an equitable partition, we have the following well-known and useful result ([1], Section 2.3).

Lemma 1. *Let $\boldsymbol{A}$ be a symmetric matrix of order n, and suppose $\mathcal{P}$ is a partition of $\{1, \ldots, n\}$ such that the corresponding partition of $\boldsymbol{A}$ is equitable with quotient matrix $\boldsymbol{B}$. Then the spectrum of $\boldsymbol{B}$ is a sub(multi)set of the spectrum of $\boldsymbol{A}$, and all corresponding eigenvectors of $\boldsymbol{A}$ are in the column space of the characteristic matrix $\boldsymbol{C}$ of $\mathcal{P}$ (this means that the entries of the eigenvector are constant on each partition class U_i). The remaining eigenvectors of $\boldsymbol{A}$ are orthogonal to the columns of $\boldsymbol{C}$ and the corresponding eigenvalues remain unchanged if the blocks $\boldsymbol{A}_{i,j}$ are replaced by $\boldsymbol{A}_{i,j} + c_{i,j}\boldsymbol{J}$ for certain constants $c_{i,j}$ (as usual, $\boldsymbol{J}$ is the all-one matrix).*

Assuming that G has n vertices, with degrees $d_1 \geq d_2 \geq \cdots \geq d_n$, and Laplacian matrix $\mathcal{L}$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n (= 0)$, it is known that, for $1 \leq m \leq n$,

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} d_i. \tag{2}$$

This is a consequence of Schur's theorem [7] stating that the spectrum of any symmetric, positive definite matrix majorizes its main diagonal. In particular, note that if $m = n$ we have equality in (2), because both terms correspond to the trace of $\mathcal{L}$. To prove (2) by using interlacing, let $\boldsymbol{B}$ be a principal $m \times m$ submatrix of $\mathcal{L}$ indexed by the subindeces corresponding to the m higher degrees, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then,

$$\operatorname{tr} \boldsymbol{B} = \sum_{i=1}^{m} d_i = \sum_{i=1}^{m} \mu_i,$$

and, by interlacing, $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ for $i = 1, \ldots, m$, whence (2) follows. Similarly, reasoning with the principal submatrix $\boldsymbol{B}$ (of $\mathcal{L}$) indexed by the m vertices with lower degrees we get:

$$\sum_{i=1}^{m} \lambda_{n-m+i} \leq \sum_{i=1}^{m} d_{n-m+i}. \tag{3}$$

The next result, which is an improvement of (2), is due to Grone [3], who proved that if G is connected and $m < n$ then,

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} d_i + 1. \tag{4}$$

In [1], Brouwer and Haemers gave two different proofs of (4), both using eigenvalue interlacing. In this paper we extend the ideas of these two proofs and find a generalization of Grone's result (4), and another lower bound on the sum of the largest Laplacian eigenvalues, which is closely related to a bound of Grone and Merris [4].

2 A generalization of Grone's result

We begin with a basic result from where most of our bounds derive. Given a graph G with a vertex subset $U \subset V$, let ∂U be the *vertex-boundary* of U, that is, the set of vertices in $\overline{U} = V \backslash U$ with at least one adjacent vertex in U. Also, let $\partial(U, \overline{U})$ denote the *edge-boundary* of U, which is the set of edges which connect vertices in U with vertices in $\overline{U}$.

Theorem 2. *Let G be a connected graph on $n = |V|$ vertices, having Laplacian matrix $\mathcal{L}$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n (= 0)$. For any given vertex subset $U = \{u_1, \ldots, u_m\}$ with $0 < m < n$, we have*

$$\sum_{i=1}^{m} \lambda_{n-i} \leq \sum_{u \in U} d_u + \frac{|\partial(U, \overline{U})|}{n-m} \leq \sum_{i=1}^{m} \lambda_i. \tag{5}$$

Proof. Consider the partition of the vertex set V into $m+1$ parts: $U_i = \{u_i\}$ for $u_i \in U$, $i = 1, \ldots, m$, and $U_{m+1} = \overline{U}$. Then, the corresponding quotient matrix is

$$\boldsymbol{B} = \left[\begin{array}{c|c} & b_{1,m+1} \\ \mathcal{L}_U & \vdots \\ & b_{m,m+1} \\ \hline b_{m+1,1} \cdots b_{m+1,m} & b_{m+1,m+1} \end{array}\right],$$

where $\mathcal{L}_U$ is the principal submatrix of $\mathcal{L}$, with rows and columns indexed by the vertices in U, $b_{i,m+1} = (n-m)b_{m+1,i} = -|\partial(U_i, \overline{U})|$, and $b_{m+1,m+1} = |\partial(U, \overline{U})|/(n-m)$ (because $\sum_{i=1}^{m+1} b_{m+1,i} = 0$). Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m+1}$ be the eigenvalues of $\boldsymbol{B}$. Since $\boldsymbol{B}$ has row sum 0, we have $\mu_{m+1} = \lambda_n = 0$. Moreover,

$$\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m+1} \mu_i = \operatorname{tr} \boldsymbol{B} = \sum_{u \in U} d_u + b_{m+1,m+1},$$

Then, (5) follows by applying interlacing, $\lambda_i \geq \mu_i \geq \lambda_{n-m-1+i}$ and adding up for $i = 1, 2, \ldots, m$. ∎

If equality holds on either side of (5) it follows that the interlacing is tight (see the proof of Proposition 1 for details), and therefore that the partition of G is almost equitable. In other words, in case of equality every vertex $u \in U$ is adjacent to either all or 0 vertices in $\overline{U}$, whereas each vertex $u \in \overline{U}$ has precisely $|\partial(U, \overline{U})|/(n-m)$ neighbors in U. But we can be more precise.

Proposition 1. *Let H be the subgraph of G induced by $\overline{U}$, and let $\vartheta_1 \geq \cdots \geq \vartheta_{n-m} (= 0)$ be the Laplacian eigenvalues of H. Define $b = |\partial(U, \overline{U})|/(n-m)$.*

(*a*) *Equality holds on the right hand side of* (5) *if and only if each vertex of U is adjacent to all or* 0 *vertices of $\overline{U}$, and $\lambda_{m+1} = \vartheta_1 + b$.*

(*b*) *Equality holds on the left hand side of* (5) *if and only if each vertex of U is adjacent to all or* 0 *vertices of $\overline{U}$, and $\lambda_{n-m-1} = \vartheta_{n-m} + b$.*

Proof. Since (*a*) and (*b*) have analogous proofs, we only prove (*a*). Suppose equality holds on the right hand side of (5). Then

$$\sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} \mu_i, \text{ and } \lambda_i \geq \mu_i \text{ for } i = 1, \ldots, m$$

so $\lambda_i = \mu_i$ for $i = 1, \ldots, m$. We know that $\mu_{m+1} = \lambda_n = 0$, therefore the interlacing is tight and hence the partition of G is almost equitable. Now by use of Lemma 1 we have that the eigenvalues of $\mathcal{L}$ are $\mu_1, \ldots, \mu_{m+1}$ together with the eigenvalues of $\mathcal{L}$ with an eigenvector orthogonal to the characteristic matrix $\boldsymbol{C}$ of the partition. These eigenvalues and eigenvectors remain unchanged if $\mathcal{L}$ is changed into

$$\widetilde{\mathcal{L}} = \begin{bmatrix} \Omega & \Omega \\ \Omega & \mathcal{L}_{\overline{U}} + b\boldsymbol{I} \end{bmatrix}.$$

The considered common eigenvalues of $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ are $\vartheta_1 + b \geq \cdots \geq \vartheta_{n-m-1} + b$. So $\mathcal{L}$ has eigenvalues $\lambda_1 (= \mu_1) \geq \cdots \geq \lambda_m (= \mu_m)$, and $\vartheta_1 + b \geq \cdots \geq \vartheta_{n-m-1} + b \geq \lambda_n (= \mu_{m+1} = 0)$. Hence, we have $\lambda_{m+1} = \vartheta_1 + b$. Conversely, if the partition of G is almost equitable (or equivalently, if the partition of $\mathcal{L}$ is equitable), $\mathcal{L}$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, $\vartheta_1 + b \geq \cdots \geq \vartheta_{n-m-1} + b$, and $\mu_{m+1} = \lambda_n = 0$. Since $\lambda_{m+1} = \vartheta_1 + b$, if follows that $\mu_i = \lambda_i$ for $i = 1, \ldots, m$ (tight interlacing), therefore equality holds on the right hand side of (5). ∎

Looking for examples of the above results, first observe that there is no graph with $n > 2$ satisfying equality in (4) for every $0 < m < n$. However the complete graph K_n provides an example for which both inequalities in Theorem 2 are equalities for all $0 < m < n$. In fact, this is a particular case of the following contruction (just take $q = 1$): Let us consider the *graph join* G of the complete graph K_p with the empty graph $\overline{K_q}$. (Recall that G is obtained as the graph union of K_p and $\overline{K_q}$ with all the edges connecting the vertices of one graph with the vertices of the other.) Let $V(G) = \{v_1, \ldots, v_p, v_{p+1}, \ldots, v_n\}$, where $n = p + q$ and the first vertices correspond to those of K_p. Then, the Laplacian eigenvalues of G are $\{n^p, p^{q-1}, 0^1\}$, and the following different choices for U provide some examples illustrating cases (a) and (b) of Proposition 1.

$(a1)$ Let $U = \{v_1, \ldots, v_m\}$, with $0 < m \leq p$. Then, $b = m$, and

$$\sum_{u \in U} d_u + b = m(n-1) + m = mn = \sum_{i=1}^{m} \lambda_i.$$

$(a2)$ Let $U = \{v_1, \ldots, v_m\}$, with $p < m < n$. Then, $b = p$, and

$$\sum_{u \in U} d_u + b = p(n-1) + (m-p)p + p = pn + (m-p)p = \sum_{i=1}^{m} \lambda_i.$$

(b) Let $U = \{v_{n-m+1}, \ldots, v_n\}$, with $q \leq m < n$. Then, $b = m$, and

$$\sum_{u \in U} d_u + b = qp + (m-q)(n-1) + m = (q-1)p + (m-q+1)n = \sum_{i=1}^{m} \lambda_{n-i}.$$

Another infinite family of graphs for which we do have equality on the right hand side of (5) is the complete multipartite graph (such that the vertices with largest degree lie in U).

If the vertex degrees of G are $d_1 \geq d_2 \geq \cdots \geq d_n$, we can choose conveniently the m vertices of U (that is, those with higher or lower degrees) to obtain the best inequalities in (5). Namely,

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} d_i + \frac{|\partial(U,\overline{U})|}{n-m}, \tag{6}$$

and

$$\sum_{i=1}^{m} \lambda_{n-i} \leq \sum_{i=1}^{m} d_{n-i+1} + \frac{|\partial(U,\overline{U})|}{n-m}. \tag{7}$$

Note that (7), together with (3) for $m+1$, yields

$$\sum_{i=0}^{m} \lambda_{n-m+i} = \sum_{i=1}^{m} \lambda_{n-i} \leq \sum_{i=1}^{m} d_{n-i+1} + \min\left\{ d_{n-m}, \frac{|\partial(U,\overline{U})|}{n-m} \right\}. \tag{8}$$

If we have more information on the structure of the graph, we can improve the above results by either bounding $|\partial(U,\overline{U})|$ or 'optimizing' the ratio $b = |\partial(U,\overline{U})|/(n-m)$. In fact, the right inequality in (5) (and, hence, (6)) can be improved when $\overline{U} \neq \partial U$. Simply first delete the vertices (and corresponding edges) of $\overline{U} \setminus \partial U$, and then apply the inequality. Then $d_1, \ldots, d_m$ remain the same and $\lambda_1, \ldots, \lambda_m$ do not increase. Thus we obtain:

Theorem 3. *Let G be a connected graph on $n = |V|$ vertices, with Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n (= 0)$. For any given vertex subset $U = \{u_1, \ldots, u_m\}$ with $0 < m < n$, we have*

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{u \in U} d_u + \frac{|\partial(U,\overline{U})|}{|\partial U|}. \tag{9}$$

Similarly as we did in (6), if we choose the m vertices of U such that they are those with maximum degree, then we can write:

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} d_i + \frac{|\partial(U,\overline{U})|}{|\partial U|}.$$

Notice that, as a corollary, we get Grone's result (4) since always $|\partial(U,\overline{U})| \geq |\partial U|$.

3 A variation of a bound by Grone and Merris

In [4], Grone and Merris gave another lower bound for the sum of the Laplacian eigenvalues, in the case when there is an induced subgraph consisting of isolated vertices and edges. Let G be a connected graph of order $n > 2$ with Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If the induced subgraph of a subset $U \subset V$ with $|U| = m$ consists of r pairwise disjoint edges and $m - 2r$ isolated vertices, then

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{u \in U} d_u + m - r. \tag{10}$$

An improvement of this result was given by Brouwer and Haemers in [1] (Section 3.10). Let G be a (not necessarily connected) graph with a vertex subset U, with $m = |U|$, and let h be the number of connected components of $G[U]$ that are not connected components of G. Then,

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{u \in U} d_u + h. \tag{11}$$

Following the same ideas as in [4] and using interlacing, the bound (10) of Grone and Merris can also be generalized as follows:

Theorem 4. *Let G be a connected graph of order $n > 2$ with Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Given a vertex subset $U \subset V$, with $m = |U| < n$, let $G[U] = (U, E[U])$ be its induced subgraph. Then,*

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{u \in U} d_u + m - |E[U]|. \tag{12}$$

Proof. Consider an orientation of G with all edges in $E(U, \overline{U})$ oriented from U to $\overline{U}$, and every vertex in $U \setminus \partial\overline{U}$ having some outgoing arc (this is always possible as G is connected). Let $\boldsymbol{Q}$ be the corresponding oriented incidence matrix of G, and write $\boldsymbol{Q} = [\,\boldsymbol{Q}_1\ \boldsymbol{Q}_2\,]$, where $\boldsymbol{Q}_1$ corresponds to $E[U] \cup E(U, \overline{U})$, and $\boldsymbol{Q}_2$ corresponds to $E[\overline{U}]$. Consider the matrix $\boldsymbol{M} = \boldsymbol{Q}^\top \boldsymbol{Q}$, with entries $(\boldsymbol{M})_{ii} = 2$, $(\boldsymbol{M})_{ij} = \pm 1$ if the arcs e_i, e_j are incident to the same vertex ($+1$ if both are either outgoing or ingoing, and -1 otherwise), and $(\boldsymbol{M})_{ij} = 0$ if the corresponding edges are disjoint, and define $\boldsymbol{M}_1 = \boldsymbol{Q}_1^\top \boldsymbol{Q}_1$. Then $\boldsymbol{M}$ has the same nonzero eigenvalues as $\mathcal{L} = \boldsymbol{Q}\boldsymbol{Q}^\top$, the Laplacian matrix of G, and $\boldsymbol{M}_1$ is a principal submatrix of $\boldsymbol{M}$. For every vertex $u \in U$, let E_u be the set of outgoing arcs from u. Then $\{E_u \,|\, u \in U\}$ is a partition of $E[U] \cup E(U, \overline{U})$. Consider the quotient matrix $\boldsymbol{B}_1 = (b_{ij})$ of $\boldsymbol{M}_1$ with respect to this partition. Then, $b_{uu} = d^+(u) + 1$ for each $u \in U$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the eigenvalues of $\boldsymbol{B}_1$, then

$$\operatorname{tr} \boldsymbol{B}_1 = \sum_{i=1}^{m} \mu_i = \sum_{u \in U} d_u^+ + m = \sum_{u \in U} d_u - |E[U]| + m$$

and (12) follows since the eigenvalues of $\boldsymbol{B}_1$ interlace those of $\boldsymbol{M}_1$, which in turn interlace those of $\boldsymbol{M}$. ∎

Note that (12) also follows from Equation (11). However, the result can be improved by considering the partition $\mathcal{P} = \{E_u \,|\, u \in U\} \cup \{E[\overline{U}]\}$ of the whole edge set of G.

Theorem 5. *Let G be a connected graph of order $n > 2$ with Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Given a vertex subset $U \subset V$, with $m = |U| < n$,*

let $G[U] = (U, E[U])$ and $G[\overline{U}]$ be the corresponding induced subgraphs. Let ϑ_1 be the largest Laplacian eigenvalue of $G[\overline{U}]$, then

$$\sum_{i=1}^{m+1} \lambda_i \geq \sum_{u \in U} d_u + m - |E[U]| + \vartheta_1. \tag{13}$$

Proof. First observe that the Laplacian matrix of $G[\overline{U}]$ is $\boldsymbol{Q}_2\boldsymbol{Q}_2^\top$, and therefore ϑ_1 is also the largest eigenvalue of $\boldsymbol{Q}_2^\top\boldsymbol{Q}_2$. Next we apply interlacing to an $(m+1) \times (m+1)$ quotient matrix $\boldsymbol{B} = \mathcal{S}^\top \boldsymbol{M} \mathcal{S}$, which is defined slightly different as before. The first m columns of $\mathcal{S}$ are the normalized characteristic vectors of E_u (as before), but the last column of $\mathcal{S}$ equals $\left[\begin{smallmatrix}\mathbf{0}\\ \mathbf{v}\end{smallmatrix}\right]$, where $\mathbf{v}$ is a normalized eigenvector of $\boldsymbol{Q}_2^\top\boldsymbol{Q}_2$ for the eigenvalue ϑ_1. Then $b_{m+1,m+1} = \mathbf{v}^\top\boldsymbol{Q}_2^\top\boldsymbol{Q}_2\mathbf{v} = \vartheta_1$, and we find $\operatorname{tr}\boldsymbol{B} = \sum_{u\in U} d_u + m - |E[U]| + \vartheta_1$. ∎

4 Some applications

The previous bounds on the sum of Laplacian eigenvalues are used to provide meaningful results involving the edge-connectivity of the graph, the size of a k-dominating set and the isoperimetric number.

4.1 Cuts

Given a vertex subset U of a connected graph G with $0 < |U| < n$, the edge set $\partial(U, \overline{U})$ is called a *cut* (since deletion of these edges makes G disconnected). The minimum size of a cut in G is called the *edge-connectivity* $\kappa_e(G)$ of G. By use of inequality (6) we obtain the following bound for $\kappa_e(G)$.

Proposition 2.

$$\kappa_e(G) \leq \min_{0<m<n} \left\{ (n-m) \sum_{i=1}^{m} (\lambda_i - d_i) \right\}. \tag{14}$$

Some general bounds on the size of a cut can be derived from the following lemma.

Lemma 2. *Let G be a graph with n vertices and e edges. For any m, $0 < m < n$, there exist some (not necessarily different) vertex subsets U and U' such that $|U| = |U'| = m$ and*

$$|\partial(U, \overline{U})| \geq \frac{2em(n-m)}{n(n-1)}, \qquad |\partial(U', \overline{U'})| \leq \frac{2em(n-m)}{n(n-1)}. \tag{15}$$

Proof. Choose a set S uniformly at random among all the sets of size m in V. Then the probability that an edge belong to $\partial(S, \overline{S})$ is the probability that either the first endpoint belongs to S and the second one to $\overline{S}$ or viceversa. That is,

$$\Pr(\text{edge} \in \partial(S, \overline{S})) = 2\frac{m(n-m)}{n(n-1)}.$$

Then, the expected number of edges between the two sets is,

$$\mathbb{E}\{|\partial(S, \overline{S})|\} = \frac{2em(n-m)}{n(n-1)},$$

implying that there are sets, U and U', with at least and at most this number of edges going out, respectively. ∎

Both bounds are tight for the complete graph K_n. Using bounds (15), Theorem 2 gives:

Corollary 1. *For each m $(0 < m < n)$ G has vertex sets U and U' of size m such that*

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{u \in U} d_u + \frac{2em}{n(n-1)}, \tag{16}$$

and

$$\sum_{i=1}^{m} \lambda_{n-i} \leq \sum_{u \in U'} d_u + \frac{2em}{n(n-1)}. \tag{17}$$

In particular, if G is d-regular, we have $e = nd/2$ and the above inequalities become

$$\sum_{i=1}^{m} \lambda_i \geq \frac{mdn}{n-1} \qquad \text{and} \qquad \sum_{i=1}^{m} \lambda_{n-i} \leq \frac{mdn}{n-1}, \tag{18}$$

with bounds close to md when n is large.

4.2 k-Dominating sets

A *dominating set* in a graph G is a vertex subset $D \subseteq V$ such that every vertex in $V \backslash D$ is adjacent to some vertex in D. More generally, for a given integer k, a *k-dominating set* in a graph G is a vertex subset $D \subseteq V$ such that every vertex not in D has at least k neighbors in D.

Proposition 3. *Let G be a graph on n vertices, with vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n$, and eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n (= 0)$. Let D be a k-dominating set in G of cardinality m. Then,*

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{u \in D} d_u + k. \tag{19}$$

Proof. First, the inequality (19) follows from Theorem 2 by noting that, from the definition of a k-dominating set, $|\partial(D, \overline{D})| \geq k(n-m)$. ∎

Example 1. Consider the $K_{p,\dots,p}$ regular complete multipartite graph with q classes of size p, so $n = pq$ and $d = p(q-1)$. The eigenvalues of its Laplacian matrix are

$$\{(d+p)^{q-1}, d^{n-q}, 0^1\}.$$

Observe that the union of some partition classes gives a k-dominating set of size $m = k$. If we take the first k eigenvalues, the inequality (19) becomes $(d+p)(q-1) + (k-(q-1))d \geq kd + k$, and using that $d = p(q-1)$ we get $d(k+1) \geq k(d+1)$. Note that if $k = d$ we have equality.

4.3 The isoperimetric number

Given a graph G on n vertices, the *isoperimetric number* $i(G)$ is defined as

$$i(G) = \min_{U \subset V} \left\{ |\partial(U, \overline{U})|/|U| : 0 < |U| \leq n/2 \right\}.$$

For example, the isoperimetric numbers of the complete graph, the path and the cycle are, respectively, $i(K_n) = \lceil \frac{n}{2} \rceil$, $i(P_n) = 1/\lfloor \frac{n}{2} \rfloor$, and $i(C_n) = 2/\lfloor \frac{n}{2} \rfloor$. For general graphs, Mohar [6] proved the following spectral bounds.

$$\frac{\lambda_{n-1}}{2} \leq i(G) \leq \sqrt{\lambda_{n-1}(2d_1 - \lambda_{n-1})}. \tag{20}$$

In our context we have:

Proposition 4.

$$i(G) \leq \min_{\frac{n}{2} \leq m < n} \sum_{i=1}^{m} (\lambda_i - d_i). \tag{21}$$

Proof. Apply (6) taking into account that $i(G) \leq \frac{|\partial(\overline{U}, U)|}{|\overline{U}|}$ when $0 < |\overline{U}| \leq \frac{n}{2}$. ∎

Example 2. Consider the graph join G of the complete graph K_p with the empty graph $\overline{K_q}$, so $n = p + q$. The Laplacian spectrum and the degree sequence are

$$\{n^p, p^{q-1}, 0^1\} \text{ and } \{(n-1)^p, p^q\},$$

respectively. Equation (21) gives $i(G) \leq \min\{p, \lceil \frac{n}{2} \rceil\}$, which is better than Mohar's upper bound (20) for all $0 \leq q < n$.

References

[1] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer Verlag (Universitext), Heidelberg, 2012; available online at `http://homepages.cwi.nl/~aeb/math/ipm/`.

[2] M.A. Fiol, Eigenvalue interlacing and weight parameters of graphs, *Linear Algebra Appl.* **290** (1999), 275–301.

[3] R. Grone, Eigenvalues and the degree seqences of graphs, *Linear Multilinear Algebra* **39** (1995) 133–136.

[4] R. Grone and R. Merris, The Laplacian spectrum of a graph. II, *SIAM J. Discrete Math.* **7** (1994), no. 2, 221–229.

[5] W.H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* **226-228** (1995) 593–616.

[6] B. Mohar, Isoperimetric numbers of graphs, *J. Combin. Theory Ser. B* **47** (1989), no. 3, 274–291.

[7] I. Schur, Über eine Klasse von Mittelbildungen mit Anwendungen die Determinanten, *Theorie Sitzungsber.* Berlin. Math. Gessellschaft **22** (1923) 9–20.

An algorithm to compute the primitive elements of an embedding dimension three numerical semigroup

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Abstract. We give an algorithm to compute the set of primitive elements for an embedding dimension three numerical semigroups. We show how we use this procedure in the study of the construction of L-shapes and the tame degree of the semigroup.

Key words: Numerical semigroup, L-shapes, primitive elements, tame degree.

1 Introduction

Let S be a numerical semigroup, that is, a submonoid of $(\mathbb{N},+)$ with finite complement in $\mathbb{N}$ (here $\mathbb{N}$ denotes the set of nonnegative integers). It is well known that S is finitely generated as monoid: exists $A \subset S$ finite such that $S = \langle A\rangle = \{\sum_{a\in A}\lambda_a a \mid \lambda_a \in \mathbb{N}\}$. Moreover, there is a unique minimal (with respect to set inclusion) A satisfying this condition, which is known as minimal generating system. Suppose that $A = \{a_1,\ldots,a_n\}$. Then the monoid morphism

$$\varphi : \mathbb{N}^n \to S,\ (x_1,\ldots,x_n) \mapsto \sum_{i=1}^{n} x_i a_i$$

is surjective, and so S is isomorphic as monoid to $\mathbb{N}^n/\ker(\varphi)$, where

$$\ker(\varphi) = \{(x,y) \in \mathbb{N}^n \times \mathbb{N}^n \mid \varphi(x) = \varphi(y)\}.$$

The congruence $\ker(\varphi)$ is a monoid too, and it is minimally generated by the set of minimal (with respect to the order given by the Cartesian product) nonzero elements of $\ker(\varphi)\setminus\{(0,0)\}$, which we denote by $\mathcal{I}(\ker(\varphi))$, (see [10, Chapter 8]). Let e_i be the ith row of the $n\times n$ identity matrix. Then (e_i,e_i) is always one of these minimal elements.

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Define

$$\mathrm{Prim}(S) = \{\varphi(a) \mid (a,b) \in \mathcal{I}(\ker(\varphi)) \text{ for some } b \neq a\},$$

which we call *primitive elements of* S.

In [4] an algorithm to compute $\mathcal{I}(\ker(\varphi))$ is given, and so $\mathrm{Prim}(S)$. In this work, we particularize this method to the very particular case when S is a numerical semigroup with embedding dimension three ($n = 3$). Prior to this, we present two examples where an improvement of this procedure is welcomed.

1.1 L-forms

Let S be a numerical semigroup minimally generated by $\{a_1, \ldots, a_n\}$. Given $k, m \in \mathbb{N}$, $m \neq 0$, we denote by $[k]_m$ the congruence class of k modulo m, that is, the set of integers l such that m divides $k - l$. The *Apéry set* of m in S is the set $\mathrm{Ap}(S,m) = \{s \in S \mid s - m \notin S\}$. It is well known that if $m \in S \setminus \{0\}$, then $\mathrm{Ap}(S,m) = \{w_0, \ldots, w_{m-1}\}$ with $w_i = \min([i]_m \cap S)$.

Let us denote the unitary $(n-1)$-cube $[i_1, i_1+1] \times \cdots \times [i_{n-1}, i_{n-1}+1]$ in $\mathbb{R}^{n-1}$ as $[\![i_1, \ldots, i_{n-1}]\!]$. Assign two labels to each unitary cube $[\![i_1, \ldots, i_{n-1}]\!] \in \mathbb{N}^{n-1}$, one label is the equivalence class $[i_1a_1 + \cdots + i_{n-1}a_{n-1}]_{a_n}$ and the other label is the *weight* $\mathrm{w}(i_1, \ldots, i_{n-1}) = i_1a_1 + \cdots + i_{n-1}a_{n-1}$. Equivalence classes modulo a_n and weights will be used as labels of unitary cubes in $\mathbb{N}^{n-1}$. We say that an equivalence class $[m]_{a_n}$ is *represented* by the corresponding related cubes $[\![x_1, \ldots, x_{n-1}]\!]$ with $a_1x_1 + \cdots + a_{n-1}x_{n-1} \equiv m (\mathrm{mod}\ a_n)$. We denote it by $[\![x_1, \ldots, x_{n-1}]\!] \sim [m]_{a_n}$.

Let us define the *discrete cone* of unitary cubes

$$\Delta(i_1, \ldots, i_{n-1}) = \{[\![j_1, \ldots, j_{n-1}]\!] : \ 0 \leq j_1 \leq i_1, \ldots, 0 \leq j_{n-1} \leq i_{n-1}\}$$

for $(i_1, \ldots, i_{n-1}) \in \mathbb{N}^{n-1}$, and the set of unitary cubes labeled with $[m]_{a_n}$

$$Q_m = \{[\![i_1, \ldots, i_{n-1}]\!] : \ [\mathrm{w}(i_1, \ldots, i_{n-1})]_{a_n} = [m]_{a_n}, (i_1, \ldots, i_{n-1}) \in \mathbb{N}^{n-1}\}.$$

The minimum weight of unitary cubes representing $[m]_{a_n}$ will be denoted by

$$M_m = \min\{\mathrm{w}(i_1, \ldots, i_{n-1}) : \ [\![i_1, \ldots, i_{n-1}]\!] \in Q_m\}.$$

Definition 1 (Minimum Distance Diagram). *A* Minimum Distance Diagram*, $\mathcal{H}$, related to the numerical semigroup $S = \langle a_1, \ldots, a_n \rangle$ is a set of a_n unitary cubes in $\mathbb{N}^{n-1}$ with the following three properties*

(a) If $X = [\![x_1, \ldots, x_{n-1}]\!], Y = [\![y_1, \ldots, y_{n-1}]\!] \in \mathcal{H}$ represent the same equivalence class, then $X = Y$.
(b) If $[\![x_1, \ldots, x_{n-1}]\!] \in \mathcal{H}$, then $\mathrm{w}(x_1, \ldots, x_{n-1}) = M_{\mathrm{w}(x_1, \ldots, x_{n-1})}$.
(c) If $[\![x_1, \ldots, x_{n-1}]\!] \in \mathcal{H}$, then $\Delta(x_1, \ldots, x_{n-1}) \subseteq \mathcal{H}$.

In fact, the rôle of a_n in the definition can be taken by any member of the generators set. Here we choose a_n for historical reasons.

Minimum distance diagrams were used first in the computation of distances in weighted and non-weighted Cayley digraphs ([3,9]). These diagrams teselate the $\mathbb{R}^{n-1}$ space (so they are also named *tiles*). When $n = 3$ they are well known as *L-shaped tiles* and many works appeared in the bibliography ([1,2]) on Cayley digraphs and numerical semigroups. Less is known of generic tiles when $n \geq 4$, with the exception of some particular cases ([7,9]).

An important property of minimum distance diagrams is given by the following identity

$$\mathrm{Ap}(S, a_n) = \{\mathrm{w}(x_1, \ldots, x_{n-1}) : \ [\![x_1, \ldots, x_{n-1}]\!] \in \mathcal{H}\}$$

where $\mathcal{H}$ is any minimum distance diagram related to the semigroup S. When $n = 3$, several applications have been given using this fact, for instance efficient algorithms for the computation of the *Frobenius number* and the *genus* of numerical semigroups.

Given two unitary cubes $X, Y \in \mathcal{H}$, we say that X *dominates* Y if $Y \in \Delta(X)$. A unitary cube $X \in \mathcal{H}$ is called *vertex* of $\mathcal{H}$ if no cube $Y \in \mathcal{H}$, $Y \neq X$, dominates X. The following two examples show the vertices of minimum distance diagrams for $n = 3$ and $n = 4$.

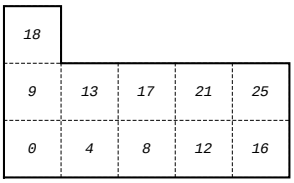

Fig. 1: Minimum distance diagram related to $\langle 4, 9, 11\rangle$

Example 1. Consider $S = \langle 4, 9, 11\rangle$, then Figure 1 shows a minimum distance diagram related to S. The labels of the cubes are their weights. The vertices are $[\![0, 2]\!] \sim [7]_{11}$ with $\mathrm{w}(0, 2) = 18$ and $[\![4, 1]\!] \sim [3]_{11}$ with $\mathrm{w}(4, 1) = 25$. Note that $\mathrm{Ap}(S, 11) = \{0, 4, 8, 9, 12, 13, 16, 17, 18, 21, 25\}$.

Now let $S = \langle 5, 7, 11, 13\rangle$. Then Figure 2 shows a minimum distance diagram $\mathcal{H}$ related to S, labeled with weights also. Now, the vertices are $[\![2, 0, 1]\!] \sim [8]_{13}$ with $\mathrm{w}(2, 0, 1) = 21$, $[\![1, 2, 0]\!] \sim [6]_{13}$ with $\mathrm{w}(1, 2, 0) = 19$ and $[\![3, 1, 0]\!] \sim [9]_{13}$ with $\mathrm{w}(3, 1, 0) = 21$. The labels of the cubes also form the Apéry set $\mathrm{Ap}(S, 13)$.

Given $s \in S$, define $\mathsf{Z}(s) = \varphi^{-1}(s)$ as the *set of factorizations* of s in S (actually in terms of $\{a_1, \ldots, a_n\}$). We study the number of different L-forms of S in terms of the factorizations of some elements in the Apéry set of a_n.

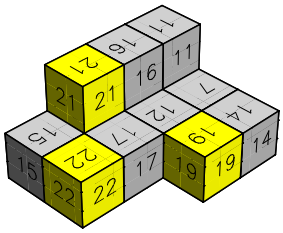

Fig. 2: Minimum distance diagram related to $\langle 5, 7, 11, 13\rangle$

For $n = 3$ there are at most two L-forms, but this is not the case for $n > 3$. For instance, for $\langle 5, 7, 11, 13\rangle$, we have

```
gap> LFormsOfNumericalSemigroup(NumericalSemigroup([5,7,11,13]));
[[[0,0,0],[1,0,0],[0,1,0],[2,0,0],[0,0,1],[1,1,0],[0,2,0],[3,0,0],[1,0,1],[2,1,0],[1,2,0],[0,3,0],[0,0,2]],
 [[0,0,0],[1,0,0],[0,1,0],[2,0,0],[0,0,1],[1,1,0],[0,2,0],[3,0,0],[1,0,1],[2,1,0],[1,2,0],[0,3,0],[3,1,0]],
 [[0,0,0],[1,0,0],[0,1,0],[2,0,0],[0,0,1],[1,1,0],[0,2,0],[3,0,0],[1,0,1],[2,1,0],[1,2,0],[2,0,1],[0,0,2]],
 [[0,0,0],[1,0,0],[0,1,0],[2,0,0],[0,0,1],[1,1,0],[0,2,0],[3,0,0],[1,0,1],[2,1,0],[1,2,0],[2,0,1],[3,1,0]]]
```

The implementation has been performed using the `numericalsgps` ([5]) `GAP` package.

In the above representation, each L-form is given by a list containing a factorization for each of the elements in $\mathrm{Ap}(S, 13)$ with respect to the semigroup $\langle 5, 7, 11\rangle$. Thus in order to compute all possible L-forms, we first compute $\mathrm{Ap}(S, 13)$, which has 13 elements, and then we compute for each of these elements its set of factorizations in $\langle 5, 7, 11\rangle$; finally we have to find a factorization for each element and arrange the chosen factorizations coherently with the definition of L-form. In this example we see that the eleven first elements are always the same, while those at the end change: $[0, 3, 0]$ can be replaced with $[2, 0, 1]$, whilst $[0, 0, 2]$ with $[3, 1, 0]$. This is due to the fact that $21 = 3 \cdot 7 = 2 \cdot 2 + 1 \cdot 11$ and $22 = 3 \cdot 5 + 1 \cdot 7 = 2 \cdot 11$. Notice that these two elements, 21 and 22, are primitive elements that belong to the Apéry set. The L-shape in Figure 2, corresponds to the last element in the list.

The first problem we encounter is when we try to obtain the Apéry set when a_4 is "big". We have some evidences that the L-shapes are determined by the set $\mathrm{Prim}(\langle a_1, a_2, a_3\rangle) \cap \mathrm{Ap}(S, a_4)$ (and for this we do not need to determine the Apéry set, since we only have to filter those primitive elements s with the condition $s - a_4 \notin S$). With this approach our execution times passed from hours to a bunch of seconds.

1.2 The tame degree

Let $x = (x_1, \ldots, x_n) \in \mathsf{Z}(s)$ be a factorization of $s \in S = \langle a_1, \ldots, a_n\rangle$. We define its length as $|x| = \sum_{i=1}^{n} x_i$. If $y = (y_1, \ldots, y_n)$ is another factorization, we denote by $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$. The distance between x and y is defined as

$$\mathrm{d}(x,y) = \max\{|x - (x \wedge y)|, |y - (x \wedge y)|\},$$

that is, the maximum of the lengths of the factorizations once we remove the common part of x and y.

An N-chain, with $N \in \mathbb{N}$, of factorizations joining x with y is a sequence $x_1, \ldots, x_k \in \mathsf{Z}(s)$ such that: $x_1 = x$, $x_k = y$, and for all i, $\mathrm{d}(x_i, x_{i+1}) \leq N$.

The *catenary degree* of s, $\mathsf{c}(s)$, is the minimum of the $N \in \mathbb{N}$, such that for any two factorizations of S there exists an N-chain joining them. The catenary degree of S is the supremum of the catenary degrees of all the elements of S, and we denote it by $\mathsf{c}(S)$. This supremum is indeed a maximum, that is, it is reached in a particular element of S.

The *tame degree* of s is the minimum $N \in \mathbb{N}$ such that whenever $x - a_i \in S$ for some $i \in \{1, \ldots, n\}$, for any $x \in \mathsf{Z}(s)$, there exists $y = (y_1, \ldots, y_n) \in \mathsf{Z}(s)$ such that $\mathrm{d}(x, y) \leq N$. The tame degree of s is denoted by $\mathsf{t}(s)$, and the tame degree of S, as occurs with the catenary degree, it is defined as the supremum (once more a maximum in our setting) of the tame degrees of the elements of S, and we denote it by $\mathsf{t}(S)$.

In [1], formulas for $\mathsf{c}(S)$, for the case $n = 3$ (embedding dimension three) are presented in terms of the parameters of any of the L-shapes associated to S. For arbitrary embedding dimension we know ([3]) that the catenary degree of S is attained in the elements that are used to construct a minimal presentation of S (which is a minimal generating set of $\ker(\varphi)$ as a congruence). In contrast to this, the tame degree is attained in the primitive elements of S, which as we have seen above are those that appear related to the minimal generating set of $\ker(\varphi)$ as a monoid.

A minimal presentation of S is *generic* if in all its elements every minimal generator of S appears. In this case, $\mathsf{c}(S) = \mathsf{t}(S)$ (see [2]). For $n = 3$, S has a generic presentation if and only if it is not symmetric (where symmetric means that $x \in \mathbb{Z} \setminus S$ implies $\mathrm{F}(S) - x \in S$, with $\mathrm{F}(S)$ the Frobenius number of S; see for instance [4, Chapter 9]). In general, $\mathsf{c}(S) \leq \mathsf{t}(S)$, and thanks to the study of the primitive elements in embedding dimension three, we have been able, with C. Viola, to characterize those embedding dimension three numerical semigroups for which the inequality is strict ([12]).

2 Computing primitive elements for embedding dimension three numerical semigroups

As we have seen above, primitive elements are of great help for the computation of L-shapes and the tame degree. In this section we sketch how we have particularized the general procedure explained in [4] to the particular case of embedding dimension three. So assume in what follows that S is minimally generated by $\{n_1, n_2, n_3\}$. Associated to the kernel congruence of φ (defined in the introduction), we define the subgroup of $\mathbb{Z}^3$

$$M = \{(x_1 - y_1, x_2 - y_2, x_3 - y_3) \colon ((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \ker(\varphi)\}$$

(see [10] for more details on this subgroup).

The method proposed in [4], starting from a basis of M, constructs all possible combinations by changing the sign of its elements, and then "saturates" these sets (adds two elements until the new ones can be reduced by the ones already obtained). Let us precise the concept of saturation a bit more.

Definition 2. *Let $(x_1, \dots x_n)$ and $(y_1, \dots y_n)$ be in $\mathbb{Z}^n$. We write $(x_1, \dots x_n) \sqsubseteq (y_1, \dots y_n)$ if $x_i y_i \geq 0$ and $|x_i| \leq |y_i|$ for all $i \in \{1, \dots, n\}$.*

A set A of $\mathbb{Z}^n$ is saturated *if for any $a_i, a_j \in A$ there exists $a_h \in A$ such that $a_h \sqsubseteq a_i + a_j$.*

We study the saturation process of $B_s = \{(0, n_3/g, -n_2/g), (g, -y_2, y_3)\}$, which is a base of M, where $g = \gcd(n_2, n_3)$.

Definition 3. *Let $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$. The signature of x, $\operatorname{sg}(x)$, is the tern formed by the signs of x_1, x_2 and x_3. Zero is assumed to have both positive and negative sign.*

Remark 1. If the signatures of a and b coincide, then the sum is comparable with both elements, that is, $\operatorname{sg}(a) = \operatorname{sg}(b)$ implies $a \sqsubseteq a + b$ and $b \sqsubseteq a + b$. Hence, in order to "saturate" we must consider taking elements with different signature.

Remark 2. In order to check if an element is incomparable with the rest, we simply have to compare it with those having its same signature.

Thus, two elements are incomparable with respect to $\sqsubseteq$ if either they have different signatures, or they have the same signature and some of their coordinates increase while other decrease (in absolute value).

We start with B_s, whose elements are

$$x_0 = (0, n_3/g, -n_2/g) = (0, +, -), \qquad x_1 = (g, -y_2, y_3) = (g, -, +).$$

Since they have different signature, by Remark 1 their addition is a possible new candidate for the saturation. We obtain

$$a_1 = x_0 + x_1 = (g, n_3/g - y_2, y_3 - n_2/g) = (g, +, -).$$

This new element has the same signature as x_0 but the first coordinate is larger, while the other two are smaller in absolute value. Hence by Remark 2, we obtain a new incomparable element that we add to the saturation of B_s.

We denote by A_i the ith step in the saturation of A. We write $A = B_s$, and thus $A_1 = \{x_0, x_1, a_1\}$.

We are going to saturate now A_1. Since we only have added a_1 in the preceding saturation, we have to add this new element with the older ones.

As a_1 has the same signature as x_0, we do not take into account $a_1 + x_0$. Thus we only add with x_1 obtaining

$$a_2 = a_1 + x_1 = x_0 + 2x_1 = (2g, n_3/g - 2y_2, 2y_3 - n_2/g) = (2g, ?, ?).$$

Lemma 1. *No mater what the signature of a_2 is, a_2 is incomparable with x_0, x_1 and a_1.*

Thus again, we have only added a new element in the saturation procedure. We see next that this is the case until an element with signature $(+, -, -)$ shows up.

Lemma 2. *In the saturation process, while there is no element with signature $(+, -, -)$, only a new element will be added. If $a_i = a_{i-1} + a_j$, then $a_{i+1} = a_i + a_j$ if $sg(a_i) = sg(a_{i-1})$; or $a_{i+1} = a_i + a_{i-1}$ if $sg(a_i) = sg(a_{i-1})$. Moreover if we compare a_{i+1} with the closest in the list with the same signature, then we observe that the first coordinate is larger, while the other two are smaller in absolute value.*

When the first element with signature $(+, -, -)$ appears, two new elements are added in the next saturation step.

Lemma 3. *If a_i is the first element with signature $(+, -, -)$, then $b_1 = a_i + a_j$ y $c_1 = a_i + a_{i-1}$ are incomparable with the preceding a's.*

Next we apply Lemma 2 focusing only in a given coordinate of b_1 and another from c_1. Then after the ith step we will start adding elements b_2 and c_2 raising from b_1 and c_1, respectively, where $b_2 = b_1 + a_j$, or $b_2 = b_1 + a_i$; while $c_2 = c_1 + a_{i-1}$, or $c_2 = c_1 + a_i$.

The following lemma asserts that $b_x + c_y$ is comparable, and thus we do not add it in any saturation.

Lemma 4. *Let b_x be an incomparable element obtained from b_1 (that is, adding to b_1 either a_j or some previous b_i), and let c_y be another incomparable element obtained from c_1 (that is, adding a_{i-1} or some previous c_j). Then $a_i \sqsubseteq b_x + c_y$.*

Theorem 1. *The primitive elements of $S = \langle n_1, n_2, n_3 \rangle$ can be computed as follows. Start with $B_s = \{(0, n_3/g, -n_2/g), (g, -y_2, y_3)\}$ and applying twice the division algorithm first with n_3/g and y_2, and then with n_2/g and y_3, we obtain two list of elements in which the second coordinate will decreasing until we reach $(n_3/g_2, 0, -n_1/g_2)$ where $g_2 = gcd(n_1, n_3)$, and in the second list the third coordinate will be decreasing until we obtain $(n_2/g_3, -n_1/g_3, 0)$ where $g_3 = gcd(n_1, n_2)$. Both list will have the same elements until a_i is attained, and then we will get the series of b's and c's.*

Observe that according to the procedure presented in [4], we must saturate B_s and $B_{-s} = \{(0, n_3/g, -n_2/g), (-g, y_2, -y_3)\}$, but this last set is already saturated.

Example 2. Take $S = \langle 17, 31, 41\rangle$.

$$x_0 = (0, 41, -31), x_1 = (1, -27, 20),$$
$$a_1 = (1, 14, -11), a_2 = (2, -13, 9), a_3 = (3, 1, -2) = 82, a_4 = (5, -12, 7)$$
$$a_5 = (8, -11, 5), a_6 = (11, -10, 3), a_7 = (14, -9, 1), a_8 = (17, -8, -1),$$
$$c_1 = (31, -17, 0),$$
$$b_1 = (20, -7, -3), b_2 = (23, -6, -5), b_3 = (26, -5, -7), b_4 = (29, -4, -9)$$
$$b_5 = (32, -3, -11), b_6 = (35, -2, -13), b_7 = (38, -1, -15), b_8 = (41, 0, -17).$$

One of the advantages of this procedure is that from the output one can easily compute a minimal presentation for S.

References

[1] F. Aguiló-Gost and P.A. García-Sánchez. Factoring in embedding dimension three numerical semigroups, *Electronic. Journ. of Combinat.*, 17, R138: 21 pp. 2010.

[2] V. Blanco, P.A. García-Sánchez, A. Geroldinger, Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids, *Illinois J. Math.* 55:1385–1414, 2011.

[3] S. T. Chapman, P.A. García-Sánchez, D. Llena, V. Ponomarenko, and J. C. Rosales, The catenary and tame degree in finitely generated commutative cancellative monoids, *Manuscripta Math.* 120:253–264, 2006.

[4] S.T. Chapman, P.A. García-Sánchez, D. Llena, J.C. Rosales. Presentations of finitely cancellative commutative monoids and nonnegative solutions of systems of linear equations, *Discrete Applied Mathematics*, 154:1947–1959, 2006.

[5] M. Delgado, P. A. García-Sánchez and J. Morais, "NumericalSgps", *A GAP package for numerical semigroups*. Available via `http://www.gap-system.org/`.

[6] M.A. Fiol, J.L.A. Yebra, I. Alegre and M.Valero. A discrete optimization problem in local networks and data alignment, *IEEE Trans. of Comp.*, C-36:702–713, 1987.

[7] H.G. Killingbergtrø. Betjening av figur i Frobenius' problem, *Normat* 2: 75–82, 2000.

[8] O.J. Rodseth, Weighted multi-connected loop networks, *Discrete Mathematics* 148:161–173, 1996.

[9] P. Sabariego, P. and F. Santos. Triple-loop networks with arbitrarily many minimum distance diagrams, *Discrete Mathematics* 309(6):1672–1684, 2009.

[10] J. C. Rosales and P. A. García-Sánchez. Finitely Generated Commutative Monoids, *Nova Science Publishers,* New York, 1999.

[11] J. C. Rosales and P. A. García-Sánchez. Numerical Semigroups, *Developments in Mathematics* 20. Springer, New York, 2009.

[12] C. Viola, Catenary and tame degre for embedding dimension three numerical semigroups, Master Thesis, Università di Catania, in preparation.

Denumerants of 3-numerical semigroups

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Abstract. Denumerants of numerical semigroups are known to be difficult to obtain, even with small embedding dimension of the semigroups. In this work we give some results on denumerants of 3-semigroups $S = \langle a, b, c\rangle$. The time efficiency of the resulting algorithms range from $O(1)$ to $O(c)$. Closed expressions are obtained under certain conditions.

Key words: Denumerant, L-shapes, numerical semigroup, factorization.

1 Introduction

Let $\mathbb{N}_0$ be the set of non negative integers. Given a set $A = \{a_1, \ldots, a_n\} \subset \mathbb{N}_0$, $\gcd(a_1, \ldots, a_n) = 1$, the *n-numerical semigroup* $S = S(A)$ generated by A is defined by $\langle a_1, \ldots, a_m\rangle = \{x_1a_1 + \cdots + x_na_n : \ x_1, \ldots, x_n \in \mathbb{N}_0\}$. If A is a minimal set of generators then S has *embedding dimension* $\mathrm{e}(S) = n$. An element $m \in S$ has a *factorization* $(t_1, \ldots, t_n)$ in S if $m = t_1a_1 + \cdots + t_na_n$. The set of factorizations of m in S is denoted by $\mathcal{F}(m, S) = \{(x_1, \ldots, x_n) \in \mathbb{N}_0^n : \ x_1a_1 + \cdots + x_na_n = m\}$. The *denumerant* of m in S is the cardinality $\mathrm{d}(m, S) = |\mathcal{F}(m, S)|$.

In this work we give some results on denumerants of generic elements of embedding dimension three numerical semigroups $S = \langle a, b, c\rangle$. Algorithms of time–cost ranging from $O(1)$ to $O(c)$ (in the worst case) are also derived. We use *minimum distance diagrams* related to S as a main tool. In particular, we use the main results given in [1].

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2 Tools

From now on we denote the equivalence class of n modulo N as $[n]_N$. Given $m \in S \setminus \{0\}$, the *Apéry set* of m in S is $\mathrm{Ap}(m,S) = \{s \in S : \ s-m \notin S\}$. This set encodes many properties of semigroup. It can be shown that $|\mathrm{Ap}(m,S)| = m$ and $\mathrm{Ap}(m,S) = \{l_0, \ldots, l_{m-1}\}$ with $l_k \in [k]_m$ for $0 \leq k < m$.

Minimum Distance Diagrams, $\mathcal{H}$, of numerical semigroups are used to study many distance–related and distribution of the semigroup elements. The minimum distance diagrams related to $S = \langle a_1, \ldots, a_n \rangle$ are sets of cardinality $|\mathcal{H}| = a_n$ of unitary cubes $[\![i_1, \ldots, i_{n-1}]\!] = [i_1, i_1+1] \times \cdots \times [i_{n-1}, i_{n-1}+1]$ in the first orthant of $\mathbb{R}^{n-1}$ with coordinates $(i_1, \ldots, i_{n-1}) \in \mathbb{N}_0^{n-1}$ representing factorizations of the elements of $\mathrm{Ap}(a_n, S)$, where $[s,t] = \{r \in \mathbb{R} : \ s \leq r \leq t\}$. Each element $l_k \in \mathrm{Ap}(a_n, S)$ is represented by exactly one cube $[\![i_1, \ldots, i_{n-1}]\!]$ in $\mathcal{H}$ with $i_1a_1 + \cdots + i_{n-1}a_{n-1} = l_k$. Given any $m \in S$, with $m \in [l_k]_{a_n}$, we call the *basic coordinates* of m with respect to $\mathcal{H}$ to $(x_1, \ldots, x_{n-1}) \in \mathbb{N}_0^{n-1}$ if $[\![x_1, \ldots, x_{n-1}]\!] \in \mathcal{H}$ and $x_1a_1 + \cdots + x_{n-1}a_{n-1} = l_k$. We also call the *basic factorization* of m in S with respecto to $\mathcal{H}$ to $(x_1, \ldots, x_{n-1}, \frac{m-l_k}{a_n}) \in \mathcal{F}(m,S)$.

These diagrams were used in [1] to study some aspects on factorization and catenary degree of 3-numerical semigroups $S = \langle a, b, c \rangle$. In particular, we recall here some nomenclature and results. Diagrams related to S are L-shaped or rectangles (considered as degenerated L-shapes with $wy = 0$) and are denoted by $\mathcal{H} = \mathsf{L}(l, h, w, y)$, where these entries are the lengths of the sides $0 \leq w < l$, $0 \leq y < h$, $\gcd(l, h, w, y) = 1$ and $lh - wy = c$. The diagram $\mathcal{H}$ tessellates $\mathbb{R}^2$ by translation through the vectors $\boldsymbol{u} = (l, -y)$ and $\boldsymbol{v} = (-w, h)$. See [2,3] for more details.

For the parameters $\delta = (la - yb)/c$ and $\theta = (hb - wa)/c$, we have $\delta \geq 0$, $\theta \geq 0$, $\delta + \theta > 0$, $\delta + y > 0$ and $\theta + w > 0$ whenever $\mathcal{H} = \mathsf{L}(l, h, w, y)$ is a minimum distance diagram of $S = \langle a, b, c \rangle$.

Theorem 1 ([1, Th. 2]). *Given $\mathcal{H} = \mathsf{L}(l, h, w, y)$ a minimum distance diagram of $S = \langle a, b, c \rangle$ and $m \in S$, let (x_0, y_0, z_0) be the basic factorization of m in S respect to $\mathcal{H}$, then*

$$\mathrm{d}(m,S) = 1 + \left\lfloor \frac{z_0}{\delta + \theta} \right\rfloor + \sum_{k=0}^{\left\lfloor \frac{z_0}{\delta+\theta} \right\rfloor} (S_k + T_k)\,,$$

with

$$S_k = \begin{cases} \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor & \text{if } \delta = 0, \\ \left\lfloor \frac{z_0 - k(\delta+\theta)}{\delta} \right\rfloor & \text{if } y = 0, \\ \min\{ \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor, \left\lfloor \frac{z_0 - k(\delta+\theta)}{\delta} \right\rfloor \} & \text{if } \delta y \neq 0, \end{cases}$$

and

$$T_k = \begin{cases} \left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor & \text{if } \theta = 0, \\ \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor & \text{if } w = 0, \\ \min\{ \left\lfloor \frac{x_0+k(l-w)}{w} \right\rfloor, \left\lfloor \frac{z_0-k(\delta+\theta)}{\theta} \right\rfloor \} & \text{if } \theta w \neq 0. \end{cases}$$

This theorem does not give an efficient algorithm for computing denumerants. The reason is not the expression of minima appearing in S_k when $\delta y \neq 0$ or in T_k when $\theta w \neq 0$, these minima can be left out for $k \geq \left\lceil \frac{yz_0-\delta y_0}{\delta(h-y)+y(\delta+\theta)} \right\rceil$ in S_k when $\delta y \neq 0$ and for $k \geq \left\lceil \frac{wz_0-\theta x_0}{\theta(l-w)+w(\delta+\theta)} \right\rceil$ in T_k when $\theta w \neq 0$. The reason is the expression of the sum. Essentially we have to compute sums like

$$\sum_{k=0}^{N} \left\lfloor \frac{\alpha \pm k\beta}{q} \right\rfloor = \sum_{k=0}^{N} (\overline{\alpha} \pm k\overline{\beta}) + \sum_{k=0}^{N} \left\lfloor \frac{s \pm kt}{q} \right\rfloor,$$

for $\alpha, \beta, \overline{\alpha}, \overline{\beta}, q \in \mathbb{N}$, with $q \neq 0$, $\alpha = \overline{\alpha}q + s$, $0 \leq s < q$, and $\beta = \overline{\beta}q + t$, $0 \leq t < q$.

3 Trying $\sum_{k=0}^{N} \left\lfloor \frac{s \pm kt}{q} \right\rfloor$

The efficiency of computing denumerants using these sums is compromised because of the number of terms to be added N, $O(N) = O(z_0)$. The coordinate z_0 of a basic factorization can become very large. Thus, we need to save add up all those terms.

In this section we try the sum $\sum_{k=0}^{N} \left\lfloor \frac{s+kt}{q} \right\rfloor$ with $0 \leq s, t < q$. The sum with the minus sign can be solved using an analogous method by symmetry. The idea of the method we propose here is making this addition as a Lebesgue–like discrete integration. Let us consider the graph of the function $f(x) = \left\lfloor \frac{s+tx}{q} \right\rfloor$ and pay attention to the constant parts of the graph. Let us denote the interval $I_m = [x_m, x_{m+1})$ where de function f has constant value m, that is $x_m = \frac{mq-s}{t}$ (if $t = 0$ the entire sum equals zero). Then, some main properties hold:

(1) There are $M + 1$ different intervals, with $M = \left\lfloor \frac{s+Nt}{q} \right\rfloor$. Each interval I_m has length $x_{n+1} - x_m = \frac{q}{t}$ except, possibly, I_0 and/or I_M.
(2) $\left\lfloor \frac{q}{t} \right\rfloor \leq |I_m \cap \mathbb{N}| \leq \left\lceil \frac{q}{t} \right\rceil$ except, possibly, I_0 and/or I_M.
(3) Those intervals I_m with $x_m \in \mathbb{N}_0$ are iS–type intervals. Intervals with $|I_m \cap \mathbb{N}| = \left\lceil \frac{q}{t} \right\rceil$ are mS–type intervals. Intervals iS are also mS except, possibly, I_M. There can be mS–intervals that are not iS–intervals.
(4) The behaviour of intervals at the same relative position between two consecutive iS–intervals is the same. There are $t - 1$ intervals between two

consecutive iS–intervals. The behaviour of intervals I_m is periodic of period t. Depending on the values s,t and q, this period is not the minimum one.

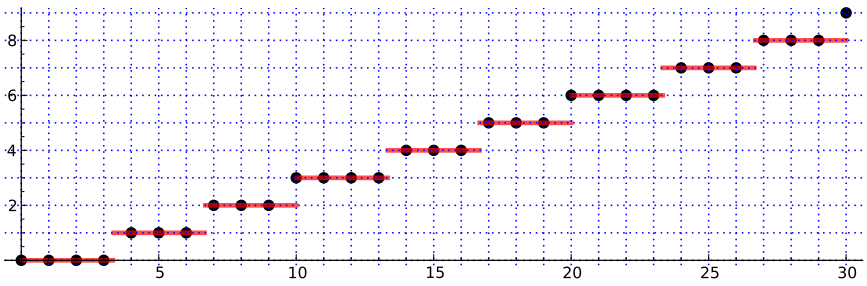

Fig. 1: Graph of f with $s = 0$, $t = 3$, $q = 10$

Figure 1 shows the graph of f with parameters $s = 0$, $t = 3$ and $q = 10$. Note the 3–periodic behaviour of the intervals I_m, $m \in \{0, \ldots, 9\}$. In this case, only the iS–intervals contain $\lceil \frac{q}{t} \rceil = 4$ integral values, except I_M (with $M = 9$) for $N = 30$ that is only one integral point $I_9 = \{9\}$.

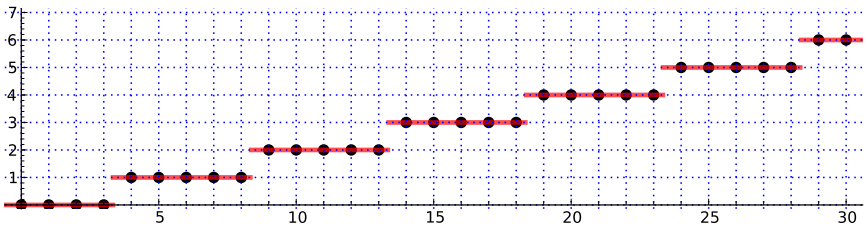

Fig. 2: Graph of f with $s = 5$, $t = 3$, $q = 15$

We study two main cases, $\gcd(t, q) > 1$ and $\gcd(t, q) = 1$. Figure 1 is the simple behaviour of the second case. The first case can be studied for $t \mid q$ and $t \nmid q$. When $t \mid q$, all intervals have the same behavior except, possibly, the first and the last ones; this is the case of Figure 2. Note that all intervals are of type mS and there is no iS–intervals, essentially the existence of iS–intervals depends on the value s. When $t \nmid q$ and $\gcd(t, q) = g > 1$, depending on the value of s appear iS–intervals and the period is not t but t/g; Figure 3 shows this case.

Depending on the behavior of intervals I_m the value of the sum can be arranged in an almost closed expression that sometimes results in a closed expression. The following results take into account all cases (that are more than the examples appearing in the three figures mentioned above).

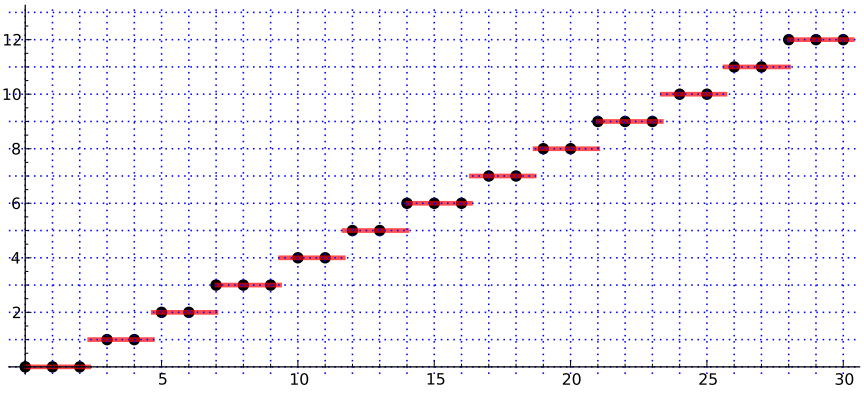

Fig. 3: Graph of f with $s = 0$, $t = 6$, $q = 14$

From now on set $M = \left\lfloor \frac{s+Nt}{q} \right\rfloor$ and $x_M = \frac{Mq-s}{t}$.

Theorem 2 (Case $t \mid q$). *Assume $q = \overline{q}t$. Then,*

$$\sum_{k=0}^{N} \left\lfloor \frac{s+kt}{q} \right\rfloor = \overline{q}\frac{M(M-1)}{2} + M(N - \lceil x_M \rceil + 1).$$

Theorem 2 is a simple case for applying the Lebesgue–like discrete summation. Thi theorem give closed expressions for denumerants.

Now we give the cases $t \nmid q$ with $\gcd(t,q) = 1$ or $\gcd(t,q) = g > 1$. To describe it we must define another type of intervals. Define $\hat{s}$ and $\hat{q}$ by

$$q = \overline{q}t + \hat{q},\ s = \overline{s}t + \hat{s}, \quad 0 \le \hat{s}, \hat{q} < t.$$

We say that I_m is an hS–interval if $(\hat{s} - m\hat{q}) (\mathrm{mod}\ t) < \hat{q}$. We denote the sets $I = \{i_1, \ldots, i_\alpha\}$, $J = \{j_1, \ldots, j_n\}$, $K = \{k_1, \ldots, k_v\}$ of ordered indices of all hS-type intervals taken from some region to be defined. We also denote the sums $S_I = \sum_{l=1}^{\alpha} i_l$, $S_J = \sum_{l=1}^{n} j_l$, $S_K = \sum_{l=1}^{v} k_l$, $S = \overline{q}\frac{(M-1)M}{2} + M(N - \lceil x_M \rceil + 1)$ and $\mathbb{S} = \sum_{k=0}^{N} \left\lfloor \frac{s+kt}{q} \right\rfloor$.

Theorem 3 (Case $t \nmid q$ and $\gcd(t,q) = 1$). *Set $m_0 \equiv q^{-1}s (\mathrm{mod}\ t)$ with $m_0 \in \{0, \ldots, t-1\}$ and $u = \left\lfloor \frac{M-m_0-1}{t} \right\rfloor$.*

(a) If $m_0 = 0$ and $u \le 0$, or $m_0 = M$, take $K \subset [1, M-1]$. Then $\mathbb{S} = S + S_K$.

(b) If $m_0 = 0$ and $u > 0$, take $J \subset [0, t-1]$ and $K \subset [ut, M-1]$. Then $\mathbb{S} = S + uS_J + nt\frac{(u-1)u}{2} + S_K$.

(c) If $0 < m_0 < M$ and $m_0 + t \ge M$, take $I \subset [1, m_0 - 1]$ and $K \subset [m_0, M-1]$. Then $\mathbb{S} = S + S_I + S_K$.

(d) If $0 < m_0 < M$ and $m_0 + t < M$, take the sets $I \subset [1, m_0 - 1]$ and $J \subset [m_0, m_0 + t - 1]$. Take also $K \subset [m_0 + ut, M-1]$ whenever $u > 0$ and $K = \emptyset$ otherwise (thus $S_K = 0$). Then $\mathbb{S} = S + S_I + uS_J + nt\frac{(u-1)u}{2} + S_K$.

Theorem 4 (Case $t \nmid q$ and $\gcd(t,q) = g > 1$). *Set $\tilde{t} = t/g$. Take $J \subset [0, \tilde{t}-1]$. Set $m_0 = j_1$ and $u = \left\lfloor \frac{M-m_0-1}{\tilde{t}} \right\rfloor$.*

(a) If $m_0 \geq M$, then $\mathbb{S} = S$.

(b) If $0 \leq m_0 < M$, take $K \subset [m_0 + u\tilde{t}, M-1]$. Then $\mathbb{S} = S + uS_J + n\tilde{t}\frac{(u-1)u}{2} + S_K$.

Theorems 3 and 4 do not give closed expressions for denumerants because of the computation of the sets of indices. This task has order $O(t)$ in the worst case. However, the resulting algorithm is efficient. Similar results can be obtained to compute the sum $\sum_{k=0}^{N} \lfloor \frac{s-kt}{q} \rfloor$.

4 Application example

Each 3-semigroup has its own denumerant's information encoded in the related L-shapes. As an example, let us consider for instance $S = \langle 121, 1111, 2323 \rangle$ with related L-shape $\mathsf{L}(101, 23, 0, 11)$. Thus, $\delta = 0$ and $\theta = 25553$.

This semigroup belongs to the general case $\delta = w = 0$, according with Theorem 1 (there are exactly five generic cases $\{\delta = 0, w = 0\}$, $\{\delta = 0, w \neq 0\}$, $\{\theta = 0, y = 0\}$, $\{\theta = 0, y \neq 0\}$ and $\{\delta \neq 0, \theta \neq 0\}$). Let us consider the case $\{\delta = 0, w = 0\}$ (then $y\theta \neq 0$). Assume $S = \langle a, b, c \rangle$ and $\mathcal{H} = \mathsf{L}(l, h, 0, y)$ belongs to this case. Then, the denumerant is given by

$$\mathrm{d}(m, S) = 1 + \left\lfloor \frac{z_0}{\theta} \right\rfloor + \sum_{k=0}^{\left\lfloor \frac{z_0}{\theta} \right\rfloor} \left(\left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor + \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor \right),$$

where (x_0, y_0, z_0) is the basic factorization of $m \in S$ with respect to $\mathcal{H}$. Set $y_0 = \overline{y_0}y + \hat{y_0}$ and $h - y = \overline{n}y + \hat{n}$ with $0 \leq \hat{y_0}, \hat{n} < y$. Then,

$$\sum_{k=0}^{\lambda_0} \left\lfloor \frac{y_0 + k(h-y)}{y} \right\rfloor = (1+\lambda_0)[\overline{y_0} + \frac{1}{2}\lambda_0 \hat{n}] + \sum_{k=0}^{\lambda_0} \left\lfloor \frac{\hat{y_0} + k\hat{n}}{y} \right\rfloor,$$

with $\lambda_0 = \left\lfloor \frac{z_0}{\theta} \right\rfloor$. Similarly,

$$\sum_{k=0}^{\lambda_0} \left\lfloor \frac{z_0 - k\theta}{\theta} \right\rfloor = \sum_{k=0}^{\lambda_0} (\lambda_0 - k) = \frac{1}{2}\lambda_0(1+\lambda_0).$$

Therefore, the denumerant is given by

$$\mathrm{d}(m, S) = (1+\lambda_0)[1 + \overline{y_0} + \frac{1}{2}\lambda_0(1+\hat{n})] + \sum_{k=0}^{\lambda_0} \left\lfloor \frac{\hat{y_0} + k\hat{n}}{y} \right\rfloor.$$

For $S = \langle 121, 1111, 2323 \rangle$, let us take the family of elements of $m_t \in S$ given by the basic factorization $(x_0, y_0, z_0) = (1, 1, t)$, i.e. $m_t = 1232 + 2323t$. Then,

in this case, the parameters are $\overline{y_0} = 0$, $\hat{y_0} = 1$, $\overline{n} = 1$, $\hat{n} = 1$, $y = 11$ and $\lambda_0 = \lfloor \frac{t}{11} \rfloor$. Thus, the denumerant is

$$\mathrm{d}(m_t, S) = (1 + \lfloor \frac{t}{11} \rfloor)^2 + \sum_{k=0}^{\lfloor \frac{t}{11} \rfloor} \left\lfloor \frac{1+k}{11} \right\rfloor .$$

Applying the results given in Section 2, we can take large values for t. For instance, when $t = 10^{10^3}$, the denumerant $\mathrm{d}(m_t, \langle 121, 1111, 2323 \rangle)$ is

86401202103681442373725632370418508815236113947008384865900
18355482581937070575687706067297170372656872124127106355013
83705399205261030784757611250382814450458614863741996576336
04381350933152481314521762235166717566155055157623293969446
08615786367662651758827936885013572929910812562936838277177
28615652451360231647622463525737529205087066732129366369694
10298518906363268082467489122400475776673691012914136765846
09475113507696144835778060008209708220861265294901965031701
37343295778759267316777895472559376756325707720000071855742
17322046447831982160259072389696123371066939277711855383683
57767721175049376552251604388714812532134415445726788119363
68575850635618156081169145034427047492856768128250632899016
17212129580044803105058702050223244750468912218659498851826
79421268043289617415276449211375778795179442471940300299127
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02863506089428812738102330203742063649097015260203032359621
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62618991722039200052628030197744818517237285777788689442446
19935553020251798305234674292358662883696542490554205560718
08819266153700438284637594415248278846875002464266544621032
88133854180289115673176779597911111148376552939482291855 72
44097229517541555204729647852029427426683172559095558399454
98778302979322010609690412121408608231615838397400125843956
68593577168522246218425317381671565212196629533706324699481
36819653825790303884550101940617170942701351142361553525485
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62687036855313366690934052679792085475244714662187442952179
12806317148265826264562297662155082803832709199101981830544
85083034700705484823478715229336358831442400219780

References

[1] F. Aguiló-Gost and P. A. García-Sánchez. Factoring in embedding dimension three numerical semigroups, *The Electronical Journal of Combinatorics*, 17, R138: 21 pp. 2010.

[2] M. A. Fiol, J. L. A. Yebra, I. Alegre and M.Valero. A discrete optimization problem in local networks and data alignment, *IEEE Transaction of Computations*, C-36:702–713, 1987.

[3] O. J. Rodseth, Weighted multi-connected loop networks, *Discrete Mathematics* 148:161–173, 1996.

[4] J. C. Rosales and P. A. García-Sánchez. Numerical Semigroups, *Developments in Mathematics* 20. Springer, New York, 2009.

On the Roman domination contraction number of trees with small diameter

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Abstract. The *weight of a vertex-labeling* of a graph is the sum of all the vertex-labels. The *Roman domination number* of a graph is the minimum value over all the weights of a vertex-labeling that satisfy the following two conditions: (i) the available labels are $\{0, 1, 2\}$; (ii) the set of vertices having a positive label is a dominating set of vertices in the graph. *Contracting an edge*, $e = vw$, of a graph G is an operation that results in a new graph G_e such that $V(G_e) = (V(G) - \{v, w\}) \cup \{z\}$, where z is a new vertex which is adjacent in G_e to every neighbor of $\{v, w\}$ in G. In this work we introduce the concept of *Roman domination contraction number* which is the minimum number of edges that is needed to be removed in a graph in order to decrease its Roman domination number. General results are proven and the exact values are obtained for trees of diameter 4 and 5.

Key words: Domination, Roman domination, edge contraction, Roman domination number.

1 Introduction

To begin with, we introduce the main terminology and notation which it is used throughout this paper. To different symbols, notation and terminology not explicitly given here we refer the reader to [4]. Hereafter $G = (V, E)$ represents a simple undirected finite graph having neither loops nor multiple edges. The set of vertices of the graph G is denoted by $V(G)$ or simply V, for the sake of simplicity. Analogously, the set of edges of the graphs is $E = E(G)$. The order of G is the number n, of vertices of the graph, that is to say, $n = |V|$. We denote two adjacent vertices $u, v \in V$ by $u \sim v$ and, in this case, we say that uv is an edge of G or, equivalently, we say that $uv \in E$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, let $N_X(v)$ denote the set of neighbors that v has in X, that is to say, $N_X(v) = \{u \in X : u \sim v\}$. In the case

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$X = V$ we use only $N(v)$, instead of $N_G(v)$, which is also called the open neighborhood of the vertex $v \in V$. The close neighborhood of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. Given a set of vertices $S \subseteq V$, the open neighborhood of S is the set $N(S) = \{w \in V(G) \setminus S : w \sim v, \text{ for some } v \in S\}$. The closed neighborhood is $N[S] = N(S) \cup S$. A universal vertex of G is a vertex which is adjacent to every other vertex of G.

Let u and v be two (not necessarily distinct) vertices of a graph G. A uv-path in G is a finite alternating sequence: $u_0 = u, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of different vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$ for all $i = 1, 2, \ldots, k$. The number of edges in a path is called the *length* of the path. The length of a shortest uv-path is the distance between the vertices u and v, and it is denoted by $d(u, v)$. A cycle is a uu-path. The length of a shortest cycle in the graph, if any, is called the girth of the graph. Otherwise, the girth of G is not finite.

The set of vertices $D \subset V$ is a *dominating set* if every vertex v not in D is adjacent to at least one vertex in D. The minimum cardinality of any dominating set of G is the *domination number* of G and it is denoted by $\gamma(G)$. A dominating set D in G with $|D| = \gamma(G)$ is called a $\gamma(G)$-set. Notice that a graph having a universal vertex has domination number equal to one.

The concepts about Roman domination were suggested by Stewart in [8]. Nevertheless the formal definition was presented first in [2]. Some authors have approach this problem, for example, see [3], [4], [5], [6] and [9]. A map $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* for a graph G if for every vertex v with $f(v) = 0$, there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The *weight* of a Roman dominating function is given by $w(f) = f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G and it is denoted by $\gamma_R(G)$. A Roman dominating function in G of minimum weight is called a $\gamma_R(G)$-function.

Let f be a Roman dominating function for G and let $V(G) = \{B_0, B_1, B_2\}$ be the sets of vertices of G induced by f, where $B_i = \{v \in V \ : \ f(v) = i\}$, for all $i \in \{0, 1, 2\}$. It is clear that for any Roman dominating function f for G of order n we have that $f(V) = \sum_{u \in V} f(u) = 2|B_2| + |B_1|$. Usually a Roman dominating function f is denoted by (B_0, B_1, B_2). In [2] was proved that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ and in this paper the authors called *Roman graphs* those graphs G satisfying that $\gamma_R(G) = 2\gamma(G)$. Note that if G is not connected graph with connected components $C_1, C_2, ..., C_t$, then $\gamma_R(G) = \sum_{i=1}^{t} \gamma_R(C_i)$. Therefore, from now on we will only consider connected graphs.

Given a Roman dominating function in G, $f = (B_0, B_1, B_2)$, we say that a vertex $z \in B_2$ has a linked neighbor if there exists a unique vertex $v \in B_0$ such that $N(v) \cap B_2 = \{z\}$. In this situation, we write $z \sim_l v$. Let us denote by $LN(B_2)$ the set of vertices in B_2 that have a linked neighbor in B_0.

Motivated by several published works related to subdivision and the recent paper [7], about domination contraction, we consider here the edge contraction related with Roman domination and we introduce a similar concept which we call Roman domination contraction number. Given a graph G we define the *Roman domination contraction number*, $ct_{\gamma_R}(G)$, as the minimum number of edges which must be contracted in order to decrease the Roman domination number of G.

Given a graph G and an edge $uv \in E(G)$ let us denote by $G_{uv}(z)$ the graph obtained by contracting the edge uv, that is to say, $V(G_{uv}(z)) = (V(G) \setminus \{u, v\}) \cup \{z\}$ such that $N_{G_{uv}(z)}(w) = N_G(w)$ for all $w \neq z$ and $N_{G_{uv}(z)}(z) = N_G(u) \cup N_G(v)$. Let f be a Roman dominating function in G. Associated to f we can define three different roman dominating functions $f_{uv,z}$ in $G_{uv}(z)$ by choosing a value for $f_{uv,z}(z)$ in $\{0, 1, 2\}$ and assuming that $f_{uv,z}(w) = f(w)$ for all $w \neq z$.

Notice that if $\gamma_R(G) = 2$, then G has a universal vertex v. Thus, the contraction of any number of edges of G results in a graph G' satisfying that either G' has order at least two and it has Roman domination number greater than or equal to two or G' has order one and in this case the Roman domination number is one. Therefore, hereafter we will only study those connected graphs having Roman domination number greater than two.

Let T be a tree with diameter 4 (respectively, diameter 5) and let $\Gamma_0(u) = \{u\}$ (respectively, $\Gamma_0(u, v) = \{u, v\}$) be the center of T. Let us denote by Γ_1 the set of neighbors of the center of T and by $\Gamma_2 = V(G) \setminus (\Gamma_0 \cup \Gamma_1)$. Moreover, we denote by $\Gamma_1 = N^0 \cup N^1 \cup N^2$ where $N^2 = \{w \in \Gamma_1 : d_G(w) \geq 3\}$ and $N^i = \{w \in \Gamma_1 : d_G(w) = i+1\}$ whenever $i \in \{0, 1\}$. For the sake of simplicity, we assume that $\Gamma_1(u, v) = \Gamma_1(u) \cup \Gamma_1(v)$, $N^i(u, v) = N^i(u) \cup N^i(v)$ (see Fig. 1).

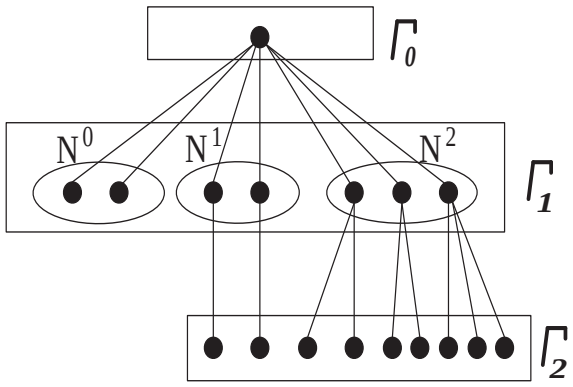

Fig. 1: A partition of the set of vertices in a diameter 4 tree.

Without lost of generality we denote by N^i either the set $N^i(u)$ when Diam(T)= 4, or the set $N^i(u,v)$ when Diam(T)= 5, i$\in \{0,1,2\}$.

2 Main results

First of all, we prove that contracting an edge uv in a graph G lead us to a new graph $G_{uv}(z)$ for which the Roman domination contraction number is, at most, the one of the original graph G.

Remark 1. Observe that if $f = (B_0, B_1, B_2)$ is a γ_R-function in a graph G then there is no edge joining a vertex $u \in B_1$ with a vertex $v \in B_2$. Otherwise, we may define a new function g in such a way that $g(u) = 0$ and $g(w) = f(w)$ for all $w \neq u$. Since g is a Roman dominating function and $w(g) < w(f)$ hence f is not a γ_R-function. A contradiction.

Lemma 1. *Let G be a connected graph and let f be a $\gamma_R(G)$-function in G. For every edge $uv \in E(G)$, there exists a Roman dominating function, $f_{uv,z}$, associated to f in $G_{uv}(z)$, such that $f_{uv,z}(V(G_{uv}(z))) \leq f(V)$.*

Proof. Let uv be an edge in G. Since f is a $\gamma_R(G)$-function in G hence $\{f(u), f(v)\} \neq \{1,2\}$, by applying the Remark 1. If $\{f(u), f(v)\} \neq \{0,2\}$, we define a Roman dominating function in $G_{uv}(z)$ by means of $f_{uv,z}(z) = \min\{f(u), f(v)\}$ and $f_{uv,z}(w) = f(w)$ for all $w \neq z$. As no 2-label disappears, the resulting is a Roman dominating function and

$$f_{uv,z}(V(G_{uv}(z))) = f_{uv,z}(z) + \sum_{w\neq z} f_{uv,z}(w) =$$

$$= min\{f(u), f(v)\} + \sum_{w\notin\{u,v\}} f(w) \leq f(V)$$

If $\{f(u), f(v)\} = \{0,2\}$, then we define $f_{uv,z}(z) = 2$. It is easy to check that $f_{uv,z}$ is a Roman dominating function and

$$f_{uv,z}(V(G_{uv(z)})) = f(V).$$

So the result holds. ∎

Since the previous result shows, the more edges are contracted in a graph, the more γ_R decrease. So, a question that arise naturally is 'how many edges must be contracted in a graph in order to decrease its Roman domination number?" The following lemma lead us to find an explicit answer to this question.

Lemma 2. *Let G be a Roman graph. If $f = (B_2, B_1, B_0)$ is a $\gamma_R(G)$-function such that $|B_2| \geq 2$ and $|B_1| = 0$, then for every $u \in B_2$ there exists $v \in B_2$, $v \neq u$, such that $d(u,v) \leq 3$.*

Proof. Suppose on the contrary that there exists $u \in B_2$ such that for every $v \in B_2$, $v \neq u$, it follows $d(u,v) > 3$. So, for every $v \in B_2$, $v \neq u$, there exists a shortest path of length at least four between u and v. Let $ux_1x_2...x_jv$, $j \geq 3$, be one of such a paths. Since $B_1 = \emptyset$, we have that $f(x_2) = 0$ and $N(x_2) \cap B_2 = \emptyset$, which is a contradiction. This finishes the proof. ∎

Next, we show the first of our main results. In this theorem we prove that the maximum number of edge-contractions that it is needed to perform in a graph in order to decrease its Roman domination contraction number is 3.

Theorem 1. *For any graph G without isolated vertices such that $\gamma_R(G) \geq 3$,*

$$ct_{\gamma_R}(G) \leq 3.$$

Another interesting question would be "what are the families of graphs $\mathcal{F}_j$ for which the Roman domination contraction number ct_{γ_R} is equal to j, for $j = 1, 2, 3$?" The following two results enable us to shed some light on the matter of this question.

Proposition 1. *G is not a Roman graph if and only if $ct_{\gamma_R}(G) = 1$.*

Proposition 2. *Let G be a graph without isolated vertices. Then $ct_{\gamma_R}(G) = 3$ if and only if for every $\gamma_R(G)$-function $f = (B_2, B_1, B_0)$ the following conditions are satisfied.*

- *$|B_2| \geq 2$ and $|B_1| = 0$.*
- *For every $u, v \in B_2$, $N(u) \cap N(v) = \emptyset$.*
- *For every $u \in B_2$, $|N(u)| \geq 3$.*

Both previous results give us some structural properties of the graphs. However, despite this fact, the problem is that there is not a general characterization of Roman graphs (for a characterization of Roman trees see [5]). In the following, we try to characterize some particular families of graphs with given ct_{γ_R}. From now on, we focus our interest to small trees with diameter 4 or 5.

We prove some technical results that permit us to determine the structure of a γ_R-function in a tree of diameter at most 5.

Lemma 3. *Let T be a tree so that $4 \leq Diam(T) \leq 5$, and let f be a $\gamma_R - function$. The following two conditions holds,*

i) If $z \in \Gamma_1$, $d(z) = 2$, $N(z)=\{u,t\}$ with $u \in \Gamma_0, t \in \Gamma_2$ with $f(u) = 0$ then there is a $\gamma_R - function$, g, so that $g(w) = f(w)$ for all $w \neq z, t$, $g(z) = 2$ and $g(t) = 0$.

ii) If $z \in \Gamma_1$, $d(z) \geq 3$, $N(z) = \{u, t_1, ..., t_{d(z)-1}\}$ there is a $\gamma_R - function$ so that $g(w) = f(w)$ for all $w \notin N[z] \setminus u$, $g(z) = 2$ and $g(t_i) = 0$.

Lemma 4. *Let T be a tree so that $4 \leq Diam(T) \leq 5$. Then*

$$\gamma_R(T) \leq |N^0| + 2|N^1| + 2|N^2|.$$

Lemma 5. *Let T be a tree with $Diam(T) = 4$ (or with $Diam(T) = 5$, respectively). Let $v \in V(T)$ (or $uv \in E(T)$, respectively) be the center of T. If f is a γ_R-function then $f(v) \neq 1$ ($f(u) \neq 1 \neq f(v)$, respectively).*

Lemma 6. *Let T be a tree with $Diam(T) = 4$ and let v be the center of T. Then, $f(v) = 0$ for all γ_R-function if and only if $|N^0(v) \cup N^1(v)| \leq 1$.*

Finally, we obtain the value of the Roman domination contraction number of trees of diameter 4 or 5.

Theorem 2. *Let T be a tree with $Diam(T) = 4$ and let v be the center of T. Then the following conditions holds,*

$$i)\ 1 \leq ct_{\gamma R}(T) \leq 2, \text{ and}$$

$$ii)\ ct_{\gamma R}(T) = 2 \text{ if and only if } |N^0(v) \cup N^1(v)| = 0.$$

Lemma 7. *Let T be a tree with $Diam(T) = 5$, if f is a γ_R-function in T, then f_{T_u} and f_{T_v} are γ_R-functions in T_u and T_v, respectively.*

Theorem 3. *Let T be a tree with $Diam(T) = 5$, and let $\{u, v\}$ be the center of T, then $ct_{\gamma R}(T) = min\{ct_{\gamma R}(T_u), ct_{\gamma R}(T_u)\}$*

Corollary 1. *Let T be a tree with $Diam(T) = 5$ and let $\{u, v\}$ be the center of T. Then the following conditions holds,*

$$i)\ 1 \leq ct_{\gamma R}(T) \leq 2, \text{ and}$$

$$ii)\ ct_{\gamma R}(T) = 2 \text{ if and only if } |N^0(u, v) \cup N^1(u, v)| = 0.$$

References

[1] G. Chartrand and L. Lesniak. *Graphs and Digraphs.* Chapman and Hall/CRC, 2005.

[2] E. J. Cockayne, P.A. Dreyer, S. M. Hedetniemi, S. T. Hedetniemi. Roman domination in graphs. *Discrete Mathematics*, 278(1-3):11–22, 2004.

[3] P. A. Dreyer. Applications and variations of domination in graphs. Ph. D. Thesis. *New Brunswick, New Jersey*, 2000.

[4] O. Favaron, H. Karamic, R. Khoeilar, S. M. Sheikholeslami. On the Roman domination number of a graph. *Discrete Mathematics*, 309: 3447–3451, 2009.

[5] M.A. Henning. A characterizatin of Roman trees. *Discuss. Math. Graph Theory*, 22(2): 325–334, 2002.

[6] M.A. Henning. Defending the Roman Empire from multiple attacks. *Discrete Mathematics*, 271: 101–115, 2003.

[7] J. Huang, J.M. Xu. Domination and total domination contraction numbers of graphs. *Ars Combinatoria*, 94: 431–433, 2010.

[8] I. Stewart. Defend the Roman Empire! *Sci. Amer.*, 116(8):136–138, 1999.

[9] F. Xueliang, Y. Yuansheng, J. Bao. Roman domination in regular graphs. *Discrete Mathematics*, 309 (6) 1528–1537, 2009.

On a variant of Roman domination in graphs.

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Abstract. Based on the history that the Emperor Constantine decreed that any undefended place (with no legions) of the Roman Empire must be protected by a neighbor 'stronger' place (having two legions), it could be obtained a graph theoretical model called *Roman domination* in graphs. A *Roman dominating function (RDF)* is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ has at least a neighbor $w \in N(v)$ for which $f(w) = 2$. The *Roman domination number* of a graph is the minimum weight, $\sum_{v \in V} f(v)$, of a RDF. In this work we introduce a variant of Roman domination, called Strong Roman domination, in which we approach the problem of a Roman-domination type defensive strategy under multiple simultaneous attacks.

Key words: Domination, Roman domination, Roman domination number, Strong Roman domination.

1 Introduction

In this section we introduce the notation and definitions that will be used throughout this paper. Unless stated on the contrary, other notation and terminology not explicitly given here could be find in [4]. Let $G = (V, E)$ be an undirected finite graph without loops and multiple edges. The set of vertices (edges) of the graph G is denoted by $V = V(G)$ (E=E(G)). The order n of a graph is the number of vertices of the graph and the size, m, correspond to the number of edges. Let us denote by $u \sim v$ two adjacent vertices $u, v \in V$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, let $N_X(v)$ denote the set of neighbors that v has in X, that is to say, $N_X(v) = \{u \in X : u \sim v\}$. In the case $X = V$ we use only $N(v)$, instead of $N_G(v)$, which is also called the open neighborhood of the vertex $v \in V$. The close neighborhood of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. Given a set of vertices $S \subseteq V$, the open neighborhood

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of S is the set $N(S) = \{w \in V(G) \setminus S : w \sim v, \text{ for some } v \in S\}$. The closed neighborhood is $N[S] = N(S) \cup S$. An universal vertex of G is a vertex which is adjacent to every other vertex of G.

A uv-path in G, joining the (end) vertices $u, v \in V(G)$, is a finite alternating sequence: $u_0 = u, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of different vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$ for all $i = 1, 2, \ldots, k$. The number of edges in a path is called the *length* of the path. The length of a shortest uv-path is the distance between the vertices u and v, and it is denoted by $d(u, v)$. The maximum among all the distances between two vertices in an graph G is denoted by $Diam(G)$, the diameter of G. A cycle is a uu-path. The length of a shortest cycle in the graph, if any, is called the girth of the graph. Otherwise, the girth of G is not finite.

The set of vertices $D \subset V$ is a *dominating set* if every vertex v not in D is adjacent to at least one vertex in D. The minimum cardinality of any dominating set of G is the *domination number* of G and it is denoted by $\gamma(G)$. A dominating set D in G with $|D| = \gamma(G)$ is called a $\gamma(G)$-set. Notice that a graph having a universal vertex has domination number equal to one.

In 1999, Steward [8] introduced the notion of Roman domination in graphs. A few years later, Cockayne et al. [2] stated the formal definition of Roman domination number of a graph. Namely, a function $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* in a graph G if for every vertex v with $f(v) = 0$ there exists at least a vertex $u \in N(v)$ for which $f(u) = 2$. The *weight* of a Roman dominating function is given by $w(f) = f(V) = \sum_{u \in V} f(u)$. The minimum weight among all the Roman dominating functions on G is called the *Roman domination number* of G and it is denoted by $\gamma_R(G)$. A Roman dominating function in G of minimum weight is called a $\gamma_R(G)$-function. There are many works in the literature trying to shed some light on this problem (see [3], [4], [5], [6] and [9]).

Let f be a Roman dominating function for G and let $V(G) = \{B_0, B_1, B_2\}$ be the sets of vertices of G induced by f, where $B_i = \{v \in V \ : \ f(v) = i\}$, for all $i \in \{0, 1, 2\}$. It is clear that for any Roman dominating function f for G of order n we have that $f(V) = \sum_{u \in V} f(u) = 2|B_2| + |B_1|$. Usually a Roman dominating function f is denoted by (B_0, B_1, B_2). In [2] was proved that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$ and in this paper the authors called *Roman graphs* those graphs G satisfying that $\gamma_R(G) = 2\gamma(G)$. Note that if G is not connected graph with connected components $C_1, C_2, ..., C_t$, then $\gamma_R(G) = \sum_{i=1}^{t} \gamma_R(C_i)$. Therefore, from now on we will only consider connected graphs.

The defensive strategy of Roman domination is based in the fact that every place in which there is established a Roman legion (1) is able to protect itself under external attacks; and that every unsecure place (0) must have at least an stronger neighbor (2). In that way, if the unsecure place is attacked then the stronger neighbor could send one of its two legions in order to defend the

(0) from the attack. Observe the two Roman dominating functions depicted in Fig. 1.

Fig. 1: Two Roman dominating functions.

Although this two functions satisfy the conditions to be Roman dominating functions, they correspond to very different real situations. Clearly, the unique strong place (2) in the graph on the left side must defend up to 12 weak places from possible external attacks. However, in the graph on the right side of Fig. 1, the task of defending the unsecure places is divided into several strong places. This situation lead us to pose the following question: how many weak places may defend a strong place having two legions? Taking into account that the strong place must leave one of its legions to defend itself, the situation

depicted on the first graph in Fig. 1 seems to be a very bad defensive strategy. If several simultaneous attacks to weak places are developed then the only stronger place will be not able to defend its neighbors efficiently. With this motivation we introduce the concept of *Strong Roman dominating function* as follows. To do that, we consider that an strong place should be able to defend itself and, at least, one half of its weak neighbors. Let G be a graph on n vertices and let Δ be the maximum degree of a vertex in G. Let $f : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta+1}{2} \rceil\}$ be a function that labels the vertices of G. Let us denote by $B_j = \{v \in V : f(v) = j\}$ for $j = 0, 1$; and let us denote by $B_2 = V \setminus (B_0 \cup B_1) = \{v \in V : f(v) \geq 2\}$. Then, f is an *Strong Roman dominating function* (SRDF)of G if for every $v \in B_0$ there exists a vertex $w \in B_2$ such that $f(w) \geq 1 + \left\lceil \frac{1}{2}|N(w) \cap B_0| \right\rceil$. In Fig. 2 we may observe an Strong Roman dominating function for each of the graph shown in Fig.1. The minimum weight, $w(f) = f(V) = \sum_{u \in V} f(u)$, over all the Stong Roman dominating functions of G is called the *Strong Roman domination number* of G and we denote it by $\gamma_{StR}(G)$. An SRDF of minimum weight is called a $\gamma_{StR}(G)$-function.

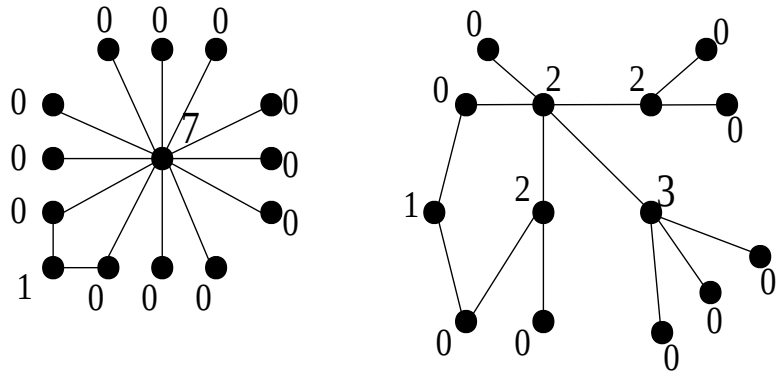

Fig. 2: Two Strong Roman dominating functions.

2 Main results

First of all, let us observe that a Strong Roman dominating defensive strategy needs, in general, more legions than a Roman dominating one, so the advantage is not to safe resources but to design an stronger empire against external attacks. Under the Strong Roman dominating strategy, any strong vertex must be able to defend itself and at least one half of its weak neighbors. The point is to design this situation by using as few resources (legions) as possible.

In our first result, we give the minimum number of legions to protect the roman empire fortifications under the Strong Roman domination strategy.

Proposition 1. *Let G a graph on n vertices. Then*

$$\gamma_{StR}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

Let us point out that the star graph permit us to prove that the bound shown in Proposition 1 is sharp.

From now on, we present some results obtaining bounds for the Strong Roman domination number.

Proposition 2. *Let G be a graph on n vertices and with maximum degree Δ. Then*

$$\gamma(G) \leq \gamma_R(G) \leq \gamma_{StR}(G) \leq \left(1 + \left\lceil \frac{\Delta}{2} \right\rceil\right) \gamma(G)$$

Corollary 1.

$$\gamma_{StR}(G) \leq \left\lceil \frac{n+1}{2} \right\rceil \gamma(G)$$

Proposition 3. *Let G be a graph on n vertices and with minimum degree δ. Then*

$$\gamma_{StR}(G) \leq \frac{\left(1 + \left\lceil \frac{\Delta}{2} \right\rceil\right) n}{\delta + 1} \left(\ln \left(\frac{1+\delta}{1 + \left\lceil \frac{\Delta}{2} \right\rceil} \right) + 1 \right)$$

Next, we approach the structure of graphs with extremal values of the Strong Roman domination.

Proposition 4. *Let G be a graph on n vertices. Then*

1. *$\gamma_{StR} = n$ if and only if $\Delta(G) = 1$*
2. *$\gamma_{StR} = n - 1$ if and only if either $G = C_q$ or $G = P_{q+1}$ with $q = 3, 4, 5$ or the following conditions holds: $n \leq 12$, $\Delta = 3$, $Diam \leq 5$ and there is no cycle in G of length at least 6.*

Finally we study the inverse problem, that is to say, given two positive integers n, p, there is a graph with n vertices and with Strong roman domination number equal to p.

Theorem 1. *Let n, p be two positive integers such that $\left\lceil \frac{n+1}{2} \right\rceil \leq p \leq \left\lfloor \frac{3n+3}{4} \right\rfloor$ There there exist a graph G with n vertices such that $\gamma_{StR}(G) = p$.*

References

[1] G. Chartrand and L. Lesniak. *Graphs and Digraphs.* Chapman and Hall/CRC, 2005.

[2] E. J. Cockayne, P.A. Dreyer, S. M. Hedetniemi, S. T. Hedetniemi. Roman domination in graphs. *Discrete Mathematics*, 278(1-3):11–22, 2004.

[3] P. A. Dreyer. Applications and variations of domination in graphs. Ph. D. Thesis. *New Brunswick, New Jersey*, 2000.

[4] O. Favaron, H. Karamic, R. Khoeilar, S. M. Sheikholeslami. On the Roman domination number of a graph. *Discrete Mathematics*, 309: 3447–3451, 2009.

[5] M.A. Henning. A characterizatin of Roman trees. *Discuss. Math. Graph Theory*, 22(2): 325–334, 2002.

[6] M.A. Henning. Defending the Roman Empire from multiple attacks. *Discrete Mathematics*, 271: 101–115, 2003.

[7] J. Huang, J.M. Xu. Domination and total domination contraction numbers of graphs. *Ars Combinatoria*, 94: 431–433, 2010.

[8] I. Stewart. Defend the Roman Empire! *Sci. Amer.*, 116(8):136–138, 1999.

[9] F. Xueliang, Y. Yuansheng, J. Bao. Roman domination in regular graphs. *Discrete Mathematics*, 309 (6) 1528–1537, 2009.

Dirichlet–to–Robin Matrix on networks ⋆

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Abstract. In this work, we define the Dirichlet–to–Robin matrix associated with a Schrödinger type matrix on general networks, and we prove that it satisfies the *alternating property* which is essential to characterize those matrices that can be the response matrices of a network. We end with some examples of the sign pattern behavior of the alternating paths.

Key words: Response matrix, Schur complements, inverse problem, Dirichlet–to–Robin matrix, network

1 Preliminaires

The Schur complement plays an important role in matrix analysis, statistics, numerical analysis, and many other areas of mathematics and its applications. Our goal is to introduce the Dirichlet–to–Robin matrix associated with a Schrödinger type matrix on general networks as the Schur complement of a Schrödinger type matrix with respect to an invertible submatrix defined troughout the interior vertices. Schur complement is a rich and basic tool in mathematical research and applications, so we display an important property that illustrates its power in solving the discrete inverse problem. A complete version of this work in terms of operators can be found in [2].

Let $\Gamma = (V, c)$ be a *network*; that is, a simple and finite connected graph where $V = \{1, 2, \ldots, \ell\}$ is the vertex set and $c : V \times V \longrightarrow \mathbb{R}^+$ is the *conductance* that defines the set of edges, E. We say that (i, j) is an edge if $c(i, j) = c_{ij} > 0$. Moreover, when $(i, j) \notin E$, then $c_{ij} = 0$, in particular $c_{ii} = 0$ for any $i = 1, \ldots, \ell$. The *(weighted) degree* of vertex i is defined as $\delta_i = \sum_{j=1}^{\ell} c_{ij}$. If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is. If F is a non–empty subset

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of V, its characteristic function is denoted by $\mathbf{1}_{_F}$. We denote by $N(i)$, the set of neighbours of $i \in V$; that is, the set of vertices adjacent to i.

Of course networks do not have boundaries by themselves, but starting from a network we can define a *network with boundary* as $\Gamma = (\bar{F}, c_{_F})$ where F is a proper subset and $c_{_F} = c \cdot \mathbf{1}_{(\bar{F}\times\bar{F})\setminus(\delta(F)\times\delta(F))}$. From now on we will work with networks with boundary and we suppose that the vertices are labelled as $\delta(F) = \{1, \ldots, n\}$ and $F = \{n+1, \ldots, n+m\}$. Moreover, for the sake of simplicity we denote $c = c_{_F}$.

Given $S = \{p_1, \ldots, p_k\}$ and $T = \{q_1, \ldots, q_k\}$ disjoint subsets of $\delta(F)$, there exist k paths, $\gamma_1, \ldots, \gamma_k$, such that γ_i starts at p_i ends at q_i and $\gamma_i \setminus \{p_i, q_i\} \subset F$, since F is connected. The pair $(S;T)$ is called *connected trough* Γ, when there exist k paths connecting S and T that are mutually disjoint.

The network $\Gamma = (\bar{F}, c)$ is called a *circular planar network* if it can be embedded in a closed disc D in the plane so that the vertices in F lie in $\overset{\circ}{D}$ and the vertices in $\delta(F)$ lie on the circumference $C = \partial D$. In this case, the vertices in $\delta(F)$ can be labelled in the clockwise circular order. The pair $(S;T)$ of boundary vertices is called a *circular pair* if the set $(p_1, \ldots, p_k; q_1, \ldots, q_k)$ is in circular order.

Given $\mathbf{u} \in \mathbb{R}^{n+m}$, the notation $\mathbf{u} \geq 0$, respectively $\mathbf{u} > 0$, means that $\mathbf{u}_i \geq 0$, respectively $\mathbf{u}_i > 0$, for any $i = 1, \ldots, n+m$. Any vector $\sigma \in \mathbb{R}^{n+m}$ such that $\sigma > 0$ and moreover $\sum_{i=1}^{n+m} \sigma_i^2 = 1$ is called *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. If $\sigma \in \Omega(\bar{F})$, $\sigma^{-1} \in \mathbb{R}^{n+m}$ is the vector whose entries are σ_i^{-1}, $i = 1, \ldots, n+m$.

Given $\mathsf{q} \in \mathbb{R}^{n+m}$ the *Schrödinger type matrix* on Γ with *potential* q is the matrix whose entries are $\mathcal{L}_{ij} = -c_{ij}$ for all $i \neq j$ and $\mathcal{L}_{ii} = \delta_i + q_i$. Therefore, for each vector $\mathbf{u} \in \mathbb{R}^{n+m}$ and for each $i = 1, \ldots, n+m$,

$$(\mathcal{L}\mathbf{u})_i = (\delta_i + \mathsf{q}_i)\mathbf{u}_i - \sum_{j=1}^{n+m} c_{ij}u_j = \sum_{j=1}^{n+m} c_{ij}(\mathbf{u}_i - \mathbf{u}_j) + \mathsf{q}_i\mathbf{u}_i.$$

Observe that $\mathsf{q} = \mathbf{0}$ corresponds with the so–called *combinatorial Laplacian* that will be denoted by $\mathcal{L}^0$ throughout this work. Moreover,

$$\mathsf{L} = \begin{bmatrix} \mathsf{D} & -\mathsf{C}(\delta(F);F) \\ -\mathsf{C}(\delta(F);F)^\top & \mathcal{L}(F;F) \end{bmatrix}$$

where D is the diagonal matrix of order n whose diagonal entries are given by $\delta + \mathsf{q}$ on $\delta(F)$ and $\mathsf{C}(\delta(F);F) = (c_{ij})_{i\in\delta(F), j\in F}$. In general, given a matrix M and A, B sets of indexes, the matrix $\boldsymbol{M}(A;B)$ will denote the matrix obtained from M with rows indexed by A and columns indexed by B.

For any weight $\sigma \in \Omega(\bar{F})$, the so–called *potential associated with* σ is the vector defined as $\big(\mathsf{q}_\sigma\big)_i = -\sigma_i^{-1}\Big(\mathcal{L}^0\sigma\Big)_i$.

The authors proved in [2] the following result.

Corollary 1. *If there exist* $\sigma \in \Omega(\bar{F})$ *and* $\lambda \geq 0$ *such that* $\mathsf{q} = \mathsf{q}_\sigma + \lambda 1_{\delta(F)}$, *then the corresponding Schrödinger type matrix is positive semi–definite. Moreover, it is not strictly definite iff* $\lambda = 0$, *in which case the eigenvectors are* $\mathbf{v} = a\sigma$, $a \in \mathbb{R}$.

From now on, we will work with potentials given by a weight $\sigma \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$ such that $\mathsf{q} = \mathsf{q}_\sigma + \lambda \mathbf{1}_{\delta(F)}$; so that the corresponding Schrödinger type matrix is positive semi–definite. Observe that in this case

$$\big(\mathcal{L}\sigma\big)_i = 0, i = n+1, \cdots, n+m \quad \text{and} \quad \big(\mathcal{L}\sigma\big)_i = \lambda\sigma_i, i = 1, \ldots, n. \tag{1}$$

In [3, Proposition 4.10], some of these authors proved the following version of the minimum principle that will be useful in what follows.

Proposition 1 (Monotonicity Property). *If* $\mathbf{u} \in \mathbb{R}^{n+m}$ *is such that* $\mathcal{L}\mathbf{u} \geq 0$ *on* F *and* $\mathbf{u} \geq 0$ *on* $\delta(F)$, *it is verified that either* $\mathbf{u} > 0$ *on* F *or* $\mathbf{u} = \mathbf{0}$ *on* $\bar{F}$.

2 Dirichlet–to–Robin matrix

Let us consider the following *Dirichlet problem*: Given $\mathsf{f} \in \mathbb{R}^m$ and $\mathsf{g} \in \mathbb{R}^n$ find $\mathbf{u} \in \mathbb{R}^{n+m}$ satisfying

$$\begin{bmatrix} \mathsf{I} & \mathsf{0} \\ -\boldsymbol{C}(\delta(F);F)^\top & \mathcal{L}(F;F) \end{bmatrix} \begin{bmatrix} \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathsf{g} \\ \mathsf{f} \end{bmatrix}. \tag{2}$$

The existence and uniqueness of solution for System (2) were proved in [3]. In fact, the *Dirichlet Principle* tell us that for any data $\mathsf{f} \in \mathbb{R}^m$ and $\mathsf{g} \in \mathbb{R}^n$, Problem (2) has a unique solution.

Associated with the Dirichlet problem we can consider the following semi homogenous problems, that allow us to introduce the concept of Green and Poisson matrices.

Given $\mathsf{f} \in \mathbb{R}^m$ find $\mathbf{u}_f \in \mathbb{R}^m$ satisfying

$$\mathcal{L}(F;F)\mathbf{u}_f = \mathsf{f} \tag{3}$$

and given $\mathsf{g} \in \mathbb{R}^n$ find $\mathbf{v}_g \in \mathbb{R}^{n+m}$ satisfying

$$\begin{bmatrix} \mathsf{I} & \mathsf{0} \\ -\boldsymbol{C}(\delta(F);F)^\top & \mathcal{L}(F;F) \end{bmatrix} \begin{bmatrix} \mathbf{v}_g \end{bmatrix} = \begin{bmatrix} \mathsf{g} \\ \mathsf{0} \end{bmatrix}. \tag{4}$$

The existence and uniqueness of solution for System (2) implies that matrix $\mathcal{L}(F;F)$ is invertible and its inverse is called *Green matrix* for F and it is denoted by Γ. Observe that Γ is a symmetric matrix.

On the other hand, we define the *Poisson matrix* for F as the matrix of order $(n+m)\times n$ given by

$$\boldsymbol{P}(F;\delta(F)) = \Gamma\cdot\boldsymbol{C}(\delta(F);F)^{\top} \quad \text{and} \quad \boldsymbol{P}(\delta(F);\delta(F)) = \boldsymbol{I}.$$

Notice that for any $\mathsf{g}\in\mathbb{R}^n$, the unique solution of System (4) is $\mathbf{v}_g = \boldsymbol{P}\mathsf{g}$. Moreover, from Equation (1), we get that $\boldsymbol{P}\sigma_{\delta(F)} = \sigma$.

Kirkhoff's law say that the sum of the currents flowing out of each interior vertex is zero, as state by System (4). If a vector g is assigned at the boundary vertices, the network Γ will acquire a unique harmonic vector $\mathbf{v}_g$, with $(\mathbf{v}_g)_i = \mathsf{g}_i$ for each $i=1,\dots,n$. The vector $\mathbf{v}_g$ is called the *potential due* to g.

The function $\mathbf{v}_g$ determines a current through each boundary node,

$$\big(\mathcal{L}\mathbf{v}_g\big)_i = \sum_{j=n+1}^{n+m} c_{ij}\big[\mathsf{g}_i - \big(\mathbf{v}_g\big)_j\big].$$

Now, we are ready to define the *Dirichlet–to–Robin matrix* on general networks and to study its main properties. This map is naturally associated to a Schrödinger type matrix, and generalizes the concept of *Dirichlet–to–Neumann map* for the case of the combinatorial Laplacian matrix.

The *Dirichlet–to–Robin matrix*, denoted by Λ, is the Schur complement of $\mathcal{L}(F;F)$ in $\mathcal{L}$; that is,

$$\Lambda = \mathsf{L}/\mathcal{L}(F;F) = \mathsf{D} - \mathsf{C}(\delta(F);F)\cdot\Gamma\cdot\mathsf{C}(\delta(F);F)^{\top}.$$

Observe that for any $\mathsf{g}\in\mathbb{R}^n$,

$$\Lambda\mathsf{g} = \mathsf{D}\mathsf{g} - \mathsf{C}(\delta(F);F)\mathbf{v}_{g_{|F}} = \mathcal{L}(\delta(F);\bar{F})\mathbf{v}_g.$$

Hence, Λ sends boundary Dirichlet date g to boundary Robin currents $\mathcal{L}\mathbf{v}_g$. The inverse problem is to recover the conductances $\boldsymbol{C}$ form Λ, see [2,5,6]. In this work we are not worried about this problem, but in studying some properties of Λ. The following ones are a direct consequence of the expression of Λ and of some properties for Schur complements of symmetric matrices, see [7, Theorem 1.12]

Proposition 2. *The Dirichlet–to–Robin matrix is symmetric, negative off–diagonal, positive on the diagonal and positive semi–definite. Moreover, λ is the lowest eigenvalue of Λ and its associated eigenvectors are multiple of $\sigma_{|_{\delta(F)}}$.*

Now we show that the Dirichlet–to–Robin matrix has the alternating property, which may be considered as a generalization of the monotonicity property; see [6, Theorem 2.1] for the continuous version of this property.

Theorem 1 (Alternating paths). *Suppose that $\delta(F) = A \cup B$, where A and B are disjoint subsets. Let $\mathsf{g} \in \mathbb{R}^n$ such that $\mathsf{g}_i \neq 0$ iff $i \in B$ and $p_1, \ldots, p_k \in A$ such that*

$$(-1)^{i+1}\Big(\Lambda\mathsf{g}\Big)_{p_i} > 0. \tag{5}$$

Then, there exist $q_1, \ldots, q_k \in B$ such that

$$\Big(\Lambda\mathsf{g}\Big)_{p_i} \mathsf{g}_{q_i} < 0. \tag{6}$$

Moreover, for any $i = 1, \ldots, k$, there exists a path from p_i to q_i such that $\gamma_i \setminus \{p_i, q_i\} \subset F$ and $\mathsf{g}_{q_i} \mathbf{v}_{g|_{(\gamma_i \setminus p_i)}} > 0$, where $\mathbf{v}_g = \boldsymbol{P}\mathsf{g}$.

Proof. As $p_1 \in A$, from (5), we have that $0 < \Big(\Lambda\mathsf{g}\Big)_{p_1} = -\sum_{i=1}^{m} c_{p_1 n+i} (\mathbf{v}_g)_{n+i}$. Then, there exists $t \in F \cap N(p_1)$ such that $(\mathbf{v}_g)_t < 0$.

Let W be the connected component of $\{k \in F : (\mathbf{v}_g)_k < 0\}$ containing t. Suppose that $\overline{W} \cap B = \emptyset$; that is, $\overline{W} \subset F \cup A$. We consider $\mathbf{u} = \big(\mathbf{v}_g\big)_{|_{\overline{W}}}$; then $\mathcal{L}\mathbf{u} = 0$ on W, $\mathbf{u} \geq 0$ on $\delta(W)$, then from the monotonicity principle $\mathbf{u} \geq 0$ on W which is a contradiction. Therefore, $\overline{W} \cap B \neq \emptyset$ and hence $\mathbf{v}_g \geq 0$ on $\delta(W) \cap F$. If $\mathbf{v}_g \geq 0$ on $\delta(W) \cap B$, we get that $\mathcal{L}\mathbf{v}_g = 0$ on W, $\mathbf{v}_g \geq 0$ on $\delta(W)$, so $\mathbf{v}_g \geq 0$ on W applying again the monotonicity principle which is a contradiction. So, there exists $q_1 \in \delta(W) \cap B$ such that $(\mathbf{v}_g)_{q_1} < 0$. As $q_1 \in \delta(W)$, there exists $z_1 \in W$, so $(\mathbf{v}_g)_{z_1} < 0$, such that $q_1 \sim z_1$. As W is a connected subset we can join q_1 and j_1 by a path $\gamma_1 = \{p_1 \sim t \sim \ldots \sim z_1 \sim q_1\}$ such that $\{t, \ldots, z_1\} \subset W$ and hence $\mathbf{v}_{g|_{(\gamma_1 \setminus p_1)}} < 0$.

We can repeat this argument to produce paths γ_j such that γ_j joins p_j to a point $q_j \in B$ such that $\gamma_j \setminus \{p_j, q_j\} \subset F$ and $(-1)^j (\mathbf{v}_g)_z < 0$ for all $z \in \gamma_j \setminus p_j$. □

Corollary 2. *Suppose that the network is circular planar and $\delta(F) = A \cup B$, where A and B are disjoint subsets. Let $\mathsf{g} \in \mathbb{R}^n$ such that $\mathsf{g}_i \neq 0$ iff $i \in B$ and $p_1, \ldots, p_k \in A$ in circular order such that*

$$(-1)^{i+1}\Big(\Lambda\mathsf{g}\Big)_{p_i} > 0. \tag{7}$$

Then there exist $q_1, \ldots, q_k \in B$ in circular order such that

$$\Big(\Lambda\mathsf{g}\Big)_{p_i} \mathsf{g}_{q_i} < 0. \tag{8}$$

Moreover, for any $i = 1, \ldots, k$, there exists a path from p_i to q_i such that $\gamma_i \setminus \{p_i, q_i\} \subset F$ and $\mathsf{g}_{q_i} \mathbf{v}_{g|_{(\gamma_i \setminus p_i)}} > 0$ in such a way that $(p_1, \ldots, p_k; q_1, \ldots, q_k)$ is connected through Γ.

Proof. The paths built on the proof of Theorem 1 do not intersect if the network is planar and the vertices $p_j \in A$ are in circular order, since the values for the vector $\mathbf{v}_g$ in γ_j have different sign than the ones on γ_{j-1} and γ_{j+1}. Then, $q_j \in B$ are the points given by the last theorem and are also in circular order. □

The following result can be obtained from Theorem 1 by a slightly modification of the proof.

Theorem 2 (Strong alternating paths). *Suppose that $\delta(F) = A \cup B$, where A and B are disjoint subsets. Let $\mathsf{g} \in \mathbb{R}^n$ such that $\mathsf{g}_i \neq 0$ iff $i \in B$ and $p_1, \ldots, p_k \in A$ such that*

$$\Big(\Lambda \mathsf{g}\Big)_{p_i} = 0 \tag{9}$$

then there is a sequence of points $q_1, \ldots, q_k \in B$ such that

$$(-1)^i \mathsf{g}_{q_i} \geq 0. \tag{10}$$

Moreover, for any $i = 1, \ldots, k$, there exists a path from $p_i \sim x_1^i \sim \ldots \sim x_{n_i}^i \sim q_i$ such that $P_i \setminus \{p_i, q_i\} \subset F$ and there exists $m_i \in \{1, \ldots, n_i + 1\}$ such that $(\mathbf{v}_g)_{x_\ell} = 0$ for all $\ell = 0, \ldots, m_i - 1$ and $\mathsf{g}_{\ell_i}(\mathbf{v}_g)_{x_\ell} > 0$ for all $\ell = m_i, \ldots, n_i$.

The following examples show the behavior of the paths described in the above results.

1. Consider the Spider graph displayed in Figure 1, see [2] for the definition. With the following weights and parameters: $\sigma = \dfrac{1}{10}$ on $\delta(F) \cup \{x_{00}\}$ and $\sigma = \dfrac{1}{5}$ on $F \setminus \{x_{00}\}$, where x_{00} is the central vertex and $\lambda = 2$. Moreover, all the conductances equal 1 on the edges in the radius and equal 2 on the edges of the circles. Then, the Dirichlet–to–Robin matrix is

$$\Lambda = \frac{1}{110384474959} \begin{pmatrix} a & b & c & d & e & f & f & e & d & c & b \\ b & a & b & c & d & e & f & f & e & d & c \\ c & b & a & b & c & d & e & f & f & e & d \\ d & c & b & a & b & c & d & e & f & f & e \\ e & d & c & b & a & b & c & d & e & f & f \\ f & e & d & c & b & a & b & c & d & e & f \\ f & f & e & d & c & b & a & b & c & d & e \\ e & f & f & e & d & c & b & a & b & c & d \\ d & e & f & f & e & d & c & b & a & b & c \\ c & d & e & f & f & e & d & c & b & a & b \\ b & c & d & e & f & f & e & d & c & b & a \end{pmatrix},$$

where $a = 395732805366$, $b = -28317414524$, $c = -19609504324$, $d = -15073456676$, $e = -12739926180$ and $f = -11741626020$. Finally, for $\mathsf{g} = (-4, 21, -37.2, 26.38, -6.29519)^T$, we get the following sign pattern for $\mathbf{v}_g$ depicted in Figure 1.

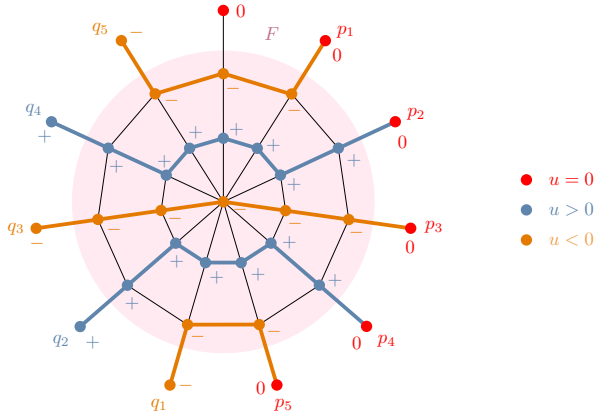

Fig. 1: Sign pattern in a Spider.

2. Consider now the spider network displayed in Figure 2. In this case, $\sigma = \dfrac{1}{6}$ on $\delta(F) \cup \{x_{00}\}$ and $\sigma = \dfrac{1}{3}$ on $F \setminus \{x_{00}\}$ and $\lambda = 2$. Moreover, the conductances equal 1 on the edges in the radius and equal 2 on the edges of the circles. Then, the Dirichlet–to–Robin matrix is

$$\Lambda = \frac{1}{889} \begin{pmatrix} a & b & c & d & d & c & b \\ b & a & b & c & d & d & c \\ c & b & a & b & c & d & d \\ d & c & b & a & b & c & d \\ d & d & c & b & a & b & c \\ c & d & d & c & b & a & b \\ b & c & d & d & c & b & a \end{pmatrix},$$

where $a = 3128$, $b = -281$, $c = -211$ and $d = -183$. Finally, for $\mathsf{g} = (-1, 3.5, -3.5, 1)^T$ we get the following sign pattern for $\mathbf{v}_g$ depicted in Figure 2.

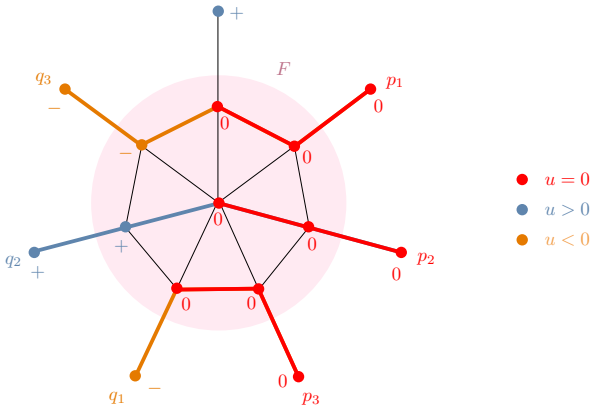

Fig. 2: Sign pattern in a Spider.

3. Consider the network displayed in Figure 3. With the following weights and parameters: $\sigma = \dfrac{1}{4}$ on $A \cup F$ and $\sigma = \dfrac{1}{2}$ on B and $\lambda = 2$. Moreover, the conductances equal 2 on $F \times F$ and equal 1 otherwise. Then, the Dirichlet–to–Robin matrix is

$$\Lambda = \frac{1}{24}\begin{pmatrix} 105 & -13 & -15 & -13 & -3 & -5 \\ -13 & 105 & -13 & -15 & -5 & -3 \\ -15 & -13 & 105 & -13 & -3 & -5 \\ -13 & -15 & -13 & 105 & -5 & -3 \\ -3 & -5 & -3 & -5 & 57 & -1 \\ -5 & -3 & -5 & -3 & -1 & 57 \end{pmatrix}.$$

Finally, for $\mathsf{g} = (1, -1.5)^T$ we get the following sign pattern for $\mathbf{v}_g$ depicted in Figure 3.

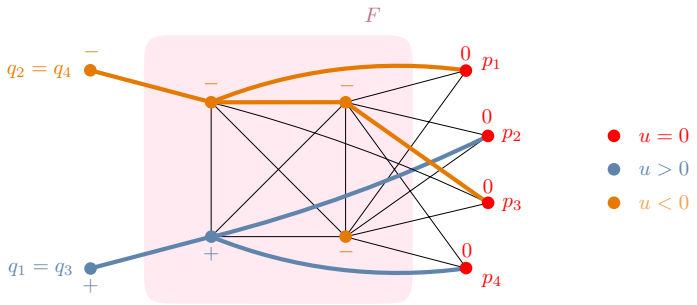

Fig. 3: Sign pattern in a non–planar network

References

[1] C. Araúz, A. Carmona, and A. M. Encinas: Dirichlet–to–Robin map on finite networks. *submmitted*, 2014.

[2] E. Bendito, A. Carmona, and A. M. Encinas: Potential Theory for Schrödinger operators on finite networks. *Rev. Mat. Iberoamericana*, 21(3): 771–818, 2005.

[3] E. B. Curtis, D. Ingerman, and J. A. Morrow: Circular planar graphs and resistor networks. *Linear Algebra Appl.*, 283(1–3): 115–150, 1998.

[4] E. Curtis, and J. Morrow: Inverse Problems for Electrical Networks. *Series on Applied Mathematics*, vol. 13. World Scientific 2000.

[5] R.A. Horn, and F. Zhang, Basic Properties of the Schur Complement, *The Schur Complement and Its Applications*, *Numerical Methods and Algorithms*, vol. 4: 17–46. Springer 2005.

[6] D. Ingerman, and J. Morrow: On a characterization of the kernel of the Dirichlet–to–Neumann map for a planar region. *SIAM J. Math. Anal.*, 29(1): 106–115, 1998.

Recovering the conductances on grids *

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Abstract. In this work, we present an algorithm to the recovery of the conductance of a 2–dimensional grid. The algorithm is based in the solution of some overdetermined partial boundary value problems defined on the grid.

Key words: Conductance recovery, inverse problem, overdetermined boundary value problem, network, response matrix.

1 Preliminaires

Our goal is to recover the conductance of a grid network with boundary using only boundary measurements and global equilibrium conditions. This problem is known as *inverse boundary value problem*. In general, inverse problems are exponentially ill–posed, since they are highly sensitive to changes in the boundary data. However, in this work we deal with a situation where the recovery of the conductance is feasible: grid networks.

The recovery of the conductances of a grid network is performed here using its Schrödinger matrix and boundary value problems associated to it, which have been extensively studied in [2]. We will use some important results of this paper. Moreover, we use the Dirichlet–to–Robin matrix, also known as response matrix of the network, which contains the boundary information. It is a certain Schur complement of the SchrÃ**P**dinger matrix, and it was introduced in [1]. The Schur complement plays an important role in matrix analysis, statistics, numerical analysis, and many other areas of mathematics and its applications.

First, we present some definitions, notations and results that will be necessary in the sequel. Let $\Gamma = (V, c)$ be a *network*; that is, a simple and finite connected graph where $V = \{1, 2, \ldots, \ell\}$ is the vertex set and $c : V \times V \longrightarrow \mathbb{R}^+$ is the *conductance* that defines the set of edges, E. We say that (i, j) is an edge if $c(i, j) = c_{ij} > 0$. Moreover, when $(i, j) \notin E$, then $c_{ij} = 0$, in particular

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$c_{ii} = 0$ for any $i = 1, \dots, \ell$. The *(weighted) degree* of vertex i is defined as $\delta_i = \sum_{j=1}^{\ell} c_{ij}$. If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is. If F is a non–empty subset of V, its characteristic function is denoted by $\mathbf{1}_{F}$. We denote by $N(i)$, the set of neighbours of $i \in V$; that is, the set of vertices adjacent to i.

Of course networks do not have boundaries by themselves, but starting from a network we can define a *network with boundary* as $\Gamma = (\bar{F}, c_{_F})$ where F is a proper subset and $c_{_F} = c \cdot \mathbf{1}_{(\bar{F}\times\bar{F})\setminus(\delta(F)\times\delta(F))}$. From now on we will work with networks with boundary and we suppose that the vertices are labelled as $\delta(F) = \{1, \dots, n\}$ and $F = \{n+1, \dots, n+m\}$. Moreover, for the sake of simplicity we denote $c = c_{_F}$.

Given $\mathbf{u} \in \mathbb{R}^{n+m}$, the notation $\mathbf{u} \geq 0$, respectively $\mathbf{u} > 0$, means that $\mathbf{u}_i \geq 0$, respectively $\mathbf{u}_i > 0$, for any $i = 1, \dots, n+m$. Any vector $\sigma \in \mathbb{R}^{n+m}$ such that $\sigma > 0$ and moreover $\sum_{i=1}^{n+m} \sigma_i^2 = 1$ is called *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. If $\sigma \in \Omega(\bar{F})$, $\sigma^{-1} \in \mathbb{R}^{n+m}$ is the vector whose entries are σ_i^{-1}, $i = 1, \dots, n+m$.

Given $\mathsf{q} \in \mathbb{R}^{n+m}$ the *Schrödinger type matrix* on Γ with *potential* q is the matrix whose entries are $\mathcal{L}_{ij} = -c_{ij}$ for all $i \neq j$ and $\mathcal{L}_{ii} = \delta_i + q_i$. Therefore, for each vector $\mathbf{u} \in \mathbb{R}^{n+m}$ and for each $i = 1, \dots, n+m$,

$$(\mathcal{L}\mathbf{u})_i = (\delta_i + \mathsf{q}_i)\mathbf{u}_i - \sum_{j=1}^{n+m} c_{ij}u_j = \sum_{j=1}^{n+m} c_{ij}(\mathbf{u}_i - \mathbf{u}_j) + \mathsf{q}_i\mathbf{u}_i.$$

Observe that $\mathsf{q} = \mathbf{0}$ corresponds with the so–called *combinatorial Laplacian* that will be denoted by $\mathcal{L}^0$ throughout this work. Moreover,

$$\mathsf{L} = \begin{bmatrix} \mathsf{D} & -\mathsf{C}(\delta(F); F) \\ -\mathsf{C}(\delta(F); F)^{\top} & \mathcal{L}(F; F) \end{bmatrix}$$

where D is the diagonal matrix of order n whose diagonal entries are given by $\delta + \mathsf{q}$ on $\delta(F)$ and $\mathsf{C}(\delta(F); F) = (c_{ij})_{i\in\delta(F), j\in F}$. We also define here the diagonal matrix $\widetilde{\mathsf{D}}$ of order n whose diagonal entries are the values of δ on $\delta(F)$. In general, given a matrix M and A, B sets of indexes, the matrix $\boldsymbol{M}(A; B)$ will denote the matrix obtained from M with rows indexed by A and columns indexed by B. Also, given a vector v and a set of indices A, the vector v_A denotes the vector obtained from v with entries indexed by A.

For any weight $\sigma \in \Omega(\bar{F})$, the so–called *potential associated with* σ is the vector defined as $\left(\mathsf{q}_\sigma\right)_i = -\sigma_i^{-1}\left(\mathcal{L}^0\sigma\right)_i$. The authors proved in [1] the following result.

Corollary 1. *If there exist* $\sigma \in \Omega(\bar{F})$ *and* $\lambda \geq 0$ *such that* $\mathsf{q} = \mathsf{q}_\sigma + \lambda \mathsf{1}_{\delta(F)}$*, then the corresponding Schrödinger type matrix is positive semi–definite. Moreover, it is not strictly definite iff* $\lambda = 0$*, in which case the eigenvectors are* $\mathbf{v} = a\sigma$*,* $a \in \mathbb{R}$*.*

From now on, we will work with potentials given by a weight $\sigma \in \Omega(\bar{F})$ and a real value $\lambda \geq 0$ such that $\mathsf{q} = \mathsf{q}_\sigma + \lambda \mathbf{1}_{\delta(F)}$; so that the corresponding Schrödinger type matrix is positive semi–definite. Observe that in this case

$$\big(\mathcal{L}\sigma\big)_i = 0, i = n+1, \cdots, n+m \quad \text{and} \quad \big(\mathcal{L}\sigma\big)_i = \lambda\sigma_i, i = 1, \ldots, n. \tag{1}$$

Let us consider the following *Dirichlet problem*: Given $\mathsf{f} \in \mathbb{R}^m$ find $\mathbf{u}_f \in \mathbb{R}^m$ satisfying

$$\mathcal{L}(F;F)\mathbf{u}_f = \mathsf{f} \tag{2}$$

The existence and uniqueness of solution for System (2) implies that matrix $\mathcal{L}(F;F)$ is invertible and its inverse is called *Green matrix* for F and it is denoted by $\mathit{\Gamma}$. Observe that $\mathit{\Gamma}$ is a symmetric matrix.

The *Dirichlet–to–Robin matrix*, denoted by Λ, is the Schur complement of $\mathcal{L}(F;F)$ in $\mathcal{L}$; that is,

$$\Lambda = \mathsf{L}/\mathcal{L}(F;F) = \mathsf{D} - \mathsf{C}(\delta(F);F) \cdot \mathit{\Gamma} \cdot \mathsf{C}(\delta(F);F)^\top.$$

This map is the response matrix to a Schrödinger type matrix, and therefore it can be assumed to be known, since it provides boundary reactions to boundary actions, the type of data that we can measure.

2 Overdetermined partial boundary value problems

We define a partition of the boundary of the network $\mathit{\Gamma} = (V, c)$: let $A, B, R \subset \delta(F)$ such that $A = \{1, \ldots, a\}$, $R = \{a+1, \ldots, a+r\}$ and $B = \{a+r+1, \ldots, n\}$. We say that $b = |B| = n - a - r$ and clearly $a = |A|$, $r = |R|$. Note that $\delta(F) = A \cup R \cup B$ and that these three boundary sets do not intersect each other.

Let us consider the following *overdetermined partial boundary value problem*: given $\mathsf{f} \in \mathbb{R}^m$, $\mathsf{g} \in \mathbb{R}^{a+r}$ and $\mathsf{h} \in \mathbb{R}^a$, find $\mathbf{u} \in \mathbb{R}^{n+m}$ satisfying

$$\begin{bmatrix} \widetilde{\mathsf{D}}(A; A\cup R) & \mathsf{0} & -\boldsymbol{C}(A;F) \\ \\ -\boldsymbol{C}(F; A\cup R) & -\boldsymbol{C}(F;B) & \mathcal{L}(F;F) \end{bmatrix} \begin{bmatrix} \\ \mathbf{u} \\ \\ \end{bmatrix} = \begin{bmatrix} \mathsf{h} \\ \\ \mathsf{f} \end{bmatrix}. \tag{3}$$

with the additional condition

$$\begin{bmatrix} \mathsf{I} & \mathsf{0} & \mathsf{0} \end{bmatrix} \begin{bmatrix} \\ \mathbf{u} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \mathsf{g} \\ \\ \end{bmatrix}, \tag{4}$$

where I is the identity matrix of order $a+r$. Note that there is a double condition on A but there does not exist any condition on B. In [3] the authors proved the existence and uniqueness of solution of this system for any data $\mathsf{f} \in \mathbb{R}^m$, $\mathsf{g} \in \mathbb{R}^{a+r}$, $\mathsf{h} \in \mathbb{R}^a$ if and only if $|A| = |B|$ and $\Lambda(A;B)$ is invertible. In this case, the following result holds.

Proposition 1 ([3, Corollary 4.5]). *The unique solution* $\mathbf{u} \in \mathbb{R}^{n+m}$ *of the overdetermined partial Dirichlet–Neumann boundary value problem* (3)&(4) *is given by the equalities*

$$\mathsf{u}_B = \Lambda(A;B)^{-1} \cdot \Big(\boldsymbol{C}(A;F) \cdot \Gamma \cdot \mathsf{f} - \Lambda(A; A \cup R) \cdot \mathsf{g} + \mathsf{h} \Big),$$

$$\mathbf{u}_F = \Gamma \cdot \Big(\mathsf{f} + \boldsymbol{C}(F;B) \cdot \mathbf{u}_B + \boldsymbol{C}(F; A \cup R) \cdot \mathsf{g} \Big)$$

and, clearly, $\mathbf{u}_{A \cup R} = \mathsf{g}$.

3 Recovering the conductance on grids

Our purpose is to use the results on last section in order to recover the conductance on grids. First, we define this family of networks. Let us take two integers $a, \ell \in \mathbb{N}$ such that $a \leq \ell$. We call $\Gamma = (V, c)$ the *grid with length* $\ell + 2$ *and height* $a + 2$ the network given in Figure 1.

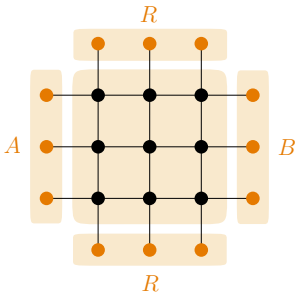

Fig. 1: An example of 2 dimensional grid with $a = \ell = 3$.

The vertices of V are labelled now as x_{ij}, where $i = 0, \ldots, a+1$ is the row and $j = 0, \ldots, \ell + 1$ is the column. We define

$$F = \{x_{ij} \in V \,:\, i = 1, \ldots, a,\ j = 1, \ldots, \ell\},$$

$A = \{x_{i0} \in V \,:\, i = 1, \ldots, a\}$, $B = \{x_{i\,\ell+1} \in V \,:\, i = 1, \ldots, a\}$ and $R = V \setminus (F \cup A \cup B)$. Notice that $\delta(F) = A \cup B \cup R$ and $\bar{F} = V$. Moreover, $|A| = |B|$ and since there exists an a–connection through Γ between A and B, the matrix $\Lambda(A;B)$ is invertible; see [6] for the proof of this statement. This means that any overdetermined partial boundary value problem has solution on a grid.

Given an index $i \in \{0, \dots, a+1\}$, we consider the *partial layers* of vertices $D_i = \left\{x_{ij} \in \bar{F} : j = i, \dots, \ell+1\right\}$. In particular, $D_0 \subset R$.

The recovery of conductances on a 2 dimensional grid is an iterative process, for we are not able to give explicit formulae for all the conductances at the same time but we can give a recovery algorithm instead. Hence, we describe the algorithm in steps, each of them requiring the information obtained in the last one.

Step 1. In this step we do not recover any conductance. However, we set the necessary tools to obtain them in future steps. Having fixed an index $k \in \{1, \dots, \ell\}$, we consider the overdetermined partial Dirichlet–Neumann boundary value problem (3) & (4) with data $\mathsf{f} = 0$, $\mathsf{h} = 0$ and $\mathsf{g} = \mathsf{e}^{x_{0k}}$, where $\mathsf{e}^{x_{0k}}$ stands for the vector of $\mathbb{R}^{a+r}$ with entries 0 except for the entry corresponding to x_{0k}, which equals 1:

$$\begin{bmatrix} \widetilde{\mathsf{D}}(A; A \cup R) & \mathsf{0} & -\boldsymbol{C}(A; F) \\ \\ -\boldsymbol{C}(F; A \cup R) & -\boldsymbol{C}(F; B) & \mathcal{L}(F; F) \end{bmatrix} \begin{bmatrix} \\ \mathbf{u} \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}. \tag{5}$$

with the additional condition $\mathbf{u}_{A \cup R} = \mathsf{e}^{x_{0k}}$.

There exists a large set of vertices of the grid Γ where the entries of $\mathbf{u}$ are 0. We denote this set by $Z(\mathbf{u}) = \left\{x \in \bar{F} : u(x) = 0\right\} = \mathsf{supp}(u)^c \subseteq \bar{F}$. Clearly, $A \subseteq Z(\mathbf{u})$. The size of $Z(\mathbf{u})$, however, is much bigger than the size of A.

Proposition 2. *The following equality is satisfied:*

$$Z(\mathbf{u}) = \{x_{ij} \in V : i = 1, \dots, a,\ j = 1, \dots, k+i-1\}.$$

Actually, the set $Z(\mathbf{u})$ has a very characteristic shape. In Figure 2(a) we show this pattern.

Now let us gather more information on $\mathbf{u}$. We already know that $\mathbf{u}_A = 0$, $\mathbf{u}_{R \smallsetminus \{x_{0k}\}} = 0$ and $\mathbf{u}_{x_{0k}} = 1$. Moreover, the values of $\mathbf{u}_B$ are given by $\mathsf{u}_B = -\Lambda(A; B)^{-1} \cdot \Lambda(A; x_{0k})$ because of Corollary 1. This means that $\mathbf{u}_{\delta(F)}$ is known, for the Dirichlet–to–Robin map is known. In Figure 2(b) we show all the information obtained at the end of this step.

Step 2. In this step recover the conductances of all the edges having one end of the form $x_{0j} \in R$.

Corollary 2. *The conductances of the edges joining the vertices of D_0 with the vertices of D_1 are given by*

$$c(x_{0j}, x_{1j}) = \sigma(x_{0j})\sigma^{-1}(x_{1j}) \left(\Lambda(x_{0j}; x_{0j}) - \Lambda(x_{0j}; B) \cdot \Lambda(A; B)^{-1} \cdot \Lambda(A; x_{0j}) - \lambda\right)$$

for all $j = 1, \dots, \ell$.

In Figure 2(c) we show all the information obtained at the end of this step.

Step 3. Again, let us fix the index $j \in \{1, \dots, \ell\}$ in this step and let us consider the unique solution $\mathbf{u}$ of problem (5). Then, we know all the values of $\mathbf{u}_{D_1}$, as the following result shows.

Lemma 1. *The values of $\mathbf{u}_{D_1}$ are known. They are given by*

$$\mathbf{u}_{x_{1j}} = \frac{1}{c(x_{0j}, x_{1j})}\Big(\lambda \mathbf{u}_{x_{0j}} - \Lambda(x_{0j}; x_{0k}) - \Lambda(x_{0j}; B) \cdot \mathbf{u}_B\Big) + \sigma_{x_{1j}} \sigma_{x_{0j}}^{-1} \mathbf{u}_{x_{0j}}$$

for all $k = 1, \dots, \ell$.

In Figure 2(d) we show all the data gathered from the 2 dimensional grid at the end of this step.

Step 4. Here we find the conductances of all the edges with both ends in D_1. However, we state a more general result.

Proposition 3. *Let $i \in \{0, \dots, a-1\}$. For every $k = 1, \dots, \ell$, let us suppose that we know the values of $\mathbf{u}_{D_{i-2}}$ and $\mathbf{u}_{D_{i-1}}$. Also, we suppose that the conductances of all the edges joining vertices from D_{i-2} and D_{i-1} are known. Now fix the index $k \in \{1, \dots, \ell\}$. Then, the conductances $c(x_{i-1\,k+i-2}, x_{i-1\,k+i-1})$ are also known. They are given by*

$$c(x_{i-1\,k+i-2}, x_{i-1\,k+i-1}) = -\frac{\mathbf{u}_{x_{i-2\,k+i-2}}}{\mathbf{u}_{x_{i-1\,k+i-1}}}\, c(x_{i-1\,k+i-2}, x_{i-2\,k+i-2}).$$

When $i = 2$, Propositions 2 and 3 show that $c(x_{1k}, x_{1\,k+1})$ is known for all $k = 1, \dots, \ell$. See Figure 2(e) in order to see all the known information at the end of this step.

Step 5. Let us define the values $\rho_{ij} = c(x_{ij}, x_{i\,j-1})\mathbf{u}_{x_{i\,j-1}} + c(x_{ij}, x_{i\,j+1})\mathbf{u}_{x_{i\,j+1}} + c(x_{ij}, x_{i-1\,j})\mathbf{u}_{x_{i-1\,j}}$ and

$$\omega_{ij} = c(x_{ij}, x_{i\,j-1})\sigma_{x_{i\,j-1}} + c(x_{ij}, x_{i\,j+1})\sigma_{x_{i\,j+1}} + c(x_{ij}, x_{i-1\,j})\sigma_{x_{i-1\,j}}$$

In this step we give the conductances of all the edges joining the vertices from D_1 and D_2. Furthermore, we state a more general result.

Proposition 4. *Let $i \in \{0, \dots, a-1\}$. For every $k = 1, \dots, \ell$, let us suppose that we know the values of $\mathbf{u}_{D_{i-2}}$ and $\mathbf{u}_{D_{i-1}}$. Also, let us suppose that we know the conductances of all the edges joining vertices from D_{i-2} and D_{i-1}, and the ones of the edges with both ends in D_{i-1}. Now fix the index $k \in \{1, \dots, \ell\}$. Then, the conductances $c(x_{i-1\,k+i-1}, x_{i\,k+i-1})$ are also known. They are given by*

$$c(x_{i-1\,k+i-1}, x_{i\,k+i-1}) = \frac{\rho_{i-1\,k+i-1}\sigma_{x_{i-1\,k+i-1}}}{\mathbf{u}_{x_{i-1\,k+i-1}}\sigma_{x_{i\,k+i-1}}} - \frac{\omega_{i-1\,k+i-1}}{\sigma_{x_{i\,k+i-1}}}.$$

In particular, when $i = 2$, Propositions 2 and 4 show that $c(x_{1\,k+1}, x_{2\,k+1})$ is known for all $k = 1, \ldots, \ell$. See Figure 2(f) in order to see all the information gathered at the end of this step.

Step 6. In this step we are able to obtain the unknown values of $\mathbf{u}_{D_2}$ for all $k = 1, \ldots, \ell$. In fact, let us state a more general result.

Proposition 5. *Let $i \in \{0, \ldots, a-1\}$. For every $k = 1, \ldots, \ell$, let us suppose that we know the values of $\mathbf{u}_{D_{i-2}}$ and $\mathbf{u}_{D_{i-1}}$. Also, let us suppose that we know the conductances of all the edges joining vertices from D_{i-2} and D_{i-1}, from D_{i-1} and D_i and the ones of the edges with both ends in D_{i-1}. Now fix the index $k \in \{1, \ldots, \ell\}$. Then, the values of $\mathbf{u}_{D_i}$ are also known. They are given by*

$$\mathbf{u}_{x_{ij}} = \frac{\rho_{i-1\,j}}{c(x_{i-1\,j}, x_{ij})} - \frac{\omega_{i-1\,j}}{\sigma_{x_{i-1\,j}} c(x_{i-1\,j}, x_{ij})} \mathbf{u}_{x_{i-1\,j}} - \frac{\sigma_{x_{ij}}}{\sigma_{x_{i-1\,j}}} \mathbf{u}_{x_{i-1\,j}}$$

for all $j = k + i, \ldots, \ell$.

In particular, when $i = 2$, Propositions 2 and 5 show that $\mathbf{u}_{D_2}$ is known for all $k = 1, \ldots, \ell$. Observe that we already knew some of the values of $\mathbf{u}_{D_2}$, which are those corresponding to the vertices of $Z(\mathbf{u})$. Figure 2(g) shows the information obtained until this step.

Step 7 and beyond. We keep repeating the same process to obtain more conductances, that is, we keep applying Proposition 3 from Step 4, then Proposition 4 from Step 5 and then Proposition 5 from Step 6 for each $i = 3, \ldots, a$. We stop when we have obtained all the conductances between, and all the values of u corresponding to, the vertex set $\bigcup_{i=1}^{a} D_i$, see Figure 2(j).

The final step left is to rotate the grid π radians (see Figure2(k)), that is, to consider the new boundary sets $A = \{x_{i\,\ell+1} \in V : i = 1, \ldots, a\}$ and $B = \{x_{i0} \in V : i = 1, \ldots, a\}$ instead of the previous ones, and consider the overdetermined partial Dirichlet–Neumann boundary value problem (5) now on these new sets. By proceeding analogously to the last steps, we obtain the lacking conductances of all the grid, see Figure 2(l).

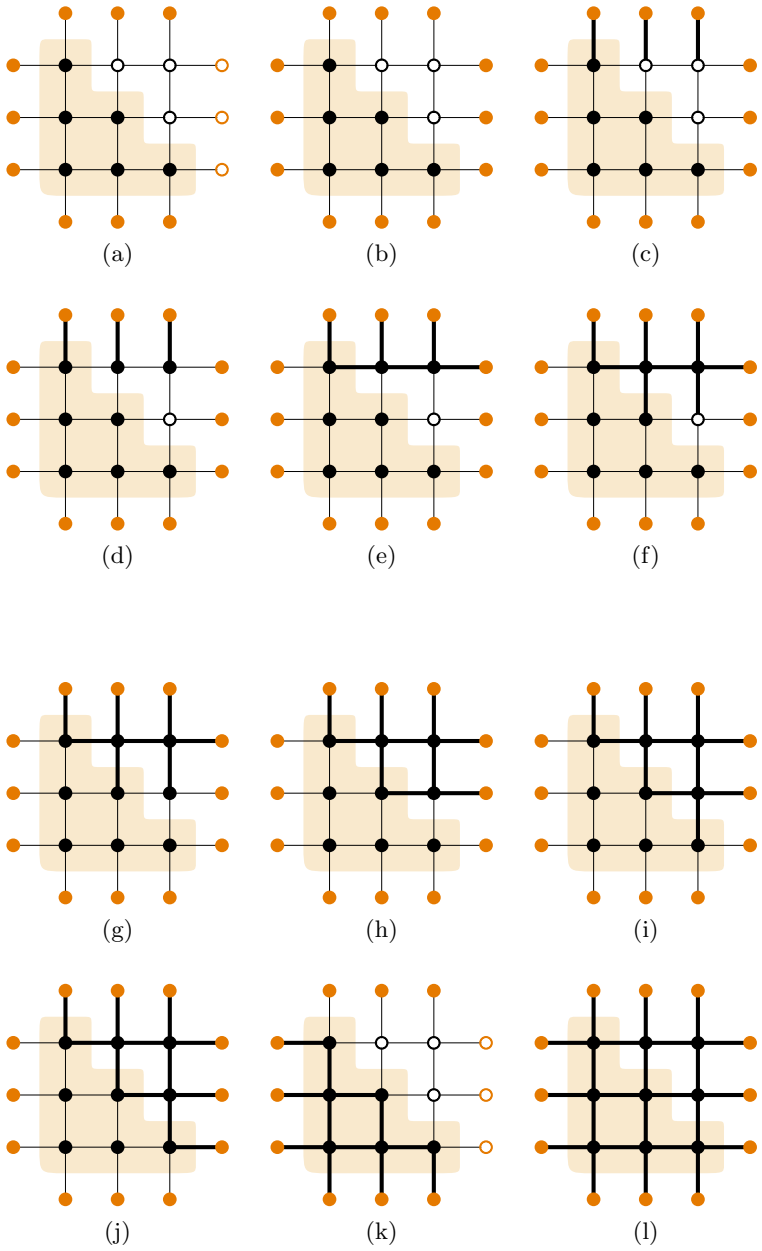

Fig. 2: The bold items are the ones known at the end of each step for the case $k = 1$.

References

[1] C. Araúz, A. Carmona, and A. M. Encinas: Dirichlet–to–Robin map on finite networks. *submmitted*, 2014.

[2] C. Araúz, A. Carmona, and A. M. Encinas: Overdetermined partial boundary value problems on finite networks. *submmitted*, 2014.

[3] C. Araúz, A. Carmona, and A. M. Encinas: Recovering the conductance in 3–dimensional grids. *submmitted*, 2014.

[4] E. Bendito, A. Carmona, and A. M. Encinas: Potential Theory for Schrödinger operators on finite networks. *Rev. Mat. Iberoamericana*, 21(3): 771–818, 2005.

[5] E. B. Curtis, D. Ingerman, and J. A. Morrow: Circular planar graphs and resistor networks. *Linear Algebra Appl.*, 283(1–3): 115–150, 1998.

[6] E. Curtis, and J. Morrow: Inverse Problems for Electrical Networks. *Series on Applied Mathematics*, vol. 13. World Scientific 2000.

[7] R.A. Horn, and F. Zhang, Basic Properties of the Schur Complement, *The Schur Complement and Its Applications*, *Numerical Methods and Algorithms*, vol. 4: 17–46. Springer 2005.

On the ascending subgraph decomposition problem for bipartite graphs ⋆

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Abstract. The Ascending Subgraph Decomposition (ASD) Conjecture asserts that every graph G with $\binom{n+1}{2}$ edges admits an edge decomposition $G = H_1 \oplus \cdots \oplus H_n$ such that H_i has i edges and is isomorphic to a subgraph of H_{i+1}, $i = 1, \ldots, n-1$. We show that every bipartite graph G with $\binom{n+1}{2}$ edges such that the degree sequence $d_1, \ldots, d_k$ of one of the stable sets satisfies $d_i \geq n - i + 2$, $1 \leq i < k$, admits an ascending subgraph decomposition with star forests. We also give a necessary condition on the degree sequence which is not far from the above sufficient one.

Key words: Graph decomposition, Ascending graph decomposition problem.

1 Introduction

A graph G with $\binom{n+1}{2}$ edges has an Ascending Subgraph Decomposition (ASD) if it admits an edge–decomposition $G = H_1 \oplus \cdots \oplus H_n$ such that H_i has i edges and is isomorphic to a subgraph of H_{i+1}, $1 \leq i < n$. It was conjectured by Alavi, Boals, Chartrand, ErdHos and Oellerman [1] that every graph of size $\binom{n+1}{2}$ admits an ASD. The conjecture has been proved for a number of particular cases, including forests [5], regular graphs [9], complete multipartite graphs [8] or graphs with maximum degree $\Delta \leq n/2$ [6].

In the same paper Alavi et al. [1] conjectured that every star forest of size $\binom{n+1}{2}$ in which each connected component has size at least n admits an ASD in which every graph in the decomposition is a star. This conjecture was proved by Ma, Zhou and Zhou [13], and the condition was later on weakened to the effect that the second smaller component of the star forest has size at least n by Chen, Fu , Wang and Zhou [4]. The above two results are connected to the Sumset Partition Problem (SPP): given an integer sequence $d_1 \geq \cdots \geq d_k > 0$ such that $\sum_i d_i = \binom{n+1}{2}$, the SPP asks for a partition $\{X_1, \ldots, X_k\}$ of the integer interval $[1, n]$ such that the sum of the elements in X_i is precisely d_i.

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If the answer is positive we say that the sequence $d_1, \ldots, d_k$ is *n–realizable*. The result of Chen et al. [4] states that every sequence with $\sum_i d_i = \binom{n+1}{2}$ and $d_{k-1} \geq n$ is n–realizable.

This result can also be reformulated in terms of ASD of bipartite graphs. Let G be a bipartite graph with bicoloration $\{A, B\}$ and size $\binom{n+1}{2}$. Suppose that the degree sequence $d_1 \geq \cdots \geq d_k$ of the vertices in A is n–realizable. Then G admits a star ASD. This motivates the study of ASD for bipartite graphs in terms of the degree sequence of one of the stable sets, which is the purpose of this paper.

The condition $d_{k-1} \geq n$ given by Chen et al. [4] for a sequence to be n–realizable can not be weakened in the sense that the result fails to be true for sequences with $d_{k-1} = d_k = 1$ or $d_{k-1} = d_k = 2$, say. The two examples above belong to a family of natural obstructions for a sequence to be n–realizable. A sequence $a_1 \geq \cdots \geq a_t$ is a *forbidden sequence* if there is no family of pairwise disjoint subsets $Y_1, \ldots, Y_t$ of the integer interval $[1, a_1]$ such that $\sum_{x \in Y_1} = a_1, \ldots, \sum_{x \in Y_t} = a_t$. Clearly, a sequence $d_1, \ldots, d_k$ which contains a forbidden sequence can not be n–realizable. The main result in [12] states that, for n large enough, a sequence $d_1 \geq \cdots \geq d_k$ with $\sum_i d_i = \binom{n+1}{2}$ is n–realizable if and only if its subsequence of terms smaller than n does not contain a forbidden subsequence. The full characterization of forbidden sequences seems to be a difficult problem. In this paper we give some sufficient conditions which ensure the existence of a star ASD of bipartite graphs in terms of the degree sequence.

Theorem 1. *Let G be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d_1 \geq \cdots \geq d_k$ be the degree sequence of the vertices in one stable set of G. If $k \leq (n+1)/4$ and $d_k \geq 4k$ then G admits a star ASD.* □

Most of the results on the ASD conjecture use families of graphs for which the isomorphic contention is easily checked, like matchings or stars. The main contribution of the paper is to study the conjecture for bipartite graphs in terms of the dfegree sequence of one of the stable sets by using star forests as building blocks. Our main result is the following one.

Theorem 2. *Let G be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d_1 \geq d_2 \geq \cdots \geq d_k$ be the degree sequence of one of the stable sets of G. If $d_i \geq n - i + 2$ then there is a star forest ASD of G.* □

The proof of Theorem 2 is based in a reduction lemma, which uses a deep result of Häggkvist [10] on edge–colorings, and on results on the Sumset Partition Problem [12]. The sufficient condition in Theorem 2 is not far from necessary.

Lemma 1. *Let G be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d_1 \geq d_2 \geq \cdots \geq d_k$ be the degree sequence of the stable set A of G. If G admits a star forest ASD with the centers of the stars in A then*

$$\sum_{i=1}^{t} d_i \geq \sum_{i=1}^{t}(n-i+1),\ t=1,\ldots,k.$$

□

The paper is organized as follows. We first consider star ASD in section 2, where we recall some results derived from the Sumset Partition Problem and prove Theorem 1. Section 3 contains a reduction lemma for the star forest ASD of bipartite graphs and the proof of Lemma 1. The proof of Theorem 2 is contained in section 4.

2 Star ASD of bipartite graphs

A sequence $d_1 \geq \cdots \geq d_k$ of nonnegative integers with $d_1 + \cdots + d_k = \binom{n+1}{2}$ is *n–realizable* if there is a partition $\{X_1, \ldots, X_k\}$ of $[n]$ such that the sum of the elements in X_i is d_i for each i.

Let G be a bipartite graph with stable sets $A = \{a_1, \ldots, a_k\}$ and B. Let $d_1 \geq \ldots \geq d_k$ be the degrees of the vertices $a_1, \ldots, a_k$. If the sequence $(d_1, \ldots, d_k)$ is n–realizable then G clearly admits a star ASD. Known results on the Sumset Partition Problem [4,12] imply the following results on the ASD problem

Proposition 1. *Let $G(A,B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. If $|A| \leq 4$ and $n \geq 7$ then $G(A,B)$ admits an ASD.* □

Chen et al. [4] proved that a sequence $(d_1 \geq \cdots \geq d_k)$ such that $\sum_i d_i = \binom{n+1}{2}$ and $d_{k-1} \geq n$ is n–realizable. The result can be rephrased in terms of ASD of bipartite graphs as follows.

Corollary 1. *Let $G(A,B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d(A) = (d_1 \geq \cdots \geq d_k)$ be the degree sequence of the elements in A. If $d_{k-1} \geq n$ then $G(A,B)$ admits a star ASD_n.*

Proposition 2. *Let $G(A,B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. If $k \leq (n+1)/4$ and $d_k \geq 4k$ then $G(A,B)$ admits a star ASD_n.*

3 A reduction Lemma

Let G be a bipartite graph with stable sets $A = \{a_1, \ldots, a_k\}$ and B. Let $d = (d_1 \geq \cdots \geq d_k)$ be the degree sequence of the vertices in A, $d_i = d(a_i), i = 1, \ldots, k$. We denote by G_0 the bipartite graph with stable sets A and $B' = \{b_1, \ldots, b_{d_1}\}$ where a_i is adjacent to the vertices $b_1, \ldots, b_{d_i}$, $i = 1, \ldots, k$, and call G_0 the *compression* of G. In this section we prove that, if G_0 admits a star forest decomposition then so does G. This reduces the problem of giving

sufficient conditions on the degree sequence of one stable set to ensure the existence of a star forest ASD to bipartite compressed graphs. For the proof of our reduction Lemma we use the following result [10] on edge list–colorings of bipartite multigraphs.

Theorem 3. *[10, Theorem 2.5] Let H be a bipartite multigraph with stable sets A and B. If H admits a proper edge–coloring such that each vertex $a \in A$ is incident with edges colored $\{1, 2, \ldots, d(a)\}$, then H can be properly edge–colored for an arbitrary assignment of lists $\{L(a) : a \in A\}$ such that $|L(a)| = d(a)$ for each $a \in A$.* □

Lemma 2 (Reduction Lemma). *Let G be a bipartite graph with bipartition $A = \{a_1, \ldots, a_k\}$ and B and degree sequence $d = (d_1 \geq \cdots \geq d_k)$, $d = d(a_i)$, of the vertices in A. If the compression G_0 of G admits a decomposition*

$$G_0 = F_1' \oplus \cdots \oplus F_t',$$

where each F_i' is a star forest, then G has an edge decomposition

$$G = F_1 \oplus \cdots \oplus F_t$$

where $F_i \cong F_i'$ for each $i = 1, \ldots, t$.

Proof. Let C be the $(k \times t)$ matrix whose entry c_{ij} is the number of edges incident to a_i in the star forest F_j' of the edge decomposition of G_0.

Consider the bipartite multigraph H with A and $U = \{u_1, \ldots, u_t\}$ as stable sets, where a_i is joined with u_j with c_{ij} parallel edges. Now, for each pair (i, j), color the c_{ij} parallel edges of H with the neighbors of a_i in the forest F_j' bijectively. Note that in this way we get a proper edge–coloring of H: two edges incident with a vertex a_i receive different colors since the bipartite graph G_0 has no multiple edges, and two edges incident to a vertex u_j receive different colors since F_j' is a star forest.

By the definition of the bipartite graph G_0, each vertex $a_i \in A$ is incident in the bipartite multigraph H with edges colored $1, 2, \ldots, d_i$. Let $L(a_i)$ be the list of neighbours of a_i in the original bipartite graph G. By Theorem 3, there is a proper edge–coloring χ' of H in which the edges incident to vertex a_i in A receive the colors from the list $L(a_i)$ for each $i = 1, \ldots, k$. Now construct F_s by letting the edge $a_i b_j$ be in F_s whenever the edge $a_i u_s$ is colored b_j in the latter edge–coloring of H. Thus F_s has the same number of edges than F_s' and the degree of a_i in F_s is c_{is}, the same as in F_s'. Moreover, since the coloring is proper, F_s' is a star forest. This concludes the proof. □ ∎

4 Ascending Star forest decompositions

Let $G = G(A, B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. We denote by $d = (d_1 \geq \cdots \geq d_k)$ the degree sequence of the vertices in the stable set A of G.

We focus on star forest ASD with the stars of the decomposition centered at the vertices in A. We say that a degree sequence $d = (d_1 \geq \cdots \geq d_k)$ with $\sum_i d_i = \binom{n+1}{2}$ is *good* if every bipartite graph $G(A, B)$ with $A = \{a_1, \ldots, a_k\}$ and $d(a_i) = d_i$, $1 \leq i \leq k$, admits a star forest ASD with the centers of the stars in A. We next give a necessary condition for a sequence to be good.

Lemma 3. *If the sequence* $(d_1 \geq \cdots \geq d_k)$ *is good then*

$$\sum_{i=1}^{t} d_i \geq \sum_{i=1}^{t} (n - i + 1) \quad \textit{for each} \quad t = 1, \ldots, k. \tag{1}$$

Proof. Consider the compressed bipartite graph $G = G(A, B)$ such that $a_i \in A$ is adjacent to $\{b_j : \ j = 1, \ldots, d_i\}$, $i = 1, \ldots, k$. Let

$$G = F_1 \oplus \cdots \oplus F_n$$

be a star forest ASD of G. Since F_n has n leaves in B we clearly have $|B| = d_1 \geq n$. Thus (1) is satisfied for $t = 1$.

Suppose that (1) is satisfied for some $t = j-1 < k$. If $d_j \geq n-j+1$ then the inequality extends to $t = j$. Suppose that $d_j \leq n - j$. Since G is compressed, the neighborhhod of the vertices $a_{j+1}, \ldots, a_k$ is contained in the neighborhood of a_j. It follows that the forest F_n has at least $(n - d_j)$ end–vertices adjacent only to vertices from $\{a_1, \ldots, a_j\}$. Likewise, F_{n-i} has at least $(n - i + 1 - d_j)$ end–vertices adjacent only to vertices from $\{a_1, \ldots, a_t\}$, $i = 1, \ldots, t$. Hence,

$$\begin{aligned} \sum_{i=1}^{j} d_i &\geq j d_j + (n - d_j) + (n - 1 - d_j) + \cdots + (n - t + 1 - d_j) \\ &= n + (n - 1) + \cdots + (n - j + 1), \end{aligned}$$

and (1) is satisfied for $t = j$. This concludes the proof. □ ∎

We next obtain a close sufficient condition for a sequence to be good. We first introduce some definitions.

Given two k–dimensional vectors $c = (c_1, \ldots, c_k)$ and $c' = (c'_1, \ldots, c'_k)$, we say that $c \preceq c'$ if there is a permutation $\sigma \in Sym(k)$ such that $c_i \leq c'_{\sigma(i)}$ for $i = 1, 2, \ldots, k$. In other words, after reordering the components of each vector in nonincreasing order, the i–th component of c is not larger than the i–th component of c'. This definition is motivated by the following remark.

Remark 1. Let F, F' be two forests of stars with centers $x_1, \ldots, x_k$ and $x'_1, \ldots, x'_k$ respectively. Then F is isomorphic to a subgraph of F' if and only if $(d_F(x_1), \ldots, d_F(x_k)) \preceq (d_{F'}(x'_1), \ldots, d_{F'}(x'_k))$. □

Given a sequence $d = (d_1 \geq \cdots \geq d_k)$ of positive integers with $\sum_i d_i = \binom{n+1}{2}$, we say that a $(k \times n)$ matrix C with nonnegative integer entries is *d–ascending* if it satisfies the following three properties:

(A1) $\sum_j c_{ij} = d_i$, $i = 1, \ldots, k$,
(A2) $\sum_i c_{ij} = n - j + 1$, $j = 1, \ldots, n$,
(A3) $c^j \succeq c^{j+1}$, $j = 1, \ldots, n-1$, where c^j denotes the j–th column of C.

Next Lemma gives a sufficient condition for a degree sequence to be good assuming the existence of an appropriate ascending matrix.

Lemma 4. *Let $d = (d_1 \geq \cdots \geq d_k)$ be a sequence of positive integers with $\sum_i d_i = \binom{n+1}{2}$. Suppose that there is a d–ascending matrix C such that $c_{ij} \geq 1$ for each pair (i, j) with $i + j \leq k + 1$. If*

$$d_i \geq n - i + 1,\ i = 1, \ldots, k-1,$$

then d is good.

Proof. By the reduction Lemma it suffices to show that the compressed graph G with degree sequence d admits a star forest decomposition.

Let H be the bipartite multigraph with stable sets $A = \{a_1, \ldots, a_k\}$ and $U = \{u_1, \ldots, u_n\}$ and with c_{ij} parallel edges joining $a_i \in A$ with $u_j \in U$. We next show that H can be properly edge–colored in such a way that the edges incident to a_i receive colors from the set $\{1, \ldots, d_i\}$, $i = 1, \ldots, k$.

For each $s = 1, \ldots, k$ denote by M_s the matching in H formed by the s edges

$$a_1u_s, a_2u_{s-1}, \ldots, a_su_1.$$

Such matchings exist by the condition $c_{ij} \geq 1$ for each pair (i, j) with $i + j \leq k + 1$. We color the edges of the matching M_s with $k - s + 1$. In this way the vertex a_i is incident in $M_1 \oplus \cdots \oplus M_k$ with edges colored $\{1, \ldots, k - i + 1\}$.

Let H' denote the bipartite multigraph obtained from H by removing the edges in $M_1 \oplus \cdots \oplus M_k$. Let $d'_A = (d'_1 \geq \cdots \geq d'_k)$ be the degree sequence of A in H'. Since $d_i \geq n - i + 1$, we have $d'_i = d_i - (k - i + 1) \geq n - k$, $i = 1, \ldots, k$. On the other hand, each vertex u_i has degree $n - i + 1$ in H and, for $i \leq k$, it is incident to the matchings $M_1, \ldots, M_{k+1-i}$. Hence, every vertex in U has degree at most $n - k$ in H'.

Let $\Delta'(A)$ be the maximum degree in H' of the vertices in A. If $\Delta'(A) > n - k$ then there is a matching M'_1 in H' from the vertices of maximum degree in A to U. Color the edges of this matching with $\Delta'(A)$. By removing this matching from H' we obtain a bipartite multigraph in which the maximum degree of vertices in A is $\Delta'(A) - 1$. By iterating this process we reach a bipartite multigraph H'' with $\Delta''(A) = n - k$, while the maximum degree of the vertices in U still satisfies $\Delta''(U) \leq n - k$. By König's theorem, the edge–chromatic number of H'' is $n - k$. Hence H' can be properly edge–colored in such a way that vertex a_i is incident in H' with colors $\{1, \ldots, d'_i = d_i - (k - i + 1)\}$. By Theorem **??**, there is also a proper edge–coloring of H' in which each vertex a_i is incident with edges colored $\{k - i + 1, k - i + 2, \ldots, d_i\}$. By combining this coloring with the one of $M_1 \oplus \cdots \oplus M_k$ defined above we

get a proper edge–coloring of the original bipartite multigraph H in which the vertex a_i is incident with edges colored $\{1,\ldots,d_i\}$ for each $i=1,\ldots,k$.

We use this coloring to obtain a star forest decomposition

$$G = F_1 \oplus \cdots \oplus F_n,$$

of the compressed bipartite graph with stable sets A and $B=\{b_1,\ldots,b_{d_1}\}$ by letting F_s consist of the edges a_ib_j such that a_iu_s is colored b_j in the edge-colored multigraph H. Thus F_s has degree sequence $d_A(F_s)=(c_{s1},\ldots,c_{sk})$. By the column sum property of the matrix C, the star forest F_s has $\sum_i c_{is} = n-s+1$ edges and, by the ascending column property, it is isomorphic to a subgraph of F_{s-1}. This completes the proof. □ ∎

We are now ready to prove our main result.

Proof. Consider the matrix C' whose first row vector is $(n-k+1,\ldots,n-k+1,n-k,n-k-1,\ldots 3,2,1)$ and, for $j>1$ the i-th row of C' has $k-i+1$ ones followed by zeros. The sum of the entries of the j–th column of C' is $n-i+1$. Thus C' is an ascending matrix for the degree sequence $d'_1=\binom{n-k}{2}+(n-k+1)k$ and $d'_i=k-i+1$ for $i=2,\ldots,k$.

Hence,

$$d_i - d'_i \geq n-i+2-(k-i+1) = n-k+1,\ i=2,\ldots,k.$$

Consider the sequence $(d_2-d'_2,\ldots,d_k-d'_k,\alpha)$ where α is such that $\sum_{i=2}^{k}(d_i-d'_i)+\alpha=\binom{n-k+1}{2}$. Since all but perhaps one elements of the sequence are larger than $n-k+1$, the sequence is $(n-k+1)$–realizable. Thus there is a partition $\{X_2,\ldots,X_k,X_{k+1}\}$ of the interval $[1,n-k+1]$ with $\Sigma(X_i)=d_i-d'_i$, $i=2,\ldots,k$.

By permuting the corresponding entries from the first column to the i–th we get an ascending matrix which has row sum d_A. □ ∎

The next Figure illustrates an example.

$$C=\begin{pmatrix} 5&5&5&5&4&3&2&1\\ 1&1&1&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}\quad C=\begin{pmatrix} 5&5&5&0&0&0&0&1\\ 1&1&1&5&0&0&0&0\\ 1&1&0&0&0&3&2&0\\ 1&0&0&0&4&0&0&0 \end{pmatrix}$$

Fig. 1: An example with $n=8$ and the sequence $d_A=(16,8,7,5)$.

References

[1] Y. Alavi, A. J.Boals, G. Chartrand, P. Erdös, O. Oellerman, The ascending subgraph decomposition problem, *Congressus Numeratium* **58** (1987), 7–14.

[2] K. Ando, S. Gervacio, M. Kano, Disjoint integer subsets having a constant sum, *Discrete Mathematics* **82** (1990), 7–11.

[3] Chen Huaitang and Ma Kejie, On the ascending subgraph decompositions of regular graphs, *Applied Mathematics - A Journal of Chinese Universities* Volume 13, Number 2 (1998) 165–170

[4] F. L. Chen, H. L. Fu, Y. Wang and J. Zhou. Partition of a set of integers into subsets with prescribed sums. *Taiwanese Journal of Mathematics*, **9** (4) (2005), 629–638.

[5] R. J. Faudree, A. Gyarfas, R. H. Schelp, Graphs which have an ascending subgraph decomposition, *Congressus Numeratium* **59** (1987), 49–54.

[6] H.L.Fu,A note on the ascending subgraph decomposition problem, Discrete Math. **84** (1990) 315–318.

[7] H. L. Fu, W. H. Hu, A special partition of the set I_n, *Bulletin of ICA* **6** (1992), 57–61.

[8] H. L. Fu, W. H. Hu, A Note on ascending subgraph decompositions of complete multipartite graphs *Discrete Mathematics* **226** (2001) 397–402.

[9] H. L. Fu, W. H. Hu, Ascending subgraph decomposition of regular graphs *Discrete Mathematics* **253** (2002) 11–18.

[10] Häggkvist, Roland Restricted edge-colourings of bipartite graphs. Combin. Probab. Comput. 5 (1996), no. 4, 385–392.

[11] P. Hamburger, W. Cao, Edge Disjoint Paths of Increasing Order in Complete Bipartite Graphs, Electronic Notes in Discrete Mathematics 22 (2005) 61–67.

[12] Anna Lladó, Jordi Moragas, On the modular sumset partition problem, European Journal of Combinatorics, Volume 33, Issue 4, May 2012, 427–434, http://dx.doi.org/10.1016/j.ejc.2011.09.001.

[13] K. Ma, H. Zhou, J. Zhou, On the ascending star subgraph decomposition of star forests, *Combinatorica* **14** (1994), 307–320.

Edge neighbor connectivity in graphs with given girth *

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Abstract. Let G be a connected graph. An edge e of G is subverted if the set of adjacent edges to e is deleted from G, together with the two ends of e. An edge set S is subverted in a graph G, and it is denoted by $G \ominus S$, if every edge in S is subverted. An edge cut strategy is a set of edges S such that $G \ominus S$ is either disconnected, or empty, or trivial (a graph is trivial if it contains only one vertex). The edge neighbor connectivity of G is the cardinality of a minimum edge cut strategy, denoted by $\lambda_{NB}(G)$. It is easy to check that $\lambda_{NB}(G) \leq \delta(G)$, being $\delta(G)$ the minimum degree of G. We prove that the previous inequality is, in fact, an equality for every connected graph of girth $g \geq 5$ and diameter at most $g - 3$.

Key words: Connectivity, edge neighbor connectivity, diameter, girth.

1 Introduction

Throughout this paper, all the graphs are simple, that is, without loops and multiple edges. Notations and terminology not explicitly given here can be found in the book by Chartrand and Lesniak [4].

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The cardinalities of these sets are denoted by $|V(G)|$ and $|E(G)|$. For any subset W of vertices of G, the induced subgraph by W in G, denoted by $G[W]$, has vertex set W and edge set formed by all the edges of G with the two ends in W. A subset W of vertices of G is called an independent set if $G[W]$ has no edges. The maximal size of an independent subset of vertices of G is called the *independence number* of G and it is denoted by $\beta(G)$. Let u and v be two distinct vertices of G. A path from u to v, also called an uv-path in G, is a subgraph P with vertex set $V(P) = \{u = x_0, x_1, \ldots, x_r = v\}$

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and edge set $E(P) = \{x_0x_1, \ldots, x_{r-1}x_r\}$. This path is usually denoted by $P : x_0x_1 \cdots x_r$ and r is the length of P. A cycle in G of length r is a path $C : x_0x_1 \cdots x_r$ such that $x_0 = x_r$. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G, if any. Otherwise, we set $g(G) = \infty$. The set of adjacent vertices to $v \in V(G)$ is denoted by $N_G(v)$ and it is called the (open) neighborhood, whereas the closed neighborhood, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. For any subset $W \subseteq V(G)$, $N_G(W) = \left(\bigcup_{v \in W} N_G(v)\right) \setminus W$ and $N_G[W] = W \cup \bigcup_{v \in W} N_G(v)$. The *degree* of v is $d_G(v) = |N_G(v)|$, whereas $\delta(G) = \min\{d_G(v) : v \in V(G)\}$ and $\Delta(G) = \max\{d_G(v) : v \in V(G)\}$ stand for the *minimum degree* and the *maximum degree* of G, respectively.

A graph is said to be *connected* if for every pair of vertices there is a path connecting them. The distance between two vertices u, v in G, $d_G(u, v)$, is the length of a shortest uv-path. The *diameter* of G, written as $D = D(G)$, is the maximum among all the distances between pairs of vertices of G. Observe that $D(G)$ is finite whenever G is connected. A *cut set* of a connected graph G is a set X of vertices such that $G - X$ is not connected or is an isolated vertex. Each connected subgraph of $G - X$ is called a *component* of $G - X$. The *(vertex)-connectivity* of G, denoted $\kappa(G)$, is the minimum cardinality of a cut set, and it is widely known that $\kappa(G) \leq \delta(G)$.

If we think of a graph as a model of a network, vulnerability parameters [1] measure the resistance of the network to disruption of operations after the failure of certain stations or communication links. There are different parameters to analyze the efficiency of a network, as connectivity [8], toughness [2], binding number [9], integrity [3] and tenacity [5]. In a Spy Network, vertices correspond to stations or operatives, and edges represent lines of communication. If a station or an operative is captured, the adjacent stations will be betrayed and are therefore useless in the whole network. Therefore, instead of removing only vertices from a graph, the analysis is focused in the resulting structure by removing a set of vertices or edges and all of their adjacent ones. The search of nodes and/or edges whose failure could cause a collapse of the system is very important in network design, whether the goal is to break the network as if the aim is to maintain its efficiency. The previous case of spy network is an example where the objective is to cause maximum damage to the network, while in the case of transport logistics network the aim is just to analyze what sites are most critical in the efficiency of the network.

Gunter and Hartnell [7] introduced the concept of *neighbor connectivity* of a graph, in which a vertex is said to be *subverted* when the complete closed neighborhood of the vertex is removed from the graph. The concept was extended to an edge version by Cozzens and Wu [6]. Given an edge $e = uv \in E(G)$, the open neighborhood of e is the set $\omega_G(e) = \{e' \in E(G) : e'$ is adjacent with e in $G\}$ and the closed neighborhood is the set $\omega_G[e] = \omega_G(e) \cup \{e\}$. For a set of edges S, $\omega_G(S) = \{e' \in E(G) \setminus S : e'$ is adjacent with some edge of S in $G\}$ and $\omega_G[S] = \omega_G(S) \cup S$. A set of

edges S is said to be *subverted* if the edges of S together with its end vertices are deleted from the graph. We denote the subversion of an edge-set S in the graph G by $G \ominus S$. Let us observe that $G \ominus S = G[V(G) \setminus V(G[S])] = G - V(G[S])$.

An *edge cut strategy* is a set of edges of G such that $G \ominus S$ is either empty, or the trivial graph K_1, or a disconnected graph. The *edge neighbor connectivity* of a graph is the cardinality of an edge cut strategy of minimum cardinality, and it is denoted by $\lambda_{NB}(G)$.

In [6] Cozzens and Wu obtained some bounds for the edge neighbor connectivity and they gave some classes of graph achieving these bounds. They determined λ_{NB} for complete bipartite and Harary graphs. In [10] Zhao et al. approached the study of the edge neighbor connectivity of the Cartesian product graph $G \times K_2$, being G a regular graph. Moreover, they proved a more general result for some bijective connection graphs.

Observe that given a vertex $v \in V(G)$ of degree $d_G(v) = \delta(G)$, the set $S = \omega_G(N_G(v)) \setminus \omega_G(v)$ is an edge cut strategy, since v is isolated when $X = V(G[S])$ is removed from G. Furthermore, $N_G(v) \subseteq X$, yielding that $\lambda_{NB} \leq |S| \leq |N_G(v)| = d_G(v) = \delta(G)$. Thus, $\lambda_{NB}(G) \leq \delta(G)$ clearly holds for every connected graph G.

In this work we give a sufficient condition for a connected graph G to have $\lambda_{NB}(G) = \delta(G)$. Namely, we prove that equality holds if $D(G) \leq g(G) - 3$.

2 Main result

Since every edge has exactly two ends, we see that $|V(G[S])| \leq 2|S|$ holds for every edge set S. Moreover, denoting by $X = V(G[S])$, we also have $|S| \geq \beta(G[X])$.

In order to prove the main result, for a given edge cut strategy S in a connected graph G, we previously relate the maximum distance from a vertex of $G \ominus S$ to S, to the independence number of $G[X]$, being $X = V(G[S])$.

Lemma 1. *Let G be a connected graph with girth $g \geq 6$ and minimum degree δ. Let S be an edge cut strategy of G and let X be the set of vertices of $G[S]$. If $\max\{d(v, X) : v \in V(C)\} \leq \lfloor (g-4)/2 \rfloor$ for some component C of $G \ominus S$, then $\beta(G[X]) \geq \delta$.*

The second technical lemma relates the structure of a component C of $G \ominus S$ whose maximum distance to S is $(g(G) - 3)/2$, to the independence number of $G[X]$, being $X = V(G[S])$.

Lemma 2. *Let G be a connected graph with odd girth $g \geq 5$ and minimum degree δ. Let S be an edge cut strategy of G and let X be the set of vertices of $G[S]$. Denote by $\mu(C) = \max\{d(v, X) : v \in V(C)\}$ and by $\mathcal{F}(C) = \{v \in V(C) : d(v, X) = \mu(C)\}$, for every component C of $G \ominus S$. If there exists a*

component C of $G \ominus S$ such that $\mu(C) = (g-3)/2$, then $\beta(G[X]) \geq \delta$ or $G[\mathcal{F}(C)]$ contains a path of length at least 2.

As a consequence of Lemma 1 and Lemma 2 we give a sufficient condition for a connected graph G to have $\lambda_{NB}(G) = \delta(G)$.

Theorem 1. *Let G be a connected graph of girth $g \geq 5$ and minimum degree δ. Then $\lambda_{NB}(G) = \delta$ if $D(G) \leq g - 3$.*

An interesting open question is to prove if Theorem 1 is best possible in the sense that the hypothesis on the diameter-girth cannot be relaxed. At this moment, we have no answer to this question. It is easy to find graphs G with girth $g \geq 5$ and diameter $D(G) = g - 1$ such that $\lambda_{NB}(G) < \delta(G)$. For instance, the graph G formed by two hexagons which share an edge has minimum degree 2, girth 6 and diameter 5 (that is, $D(G) = g(G)-1$). Clearly, the set S formed by the common edge of both hexagons is an edge cut strategy, which means that $\lambda_{NB}(G) = 1 < 2 = \delta(G)$. However, at this moment, we have not found a graph with girth g at least 5 and diameter $D = g - 2$ such that $\lambda_{NB} < \delta$.

References

[1] C.A. Barefoot, R. Entringer and H. Swart. Vulnerability in graphs-a comparative survey. *J. Comb. Math. Comb. Comput.*, 1:13–22, 1987.

[2] D. Bauer, H. Broersma and E. Schmeichel. Toughness in graphs–a survey. *Graphs Combin.*, 22(1):1–35, 2006.

[3] D. Benko, C. Ernst and D. Lanphier. Asymptotic Bounds on the Integrity of Graphs and Separator Theorems for Graphs. *SIAM J. Discrete Math.*, 23(1):265–277, 2009.

[4] G. Chartrand and L. Lesniak. Graphs and Digraphs. *Chapman and Hall/CRC*, 2005.

[5] M.B. Cozzens. Stability measures and data fusion networks. *Graph Theory of New York*, 26:8–14, 1994.

[6] M.B. Cozzens and S.-S.Y. Wu. Extreme values of the edge neighbor connectivity. *Ars Combin.*, 39:199–210, 1995.

[7] G. Gunter and B. Hartnell. On minimizing the effects of betrayals in a resistance movement. *Proceeding of the eight Manitoba Conference on Numerical Mathematics and Computing*, Winnipeg, Manitoba, Canada, 285–306, 1978.

[8] A. Hellwig and L. Volkmann. Maximally edge-connected and vertex-connected graphs and digraphs: a survey. *Discr. Math.*, 308(15):3265–3296, 2008.

[9] J. Lyle and W. Goddard. The binding number of a graph and its cliques. *Discr. Appl. Math.*, 157(15):3336–3340, 2009.

[10] X. Zhao, Z. Zhang and Q. Ren. Edge neighbor connectivity of Cartesian product graph $G \times K_2$. *Appl. Math. Comp.*, 217(12):5508–5512, 2011.

Closed formulae for the local metric dimension of corona product graphs

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Abstract. A vertex $v \in V(G)$ is said to distinguish two vertices $x, y \in V(G)$ of a nontrivial connected graph G if the distance from v to x is different from the distance from v to y. A set $S \subset V(G)$ is a *local metric generator* for G if every two adjacent vertices of G are distinguished by some vertex of S. A local metric generator with the minimum cardinality is called a *local metric basis* for G and its cardinality, the *local metric dimension* of G. In this paper we study the problem of finding exact values for the local metric dimension of corona product of graphs.

Key words: Metric generator; metric dimension; local metric set; local metric dimension, corona product graph.

1 Introduction

A generator of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by the distances from the elements of S. Given a simple and connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{R}^+$, where $d_G(x, y)$ is the length of a shortest path between u and v. Clearly, (V, d_G) is a metric space, i.e., d_G satisfies $d_G(x, x) = 0$ for all $x \in V$, $d_G(x, y) = d_G(y, x)$ for all $x, y \in V$ and $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$ for all $x, y, z \in V$. A vertex $v \in V$ is said to distinguish two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subset V$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. A metric generator with the minimum cardinality is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [8], where the metric generators were called *locating sets*. The concept of metric dimension of a graph was also introduced by Harary and Melter in [3], where metric generators were called *resolving sets*. Applications of this

invariant to the navigation of robots in networks are discussed in [7] and applications to chemistry in [5,6]. Several variations of metric generators including resolving dominating sets [1], independent resolving sets [3], local metric sets [7], strong resolving sets [8], etc. have since been introduced and studied.

In this article we are interested in the study of local metric generators, also called local metric sets [7]. A set S of vertices in a connected graph G is a *local metric generator* for G if every two adjacent vertices of G are distinguished by some vertex of S. A local metric generator with the minimum cardinality is called a *local metric basis* for G and its cardinality, the *local metric dimension* of G, is denoted by $\dim_l(G)$.

In this paper we study the local metric dimension of corona product graphs. We begin by giving some basic concepts and notations. For two adjacent vertices u and v of $G = (V, E)$ we use the notation $u \sim v$ and for two isomorphic graphs G and G' we use $G \cong G'$. For a vertex v of G, $N_G(v)$ denotes the set of neighbors that v has in G, i.e., $N_G(v) = \{u \in V : \ u \sim v\}$. The set $N_G(v)$ is called the *open neighborhood of* v in G and $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood of* v in G. The degree of a vertex v of G will be denoted by $\delta_G(v)$, i.e., $\delta_G(v) = |N_G(v)|$. Given a set $S \subset V$, we denote by $\langle S\rangle_G$ the subgraph of G induced by S and by $N_G(S) = \cup_{v\in S} N_G(v)$ the *open neighborhood of* S. In particular, if $S = \{x\}$ we will use the notation $\langle x\rangle$ instead of $\langle\{x\}\rangle$.

We will use the notation K_n, $K_{r,s}$, C_n, N_n and P_n for complete graphs, complete bipartite graphs, cycle graphs, empty graphs and path graphs, respectively.

Let G and H be two graphs of order n and n_1, respectively. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G, [4]. We will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the copy of H such that $v_i \sim x$ for every $x \in V_i$.

The join $G + H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. For instance, the graph $K_1 + C_t$ is a wheel graph, $K_1 + K_r \cong K_{r+1}$ and $K_1 + N_t$ is isomorphic to a star graph whose central vertex is the vertex of K_1 and whose t leaves are the vertices of the empty graph N_t. From now on, the vertex of K_1 will be denoted by v.

The metric dimension of and related parameters have been recently studied for the case of corona graphs. For instance, the metric dimension was studied in [6] and [9], the strong metric dimension was studied in [5] and the partition dimension was studied in [12]. In this article we study the local metric dimension. The article is organized as follows: In section 2 we give closed formulae for $\dim_l(G \odot H)$ in terms of $\dim_l(G)$ and $\dim_l(K_1 \odot H)$. Then, we establish

lower and upper bounds for $\dim_l(G \odot H)$ by using the orders of G and H, and we characterize all graphs when the bounds are attained. Finally, in Section 3 we investigate the value of $\dim_l(G \odot H)$ when H is a bipartite graph of radius three, and in particular, we compute $\dim_l(G \odot T)$ when T is a tree.

2 General results

To begin with, we consider some straightforward cases. If H is an empty graph, then $K_1 \odot H$ is a star graph and $\dim_l(K_1 \odot H) = 1$. Moreover, if H is a complete graph of order n, then $K_1 \odot H$ is a complete graph of order $n+1$ and $\dim_l(K_1 \odot H) = n$.

Theorem 1. *Let G be a connected nontrivial graph. For any empty graph H,*

$$\dim_l(G \odot H) = \dim_l(G).$$

We present now the main result on the local metric dimension of corona graphs $G \odot H$ for the case where H is a non-empty graph.

Theorem 2. *Let H be a non-empty graph. The following assertions hold.*

(i) *If the vertex of K_1 does not belong to any local metric basis for $K_1 + H$, then for any connected graph G of order n,*

$$\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H).$$

(ii) *If the vertex of K_1 belongs to a local metric basis for $K_1 + H$, then for any connected graph G of order $n \geq 2$,*

$$\dim_l(G \odot H) = n(\dim_l(K_1 + H) - 1).$$

As a direct consequence of Theorem 2 we obtain the following results.

Corollary 1. *The following assertions hold for any connected graph G of order $n \geq 2$.*

(i) *For any integer $t \geq 2$, $\dim_l(G \odot K_t) = n(t-1)$.*
(ii) *For any positive integers r and s, $\dim_l(G \odot K_{r,s}) = n$.*
(iii) *Let $t \geq 4$ be an integer. If $t \equiv 1(4)$, then $\dim_l(G \odot P_t) = n \left\lfloor \frac{t}{4} \right\rfloor$ and if $t \not\equiv 1(4)$, then $\dim_l(G \odot P_t) = n \left\lceil \frac{t}{4} \right\rceil$.*
(iv) *For any integer $t \geq 4$, $\dim_l(G \odot C_t) = n \left\lceil \frac{t}{4} \right\rceil$.*

Corollary 2. *For any connected graph H and any connected graph G of order $n \geq 2$, $\dim_l(G \odot H) \geq n \cdot \dim_l(H)$.*

Now we will give some results involving the diameter $D(H)$ or the radius $r(H)$ of H.

Corollary 3. *For any graph H of diameter two and any connected graph G of order $n \geq 2$,* $\dim_l(G \odot H) = n \cdot \dim_l(H)$.

Lemma 1. *Let H be a graph of radius $r(H)$. If $r(H) \geq 4$ then the vertex of K_1 does not belong to any local metric basis for $K_1 + H$.*

The converse of Lemma 1 is not true. In Figure 1 we show a graph H of radius three where the vertex of K_1 does not belong to any local metric basis for $K_1 + H$.

Fig. 1: A graph H and the join graph $K_1 + H$. White vertices form a local metric basis for $K_1 + H$.

Theorem 3. *For any connected graph G of order n and any graph H of radius $r(H) \geq 4$,* $\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H)$.

Another consequence of Theorem 2 is the following result.

Corollary 4. *For any non-empty graph H of order n' and any connected graph G of order $n \geq 2$,* $n \leq \dim_l(G \odot H) \leq n(n' - 1)$.

Theorem 4. *Let H be a graph of order n' and let G be a connected graph of order $n \geq 2$. Then* $\dim_l(G \odot H) = n(n' - 1)$ *if and only if* $H \cong K_{n'}$ *or* $H \cong K_1 \cup K_{n'-1}$.

The radius $r(G)$ of a graph G is the minimum eccentricity of any vertex of G. The center of G, denoted by $C(G)$, is the set of vertices of G with eccentricity equal to $r(G)$.

Theorem 5. *Let H be a non-empty graph and let G be a connected graph of order $n \geq 2$. Then* $\dim_l(G \odot H) = n$ *if and only if H is a bipartite graph having only one non-trivial connected component H^* and $r(H^*) \leq 2$.*

3 The value of $\dim_l(G \odot H)$ when H is a bipartite graph of radius three

Theorems 3 and 5 and bipartite graphs of radius three. To do that, we need the following additional notation. For any $a \in V(H)$, we denote

$$N_H^{(i)}(a) = \{w \in V(H) : \ d_H(w, a) = i\}.$$

We also define $N_H^{(i)}[a] = N_H^{(i)}(a) \cup \{a\}$. Note that $N_H^{(1)}(a) = N_H(a)$ and $N_H^{(1)}[a] = N_H[a]$. Given two sets $A, B \subset V(H)$ we say that A dominates B if every vertex in $B - A$ is adjacent to some vertex belonging to A. From now on we will use the notation $A \succ B$ to indicate that A dominates B. For every $x \in C(H)$, let $\beta(x) = \min\left\{|A| : \ A \subseteq N_H(x) \text{ and } A \succ N_H^{(2)}(x)\right\}$ and let $\delta'(H) = \min_{x \in C(H)} \{\beta(x)\}$.

Lemma 2. *For any bipartite graph H of radius three,* $\dim_l(K_1+H) \leq \delta'(H)+1$. *Moreover,* $\dim_l(K_1+H) = \delta'(H)+1$ *if and only if the vertex of K_1 belongs to a local metric basis for $K_1 + H$.*

Theorem 6. *Let H be a bipartite graph of radius three and let G be a connected graph of order $n \geq 2$. Then* $\dim_l(G \odot H) \leq n \cdot \delta'(H)$.

Now we show that the above bound is attained for a subfamily of bipartite graphs of diameter three that does not contain a square (a subgraph isomorphic to $K_{2,2}$). In such a case, the girth of H must be six and $H = (U_1 \cup U_2, E)$ satisfies the following property:

- For any $i \in \{1, 2\}$ and any two distinct vertices $a, b \in U_i$, $|N_H(a) \cap N_H(b)| = 1$.

Therefore, H is the incidence graph of a finite projective plane. So, we have two possibilities (see, for instance, [1]):

(P1) $H = (U_1 \cup U_2, E)$ is the incidence graph of a degenerate projective plane. In this case $|U_1| = |U_2| = t$, $t \geq 3$, and H is a pseudo sphere graph S_t (also called near pencil) defined as follows: we consider $t-1$ path graphs of order 4 and we identify one extreme of each one of the $t-1$ path graphs in one pole a and all the other extreme vertices of the paths in a pole b. In particular, S_3 is the cycle graph C_6.

(P2) $H = (U_1 \cup U_2, E)$ is the incidence graph of a non-degenerate projective plane of order q. In this case H is a regular graph of degree $\delta_H = q + 1$ and $|U_1| = |U_2| = q^2 + q + 1$. Note that $|U_1| = |U_2| = \delta_H^2 - \delta_H + 1$.

In the case (P1) the set $B = \{a, b\}$ composed by both poles of the pseudo sphere is a dominating set of S_t. Thus, B is a local metric basis for $\langle v \rangle + S_r$ and $N_{S_t}(a) \cap N_{S_t}(b) = \emptyset$. Also, there are no local metric generators composed by two vertices at distance two, so the vertex v does not belong to any local metric basis for $\langle v \rangle + S_t$ and, by Theorem 2 (i), we obtain that for any connected graph G of order $n \geq 2$, $\dim_l(G \odot S_t) = 2n$.

The rest of this section covers the study of case (P2), *i.e.*, the case where H is the incidence graph of a non-degenerate projective plane.

Lemma 3. *For any bipartite graph $H \ncong S_t$ of diameter three and girth six,* $\delta'(H) = \delta_H$.

Lemma 4. *Let $H \ncong S_t$ be a bipartite graph of diameter three and girth six. Then the vertex of K_1 belongs to any local metric basis for $K_1 + H$.*

Theorem 7. *Let $H \ncong S_t$ be a bipartite graph of diameter three and girth six. Then for any connected graph G of order $n \geq 2$, $\dim_l(G \odot H) = n \cdot \delta_H$.*

Let $\pi = (P, L)$ be a finite non-degenerate projective plane of order q, where P is the set of points and L is the set of lines. Given two sets $P' \subset P$ and $L' \subset L$, we say that $P' \cup L'$ satisfies the property $\mathcal{G}$, if for any point p_0 and any line l_0 such that $p_0 \in l_0$ we have

- there exists $p \in P'$ such that $p \in l_0$, or
- there exists $l \in L'$ such that $p_o \in l$.

We define $\Upsilon(\pi) = \min\{|P' \cup L'|$ such that $P' \cup L'$ satisfies the property $\mathcal{G}\}$.

We have that if H is the incidence graph of π, then a set $P' \cup L'$ satisfies the property $\mathcal{G}$ if and only if $P' \cup L' \cup \{v\}$ is a local metric generator for $\langle v \rangle + H$. Therefore, according to Lemmas 2, 3 and 4 we conclude $\Upsilon(\pi) = \delta_H = q$. Note that if $P' \cup L'$ satisfies the property $\mathcal{G}$ and its cardinality is the minimum among all the sets satisfying this property, then either $P' = \emptyset$ and L' is the set of lines incident to one point or $L' = \emptyset$ and P' is the set composed by all the points laying on one line.

As a direct consequence of Theorems 2 and 5 we derive the following result.

Remark 1. For any connected graph H of radius $r(H) \geq 3$ and any connected graph G of order $n \geq 2$, $\dim_l(G \odot H) \geq 2n$.

Lemma 5. *If H is a graph of radius three and $\dim_l(K_1 + H) = 2$, then the vertex of K_1 does not belong to any local metric basis for $K_1 + H$.*

Theorem 8. *Let $H = (U_1, U_2, E)$ be a bipartite graph of radius three and let G be a connected graph of order n. Then $\dim_l(G \odot H) = 2n$ if and only if $\dim_l(K_1 + H) = 2$ or for some $i \in \{1, 2\}$, there exist $a, b \in U_i$ such that $N_H(a) \cup N_H(b) = U_j$, where $j \in \{1, 2\} - \{i\}$.*

Consider the following decision problem. The input is an arbitrary bipartite graph $H = (U_1 \cup U_2, E)$ of radius three. The problem consists in deciding whether H satisfies $\dim_l(K_1 + H) = 2$, or not. According to the next remark we deduce that the time complexity of this decision problem is at most $O(|U_1|^2|U_2|^2)$.

Remark 2. Let $H = (U_1, U_2, E)$ be a bipartite graph of radius three. Consider the following statements:

(i) For some $i \in \{1, 2\}$, there exist $a, b \in U_i$ such that $\{N_H(a), N_H(b)\}$ is a partition of U_j, where $j \in \{1, 2\} - \{i\}$.
(ii) There exist two vertices $a \in U_1$ and $b \in U_2$ such that for every edge $xy \in E$, where $x \in U_1$ and $y \in U_2$, it follows $y \in N_H(a)$ or $x \in N_H(b)$.

Then $\dim_l(K_1 + H) = 2$ if and only if (i) or (ii) holds.

Note that if $H = (U_1 \cup U_2, E)$ is a bipartite graph of diameter $D(H) = 3$, then for any $i \in \{1, 2\}$ and $x, y \in U_i$ we have $N_H(x) \cap N_H(y) \neq \emptyset$. Hence, we deduce the following consequence of Remark 2.

Corollary 5. *Let H be a bipartite graph where $D(H) = r(H) = 3$. If $B = \{a, b\}$ is a local metric basis for $K_1 + H$, the a and b belong to different color classes.*

Corollary 6. *Let $H = (U_1, U_2, E)$ be a bipartite graph of radius three. If for some $i \in \{1, 2\}$, there exist $a \in U_i$ such that $\delta_H(a) = |U_j| - 1$, where $j \in \{1, 2\} - \{i\}$, then $\dim_l(K_1 + H) = 2$.*

3.1 Closed formulae for $\dim_l(G \odot H)$ when H is a tree of radius three

Let T be a tree of radius three. For the particular case when $C(T) = \{u\}$ we consider the forest $F_u = \cup_{w \in N_T(u)} T_w$ composed of all the rooted trees $T_w = (V_w, E_w)$, of root $w \in N_T(u)$, obtained by removing the central vertex u from T. The height of T_w is $h_w = \max_{x \in V(T_w)}\{d(w, x)\}$. We denote by $\varsigma(T)$ the number of trees in F_u with h_w equal to two, i.e., $\varsigma(T) = |S(T)|$, where

$$S(T) = \{w \in N_T(u) : \ h_w = 2\}.$$

Note that if $h_w \neq 1$, for every $w \in N_T(u)$, then $\varsigma(T) = \delta'(T)$. So, as the following result shows, the bound $\dim_l(G \odot T) \leq n \cdot \delta'(T)$ is tight.

Theorem 9. *Let T be a tree of radius three and center $C(T)$. The following assertion hold for any connected graph G of order $n \geq 2$.*

(i) *If $|C(T)| = 2$, then $\dim_l(G \odot T) = 2n$*
(ii) *If $C(T) = \{u\}$, then*

$$\dim_l(G \odot T) = \begin{cases} n \cdot (\varsigma(T) + 1), & \text{if there exists } w \in N_T(u) \text{ such that } h_w = 1, \\ n \cdot \varsigma(T), & \text{otherwise.} \end{cases}$$

References

[1] J. A. Bondy, U. S. R. Murty, Graph theory, vol. 244 of Graduate Texts in Mathematics, Springer, New York, 2008.

[2] R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, Mathematica Bohemica 128 (1) (2003) 25–36.

[3] G. Chartrand, V. Saenpholphat, P. Zhang, The independent resolving number of a graph, Mathematica Bohemica 128 (4) (2003) 379–393.

[4] R. Frucht, F. Harary, On the corona of two graphs, Aequationes Mathematicae 4 (3) (1970) 322–325.

[5] F. Harary, R. A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195.

[6] H. Iswadi, E.T Baskoro, R. Simanjuntak, On the Metric Dimension of Corona Product of Graphs, Far East Journal of Mathematical Sciences 52 (2010) 155 – 170.

[7] M. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics 3 (2) (1993) 203–236, pMID: 8220404.

[8] M. A. Johnson, Browsable structure-activity datasets, in: R. Carbó-Dorca, P. Mezey (eds.), Advances in Molecular Similarity, JAI Press Inc, Stamford, Connecticut, 1998, pp. 153–170.

[9] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70 (1996) 217–229.

[10] D. Kuziak, I. G. Yero and J. A. Rodríguez-Velázquez, On the strong metric dimension of corona product graphs and join graphs, Discrete Applied Mathematics 161 (2013) 1022–1027.

[11] F. Okamoto, B. Phinezy, P. Zhang, The local metric dimension of a graph, Mathematica Bohemica 135 (3) (2010) 239–255.

[12] J. A. Rodríguez-Velázquez and I. G. Yero and D. Kuziak, Partition dimension of corona product graphs, Ars Combinatoria. To appear.

[13] A. Sebö, E. Tannier, On metric generators of graphs, Mathematics of Operations Research 29 (2) (2004) 383–393.

[14] P. J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549–559.

[15] I. G. Yero, D. Kuziak, J. A. Rodríquez-Velázquez, On the metric dimension of corona product graphs, Computers & Mathematics with Applications 61 (9) (2011) 2793–2798.

Partial permutation decoding for binary linear Hadamard codes *

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Abstract. Permutation decoding is a technique which involves finding a subset S, called PD-set, of the permutation automorphism group $\mathrm{PAut}(C)$ of a code C in order to assist in decoding. A method to obtain s-PD-sets of size $s+1$ for partial permutation decoding for the binary linear Hadamard codes H_m of length 2^m, for all $m \geq 4$ and $1 < s \leq \left\lfloor \frac{2^m-m-1}{1+m} \right\rfloor$, is described. Moreover, a recursive construction to obtain s-PD-sets of size $s+1$ for H_{m+1} of length 2^{m+1}, from a given s-PD-set of the same size for the Hadamard code of half length H_m is also established.

Key words: Permutation decoding, Hadamard codes, automorphism groups.

1 Introduction

Let $\mathbb{F}_2^n$ be the set of all binary vectors of length n. The *Hamming weight* $\mathrm{wt}(v)$ of a vector $v \in \mathbb{F}_2^n$ is the number of nonzero coordinates in v. The *Hamming distance* $d(u,v)$ between two vectors $u, v \in \mathbb{F}_2^n$ is the number of coordinates in which u and v differ, that is, $d(u,v) = \mathrm{wt}(u+v)$. Let $\mathbf{0}$ and $\mathbf{1}$ denote the all-zero and all-one vectors, respectively.

A *binary code* C of length n is a subset of $\mathbb{F}_2^n$. The vectors of a code C are called *codewords* and the *minimum (Hamming) distance*, denoted by d, is the smallest distance between any pair of different codewords in C. We said that a code C is a *t-error-correcting code* if it corrects all error vectors of weight at most t and does not correct at least one error vector of weight $t+1$, so $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ [7]. A binary code C is *linear* if it is a k-dimensional subspace of $\mathbb{F}_2^n$. A *generator matrix* for a linear code C of length n and dimension k is any $k \times n$ matrix G whose rows forms a basis of C.

Let C be a binary code of length n. For a vector $v \in \mathbb{F}_2^n$ and a set $I \subseteq \{1, \ldots, n\}$, we denote by v_I the restriction of the vector v to the coordinates in I and by C_I the set $\{v_I \mid v \in C\}$. For example, if $I = \{1, \ldots, k\}$ and

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$v = (v_1, \ldots, v_n)$, then $v_I = (v_1, \ldots, v_k)$. Suppose that C has size $|C| = 2^k$. A set $I \subseteq \{1, \ldots, n\}$ of k coordinate positions is an *information set* for C if $|C_I| = 2^k$. For each information set $I \subseteq \{1, \ldots n\}$ of k coordinates positions, the set $\{1, \ldots, n\} \backslash I$ of the remaining $n - k$ coordinate positions is a *check set* for C. If C is linear, we can label the i^{th} coordinate position by the i^{th} column of a generator matrix of C, so we will consider any information set (or check set) not only as a set of coordinate positions, but also as the set of vectors representing these positions.

Let $\mathrm{Sym}(n)$ be the symmetric group of permutations on the set $\{1, \ldots, n\}$ acting on $\mathbb{F}_2^n$ by permuting the coordinates of each vector. More specifically, for every vector $v = (v_1, \ldots, v_n) \in \mathbb{F}_2^n$ and permutation $\sigma \in \mathrm{Sym}(n)$, we define $\sigma(v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$. Then, for any binary code C, we denote by $\mathrm{PAut}(C)$ the *permutation automorphism group* of C, that is, $\mathrm{PAut}(C) = \{\sigma \in \mathrm{Sym}(n) \mid \sigma(C) = C\}$.

Permutation decoding is a technique, introduced in [7] by MacWilliams, which involves finding a subset S of the permutation automorphism group $\mathrm{PAut}(C)$ of a code C in order to assist in decoding. The method works as follows: Given a t-error-correcting linear code $C \subseteq \mathbb{F}_2^n$ with fixed information set I, we denote by $y = x + e$ the received vector, where $x \in C$ and e is the error vector. Suppose that at most t errors occur, that is, $\mathrm{wt}(e) \leq t$. The aim of permutation decoding is to move all errors in a received vector out the information positions, that is, move the nonzero coordinates of e out of I, by using an automorphism of the code. This technique is strongly based on the existence of some special subsets $S \subseteq \mathrm{PAut}(C)$, called PD-sets.

Let C be a t-error-correcting linear code with information set I. A subset $S \subseteq \mathrm{PAut}(C)$ is a *PD-set* for the code C if every t-set of coordinate positions is moved out of the information set I by at least one element of the set S. Equivalently, a subset $S \subseteq \mathrm{PAut}(C)$ is an *s-PD-set* if every s-set of coordinate positions is moved out of the information set I by at least one element of S, where $1 \leq s \leq t$.

Let S_m be the binary simplex code of length $2^m - 1$, dimension m and minimum distance 2^{m-1} with generator matrix G_{S_m} containing as column vectors the $2^m - 1$ nonzero vectors from $\mathbb{F}_2^m$, with the basis elements e_i^T, $i \in \{1, \ldots, m\}$, in the first m positions. We take the set of standard basis elements of $\mathbb{F}_2^m$ to be the information set I_m of this code, that is, $I_m = \{e_1, \ldots, e_m\} = \{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$. Let H_m be the binary linear Hadamard code of length 2^m, that is, the extended code of the simplex code S_m with generator matrix G_{H_m} constructed from G_{S_m} by adding an all-one row vector and an all-zero column vector as follows:

$$G_{H_m} = \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{0} & G_{S_m} \end{pmatrix}. \tag{1}$$

Now we consider as information set for H_m the set $\mathcal{I}_m = \{w_1, \ldots, w_{m+1}\} = \{(1,0,\ldots,0), (1,1,\ldots,0), \ldots, (1,0,\ldots,1)\}$ consisting of the first $m+1$ column vectors from the matrix G_{H_m} considered as row vectors. The check set $\mathcal{C}_m$ for H_m is the set containing the remaining column vectors from the matrix G_{H_m} considered as row vectors and denoted by $\mathcal{C}_m = \{w_{m+2}, \ldots, w_{2^m}\}$.

It is a well-know fact that $\mathrm{PAut}(S_m) = GL(m,2)$, where $GL(m,2)$ is the general linear group of degree m over $\mathbb{F}_2$. It is also known that $\mathrm{PAut}(H_m) = AGL(m,2)$ [8]. Recall that the affine group $AGL(m,2)$ consists of all mappings $\alpha : \mathbb{F}_2^m \to \mathbb{F}_2^m$ of the form $\alpha(x^T) = Ax^T + b^T$ for $x \in \mathbb{F}_2^m$, where $A \in GL(m,2)$ and $b \in \mathbb{F}_2^m$, together with the function composition as the group operation. The monomorphism

$$\begin{array}{rcl} \varphi : AGL(m,2) & \longrightarrow & GL(m+1,2) \\ (b,A) & \longmapsto & \begin{pmatrix} 1 & b \\ \mathbf{0} & A \end{pmatrix} \end{array}$$

defines an isomorphism between $AGL(m,2)$ and the subgroup of $GL(m+1,2)$ consisting of all nonsingular matrices whose first column is $(1,0,\ldots,0)$. From now on, we identify the $AGL(m,2)$ with this subgroup.

Now, we describe how to identify a permutation $\sigma \in \mathrm{PAut}(H_m) \subseteq \mathrm{Sym}(2^m)$ with a matrix $B \in AGL(m,2)$. Recall that each coordinate position can be labelled by the corresponding column of the generator matrix G_{H_m} given in (1). The first $m+1$ coordinate positions are labelled by the vectors of the information set $\mathcal{I}_m$ and the remaining coordinate positions are represented by the vectors of the check set $\mathcal{C}_m$. The vector w_i represents the i^{th} position, for all $i \in \{1, \ldots, 2^m\}$. Note that an index $i \in \{1, \ldots, m+1\}$ represents a position in the information set and an index $i \in \{m+2, \ldots, 2^m\}$ a position in the check set. By multiplying any vector w_i by the matrix $B \in AGL(m,2)$, we obtain another vector from the information set or the check set. Thus, $w_i B = w_j$ will denote that the i^{th} position of a codeword moves to the j^{th} position of that codeword. Therefore, any matrix $B \in AGL(m,2)$ can be seen as a permutation of the coordinate positions and, in fact, an automorphism of the code H_m, that is, an element of $\mathrm{PAut}(H_m) \subseteq \mathrm{Sym}(2^m)$. Along the paper, we will represent PD-sets for H_m as subsets of matrices of the affine group $AGL(m,2)$.

In [3], it is shown how to find s-PD-sets of size $s+1$ that satisfy the Gordon-Schönheim bound for partial permutation decoding for the binary simplex code S_m, for all $m \geq 4$ and $1 < s \leq \left\lfloor \frac{2^m - m - 1}{m} \right\rfloor$. In this paper, we establish similar results for the binary linear Hadamard code H_m, for all $m \geq 4$ and $1 < s \leq \left\lfloor \frac{2^m - m - 1}{1+m} \right\rfloor$, following the same techniques as the ones described in [3]. In [27], a 2-PD-set of size 5 and 4-PD-sets of size $\binom{m+1}{2} + 2$ are found for binary linear Hadamard codes H_m, for all $m > 4$. As a consequence, 3-PD-sets of size $\binom{m+1}{2} + 2$ are also found for these codes. Small PD-sets that

satisfy the Gordon-Schönheim bound have been found for binary Golay codes [4,10] and for the binary simplex code S_4 [5,6].

This work is organized as follows. In Section 2, we adapt the so-called Gordon-Schönheim bound for H_m and we define a bound that allow us to obtain s-PD-sets of size $s+1$ for H_m. In Section 3, we provide a criterion on subsets of matrices of $AGL(m,2)$ to be an s-PD-set of size $s+1$. In Section 4, we define a recursive construction to obtain s-PD-sets of size $s+1$ for H_{m+1} from a given s-PD-set of the same size for H_m. Finally, in Section 5, we show the conclusions and a further research on this topic.

2 Bound on the minimum size of s-PD-sets for H_m

There is a well-known bound on the minimum size of PD-sets for linear codes based on the length, the dimension and the minimum distance of such codes.

Proposition 1. *[4] Let C be a t-error correcting linear code of length n, dimension k and minimum distance d. Let $r = n - k$ be the redundancy of C. If S is a PD-set for C, then*

$$|S| \geq \left\lceil \frac{n}{r} \left\lceil \frac{n-1}{r-1} \left\lceil \dots \left\lceil \frac{n-t+1}{r-t+1} \right\rceil \dots \right\rceil \right\rceil \right\rceil.$$

The above inequality is often called the *Gordon-Schönheim bound*. Recall that a linear code with minimum distance d can correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors, so for the binary linear Hadamard code H_m, we have that its error-correcting capability, denoted by t_m, is

$$t_m = \left\lfloor \frac{d-1}{2} \right\rfloor = \left\lfloor \frac{2^{m-1}-1}{2} \right\rfloor = 2^{m-2} - 1.$$

For $m = 3$, since the minimum distance of H_3 is $d = 4$, the error-correcting capability is 1. Due to this fact, permutation decoding for this case becomes unnecessary, and we do not take it into account in the results.

The Gordon-Schönheim bound can be adapted to s-PD-sets for all s up to the error correcting capability of the code. We are going to compute the right side of this bound given in Proposition 1 in the particular case of the binary linear Hadamard code H_m, for all $1 \leq s \leq t_m$. Since H_m has length 2^m and dimension $m+1$, taking $t = s$, the right side becomes

$$\left\lceil \frac{2^m}{2^m - m - 1} \left\lceil \frac{2^m - 1}{2^m - m - 2} \left\lceil \dots \left\lceil \frac{2^m - s + 1}{2^m - m - s} \right\rceil \right\rceil \dots \right\rceil \right\rceil. \tag{2}$$

Let $g_m(s)$ be the function given by (2). We compute its minimum value in the following lemma.

Lemma 1. *Let m be an integer, $m \geq 4$. Let H_m be the binary linear Hadamard code. For $1 \leq s \leq t_m$,*

$$g_m(s) = \left\lceil \frac{2^m}{2^m - m - 1} \left\lceil \frac{2^m - 1}{2^m - m - 2} \left\lceil \dots \left\lceil \frac{2^m - s + 1}{2^m - m - s} \right\rceil \right\rceil \dots \right\rceil \right\rceil \geq s + 1,$$

where $t_m = 2^{m-2} - 1$ is the error-correcting capability of H_m.

The smaller the size of the PD-set is, the more efficient permutation decoding becomes. Because of this, we will focus on the case when we have that $g_m(s) = s + 1$. Let m be an integer, $m \geq 4$. For the binary linear Hadamard code H_m, we define the following integer:

$$f_{H_m} = \max\{s \mid 2 \leq s,\ g_m(s) = s + 1\}.$$

For each binary linear Hadamard code H_m, the integer f_{H_m} represents the greater s in which we can find s-PD-sets of size $s + 1$. The following result characterize this parameter from the value of m.

Lemma 2. *For $m \geq 4$, $f_{H_m} = \left\lfloor \frac{2^m - m - 1}{1 + m} \right\rfloor$.*

The integer $f_{S_m} = \max\{s \mid 2 \leq s,\ g_m(s) = s+1\} = \left\lfloor \frac{2^m - m - 1}{m} \right\rfloor$ is computed for the binary simplex code S_m, $m \geq 4$, in [3]. It is easy to check that f_{S_m} is always greater than or equal to f_{H_m}. For example, if we take $m = 4$, we obtain $f_{S_4} = f_{H_4} = 2$, but for $m = 5$, we have that $f_{S_5} = 5 > 4 = f_{H_5}$. This implies that we can find 5-PD-sets of size 6 for the binary simplex code S_5, but such set cannot be found for the binary linear Hadamard code H_5.

3 Finding s-PD-sets of size $s + 1$ for H_m

Let M be a matrix of $GL(m, 2)$. We can regard the rows of M as row vectors and consider the set $V = \{v_1, \dots, v_m\}$ consisting of such row vectors. We define M^* as the matrix with rows given by $V^* = \{v_1, v_1 + v_2, \dots, v_1 + v_m\}$. We denote by Id_m the $m \times m$ identity matrix.

An s-PD-set of size $s+1$ meets the Gordon-Schönheim bound for correction of s errors if $s \leq f_{H_m}$. The following proposition provides us a condition on sets of matrices of $AGL(m, 2)$ in order to be s-PD-sets of size $s + 1$.

Proposition 2. *Let H_m be the binary linear Hadamard code of length $n = 2^m$, with $m \geq 4$. Let $P_s = \{M_i \mid 0 \leq i \leq s\}$ be a set of $s+1$ matrices in $AGL(m, 2)$. Then, P_s is an s-PD-set of size $s + 1$ for H_m if and only if no two matrices $(M_i^{-1})^*$ and $(M_j^{-1})^*$ for $i \neq j$ have a row in common. Moreover, any subset $P_k \subseteq P_s$ of size $k + 1$ is a k-PD-set for $k \in \{1, \dots, s\}$.*

Example 1. The set of matrices $P_2 = \{Id_5, M_1, M_2\}$, where

$$M_1 = \begin{pmatrix} 1\,1\,1\,1\,1 \\ 0\,0\,0\,0\,1 \\ 0\,0\,1\,0\,0 \\ 0\,1\,0\,0\,1 \\ 0\,0\,1\,1\,0 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1\,1\,1\,1\,1 \\ 0\,1\,0\,0\,1 \\ 0\,0\,1\,1\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1 \end{pmatrix}$$

and Id_5 is the 5×5 identity matrix, is a 2-PD-set for the binary linear Hadamard code H_4 of length 16. Note that these matrices are elements of the subgroup $AGL(4,2) \subset GL(5,2)$. The inverse matrices of M_1 and M_2 are

$$M_1^{-1} = \begin{pmatrix} 1\,0\,0\,1\,1 \\ 0\,1\,0\,1\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,1\,0\,1 \\ 0\,1\,0\,0\,0 \end{pmatrix} \quad \text{and} \quad M_2^{-1} = \begin{pmatrix} 1\,1\,1\,0\,0 \\ 0\,1\,0\,0\,1 \\ 0\,0\,0\,1\,0 \\ 0\,0\,1\,1\,0 \\ 0\,0\,0\,0\,1 \end{pmatrix},$$

respectively. It is straightforward to check that the matrices Id_5^*,

$$(M_1^{-1})^* = \begin{pmatrix} 1\,0\,0\,1\,1 \\ 1\,1\,0\,0\,1 \\ 1\,0\,1\,1\,1 \\ 1\,0\,1\,1\,0 \\ 1\,1\,0\,1\,1 \end{pmatrix} \quad \text{and} \quad (M_2^{-1})^* = \begin{pmatrix} 1\,1\,1\,0\,0 \\ 1\,0\,1\,0\,1 \\ 1\,1\,1\,1\,0 \\ 1\,1\,0\,1\,0 \\ 1\,1\,1\,0\,1 \end{pmatrix}$$

have no rows in common. In addition, note that $f_{H_4} = 2$, so no s-PD-set of size $s+1$ can be found for $s \geq 3$. We can also observe that $f_{H_4} = 2 < 3 = t_4$, where t_4 is the error-correcting capability of H_4. In fact, the value of the bound f_{H_m} is always smaller than t_m, for all $m \geq 4$. Finally, also remark that the matrices of P_2 can be regarded as elements of Sym(16). In this case, we obtain $\{id, \sigma_1, \sigma_2\}$ as a 2-PD-set, where

$$\sigma_1 = (1, 14, 11, 9, 6, 10, 13, 3, 15, 5, 16, 2, 12, 8)(4, 7),$$
$$\sigma_2 = (1, 14, 11, 2, 7, 9, 5, 12, 3, 16, 13, 6)(4, 15, 8, 10).$$

Let S be an s-PD-set of size $s+1$. The set S is a *nested* s-PD-set if there is an ordering of the elements of S, $S = \{\sigma_0, \ldots, \sigma_s\}$, such that $S_i = \{\sigma_0, \ldots, \sigma_i\} \subseteq S$ is an i-PD-set of size $i+1$, for all $i \in \{0, \ldots, s\}$. Note that $S_i \subset S_j$ if $0 \leq i < j \leq s$ and $S_s = S$. From Proposition 2, we have two important consequences. The first one is related to how to obtain nested s-PD-sets and the second one provides another proof of Lemma 1.

Corollary 1. *Let m be an integer, $m \geq 4$. If P_s is an s-PD-set of size $s+1$ for the binary linear Hadamard code H_m, then any ordering of the elements of P_s gives nested k-PD-sets for $k \in \{1, \ldots, s\}$.*

Corollary 2. *Let m be an integer, $m \geq 4$. Let P_s be an s-PD-set of size $s+1$ for the binary linear Hadamard code H_m. Then, $s \leq \left\lfloor \frac{2^m - m - 1}{1+m} \right\rfloor$.*

4 Recursive construction of s-PD-sets of size $s+1$

Given an s-PD-set of size $s+1$ for the binary linear Hadamard code H_m of length 2^m, where $0 \leq s \leq f_{H_m}$, we can construct recursively an s-PD-set of the same size for $H_{m'}$ of length $2^{m'}$, for all $m' > m$.

Let $M \in AGL(m,2)$ and $v = (0, v_2 \ldots, v_{m+1})$ be the last row of the matrix M. We define the matrix $M(v) \in AGL(m+1,2)$ as

$$M(v) = \left(\begin{array}{c|c} 1 & \\ 0 & M \\ \vdots & \\ \hline 0 & 1\ v_2\ \ldots\ v_{m+1} \end{array}\right). \tag{3}$$

Since the first column of $M(v)$ is $e_1 = (1, 0, \ldots, 0)$, we can guarantee that the first column of $M(v)^{-1}$ is e_1 as well. Thus, both matrices $M(v)$ and $M(v)^{-1}$ belong to the affine group $AGL(m+1,2)$. Also note that the vector $v \in \mathbb{F}_2^{m+1}$ depends on each matrix $M \in AGL(m,2)$.

Proposition 3. *Let m be an integer, $m \geq 4$, and $P_s = \{Id_{m+1}, M_1, \ldots, M_s\}$ be an s-PD-set of size $s+1$ for the binary linear Hadamard code H_m. Let $N_i = M_i^{-1}$, for all $i \in \{1, \ldots, s\}$. Then, $Q_s = \{Id_{m+2}, (N_1(v))^{-1}, \ldots, (N_s(v))^{-1}\}$ is an s-PD-set of size $s+1$ for the binary linear Hadamard code H_{m+1}.*

Example 2. Considering the matrices from the 2-PD-set $P_2 = \{Id_5, M_1, M_2\}$ for H_4 of length 16, given in Example 1, matrices $N_1(v) = (M_1^{-1})(v)$ and $N_2(v) = (M_2^{-1})(v)$ are

$$N_1(v) = \left(\begin{array}{c|ccccc} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad N_2(v) = \left(\begin{array}{c|ccccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Note that the last row of N_1 is $v = (0, 1, 0, 0, 0)$, and the last row of N_2 is $v = (0, 0, 0, 0, 1)$. It is clear that matrices Id_6^*,

$$(N_1(v))^* = \begin{pmatrix} 1\,1\,0\,0\,1\,1 \\ 1\,1\,1\,0\,0\,1 \\ 1\,1\,0\,1\,1\,1 \\ 1\,1\,0\,1\,1\,0 \\ 1\,1\,1\,0\,1\,1 \\ 1\,0\,1\,0\,1\,1 \end{pmatrix} \quad \text{and} \quad (N_2(v))^* = \begin{pmatrix} 1\,1\,1\,1\,0\,0 \\ 1\,1\,0\,1\,0\,1 \\ 1\,1\,1\,1\,1\,0 \\ 1\,1\,1\,0\,1\,0 \\ 1\,1\,1\,1\,0\,1 \\ 1\,0\,1\,1\,0\,1 \end{pmatrix}$$

have no rows in common, so the set $\{Id_{m+2}, (N_1(v))^{-1}, (N_2(v))^{-1}\}$ becomes a 2-PD-set for the binary linear Hadamard code H_5 of length 32.

Note 1. Proposition 3 is also true if we define the matrix $M(v)$ taking as vector v any of the last m rows of the matrix M instead of the last one as it is considered in (3).

Note 2. The bound $f_{H_{m+1}}$ for the binary linear Hadamard code H_{m+1} cannot be achieve recursively from an s-PD-set for H_m. The recursive construction only works when fixing the number s of errors we want to correct and increasing the length of the Hadamard code.

5 Conclusions and further research

In this work, we studied how to find s-PD-sets for partial permutation decoding for binary linear Hadamard codes. An alternative permutation decoding algorithm for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes [2] is described in [1]. In particular, it can be applied to Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. Nevertheless, this method assumes that we know an appropriate PD-set for such codes. Further work on this topic will be study how to find s-PD-sets for Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes (not necessarily binary linear Hadamard codes) and establish the size of these s-PD-sets.

References

[1] J. J. Bernal, J. Borges, C. Fernández-Córboda, and M. Villanueva. Permutation decoding of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. *Des. Codes Cryptogr.*, DOI 10.1007/s10623-014-9946-4, 2014.

[2] J. Borges, C. Fernández-Córdoba, J. Pujol, J. Rifà, and M. Villanueva, $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality. *Des. Codes and Cryptogr.*, vol. 54: 167–179, 2010.

[3] W. Fish, J. D. Key, and E. Mwambene. Partial permutation decoding for simplex codes. *Advances in Mathematics of Comunications*, 6(4): 505–516, 2012.

[4] D. M. Gordon. Minimal permutation sets for decoding the binary Golay codes. *IEEE Trans. Inform. Theory*, vol. 28: 541–543, 1982.

[5] H.-J. Kroll and R. Vicenti. PD-sets related to the codes of some classical varieties. *Discrete Math.*, vol 301: 89–105, 2005.

[6] H.-J. Kroll and R. Vicenti. PD-sets for binary RM-codes and the codes related to the Klein quadric and to the Schubert variety of PG(5,2). *Discrete Math.*, vol 308: 408–414, 2008.

[7] F. J. MacWilliams. Permutation decoding of systematics codes. *Bell System Tech. J.*, vol. 43:485–505, 1964.

[8] F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error-Correcting Codes.* North-Holland Publishing Company, Amsterdam, 1977.

[9] P. Seneviratne. Partial permutation decoding for the first-order Reed-Muller codes. *Discrete Math.*, vol 309: 1967–1970, 2009.

[10] J. Wolfmann. A permutation decoding of the (24, 12, 9) Golay code. *IEEE Trans. Inform. Theory*, vol 29: 748–750, 1983.

The differential and the Roman domination number of a graph ⋆

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Abstract. Let $G = (V, E)$ be a graph of order n and let $B(S)$ be the set of vertices in $V \setminus S$ that have a neighbor in the vertex set S. The differential of a vertex set S is defined as $\partial(S) = |B(S)| - |S|$ and the maximum value of $\partial(S)$ for any subset S of V is the differential of G. A Roman dominating function of G is a function $f : V \to \{0, 1, 2\}$ such that every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is the Roman domination number of G, written $\gamma_R(G)$. In this work, we show that, although these two parameters have been studied intensively but separately, they have a strong and hitherto unnoticed relation, namely, $\gamma_R(G) = n - \partial(G)$.

Key words: Roman domination, differential, domination number.

1 Introduction

The Roman domination number is a variant of the domination number suggested by I. Stewart [20], motivated by a problem from military history. A *Roman dominating function* (RDF) of a graph $G = (V, E)$ is a (total) function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The definition of a Roman dominating function is given implicitly in [20] and [19]. The idea is that the values of f represent the number of legions stationed at a given location (modeled as a vertex). A location is considered to be unsecured if no legions are stationed there. An unsecured location u can be secured by

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sending a legion to u from an adjacent location v. But Emperor Constantine the Great, in the fourth century A.D., decreed that a legion cannot be sent off from a location v if doing so leaves that location unsecured (i.e., if $f(v) = 1$). Thus, two legions must be stationed at a location ($f(v) = 2$) before one of the legions can be sent to an adjacent location.

The *weight* of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G, and it is denoted by $\gamma_R(G)$. This parameter (as well as related ones) has been studied by many authors [8,9,11,13,14,15,18,21,22]. Alternatively, an RDF can be presented in terms of the ordered partition of V induced by f, i.e., $f = (V_0, V_1, V_2)$ with $V_i = \{v \in V : f(v) = i\}$. An RDF $f = (V_0, V_1, V_2)$ is a γ_R-function if $f(V) = \gamma_R(G)$.

The differential of a graph was introduced in [19] in 2006, and it has been studied by several authors [2,4,10,11,4], motivated by its applications to information diffusion problems in social networks. Informally, this problem consists in determining the most effective group of nodes to influence the rest. More formally, the *differential of a vertex set* S is defined as $\partial(S) = |B(S)| - |S|$, where $B(S)$ is the set of vertices in $V \setminus S$ that have a neighbor in the vertex set S, and the *differential of a graph* is defined as $\partial(G) = \max\{\partial(S) : S \subseteq V\}$. We will say that $S \subseteq V$ is a *∂-set* or *differential set* if $\partial(S) = \partial(G)$. If we return to the motivation of introducing this parameter, we observe the following. Since the differential of a set is the difference between the number of neighbors of a set and the number of elements in this set, the differential of a set could act as a measure of how this set can influence the rest of the elements.

These two parameters have been independently studied, as no relationship between them was known before, and until very recently, also none of us realized the strong relation they exhibit. In this paper, we show that the sum $\gamma_R(G) + \partial(G)$ equals the order of G, and we present several consequences of this relationship. This type of Gallai identity result complements what was previously known for the domination number and the enclaveless number of a graph (defined below) and several other domination parameters [1], and the best known of such relationships is probably Gallai's identity [10], stating that the sum of the independence number and the vertex cover number of a graph is equal to its order.

Moreover, we thoroughly revise what is known about both parameters, turning this paper in a bibliographical study that might be interesting in its own right.

Finally, we fix some general notation and terminology. $G = (V, E)$ denotes a simple graph of *order* $n = |V|$ and *size* $m = |E|$. We denote two adjacent vertices u and v by $u \sim v$. For a vertex $v \in V$ we write $N(v) = \{u \in V : u \sim v\}$ for its open *neighborhood*. The *degree* of a vertex $v \in V$ will be denoted by $\delta(v) = |N(v)|$. For a non-empty subset $S \subseteq V$, the subgraph in G induced by

the vertex set S will be denoted by $G\langle S\rangle$. Given a graph H, the graph G is called *H-free* if G has no induced subgraph isomorphic to H.

We will discuss several basic graph parameters: the order n and size m of a graph were already mentioned; $\delta(G)$ denotes the *minimum degree* of any vertex of G, while $\Delta(G)$ refers to the *maximum degree*. The *distance* between two vertices x, y is the length of the shortest path between x and y, and the *diameter* $D(G)$ refers to the largest distance between any two vertices from G. There is a wealth of graph parameters related to domination [12] and some of these will come into play in our exposition. Recall that a set D of vertices in $G = (V, E)$ is a *dominating set* if $B(D) = V \setminus D$. The size of the smallest dominating set is also known as the *domination number* of the graph G, written as $\gamma(G)$. Domination can be generalized in several ways. A vertex set $D \subseteq V$ is a *k-dominating set* if every vertex not in D has at least k adjacent vertices in D. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set. Clearly, a 1-dominating set is just a dominating set. The *enclaveless number* of a graph $G = (V, E)$, denoted $\Psi(G)$, is defined as $\Psi(G) := \max\{|B(D)| : D \subseteq V\}$. As mentioned above, $\gamma(G) + \Psi(G) = n$ for any graph G of order n.

2 Relating the Roman domination number and the differential of a graph

For every graph G with connected components $G_1, \ldots, G_k$, $\gamma_R(G) = \gamma_R(G_1) + \cdots + \gamma_R(G_k)$ and $\partial(G) = \partial(G_1) + \cdots + \partial(G_k)$. Therefore, it is unnecessary to consider disconnected graphs. Most of the results in this paper will be obtained by using the following theorem, which has not been observed so far, and it has quite a number of consequences that we will exhibit in the next sections.

Theorem 1. *If $G = (V, E)$ is a graph of order n, then $\gamma_R(G) = n - \partial(G)$.*

As a consequence of this theorem we obtain the following corollary.

Corollary 1. *Given a graph G, $f = (V_0, V_1, V_2)$ is a γ_R-function of G if and only if V_2 is a ∂-set of G and $V_0 = B(V_2)$.*

3 New bounds for the Roman domination number and for the differential of a graph

In this section, we show new bounds for the Roman domination number of a graph which can be directly obtained from results about the differential of a graph. Occasionally, we also get new combinatorial bounds for the differential from published results on the Roman domination number.

3.1 Relations to basic graph parameters

Here, we focus on the order, size, minimum and maximum degree and the diameter of a graph. Sometimes, other parameters come into play, as well, refining previously made assertions.

A simple lower bound on the Roman domination number appeared in [9]:

Proposition 1. *For any graph G of order n with $\Delta(G) \geq 1$, $\gamma_R(G) \geq \frac{2n}{\Delta(G)+1}$.*

Using Theorem 1 and Theorem 2.16 from [4], the same result follows. Moreover, an infinite family of graphs attaining this bound was also given in [4].

To get an upper bound for the Roman domination number in terms of the order and the minimum degree of a graph, we can consider the lower bound of the differential given in Theorem 2.15 in [2], which is $\partial(G) \geq \left\lceil \frac{n(\delta(G)-1)}{3\delta(G)-1} \right\rceil$, together with Theorem 1 to conclude:

Theorem 2. *Let G be a graph of order n. Then $\gamma_R(G) \leq \left\lfloor \frac{2n\delta(G)}{3\delta(G)-1} \right\rfloor$.*

Lower bounds on the differential of a graph in terms of its order, size and maximum degree as proved in [2] translate as follows to so far unknown upper bounds on the Roman domination number:

Theorem 3. *If G is a graph of order n and size m, then*

$$\gamma_R(G) \leq \min\left\{ \left\lfloor \frac{3\Delta(G)n - 2m}{3\Delta(G) - 1} \right\rfloor, \left\lfloor \frac{(3\Delta(G)+4)n - 2m}{3\Delta(G)+4} \right\rfloor \right\}.$$

Moreover, if G is a C_5-free graph, then $\gamma_R(G) \leq \left\lfloor \frac{(3\Delta(G)+2)n-2m}{3\Delta(G)+2} \right\rfloor$.

Conversely, we can obtain a new tight lower bound on the differential of a graph in terms of its minimum degree and its order, based on [23, Theorem 12].

Theorem 4. *For any graph G with $\delta(G) > 0$, $\partial(G) \geq \left(\frac{2^{1+1/\delta(G)} \cdot \delta(G)}{(1+\delta(G))^{1+1/\delta(G)}} - 1 \right) n$.*

In [8], it was proved that, for every sufficiently large connected graph G with $\delta(G) \geq 1$, it holds that $\gamma_R(G) \leq \frac{4n}{5}$ and, if $\delta(G) \geq 2$, then $\gamma_R(G) \leq \frac{8n}{11}$. This also follows from Theorems 3.1 and 3.4 in [10] by making use of Theorem 1. Observe that different techniques were used in the proofs of these theorems. Moreover, Liu and Chang proved in [15] that $\gamma_R(G) \leq \frac{2n}{3}$ if $\delta(G) \geq 3$, which provides the unknown lower bound $n/3$ for the differential in such a case. Let us formulate these important results explicitly:

Theorem 5. *Let G be a connected graph of order n.*

- *If $n \geq 2$, then $\gamma_R(G) \leq \frac{4n}{5}$ and $\partial(G) \geq \frac{n}{5}$.*

- *If* $n \geq 9$ *and* $\delta(G) \geq 2$, *then* $\gamma_R(G) \leq \frac{8n}{11}$ *and* $\partial(G) \geq \frac{3n}{11}$.
- *If* $\delta(G) \geq 3$, *then* $\gamma_R(G) \leq \frac{2n}{3}$ *and* $\partial(G) \geq \frac{n}{3}$.

Moreover, from results in [10] and [11], we can improve these upper bounds for the Roman domination number in two cases: (a) if we consider the number of vertices having exactly two so-called hairs connected to it as an additional parameter and (b) if we consider special pendant subgraphs in a graph of maximum degree three.

Let us turn to Case (a) first. A *hair* is a sequence of two vertices $w - v$, where w is a vertex of degree one and v has degree two. We denote by $p_5(G)$ the set of vertices in V which have exactly two hairs connected to them. Notice that these are centers of certain induced paths on five vertices. Some infinite families of graphs attaining the following bound were given in [11].

Theorem 6. *Any connected graph* G *of order* $n \geq 6$ *satisfies* $\gamma_R(G) \leq \frac{3n+|p_5(G)|}{4}$.

Now we come to Case (b). We say that a (connected) graph $G_1 = (V_1, E_1)$ is a *pendant subgraph* of $G = (V, E)$ from $v \in V_1$, if $V_1 \subseteq V$, $G_1 = G\langle V_1 \rangle$ and there exist $u \in V \setminus V_1$ and $e = uv \in E$ such that e is a bridge (an edge whose deletion disconnects the graph). For instance, a hair (described by $w - v$) can be viewed as a pendant subgraph of the form $(\{w, v\}, \{wv\})$. We are particularly interested in the graph G_1 depicted in Fig. 1 as being pendant from v.

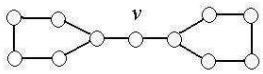

Fig. 1: A subgraph counted in Theorem 7.

Theorem 7. *Let* G *be a graph of order* $n \geq 12$ *with* $\delta(G) \geq 2$ *and* $\Delta(G) \leq 3$. *If* G *has* t *pendant subgraphs isomorphic to* G_1, *then* $\gamma_R(G) \leq \frac{5n+t}{7}$.

A consequence from [4] for Roman domination is the following one.

Proposition 2. *If* G *is a graph of order* n *and diameter* $D(G)$, *then*

$$\gamma_R(G) \leq n - \left(\left\lfloor \frac{D(G)}{3} \right\rfloor + 1 \right) (\delta(G) - 1).$$

The proposition above clearly improves Theorems 3 and 4 in [18] when $\delta(G) \geq 2$. Moreover, the authors in [4] gave some infinite families of graphs satisfying this upper bound.

Since $\partial(G) \geq \Delta(G) - 1$, it is clear that $\gamma_R(G) \leq n - \Delta(G) + 1$. This was also observed in [8]. The next theorem, obtained by Theorem 2.14 in [2] and Theorems 2.15 and 2.17 in [4], gives better upper bounds with additional conditions.

Theorem 8. *Let G be a graph of order n. Then, the following upper bounds on the Roman domination number of G are true.*

(a) *If $\delta(G) \geq 2$ and $n > \frac{\Delta(G)(\delta(G)+1)}{\delta(G)-1} + 1$, then $\gamma_R(G) \leq n - \Delta(G)$.*
(b) *If $\delta(G) \geq 2$ and $n > \frac{(\Delta(G)+k-1)(3\delta(G)-1)}{\delta(G)-1}$ for some natural number k, then $\gamma_R(G) \leq n - \Delta(G) - k$.*
(c) *If $D(G) > 4$, then $\gamma_R(G) \leq n - \Delta(G) - \delta(G) + 2$.*
(d) *If $D(G) > 4$, $\delta(G) \geq 2$ and $n > \frac{2\Delta(G)(\delta(G)+1)}{\delta(G)-1} + 2$, then*

$$\gamma_R(G) \leq n - \Delta(G) - \delta(G) + 1.$$

3.2 Relations to other graph parameters

Here, we will consider several other domination parameters and their relation to the Roman domination number. The following result appears in [9] and it has been used by most of the authors who have worked on the Roman domination number.

Proposition 3. *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Using Theorem 1 and $n - 2\gamma(G) \leq \partial(G) \leq n - \gamma(G) - 1$, which is satisfied except when G is an isolated vertex (see [19]), we can strengthen Proposition 3 towards:

Corollary 2. *For any connected graph G of order $n > 1$, $\gamma(G)+1 \leq \gamma_R(G) \leq 2\gamma(G)$.*

It was proved in [11] that, for any graph G of order $n \geq 3$, $\gamma_R(G) \leq n - \frac{\gamma(G)}{2}$, and the authors even characterized the graphs attaining this bound. The same upper bound and even a lower bound can be obtained from Theorem 1 and Theorem 2.4 in [2], where the authors proved that $\frac{\gamma(G)}{2} \leq \partial(G) \leq \gamma(G)(\Delta(G) - 1)$ using so-called big star packings. Namely, we derive:

Theorem 9. *If G is a graph of order $n \geq 3$, then*

$$n - \gamma(G)(\Delta(G) - 1) \leq \gamma_R(G) \leq n - \frac{\gamma(G)}{2}.$$

The upper bound is better than $2\gamma(G)$ only when $\frac{2n}{5} \leq \gamma(G)$. Now, let us see that this upper bound can be improved in some classes of graphs, using the following result which was proved in [2].

Lemma 1. *For any connected graph G of order n, the following is true.*

(a) *If $n \geq 3$ and $\delta(G) \geq 1$, then $\partial(G) \geq \frac{3n}{5} - \gamma(G)$.*
(b) *If $n \geq 9$ and $\delta(G) \geq 2$, then $\partial(G) \geq \frac{7n}{11} - \gamma(G)$.*

Consequently, By Theorem 1, we obtain the following.

Theorem 10. *For any connected graph G of order n, the following is true.*

(a) *If $n \geq 3$ and $\delta(G) \geq 1$, then $\gamma_R(G) \leq \frac{2n}{5} + \gamma(G)$.*
(b) *If $n \geq 9$ and $\delta(G) \geq 2$, then $\gamma_R(G) \leq \frac{4n}{11} + \gamma(G)$.*

Let us remark that McCuaig and Shephard [17] proved that $\gamma(G) \leq \frac{2n}{5}$ for every graph of order n with $\delta(G) \geq 2$ if $n \geq 8$. Hence, (b) is better than the bound given in [11] when $n \geq 9$ and $\delta(G) \geq 2$.

Moreover, using the definition of 2-dominating set, Theorem 1 and a result from [2], we can get another upper bound that is better than the one presented in [11] when $\delta(G) \geq 3$. In [2], it was proved that, for every graph G with minimum degree $\delta(G) \geq 3$, we have $\partial(G) \geq \max\left\{\gamma_2(G) - \gamma(G), \frac{\gamma_2(G)}{2}\right\}$, so we deduce the following.

Theorem 11. *If G is a graph of order n, with minimum degree $\delta(G) \geq 3$, then*

$$\gamma_R(G) \leq \min\left\{n + \gamma(G) - \gamma_2(G), n - \frac{\gamma_2(G)}{2}\right\}.$$

A set $S \subseteq V$ is a *2-packing* if for all $u, v \in S$, $d(u, v) > 2$. The *2-packing number* of G, denoted by $\alpha_2(G)$, is the maximum cardinality of a 2-packing of G. Corollary 1 in [9] relates the Roman domination number and the 2-packing number of the graph. Using that $\partial(G) \geq \alpha_2(G)(\delta(G) - 1)$, proved in [2], we give a new result relating the Roman domination number and the 2-packing number.

Proposition 4. *If G is a graph of order n, then $\gamma_R(G) \leq n - \alpha_2(G)(\delta(G) - 1)$.*

References

[1] S. Balasubramanian, J. Farr, K. Bernasconi, R. Laskar, J. Villalpando, K. Hutson, G. Stevens, M. Gairing, and S.T. Hedetniemi Gallai theorems involving domination parameters. *Congressus Numerantium*, 157: 149–157, 2002.

[2] L. A. Basilio, S. Bermudo, J. M. Sigarreta. Bounds on the differential of a graph. *Utilitas Mathematica*. To appear.

[3] S. Bermudo, H. Fernau. Lower bounds on the differential of a graph. *Discrete Mathematics*, 312: 3236–3250, 2012.

[4] S. Bermudo, H. Fernau. Computing the differential of a graph: hardness, approximability and exact algorithms. *Discrete Applied Mathematics*, 165: 69–82, 2014.

[5] S. Bermudo, H. Fernau. Combinatorics for smaller kernels: The differential of a graph. Submitted, 2012.

[6] S. Bermudo, H. Fernau, J. M. Sigarreta. The differential and the Roman domination number of a graph. *Applicable Analysis and Discrete Mathematics*, 8: 155-171, 2014.

[7] S. Bermudo, J. M. Rodríguez, J. M. Sigarreta. On the differential in graphs. *Utilitas Mathematica*. To appear.

[8] E. W. Chambers, B. Kinnersley, N. Prince, D. B. West. Extremal problems for Roman domination. *SIAM Journal of Discrete Mathematics*, 23: 1571–1586, 2009.

[9] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, S. T. Hedetniemi. Roman domination in graphs. *Discrete Mathematics*, 278: 11–22, 2004.

[10] E. J. Cockayne, S. T. Hedetniemi, R. Laskar. Gallai theorems for graphs, hypergraphs and set systems. *Discrete Mathematics*, 72: 35–47, 1988.

[11] O. Favaron, H. Karami, R. Khoeilar, S. M. Sheikholeslami. On the Roman domination number of a graph. *Discrete Mathematics*, 309: 3447–3451, 2010.

[12] T. W. Haynes, S. T. Hedetniemi, P. J. Slater. *Fundamentals of Domination in Graphs.* Vol. 208 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, 1998.

[13] M. A. Henning, S. T. Hedetniemi. Defending the Roman empire: a new strategy. *Discrete Mathematics*, 266: 239–251, 2003.

[14] C. H. Liu, G. J. Chang. Roman domination on 2-connected graphs. *SIAM Journal of Discrete Mathematics*, 26: 193–205, 2012.

[15] C. H. Liu, G. J. Chang. Upper bounds on Roman domination numbers of graphs. *Discrete Mathematics*, 312: 1386-1391, 2012.

[16] J. L. Mashburn, T. W. Haynes, S. M. Hedetniemi, S. T. Hedetniemi, P. J. Slater. Differentials in graphs. *Utilitas Mathematica*, 69: 43–54, 2006.

[17] B. McCuaig, B. Shepherd. Domination in graphs of minimum degree two. *Journal of Graph Theory*, 13: 749–762, 1989.

[18] B. P. Mobaraky, S. M. Sheikholeslami. Bounds on Roman domination numbers of graphs. *Matematitchki Vesnik*, 60: 247–253, 2008.

[19] C. S. ReVelle, K. E. Rosing. Defendens imperium Romanum: A classical problem in military strategy. *American Mathematical Monthly*, 107: 585–594, 2000.

[20] I. Stewart. Defend the Roman Empire. *Scientific American*, 136–139, 1999.

[21] H. M. Xing, X. Chen, X. G. Chen. A note on Roman domination in graphs. *Discrete Mathematics*, 306: 3338–3340, 2006.

[22] F. Xueliang, Y. Yuansheng, J. Baoqi. Roman domination in regular graphs. *Discrete Mathematics*, 309: 1528–1537, 2009.

[23] V. E. Zverovich and A. Poghosyan. On Roman, Global and Restrained Domination in Graphs. *Graphs and Combinatorics*, 27: 755-768, 2011.

Relations between the differential and parameters in graphs

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Abstract. In this work, we obtain several tight bounds for the differential of a graph. In particular, we relate the differential of a graph with known parameters of a graph, namely, its order, size, minimum and maximum degree, its girth, its domination number, its 2-domination number, its 2-packing number and its independence number.

Key words: domination number, differential, packing number, independence number.

1 preliminary considerations

Let $G = (V, E)$ be a graph of order n, for every set $D \subseteq V$ let $B(D)$ be the set of vertices in $V \setminus D$ that have a neighbor in the vertex set D, and let $C(D) = V \setminus (D \cup B(D))$. The *differential of* D is defined as $\partial(D) = |B(D)| - |D|$ and the *differential of a graph* G, written $\partial(G)$, is equal to $\max\{\partial(D) : D \subseteq V\}$. A set D satisfying $\partial(D) = \partial(G)$ is called a *∂-set* or *differential set*. The study of the mathematical properties of the differential in graphs, together with a variety of other kinds of differentials of a set, started in [18,19]. In particular, several bounds on the differential of a graph were given. This parameter has also been studied in [9,10,11,4,20], and the differential of some products of graphs has been studied in [32]. The differential of a set D was also considered in [13], where it was denoted by $\eta(D)$, and the minimum differential of an independent set was considered in [23]. The case of the B-differential of a graph or enclaveless number, defined as $\psi(G) = \max\{|B(D)| : D \subseteq V\}$, was investigated in [19,22].

We begin by stating some notation and terminology. Let $G = (V, E)$ denote a simple graph of order $n = |V|$ and size $m = |E|$. We denote two adjacent

vertices u and v by $u \sim v$. For a vertex $u \in V$ we denote $N(v) = \{u \in V : u \sim v\}$ and $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ will be denoted by $\delta(v) = |N(v)|$. We denote by δ and Δ the minimum and maximum degree of the graph, respectively. The subgraph induced by a set $S \subseteq V$ will be denoted by $G[S]$. For a non-empty subset $S \subseteq V$, and any vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors v has in S: $N_S(v) := \{u \in S : u \sim v\}$ and $\delta_S(v) = |N_S(v)|$. Finally, we denote $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

Notice that, for every graph G with connected components $G_1, ..., G_k$, $\partial(G) = \partial(G_1) + \cdots + \partial(G_k)$. Therefore, we will only consider connected graphs.

2 The differential and other parameters in graphs

First of all, let us mention that depending on the election of the differential set D of the graph, we have different properties for the partition $\{D, B(D), C(D)\}$. Let us show now some properties about $\{D, B(D), C(D)\}$ when we consider a minimum ∂-set.

Proposition 1. *If D is a minimum ∂-set, then the set $\{D, B(D), C(D)\}$ is a partition of V such that:*

(a) *for all $v \in D$, $|epn[v, D]| \geq 2$,*
(b) *for all $v \in B(D)$, $\delta_{C(D)}(v) \leq 2$,*
(c) *for all $v \in C(D)$, $\delta_{C(D)}(v) \leq 1$.*

In the first results we relate the differential with some well-known domination parameters. A set $S \subset V$ is a *dominating set* if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. For more information on domination in graphs see [16,17].

The following result was proved in [19], it relates the domination number and the differential of a graph.

Theorem 1. *For any graph G of order n without isolated vertices,*

$$n - 2\gamma(G) \leq \partial(G) \leq n - \gamma(G) - 1.$$

Let us prove the following lemma which will be used in the proof of the theorem.

Lemma 1. *If D is a ∂-set of a graph G, then $|D| \leq \gamma(G)$.*

Proof. Let A be a minimum dominating set. If $|A| < |D|$, we have

$$\partial(A) = n - 2|A| > n - |D| - |D| \geq |B(D)| - |D| = \partial(G),$$

a contradiction. Therefore, $|D| \leq \gamma(G)$. □

Theorem 2. *If G is a graph with order $n \geq 3$ and maximum degree Δ, then*

$$\partial(G) \leq \gamma(G)(\Delta - 1).$$

Proof. If D is a ∂-set of G, since $|B(D)| \leq \Delta|D|$ we have that

$$\partial(G) = |B(D)| - |D| \leq |D|(\Delta - 1).$$

By Lemma 1, the upper bound follows. □

The upper bound is sharp as we can check in the graphs $G = C_{3n}$ and $G = K_n$.

It is known that $\gamma(G) \geq \frac{n}{\Delta+1}$ for every graph G of order n and maximum degree Δ. The following result shows that the graphs attaining this lower bound can be characterized by its differential.

Proposition 2. *Let G be a graph with order n and maximum degree Δ. Then, $\gamma(G) = \frac{n}{\Delta+1}$ if and only if $\partial(G) = \frac{n(\Delta-1)}{\Delta+1}$.*

Proof. If $\gamma(G) = \frac{n}{\Delta+1}$ we have

$$\frac{n(\Delta-1)}{\Delta+1} = n - 2\left(\frac{n}{\Delta+1}\right) = n - 2\gamma(G) \leq \partial(G) \leq \gamma(G)(\Delta-1) = \frac{n(\Delta-1)}{\Delta+1},$$

so $\partial(G) = \frac{n(\Delta-1)}{\Delta+1}$.

Now, we suppose that $\partial(G) = \frac{n(\Delta-1)}{\Delta+1}$ and let D be a differential set. Since $|B(D)|-|D| = \frac{n(\Delta-1)}{\Delta+1} = n-2\left(\frac{n}{\Delta+1}\right)$, if $|D| > \frac{n}{\Delta+1}$, then $|B(D)| > n-\frac{n}{\Delta+1} > n - |D|$, a contradiction. Therefore, $|D| \leq \frac{n}{\Delta+1}$. Finally, using that

$$\frac{n(\Delta-1)}{\Delta+1} = \partial(G) \leq |D|(\Delta-1) \leq \frac{n(\Delta-1)}{\Delta+1},$$

we conclude that $|D| = \frac{n}{\Delta+1}$ and $|B(D)| = n - \frac{n}{\Delta+1}$, in consequence, D is a dominating set and $\gamma(G) = \frac{n}{\Delta+1}$. □

As we have mentioned before, $\partial(G) \geq \frac{n}{5}$ for any graph G and order $n \geq 3$ and $\partial(G) \geq \frac{3n}{11}$ for any graph G of order $n \geq 9$ and minimum degree $\delta \geq 2$. Let us see other result relating the differential, the order and the domination number of a graph.

Theorem 3. *For any graph G of order n and minimum degree δ.*

(a) *If $n \geq 3$, then $\frac{3n}{5} \leq \partial(G) + \gamma(G) \leq n - 1$.*
(b) *If $n \geq 9$ and $\delta \geq 2$, then $\frac{7n}{11} \leq \partial(G) + \gamma(G) \leq n - 1$.*

Proof. In both cases, if D is a differential set, since $\gamma(G) \leq |D| + |C(D)|$, we have

$$\partial(G)+\gamma(G) \leq |B(D)|-|D|+|D|+|C(D)| = |B(D)|+|C(D)| = n-|D| \leq n-1.$$

Now, we study the lower bound separately. (a) If $\gamma(G) \geq \frac{2n}{5}$, since $\partial(G) \geq \frac{n}{5}$, we have $\partial(G) + \gamma(G) \geq \frac{3n}{5}$. If $\gamma(G) < \frac{2n}{5}$ and we take a dominating set $S \subseteq V$ such that $|S| = \gamma(G)$, we obtain

$$\partial(G) + \gamma(G) \geq \partial(S) + \gamma(G) = n - \gamma(G) > n - \frac{2n}{5} = \frac{3n}{5}.$$

(b) Since $\partial(G) \geq \frac{3n}{11}$ if $\delta \geq 2$ and $n \geq 9$, we can use the same argument used in (a), using $\frac{4n}{11}$ instead of $\frac{2n}{5}$, to get the bound. ∎

The lower bounds given in this theorem are tight on the family of graphs given in Proposition 3.2 and 3.5 in [9], and the upper bound is tight on any graph G such that $\gamma(G) = 1$.

Next, using the definition of 2-dominating set, we can get another lower bound for the differential when the graph has a minimum degree at least 3. A set $S \subset V$, is a *k-dominating set* if every vertex not in S has at least k adjacent vertex in S. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set.

Theorem 4. *If G is a graph with minimum degree $\delta \geq 3$, then*

$$\partial(G) \geq \max\left\{\gamma_2(G) - \gamma(G), \frac{\gamma_2(G)}{2}\right\}.$$

Proof. If D is a minimum ∂-set of G, then $\partial(G) = |B(D)| - |D|$. Since the minimum degree of G is greater than or equal to 3, by Proposition 1, we have that $B(D)$ is a 2-dominating set of G. Using now that $|D| \leq \gamma(G)$, we obtain $\partial(G) \geq \gamma_2(G) - \gamma(G)$. On the other hand, since $B(D)$ is a 2-dominating set of G and using, again by Lemma 1, that $|B(D)| \geq 2|D|$, we have $\partial(G) = |B(D)| - |D| \geq \frac{|B(D)|}{2} \geq \frac{\gamma_2(G)}{2}$. ∎

Given a graph $G = (V, E)$, a subset S of V is called *independent* if its vertices are mutually non-adjacent. The *independence number* $\alpha(G)$ is the largest cardinality among all independent sets of G. A set of vertices U_c is called *critical* if $|U_c| - |N(U_c)| = \mu_c = \max\{|U| - |N(U)| : U \subseteq V\}$. In [23] was proved that $\alpha_c = \mu_c$, where $\alpha_c = \max\{|I| - |N(I)| : I \subseteq V \text{ is an independent set}\}$.

Theorem 5. *For any graph G, $\partial(G) \geq \alpha_c$.*

Proof. If an independent set I_c satisfies $|I_c| - |N(I_c)| = \alpha_c$ and we take $S = N(I_c)$, we have $|B(S)| \geq |I_c|$, therefore,

$$\alpha_c = |I_c| - |N(I_c)| \leq |B(S)| - |S| \leq \partial(G).$$

□

Note that the previous bounds are reached in $G = S_n$. Moreover, the bounds are also reached in the graphs formed by a cycle of order n with two or more leaves added to each vertex.

A set $S \subseteq V$ is a *2-packing* if for all $u, v \in S$, $d(u, v) > 2$. The *2-packing number* of G, denoted by $\alpha_2(G)$, is the maximum cardinality of a 2-packing of G (for more information see [15]). The following result appears in [19].

Proposition 3. *For every k-regular graph G, $\partial(G) \geq \alpha_2(G)(k-1)$.*

A similar result can be given for any graph G of minimum degree δ.

Theorem 6. *If G is a graph with minimum degree δ, then*

$$\partial(G) \geq \alpha_2(G)(\delta - 1).$$

Proof. If S is a 2-packing of cardinality k, any pair of vertices $u, v \in S$ have no common neighbors, thus $\partial(S) \geq k\delta - k = k(\delta - 1)$. □

We can also give a lower and upper bound of the differential using other well-known parameter of the graph, its girth. The *girth* of a graph G, denoted $g(G)$, is the length of the shortest induced cycle contained in G. The following results, which were proved in [4], will be used in the next proposition.

Lemma 2. *For paths P_n, $n \geq 1$ and cycles C_n, $n \geq 3$, $\partial(C_n) = \partial(P_n) = \left\lfloor \frac{n}{3} \right\rfloor$.*

Theorem 7. *Let G be a graph with girth $g(G)$. Then*

$$\left\lfloor \frac{g(G)}{3} \right\rfloor \leq \partial(G) \leq n - 2 \left\lfloor \frac{g(G)}{3} \right\rfloor.$$

Proof. If G' is an induced subgraph of G, it is clear that $\partial(G') \leq \partial(G)$. Since the girth of a graph is a cycle which is an induced subgraph, then

$$\left\lfloor \frac{g(G)}{3} \right\rfloor \leq \partial(G).$$

Let C be a induced cycle of length $g(G)$ in G. The inequality on the right is trivial if $g(G) = 3, 4$ or 5, then we suppose $g(G) \geq 6$. In such a case, a vertex which is not in $V(C)$ can be adjacent to at most one vertex of C, otherwise we obtain an induced cycle of length less than $g(G)$, which is a contradiction. Finally, if D is a ∂-set and $D < \left\lfloor \frac{g(G)}{3} \right\rfloor$ there exist three consecutive vertices $\{v_1, v_2, v_3\}$ in C such that $\{v_1, v_2, v_3\} \cap B(D) = \emptyset$, so $\partial(D \cup \{v_2\}) > \partial(D)$. In consequence, $|D| \geq \left\lfloor \frac{g(G)}{3} \right\rfloor$ and, therefore, $\partial(G) \leq n - 2|D| \leq n - 2 \left\lfloor \frac{g(G)}{3} \right\rfloor$. □

The lower bound is reached, for instance, in the cycle graph C_n. The upper bound is attained in cycle graphs C_{3k}, complete graphs K_n and wheel graphs W_n.

Recall that a graph consisting of one central vertex c and d neighbors that in turn have no further neighbors other than c is also known as a *star* $S_d = K_{1,d}$. We also denote an S_d star S by $S = \{c; v_1, \ldots, v_d\}$ to indicate that c is its center and $v_1, \ldots, v_d$ are its ray vertices. We will call an S_d star *big* if $d \geq 2$. Given a graph $G = (V, E)$, a *big star packing* is given by a vertex-disjoint collection $\mathcal{S} = \{X_i \mid 1 \leq i \leq k\}$ of (not necessarily induced) big stars $X_i \subseteq V$, i.e., the graph induced by X_i, written $G[X_i]$ for short, contains some S_d with $d = |X_i| - 1 \geq 2$. We will write $\mathcal{S}(D)$ when we want to specify that D is the set of vertices which are star centers of $\mathcal{S}$. The set $\mathcal{S}_t(D)$ collects all S_t stars from $\mathcal{S}(D)$ for $t \geq 2$, and $\mathcal{S}_{\geq t}(D)$ collects all S_d stars from $\mathcal{S}(D)$ such that $d \geq t$. If $\mathcal{S}$ is a big star packing of G, we denote this property by $\mathcal{S} \in SP(G)$.

In [9] it was proved that $\partial(G) = \max\{\sum_{X \in \mathcal{S}} (|X| - 2) : \mathcal{S} \in SP(G)\}$. For every $\mathcal{S} \in SP(G)$ we write $\partial(\mathcal{S}) = \sum_{X \in \mathcal{S}} (|X| - 2)$ and call this the *differential* of the big star packing $\mathcal{S}$. We call a star packing $\mathcal{S} \in SP(G)$ a *differential (star) packing* if it assumes the differential of the graph, i.e., if $\partial(\mathcal{S}) = \partial(G)$. A *maximum differential (star) packing* is a differential packing of maximum cardinality, i.e., with the maximum number of stars contained in it. Let $\max SP(G)$ collect all maximum differential packings of G.

Lemma 3. *For every big star packing $\mathcal{S}(D) \in \max SP$ it is satisfied that $\delta_{C(D)}(v) \leq 1$ for every $v \in B(D)$. Moreover, there exists a big star packing $\mathcal{S}(D) \in \max SP$ such that $B(D)$ is a dominating set in G.*

Theorem 8. *Let G be a graph of order n and minimum degree δ. Then*

$$\partial(G) \geq \left\lceil \frac{n(\delta - 1)}{3\delta - 1} \right\rceil.$$

Proof. If we take the big star packing $\mathcal{S}(D) \in \max SP$ given in Lemma 3, we have that $\delta_{C(D)}(v) \leq 1$ for every $v \in B(D)$ and $\delta_{C(D)}(v) \leq 1$ for every $v \in C(D)$, then, if $C(D) = \{c_1, ..., c_t\}$, we have that $(\delta - 1)|C(D)| \leq |B(D)|$. Thus,

$$n = |D| + |B(D)| + |C(D)| \leq |D| + |B(D)| + \frac{|B(D)|}{\delta - 1}.$$

Since $|D| \leq \partial(G)$, we conclude that $n(\delta - 1) \leq \partial(G)(3\delta - 1)$, that is, $\partial(G) \geq \frac{n(\delta-1)}{3\delta-1}$.

□

A graph G is said to be *dominant differential* if it contains a ∂-set which is also a dominating set. Some examples of dominant differential graphs are complete graphs, star graphs, wheel graphs, and path graphs P_n and cycle graphs C_n with $n = 3k$ or $n = 3k + 2$.

Theorem 9. *A graph G is dominant differential if and only if $\partial(G) = n - 2\gamma(G)$.*

Proof. If D is a ∂-set of G which is a dominating set, by Lemma 1, we have $|D| \leq \gamma(G)$, so $|D| = \gamma(G)$ and $\partial(G) = n - 2\gamma(G)$.

If A is a minimum dominating set and $\partial(G) = n - 2|A| = |B(A)| - |A|$, then A is a ∂-set and a dominating set. □

References

[1] M. Aldaz, J. Heintz, G. Matera, J. L. Montaña, and L. M. Pardo. Combinatorial hardness proofs for polynomial evaluation. *Mathematical foundations of computer science*, *Lecture Notes in Comput. Sci.*, vol. 1450:167–175. Springer, 1998.

[2] I. Bárány. Extremal problems for convex lattice polytopes: a survey. *Surveys on discrete and computational geometry*, *Contemp. Math.* vol. 453:87–103. Amer. Math. Soc., Providence, RI, 2008.

[3] J. M. Brunat, A. Guedes de Oliveira and M. Noy. Partitions of a finite Boolean lattice into intervals. *European J. Combin.*, 30(8):1801–1809, 2009.

[4] M. Baca, Y. Lin M. Miller and M.Z. Youssef. Edge-antimagic graphs. *Discrete Math.*, 307(11–12):1232–1244, 2007.

[5] D.E. Knuth. *The TEX book*. Addison-Wesley, 1989.

[6] L. Lamport. *LaTeX User's Guide and Document Reference Manual*. Addison-Wesley Publishing Company, Reading, Massachusetts, 1986.

[7] J. A. De Loera, E. D. Kim, S. Onn and F. Santos. Graphs of transportation polytopes. *Journal of Combinatorial Theory. Series A*, 116(8):1306–1325, 2009.

[8] W. A. Stein et al. *Sage Mathematics Software (Version 4.3)*. The Sage Group, 2009. `http://www.sagemath.org`.

[9] S. Bermudo and H. Fernau. *Lower bound on the differential of a graph*. Discrete Math. **312**, 3236–3250, 2012.

[10] S. Bermudo and H. Fernau. *Computing the differential of a graph: hardness, approximability and exact algorithms*. Discrete Applied Mathematics. To appear.

[11] S. Bermudo and H. Fernau. *Combinatorics for smaller kernels: the differential of a graph*. Submitted 2012.

[12] S. Bermudo J. M. Rodríguez and J. M. Sigarreta. *On the differential in graphs*, Utilitas Mathematica. To appear.

[13] W. Goddard and M. A. Henning. *Generalised domination and independence in graphs*, Congr. Numer. **123**, 161–171, 1997.

[14] A. Kaneko, A. Kelmans and T. Nishimura. *On packing 3-vertex paths in a graph*, J. Graph Theory **36** (4), 175-197, 2001.

[15] W. H. Mills and R. C. Mullin. *Coverings and packings*, in: Contemporary Design Theory: A collection of Surveys, 371-399, edited by J. H. Dinitz and D. R. Stinson. Wiley, 1992.

[16] T. W. Haynes, S. Hedetniemi and P. J. Slater. *Fundamentals of Domination in Graphs*, Marcel Dekker, 1998.

[17] T. W. Haynes, S. Hedetniemi and P. J. Slater. *Domination in Graphs: Advanced Topics*, Marcel Dekker, 1998

[18] J. R. Lewis. *Differentials of graphs*, Master's Thesis, East Tennessee State University, 2004.

[19] J.L. Mashburn, T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi and P.J. Slater. *Differentials in graphs*, Utilitas Mathematica **69**, 43-54, 2006.

[20] P. Roushini Leely Pushpam and D. Yokesh. *Differential in certain classes of graphs*, Tamkang Journal of Mathematics, **41** (2), 129-138, 2010.

[21] J. M. Sigarreta. *Differential in Cartesian Product Graphs*, Ars Combinatoria. To appear.

[22] P. J. Slater. *Enclaveless sets and MK-systems*. J. Res. Nat. Bur. Standards, **82** (3), 197-202, 1977.

[23] C. Q. Zdg. *Finding critical independent sets and critical vertex subsets are polynomial problems*, SIAM J. Discrete Math., **3** (3), 431-438, 1990.

Graphs with small hyperbolicity constant

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Abstract. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. We denote by $\delta(X)$ the sharpest hyperbolicity constant of X, i.e. $\delta(X) := \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. In this paper we study the graphs with small hyperbolicity constant.

Key words: Graphs; infinite graphs; geodesics; Gromov hyperbolicity; hyperbolicity constant.

1 Introduction

Hyperbolic spaces play an important role in geometric group theory and in geometry of negatively curved spaces(see, e.g., [16]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see, e.g., [16]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [1,2,4,9,6,11,15,9,11,21,23,15,16,29,19,20,22,22,31,14,34] and the references therein. In particular, in [19,24,14] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a very simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

The theory of Gromov's spaces was used initially for the study of finitely generated groups, where it was demonstrated to have practical importance. This theory was applied principally to the study of automatic groups (see [18]), that play an important role in computer science. Another important application of these spaces is secure transmission of information by internet (see [11,20,21]), the spread of viruses through the network (see [20,21]), or the study of DNA data (see [9]).

Now we give the basic facts about Gromov's spaces. We say that a curve in a metric space (X,d) $\gamma : [a,b] \longrightarrow X$ is a *geodesic* if it is an isometry, i.e., $L(\gamma|_{[t,s]}) = d(\gamma(t),\gamma(s)) = |t-s|$ for every $s,t \in [a,b]$. We say that X is a *geodesic metric space* if for every $x,y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If X is a graph, we use the notation $[u,v]$ for the edge of a graph joining the vertices u and v.

We consider graphs such that the length of every edge is $k \in (0,\infty)$. In order to consider a graph G as a geodesic metric space, we must identify any edge $[u,v] \in E(G)$ with the real interval $[0,k]$; therefore, any point in the interior of any edge is a point of G. Hence, if we consider the edge $[u,v]$ as a graph with just one edge, then it is isometric to $[0,k]$. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Along the paper we just consider simple (without loops and multiple edges) connected graphs whose edges have length equal to k; these properties guarantee that the graphs are geodesic metric spaces. Note that to exclude multiple edges and loops is not an important loss of generality, since [1, Theorems 8 and 10] reduce the problem of compute the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$, with $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharpest thin constant of J, i.e., $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1,x_2,x_3 \in X$, a *geodesic triangle* $T = \{x_1,x_2,x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$; it is usual to write also $T = \{[x_1x_2],[x_2x_3],[x_3x_1]\}$ and we will say that x_1,x_2,x_3 are the vertices of the triangle. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$.

The hyperbolicity constant $\delta(X)$ of a geodesic metric space can be viewed as a measure of how "tree-like" the space is, since those spaces with $\delta(X) = 0$ are precisely the metric trees. This is an interesting subject since, in many

applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with.

Since the hyperbolicity of many geodesic metric spaces is equivalent to the hyperbolicity of some graphs related to them, the study of hyperbolic graphs becomes an interesting topic. The main aim of this paper is to study the graphs with small hyperbolicity constant, i.e., the graphs which are like trees (in the Gromov sense).

We state now the main results in this work and the lemmas used in their proofs, see [3] for the detailed proofs.

2 Previous results

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex. It is known (see [24, Lemma 2.1]) that, for every graph G, it is satisfied

$$\delta(G) = \sup\{\delta(T) : T \text{ is a geodesic triangle in } G \text{ that is a cycle}\}.$$

Let G be a graph with edges of the same length k. We denote by $J(G)$ the union of the set $V(G)$ and the midpoints of the edges of G. Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles and such that the three vertices of the triangle T belong to $J(G)$, and denote by $\delta_1(G)$ the infimum of the constants λ such that every triangle in $\mathbb{T}_1$ is λ-thin.

The following results, which appear in [4, Theorems 2.5, 2.6 and 2.7], will be used throughout the paper.

Theorem 1. *For every graph G with edges of lengths k we have $\delta_1(G) = \delta(G)$.*

Theorem 2. *For every hyperbolic graph G with edges of lengths k, $\delta(G)$ is a multiple of $\frac{k}{4}$.*

The following result is a direct consequence of Theorems 1 and 2; it states that in the hyperbolic graphs with edges of length k there always exists a geodesic triangle for which the hyperbolicity constant is attained.

Theorem 3. *For any hyperbolic graph G with edges of lengths k, there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

3 Characterizations of graphs with small hyperbolicity constant

The results in this section show some characterizations for hyperbolic graphs with small hyperbolicity constant. In this sense, in [15, Theorem 11] appears the following result.

Theorem 4. *Let G be any graph with edges of length k.*

(a) $\delta(G) = 0$ *if and only if G is a tree.*

(b) $\delta(G) = \frac{k}{4}, \frac{k}{2}$ *is not satisfied for any graph G.*

(c) $\delta(G) = \frac{3k}{4}$ *if and only if G is not a tree and every cycle in G has length $3k$.*

In order to characterize the graphs with edges of length k and hyperbolicity constant greater than $\frac{3k}{4}$ it is necessary to obtain some previous results. If H is a subgraph of G and $w \in V(H)$, we denote by $\deg_H(w)$ the degree of the vertex w in the subgraph induced by $V(H)$. A subgraph H of G is said *isometric* if $d_H(x, y) = d_G(x, y)$ for every $x, y \in H$. Note that this condition is equivalent to $d_H(u, v) = d_G(u, v)$ for every vertices $u, v \in V(H)$.

Theorem 5. *Let G be any graph with edges of length k. Then $\delta(G) \geq \frac{5k}{4}$ if and only if there exist a cycle g in G with length $L(g) \geq 5k$ and a vertex $w \in g$ such that $\deg_g(w) = 2$.*

For every $m \geq 4$, we say that a graph G with edges of length 1 is *m-chordal* (see [34]) if for any cycle C in G with length $L(C) \geq m$, there exists an edge joining two non-consecutive vertices x, y of C. Given a cycle C in G, we say that a geodesic $g = [uv]$ is a *shortcut* if $u, v \in V(C)$, $L(g) = d(u, v) < d_C(u, v)$ and $g \cap C = \{u, v\}$.

Corollary 1. *Let G be any graph with edges of length 1. If $\delta(G) \leq 1$, then G is 5-chordal.*

Proposition 1. *Let G be any graph with edges of length k. If $\delta(G) = k$, then G has a cycle isomorphic to C_4.*

Lemma 1. *Let G be a graph with edges of length 1 such that $\delta(G) = \frac{5}{4}$. If G contains a cycle C of length 6 and there exist $x, y \in C$ such that $d(x, y) = 3$, then G has a cycle isomorphic to C_5.*

We will need the following results (see [1, Lemma 9] and [31, Theorem 11]) in order to prove Proposition 2 below.

Lemma 2. *If H is an isometric subgraph of G, then $\delta(H) \leq \delta(G)$.*

Lemma 3. *If C_n is the cycle graph with n vertices and edges of length k, then $\delta(C_n) = \frac{nk}{4}$.*

Corollary 2. *Let G be any graph with edges of length k. If G contains an isometric subgraph which is isomorphic to C_n, then $\delta(G) \geq \frac{nk}{4}$.*

Proposition 2. *Let G be any graph with edges of length k. If $\delta(G) = \frac{5k}{4}$, then G has a cycle isomorphic to C_5.*

Looking at Propositions 1 and 2 it seems logical to think that, if $\delta(G) = \frac{6k}{4}$, then G has a cycle isomorphic to C_6 or, more generally, if $\delta(G) = \frac{nk}{4}$, then G has a cycle isomorphic to C_n for each $n > 5$. But this is not true. Given any fixed $n > 5$, let us consider two graphs G, G' isomorphic to C_{n-2} with edges of length k and fix $e \in E(G)$ and $e' \in E(G')$; let G_n be the union of these two graphs by identifying e and e'. Choose a point $x \in G$ with $d(x, e) = \frac{n}{2} - 2$, and let y be the antipodal point of $x \in G$, i.e., the point with $d(x, y) = n - 3$. If $B = \{x, y\}$ is the geodesic bigon (i.e., a triangle with two equal vertices) with two different geodesics which do not contain e, one can check that

$$\delta(G_n) = \delta(B) = \frac{k}{2}\Big(\frac{n-2}{2} + 1\Big) = \frac{nk}{4}\,;$$

nevertheless G_n does not have any cycle isomorphic to C_n.

Theorem 6. *Let G be any graph with edges of length k. Then $\delta(G) = k$ if and only if the following conditions hold:*

(1) *There exists a cycle isomorphic to C_4.*
(2) *For every cycle γ such that $L(\gamma) \geq 5k$ and for every vertex $w \in \gamma$, it is satisfied* $\deg_\gamma(w) \geq 3$.

Proposition 3. *Let G be any graph with edges of length k. Assume that the following conditions hold:*

(1) *There exist a cycle g in G such that $L(g) \geq 5k$ and a vertex $w \in g$ satisfying* $\deg_g(w) = 2$.
(2) *For every cycle γ such that $L(\gamma) \geq 6k$, we have* $\operatorname{diam}(\gamma) \leq \frac{5k}{2}$.

Then we have $\delta(G) = \frac{5k}{4}$.

Proposition 4. *Let G be any graph with edges of length k. If $\delta(G) \geq \frac{3k}{2}$, then there exists a cycle g in G such that $L(g) \geq 6k$ and* $\operatorname{diam}(g) \geq 3k$.

4 Hyperbolicity constant and effective diameter

A graph with small hyperbolicity constant can have arbitrarily large diameter: the path graph with n vertices P_n verifies $\delta(P_n) = 0$ and $\operatorname{diam}(P_n) = \operatorname{diam} V(P_n) = n - 1$ for every n. However, there is a concept related with the diameter, the *effective diameter*, which is small when the hyperbolicity constant is small.

Definition 1. *We say that a vertex v of a graph G is a* cut-vertex *if $G \setminus \{v\}$ is not connected.*

A graph is *two-connected* if it is connected and it does not contain cut-vertices. Given any edge in G, let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal two-connected subgraphs $\{G_n\}_n$ the *canonical T-decomposition* of G.

We will need the following result, which allows to obtain global information about the hyperbolicity of a graph from local information (see [1, Theorem 3]).

Lemma 4. *Let G be any graph with canonical T-decomposition $\{G_n\}_n$. Then*

$$\delta(G) = \sup_n \delta(G_n).$$

Given a graph G and its canonical T-decomposition $\{G_n\}$, we define the *effective diameter* as

$$\text{effdiam } V(G) := \sup_n \text{diam } V(G_n), \qquad \text{effdiam } (G) := \sup_n \text{diam } (G_n).$$

We will need the following result (see [31, Theorem 8]).

Theorem 7. *In any graph G the inequality $\delta(G) \leq \frac{1}{2}\text{diam}(G)$ holds, and furthermore, it is sharp.*

Lemma 2 and Theorem 7 have the following consequence.

Lemma 5. *Let G be any graph. Then*

$$\delta(G) \leq \frac{1}{2}\text{effdiam}(G).$$

As a corollary of Theorems 2 and 4, every G_n in the canonical T-decomposition of a graph G with $\delta(G) < k$ is isomorphic to either K_2 or K_3. Therefore, we have the following result.

Proposition 5. *Let G be any graph with edges of length k. If $\delta(G) < k$, then* effdiam$V(G) = k$ *and* effdiam$(G) \leq \frac{3k}{2}$.

We are going to prove now a bound for effdiam$V(G)$ when $\delta(G) = k$. Recall that $J(G)$ denotes the union of the set $V(G)$ and the midpoints of the edges of G.

Proposition 6. *Let G be any graph with edges of length k. If there exists a cycle in G containing a geodesic $[vw]$ with $v, w \in J(G)$ and $d(v, w) \geq 3k$, then $\delta(G) \geq \frac{5k}{4}$.*

The argument in the proof of Proposition 6 also gives the following result.

Corollary 3. *Let G be any graph with edges of length k. If there exists a cycle in G containing a geodesic $[vw]$ with $v, w \in V(G)$, $d(v, w) = 2k$ and $\delta(G) = k$, then $[vw]$ is contained in a cycle with length $4k$.*

We have an example showing that the equality $\delta(G) = \frac{5k}{4}$ in Proposition 6 can be attained.

Proposition 7. *Let G be any graph with edges of length k. If G does not have cut-vertices and $\delta(G) \leq k$, then* $\mathrm{diam}V(G) \leq 2k$.

Finally, we obtain an upper bound of $\mathrm{effdiam}V(G)$ for every graph G with $\delta(G) = k$.

Theorem 8. *Let G be any graph with edges of length k. If $\delta(G) = k$, then* $\mathrm{effdiam}V(G) \leq 2k$.

Remark 1. It is not possible to bound $\mathrm{effdiam}V(G)$ if $\delta(G) \geq \frac{3k}{2}$: Let G be the Cayley graph of the group $\mathbb{Z} \times \mathbb{Z}_2$ (G has the shape of an infinite railway). We have $\delta(G) = \frac{3k}{2}$ and the canonical T-decomposition of G has just a graph $G_1 = G$; hence, $\mathrm{effdiam}V(G) = \mathrm{diam}V(G_1) = \infty$.

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References

[1] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic graphs, *Discrete Math.*, 313 (15) (2013), 1575–1585.

[2] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Touris, E., Hyperbolicity and complement of graphs, *Appl. Math. Letters*, 24 (2011), 1882-1887.

[3] Bermudo, S., Rodríguez J. M., Rosario, O. and Sigarreta, J. M., Small values of the hyperbolicity constant in graphs. Submited.

[4] Bermudo, S., Rodríguez, J. M. and Sigarreta, J. M., Computing the hyperbolicity constant, *Comput. Math. Appl.*, 62 (2011), 4592-4595.

[5] Brinkmann, G., Koolen J. and Moulton, V., On the hyperbolicity of chordal graphs, *Ann. Comb.* **5** (2001), 61-69.

[6] Carballosa, W., Pestana, D., Rodríguez, J. M. and Sigarreta, J. M., Distortion of the hyperbolicity constant of a graph, *Electr. J. Combin.*, 19 (2012), P67.

[7] Carballosa, W., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., On the hyperbolicity constant of line graphs, *Electr. J. Combin.*, 18 (2011), P210.

[8] Frigerio, R. and Sisto, A., Characterizing hyperbolic spaces and real trees, *Geom. Dedicata*, 142 (2009), 139-149.

[9] Gavoille, G. and Ly, O., Distance Labeling in Hyperbolic Graphs, *Lecture Notes in Computer Science*, 3827 (2005), 1071–1079.

[10] Ghys, E. and de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. *Progress in Mathematics 83, Birkhauser Boston Inc., Boston, MA,*, (1990).

[11] Jonckheere, E. and Lohsoonthorn, P., A hyperbolic geometry approach to multi-path routing, Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002), Lisbon, Portugal, July 2002. FA5-1.

[12] Jonckheere, E. A., Contrôle du trafic sur les réseaux à géometrie hyperbolique– Une approche mathématique a la sécurité de l'acheminement de l'information, *J. Europ. Syst. Automat.*, 37(2), (2003), 145-159.

[13] Jonckheere, E. A. and Lohsoonthorn, P., Geometry of network security, *American Control Conference*, ACC (2004), 111-151.

[14] Koolen, J. H. and Moulton, V., Hyperbolic Bridged Graphs, *Europ. J. Combin.*, 23 (2002), 683-699.

[15] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Hyperbolicity and parameters of graphs, *Ars Combin.*, 100 (2011), 43-63.

[16] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolicity in cartesian product graphs, *Proc. Indian Acad. Sci. (Math. Sci.)*, 120 (2010), 1-17.

[17] Oshika, K., *Discrete groups*, AMS Bookstore, 2002.

[18] Pestana, D., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolic cubic graphs, *Central Europ. J. Math.*, 10 (2012), 1141-1151.

[19] Portilla, A., Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.*, 14 (2004), 123-149.

[20] Portilla, A., Rodríguez, J. M. and Tourís, E., Stability of Gromov hyperbolicity, *J. Adv. Math. Studies*, 2 (2009), No. 2, 77-96.

[21] Portilla, A. and Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.*, 53 (2009), 83-110.

[22] Rodríguez, J. M. and Sigarreta, J. M., Bounds on Gromov hyperbolicity constant in graphs, *Proc. Indian Acad. Sci. (Math. Sci.)*, 122 (2012), 53-65.

[23] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M. and Villeta, M., On the hyperbolicity constant in graphs, *Discrete Math.*, 311 (2011), 211-219.

[24] Rodríguez, J. M. and Tourís, E., Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hung.*, 103 (2004), 53-84.

[25] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.*, 380 (2011), 865-881.

[26] Wu, Y. and Zhang, C., Chordality and hyperbolicity of a graph, *Electr. J. Combin.*, 18 (2011), P43.

Classification of lattice 3-polytopes with few lattice points *

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Abstract. We undertake the complete classification of lattice 3-polytopes with n lattice points, modulo unimodular equivalence, for $n = 5, 6$. We first argue that for each n there is only a finite number of (equivalence classes) of polytopes with n points and of lattice width larger than one. Since polytopes of width one are easy to classify, we concentrate on an exhaustive classification of those of width larger than one.

For $n = 4$, all empty tetrahedra have width one (White, Howe). For $n = 5$ we show that there are exactly 9 different polytopes of width 2, and none of larger width. For $n = 6$, we show that there are 74 classes of width 2, two of width 3, and none of larger width.

Our motivation comes partly from the concept of *distinct pair sum* (or dps) polytopes, which are known to have at most 8 lattice points. Among the $9+74+2$ classes mentioned above, exactly $9 + 44 + 1$ are dps.

Key words: Lattice polytopes, unimodular equivalence, lattice points.

1 Introduction

A *lattice polytope* is the convex hull of a finite set of points in $\mathbb{Z}^d$ (or in a d-dimensional lattice). We call *size* of P its number $P \cap \mathbb{Z}^d$ of lattice points and *volume* of P its volume normalized to the lattice (that is, $d+1$ points form a simplex of volume one if and only if they are an affine lattice basis). Two such polytopes P and Q are said $\mathbb{Z}$-*equivalent* or *unimodularly equivalent* if there is an affine integer unimodular transformation $f : \mathbb{Z}^d \to \mathbb{Z}^d$ with $f(P) = Q$.

We are interested in the complete classification of such polytopes in dimension 3, for small size.

The same question in dimension 2 has a relatively simple, and fully algorithmic, answer. In particular, by Pick's Theorem, there is a finite number of 2-polytopes of any given size. Figure 1 shows the full list up to size five.

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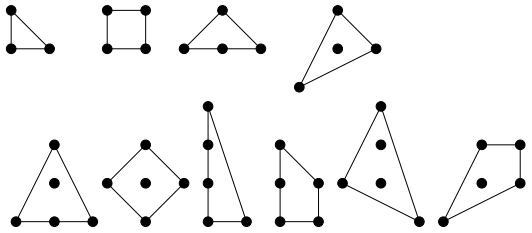

Fig. 1: The 2-dimensional polygons with up to five points

In dimension 3 the situation is completely different. In particular, there are *empty tetrahedra* with arbitrarily large volume, so no analogue of Pick's Theorem is possible. Here, an *empty tetrahedron* is the same as a 3-polytope of size four: a tetrahedron with integer vertices and no other integer point. However, the following three results are valid in arbitrary dimension. In the first one, a *hollow* lattice polytope is one with no lattice point in its interior.

Theorem 1 (Nill-Ziegler [6, Thm. 1.2]). *In each dimension d, there is only a finite number of hollow lattice d-polytopes that do not admit a lattice projection onto a hollow lattice $(d-1)$-polytope.*

Theorem 2 (Hensley [3, Thm. 3.4]). *For fixed positive integers k and d there is a number $V(k,d)$ such that every lattice d-polytope with k interior lattice points has volume bounded above by $V(k,d)$.*

Theorem 3 (Lagarias-Ziegler [5, Thm. 2]). *A family of lattice d-polytopes (for any fixed d) with bounded volume contains only a finite number of integral equivalence classes.*

Since there is a unique hollow 2-polytope of width larger than one, we get the following, where the *width* of a lattice polytope is the minimum of $\max_{x\in P} f(x) - \min_{x\in P} f(x)$, among all possible (non-constant) choices of an integer linear functional $f : \mathbb{Z}^d \to \mathbb{Z}$. In particular, P has *width one* if its vertices lie in two consecutive parallel lattice hyperplanes.

Corollary 1. *There are finitely many lattice 3-polytopes of width greater than one for each size n.*

So, it makes sense to classify, for each size n, separately the 3-polytopes of width one and those of width larger than one. Those of width larger than one are a finite list. Those of width one are infinite, but easy to describe: they consist of two 2-polytopes of sizes n_1 and n_2 ($n_1 + n_2 = n$) placed on parallel consecutive planes. For each of the two sub configurations there is a finite number of possibilities, but infinitely many ways to "rotate" (in the integer sense, that is via an element of $SL(\mathbb{Z}, 2)$) one with respect to the other.

For example, it is a now classical result that all empty tetrahedra have width one (see Theorem 5 below). The following generalization of this is very useful to us:

Theorem 4 (Howe, see [9, Thm. 1.3]). *Every lattice 3-polytope with no lattice points other than its vertices has width 1. In particular, all maximal 3-polytopes with that property consist of two empty parallelograms in consecutive parallel lattice planes.*

Our results, summarized in Theorems 7 and 8, and in Tables 1–3 are:

- There are exactly 9 3-polytopes of size 5 and width > 1, all of width 2. They are all dps.
- There are exactly 76 3-polytopes of size 6 and width > 1, 74 of width 2 and 2 of width 3. 44 and 1 of those, respectively, are dps.

Our motivation comes partially from the notion of *distinct pair-sum* lattice polytopes, defined as lattice polytopes in which all the pairwise sums $a + b, a, b \in P \cap \mathbb{Z}^d$ are distinct. Equivalently, they are lattice polytopes containing no three collinear lattice points nor the vertices of a non degenerate parallelogram [1, Lemma 1]. A dps d-polytope cannot have two lattice points in the same class modulo $(2\mathbb{Z})^d$. In particular, it cannot have more than 2^d lattice points. Reznick [8] asks:

- What is the range for the volume of dps polytopes of size 2^d in R^d?
- Is every dps d-polytope a subset of one of size 2^d?
- How many "inequivalent" dps polytopes of size 2^d are there in R^d?

Our ultimate goal would be to answer these questions in dimension three.

Acknowledgement: We thank Bruce Reznick for useful comments and references on the topic of this work.

2 Preliminaries: Empty simplices and volume vectors

Lattices 3-polytopes of size four, that is, empty tetrahedra, are well classified (see, e.g., [9,7]):

Definition 1. *We call an empty lattice simplex T of type $T(p, q)$, $1 \leq p < q$, if it has volume q and there is an affine transformation sending its vertices to $o = (0, 0, 0)$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ and sending the lattice $\mathbb{Z}^3$ to the lattice*

$$\Lambda(p, q) := (p/q, -p/q, 1/q) + \mathbb{Z}^3.$$

We call $R(p,q) := \text{conv}\{(0,0,0),(0,0,1),(1,-1,0),(1,-1,1)\}$ the *fundamental rectangle* of $T(p,q)$. Observe that all non-integer lattice points of $\Lambda(p,q)$ lie in an integer translation of $R(p,q)$.

Theorem 5 (Classification of empty tetrahedra, White [10], Howe (see [9])). *Every empty lattice simplex of volume $q \geq 2$ is of type $T(p,q)$ for some $p \in \{1, q-1\}$ with $\gcd(p,q) = 1$. Moreover, if $p' = p^{-1} \pmod q$, we have*

$$T(p,q) \cong T(p',q) \cong T(q-p,q) \cong T(q-p',q).$$

Among the methods we have used to elaborate the classification, we often need to check whether certain tetrahedra are empty and to find their type (p,q). We have implemented this in MATLAB.

One basic property of $\mathbb{Z}$-equivalence is that it preserves volume. This makes the following definition useful for our classification:

Definition 2. *Let $\{p_1, p_2, \ldots, p_n\}$ be the set of lattice points in a 3-polytope P. The* volume vector of P *is the vector*

$$\overline{w} = (w_{i,j,k,l})_{1 \leq i < j < k < l \leq n} \in \mathbb{Z}^{\binom{n}{4}}, \quad \textit{where} \quad w_{i,j,k,l} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_i & p_j & p_k & p_l \end{pmatrix}. \quad (1)$$

Observe that the definition of volume vector implicitly assumes a specific ordering for the n points in P.

Remark 1. The volume vector encodes the unique (modulo a scalar factor) dependence among each set of 5 points $\{p_i, p_j, p_k, p_l, p_r\}$, which is:

$$w_{j,k,l,r} \cdot p_i - w_{i,k,l,r} \cdot p_j + w_{i,j,l,r} \cdot p_k - w_{i,j,k,r} \cdot p_l + w_{i,j,k,l} \cdot p_r = 0$$
$$w_{j,k,l,r} - w_{i,k,l,r} + w_{i,j,l,r} - w_{i,j,k,r} + w_{i,j,k,l} = 0.$$

3 Polytopes with five lattice points

The five points $A = \{p_1, p_2, p_3, p_4, p_5\}$ in a 3-polytope of size five have a unique affine dependence. The *Radon partition* of A is obtained by looking at the signs of coefficients in this dependence. We say that $P = \text{conv}\, A$ has *signature* (i,j) if this dependence has i positive and j negative coefficients. The five possibilities for (i,j) are $(1,2)$, $(1,3)$, $(1,4)$, $(2,2)$ and $(3,2)$. (Observe that (i,j) and (j,i) are the same signature).

In order for the volume vector of P to encode its signature, taking into account Remark 1, for a size 5 polytope we modify its volume vector to be

$$(w_{2,3,4,5}, -w_{1,3,4,5}, w_{1,2,4,5}, -w_{1,2,3,5}, w_{1,2,3,4})$$

where $w_{i,j,k,l}$ is as in Equation 1. In this way, the signature of P is just the number of positive and negative entries in the volume vector, and the sum of coordinates in the volume vector vanishes.

To classify 3-polytopes of size five, we treat separately signatures $(*, 2)$ and $(1, *)$ (the case $(1, 2)$ appears in both, but that is not a problem).

Our starting point is that every polytope P of size five and signature $(*, 2)$ have width one. The proof, that we omit, is based on the fact that if T is the tetrahedron of largest volume in P then P is contained in the second dilation of T.

Theorem 6. *If P is a lattice 3-polytope of size 5 and of one of the signatures $(3, 2)$, $(2, 2)$ or $(1, 2)$, then P has width one.*

The classification of these configurations, though infinitely many, is easy and contained in the first three lines of Table 1.

So, we concentrate on the cases of signatures $(*, 1)$. Let $(v_1, v_2, v_3, v_4, v_5)$ be the volume vector of a polytope of signature $(*, 1)$, reordered so that $v_5 < 0 \leq v_i \leq v_4$, $i = 1, 2, 3$. That is, point p_5 lies in the interior of $\text{conv}\{p_1, p_2, p_3, p_4\}$, and the tetrahedron $T_4 := \text{conv}\{p_1, p_2, p_3, p_5\}$ has the maximum volume among the empty simplices in P. Also, without loss of generality, assume that T_4 has width one with respect to the pair of edges p_1p_2 and p_3p_5. That is, the affine change of coordinates that sends T_4 to be the standard simplex, with $p_5 = o$, $p_1 = e_1$, $p_2 = e_2$, $p_3 = e_3$, sends $\mathbb{Z}^3$ to the lattice $\Lambda(p, q)$ corresponding to the type of T_4.

The affine dependence

$$\sum v_i p_i = 0, \qquad \sum v_i = 0$$

implies that $p_4 = \frac{-1}{v_4}(v_1, v_2, v_3)$. Now, since $v_i \geq 0$ for $i \in \{1, 2, 3\}$, p_4 lies in the (closed) negative orthant. Also, since $v_4 = -(v_1 + v_2 + v_3 + v_5)$ we have that p_4 lies in the cube $[-1, 0]^3$. Then there are four cases for p_4:

(a) If $p_4 \in \{(-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$, then P has signature $(2, 1)$. In particular, it has width one and has been analyzed in the previous section.
(b) If $p_4 \in \{(-1, -1, 0), (0, -1, -1), (-1, 0, -1)\}$, then P has signature $(3, 1)$ and volume vector $(q, q, q, 0, -3q)$.
(c) If $p_4 = (-1, -1, -1)\}$, then P has signature $(4, 1)$ and volume vector $(q, q, q, q, -4q)$.
(d) If p_4 is not a vertex of the cube $[-1, 0]^3$, then it must be an interior lattice point in the (translated) fundamental rectangle $(-1, 0, 0)$, $(0, -1, 0)$, $(-1, 0, -1)$, $(0, -1, -1)$. The signature is again $(4, 1)$.

We omit details, but a cue by case analysis of this this division leads to:

Theorem 7. *There are exactly nine 3-polytopes of size 5 and width two and none of larger width. Eight of them have signature $(4, 1)$, and one has signature*

$(3,1)$. *Representatives and volume vectors for them are shown in Table 1. All of them are dps.*

Let us mention that part of this result was already known. Configurations of signatures $(2,2)$ and $(3,2)$ have width 1 by Howe's Theorem. Configurations of signature $(4,1)$ were classified by Reznick [7, Thm. 7] and Kasprzyk [4], who obtained exactly the same result as we do.

4 Polytopes with six lattice points

Let us here only state our classification results for width > 1. The methods combine a case by case study with the use of the classification of size five. In some of the cases we use partial results from [2], and some are solved by an exhaustive computer search implemented in MATLAB.

Theorem 8. *Among the lattice* 3*-polytopes with* 6 *lattice points, there are exactly* 74 *equivalence classes of width two,* 2 *of width three, and none of width larger than three, as shown in Tables 2 and 3.*

The tables show separately the polytopes that contain point coplanarities (44 of them, Table 2) and those which don't (32 of them, Table 3). In Table 2 the configurations are divided into four sections, related to the way we have computed them. First those containing 5 coplanar points; among the rest: first those with $(3,1)$ coplanarities; then those with some $(2,2)$ but no $(3,1)$ coplanarity; finally those with only $(2,1)$ coplanarities. In Table 3 configurations are separated according to whether they have one or two interior points.

In all tables, the points $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$ are omitted from the representative of each class, and the dps polytopes are marked with (*).

References

[1] M. D. Choi, T. Y. Lam and B. Reznick. Lattice polytopes with distinct pair-sums. *Discrete Comput. Geom.* 27 (2002), 65–72.

[2] M. Curcic. Lattice polytopes with distinct pair-sums. Ph. D. Thesis, University of Illinois at Urbana-Champaign, 2012.

[3] D. Hensley. Lattice vertex polytopes with interior lattice points. *Pacific J. Math.* 105:1 (1983), 183–191.

[4] A. Kasprzyk. Toric Fano 3-folds with terminal singularities. *Tohoku Math. J.* (2) 58:1 (2006), 101–121.

[5] J. C. Lagarias and G. M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canadian J. Math.*, 43:5 (1991), 1022–1035.

Signature	**Volume vector**	**Width**	**Representative**
$(2,2)$	$(1,-1,-1,1,0)$	1	$(1,1,0)$,$(0,0,1)$
$(2,1)$	$(-2q,q,0,q,0)$ $0 \le p \le \frac{q}{2}$ with $\gcd(p,q)=1$	1	$(-1,0,0)$,$(p,1,q)$
$(3,2)$	$(-(p+q),p,q,1,-1)$ $0 < p \le q$ with $\gcd(p,q)=1$	1 (*)	$(0,0,1)$,$(p,q,1)$
$(3,1)$	$(-3,1,1,1,0)$	1 (*)	$(-1,-1,0)$,$(0,0,1)$
	$(-9,3,3,3,0)$	2 (*)	$(-1,-1,0)$,$(1,2,3)$
$(4,1)$	$(-4,1,1,1,1)$	2 (*)	$(1,1,1)$,$(-2,-2,-1)$
	$(-5,1,1,1,2)$	2 (*)	$(1,1,2)$,$(-1,-1,-1)$
	$(-7,1,1,2,3)$	2 (*)	$(1,1,3)$,$(-1,-1,-2)$
	$(-11,1,3,2,5)$	2 (*)	$(2,1,5)$,$(-1,-1,-2)$
	$(-13,3,4,1,5)$	2 (*)	$(2,1,5)$,$(-1,-1,-1)$
	$(-17,3,5,2,7)$	2 (*)	$(2,1,7)$,$(-1,-1,-2)$
	$(-19,5,4,3,7)$	2 (*)	$(3,1,7)$,$(-2,-1,-3)$
	$(-20,5,5,5,5)$	2 (*)	$(2,1,5)$,$(-3,-2,-5)$

Table 1: Lattice 3-polytopes of size 5. Those marked with an asterisk are d. p. s. The points $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$ are omitted from each class.

[6] B. Nill and G. M. Ziegler. Projecting lattice polytopes without interior lattice points. *Math. Oper. Res.*, 36:3 (2011), 462–467.

[7] B. Reznick. Clean lattice tetrahedra, preprint, June 2006, 21 pages, `http://arxiv.org/abs/math/0606227`.

[8] B. Reznick. Problem 3 in "Let me tell you my favorite lattice-point problem" in *Integer Points in Polyhedra—Geometry, Number Theory, Representation Theory, Algebra, Optimization, Statistics*, M. Beck, C. Haase, B. Reznick, M. Vergne, V. Welker, and R. Yoshida (eds.). Contemp. Math. 452, Amer. Math. Soc. 2008.

[9] H. E. Scarf. Integral polyhedra in three space. *Math. Oper. Res.* 10 (1985), 403–438.

[10] G. K. White. Lattice tetrahedra. *Canadian J. Math.* 16 (1964), 389–396.

Volume vector	Width		Representative
$(0,0,2,0,-2,0,0,0,-4,4,0,4,-2,2,8)$	2		$(0,-1,0)$,$(2,0,0)$,$(1,1,2)$
$(0,0,2,0,-2,0,0,0,2,-2,0,4,4,-4,-4)$	2		$(0,-1,0)$,$(-1,0,0)$,$(1,1,2)$
$(0,1,-1,-1,1,-4,1,-1,1,3,3,-3,3,9,0)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,4,-1)$
$(0,1,-1,-1,1,-3,1,-1,0,3,3,-3,1,8,1)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(0,3,-1)$
$(0,1,-1,-1,1,-2,1,-1,1,1,3,-3,1,5,-2)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,2,-1)$
$(0,1,-1,-1,1,-3,1,-1,1,2,3,-3,2,7,-1)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,3,-1)$
$(0,1,-1,-1,1,-5,1,-1,1,4,3,-3,4,11,1)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,5,-1)$
$(0,1,-1,-1,1,-6,1,-1,1,5,3,-3,5,13,2)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,6,-1)$
$(0,1,-1,-1,1,-2,1,-1,0,2,3,-3,0,6,0)$	2		$(-1,-1,0)$,$(0,0,1)$,$(0,2,-1)$
$(0,1,-1,-1,1,-1,1,-1,0,1,3,-3,-1,4,-1)$	2		$(-1,-1,0)$,$(0,0,1)$,$(0,1,-1)$
$(0,1,-1,-1,1,-1,1,-1,1,0,3,-3,0,3,-3)$	2		$(-1,-1,0)$,$(0,0,1)$,$(1,1,-1)$
$(0,1,-1,-1,1,0,1,-1,0,0,3,-3,-2,2,-2)$	2		$(-1,-1,0)$,$(0,0,1)$,$(0,0,-1)$
$(0,1,-3,-1,3,-8,1,-3,1,7,3,-9,5,19,2)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,8,-3)$
$(0,1,-3,-1,3,-2,1,-3,1,1,3,-9,-1,7,-4)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,2,-3)$
$(0,1,-3,-1,3,-5,1,-3,1,4,3,-9,2,13,-1)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(1,5,-3)$
$(0,3,-3,-3,3,-9,3,-3,0,9,9,-9,3,24,3)$	2	(*)	$(-1,-1,0)$,$(1,2,3)$,$(-1,1,-3)$
$(0,3,-3,-3,3,0,3,-3,0,0,9,-9,-6,6,-6)$	2		$(-1,-1,0)$,$(1,2,3)$,$(-1,-2,-3)$
$(0,1,2,-1,-2,0,1,2,-1,1,3,6,0,0,3)$	2		$(-1,-1,0)$,$(0,0,1)$,$(-1,0,2)$
$(0,1,2,-1,-2,0,1,2,0,0,3,6,1,-1,1)$	2		$(-1,-1,0)$,$(0,0,1)$,$(0,0,2)$
$(0,3,6,-3,-6,0,3,6,0,0,9,18,3,-3,3)$	**3**		$(-1,-1,0)$,$(1,2,3)$,$(2,4,6)$
$(0,1,5,-1,-5,2,1,5,-1,-1,3,15,1,-7,4)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(-2,-1,5)$
$(0,1,7,-1,-7,1,1,7,-2,1,3,21,3,-6,9)$	2	(*)	$(-1,-1,0)$,$(0,0,1)$,$(-2,-1,7)$
$(0,3,1,-3,-1,-1,3,1,2,-1,9,3,1,2,-5)$	2	(*)	$(-1,-1,0)$,$(1,2,3)$,$(1,1,1)$
$(0,1,-1,1,-1,-4,-1,1,3,-1,-1,1,5,1,-2)$	2		$(1,1,0)$,$(0,0,1)$,$(3,4,-1)$
$(0,1,-1,1,-1,-5,-1,1,4,-1,-1,1,7,2,-3)$	2		$(1,1,0)$,$(0,0,1)$,$(4,5,-1)$
$(0,1,5,1,5,1,-1,-5,-2,-1,-1,-5,1,2,-3)$	2		$(1,1,0)$,$(0,0,1)$,$(-2,-1,5)$
$(0,1,7,1,7,2,-1,-7,-3,-1,-1,-7,1,3,-4)$	2		$(1,1,0)$,$(0,0,1)$,$(-3,-2,7)$
$(-1,1,0,-1,0,0,1,-2,2,-2,4,-1,1,-1,-7)$	2		$(1,1,1)$,$(-2,-2,-1)$,$(2,0,0)$
$(-2,1,0,-1,0,0,1,-4,2,-2,5,-2,1,-1,-9)$	2		$(1,1,2)$,$(-1,-1,-1)$,$(2,0,0)$
$(-2,1,2,-1,-2,0,1,2,0,0,5,8,1,-1,1)$	2		$(1,1,2)$,$(-1,-1,-1)$,$(-2,-2,-2)$
$(-3,2,0,-1,0,0,1,-6,4,-2,7,-3,2,-1,-13)$	2		$(1,1,3)$,$(-1,-1,-2)$,$(2,0,0)$
$(-3,2,4,-1,-2,0,1,2,0,0,7,11,2,-1,1)$	2		$(1,1,3)$,$(-1,-1,-2)$,$(-2,-2,-4)$
$(-5,2,4,-3,-6,0,1,2,0,0,11,17,2,-3,1)$	2		$(2,1,5)$,$(-1,-1,-2)$,$(-2,-2,-4)$
$(-1,1,0,-1,0,0,1,1,-1,1,4,2,-2,2,2)$	2		$(1,1,1)$,$(-2,-2,-1)$,$(-1,0,0)$
$(-2,1,0,-1,0,0,1,2,-1,1,5,4,-2,2,3)$	2		$(1,1,2)$,$(-1,-1,-1)$,$(-1,0,0)$
$(-2,1,-1,-1,1,0,1,-1,0,0,5,-1,-2,2,-2)$	2		$(1,1,2)$,$(-1,-1,-1)$,$(1,1,1)$
$(-3,2,0,-1,0,0,1,3,-2,1,7,6,-4,2,5)$	2		$(1,1,3)$,$(-1,-1,-2)$,$(-1,0,0)$
$(-3,2,-2,-1,1,0,1,-1,0,0,7,-1,-4,2,-2)$	2		$(1,1,3)$,$(-1,-1,-2)$,$(1,1,2)$
$(-5,2,-2,-3,3,0,1,-1,0,0,11,-1,-4,6,-2)$	2		$(2,1,5)$,$(-1,-1,-2)$,$(1,1,2)$
$(-1,1,0,-1,-1,1,1,-2,2,-3,4,0,0,-4,-8)$	2		$(1,1,1)$,$(-2,-2,-1)$,$(2,-1,0)$
$(-2,1,0,-1,-2,1,1,-4,2,-3,5,0,0,-5,-10)$	2		$(1,1,2)$,$(-1,-1,-1)$,$(2,-1,0)$
$(-2,1,2,-1,-2,0,1,4,-1,1,5,10,0,0,5)$	2		$(1,1,2)$,$(-1,-1,-1)$,$(-3,-2,-2)$
$(-3,2,-6,-1,0,2,1,3,-4,1,7,0,-14,0,7)$	2		$(1,1,3)$,$(-1,-1,-2)$,$(1,2,6)$
$(-3,2,4,-1,-2,0,1,5,-2,1,7,14,0,0,7)$	2		$(1,1,3)$,$(-1,-1,-2)$,$(-3,-2,-4)$

Table 2: Lattice 3-polytopes of size 6 and width > 1, part I; points with coplanarities.

Volume vector	Width		Representative
$(1,-1,1,1,-2,1,-1,3,-2,1,-4,5,-1,-3,7)$	2	(*)	$(1,1,1)$, $(-2,-2,-1)$, $(4,3,1)$
$(1,-1,1,1,-2,1,-1,5,-4,3,-4,7,-3,-1,13)$	2	(*)	$(1,1,1)$, $(-2,-2,-1)$, $(6,3,1)$
$(2,-1,-1,1,-1,1,-1,3,-2,1,-5,1,-3,-2,7)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(1,0,-1)$
$(2,-1,1,1,1,-1,-1,5,-2,3,-5,3,1,4,11)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(3,0,1)$
$(2,-1,-3,1,1,1,-1,5,-4,3,-5,-1,-7,2,13)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(1,-2,-3)$
$(2,-1,-5,1,3,1,-1,-1,-2,1,-5,-11,-7,2,3)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(-3,-4,-5)$
$(2,-1,-7,1,5,1,-1,-3,-2,1,-5,-17,-9,4,1)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(-5,-6,-7)$
$(3,-2,1,1,1,-1,-1,5,-3,2,-7,2,1,3,11)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(2,0,1)$
$(3,-2,5,1,2,-3,-1,1,1,1,-7,1,11,5,2)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(2,1,5)$
$(3,-2,7,1,4,-5,-1,-1,3,1,-7,-1,17,9,-2)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(2,1,7)$
$(3,-2,-1,1,2,-1,-1,-5,3,-1,-7,-11,5,1,-8)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(-2,-1,-1)$
$(3,-2,-5,1,1,1,-1,-4,1,-1,-7,-13,-3,-2,-5)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(-3,-2,-5)$
$(3,-2,-7,1,2,1,-1,-5,1,-1,-7,-17,-5,-1,-6)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(-4,-3,-7)$
$(3,-2,-5,1,4,-1,-1,-7,3,-1,-7,-19,1,3,-10)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(-4,-3,-5)$
$(5,-2,-1,3,4,-1,-1,-3,1,-1,-11,-13,3,1,-4)$	2	(*)	$(2,1,5)$, $(-1,-1,-2)$, $(-1,-1,-1)$
$(5,-2,-3,3,7,-1,-1,-4,1,-1,-11,-19,1,4,-5)$	2	(*)	$(2,1,5)$, $(-1,-1,-2)$, $(-2,-2,-3)$
$(5,-2,-5,3,5,1,-1,-5,1,-2,-11,-20,-3,-1,-7)$	2	(*)	$(2,1,5)$, $(-1,-1,-2)$, $(-3,-2,-5)$
$(5,-1,-1,4,-1,1,-3,7,-2,5,-13,2,-3,-1,17)$	2	(*)	$(2,1,5)$, $(-1,-1,-1)$, $(1,0,-1)$
$(5,-1,-2,4,-7,3,-3,-1,-1,-5,-13,-1,-5,-19,-2)$	2	(*)	$(2,1,5)$, $(-1,-1,-1)$, $(-1,1,-2)$
$(5,-1,-3,4,7,1,-3,-4,-1,1,-13,-19,-4,3,1)$	2	(*)	$(2,1,5)$, $(-1,-1,-1)$, $(-2,-2,-3)$
$(1,-1,-6,1,7,-1,-1,-5,-1,2,-4,-19,-5,9,-1)$	2	(*)	$(1,1,1)$, $(-2,-2,-1)$, $(-11,-13,-6)$
$(1,-1,-9,1,8,1,-1,-7,-2,1,-4,-25,-11,7,-3)$	2	(*)	$(1,1,1)$, $(-2,-2,-1)$, $(-16,-17,-9)$
$(2,-1,-7,1,9,-1,-1,-5,-1,2,-5,-23,-6,11,-1)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(-6,-8,-7)$
$(2,-1,-11,1,9,1,-1,-7,-2,1,-5,-29,-13,8,-3)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(-9,-10,-11)$
$(2,-1,7,1,1,-4,-1,-3,5,-1,-5,1,17,3,-8)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(2,3,7)$
$(2,-1,11,1,3,-7,-1,-1,6,1,-5,5,25,10,-5)$	2	(*)	$(1,1,2)$, $(-1,-1,-1)$, $(5,4,11)$
$(3,-2,-7,1,5,-1,-1,-8,3,-1,-7,-23,-1,4,-11)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(-5,-4,-7)$
$(3,-2,-11,1,10,-3,-1,-7,1,1,-7,-31,-5,13,-6)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(-6,-7,-11)$
$(3,-2,-1,1,-7,5,-1,-2,1,-3,-7,1,-3,-16,-5)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(-1,2,-1)$
$(3,-2,13,1,1,-5,-1,-4,7,-1,-7,5,27,4,-11)$	2	(*)	$(1,1,3)$, $(-1,-1,-2)$, $(3,4,13)$
$(5,-2,-9,3,11,1,-1,-7,1,-2,-11,-32,-7,5,-9)$	2	(*)	$(2,1,5)$, $(-1,-1,-2)$, $(-5,-4,-9)$
$(5,-2,11,3,1,-7,-1,-2,3,-1,-11,3,23,4,-5)$	**3**	(*)	$(2,1,5)$, $(-1,-1,-2)$, $(4,2,11)$

Table 3: Lattice 3-polytopes of size 6 and width > 1, part II; no coplanarities.

$\mathbb{Z}_2$-double cyclic codes *

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Abstract. A binary linear code C is a $\mathbb{Z}_2$-double cyclic code if the set of coordinates can be partitioned into two subsets such that any cyclic shift of the coordinates of both subsets leaves invariant the code. These codes can be identified as submodules of the $\mathbb{Z}_2[x]$-module $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$. We determine the structure of $\mathbb{Z}_2$-double cyclic codes giving the generator polynomials of these codes. The related polynomial representation of $\mathbb{Z}_2$-double cyclic codes and its duals, and the relations between the polynomial generators of these codes are studied.

Key words: Binary linear codes, Duality, $\mathbb{Z}_2$-double cyclic codes.

1 Introduction

Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Let $\mathbb{Z}_2^n$ denote the set of all binary vectors of length n. Any non-empty subset of $\mathbb{Z}_2^n$ is a binary code and a subgroup of $\mathbb{Z}_2^n$ is called a *binary linear code*. In this paper we introduce a subfamily of binary linear codes, called $\mathbb{Z}_2$*-double cyclic codes*, with the property that the set of coordinates can be partitioned into two subsets, the first r coordinates and the last s coordinates, such that any cyclic shift of the coordinates of both subsets of a codeword is also a codeword.

Notice that if one of these sets of coordinates is empty, for example $r = 0$, then we obtain a binary cyclic code of length s. So, binary cyclic codes are a special class of $\mathbb{Z}_2$-double cyclic codes. Most of the theory of binary cyclic codes can be found in [5]. Another special case is when $r = s$, where a $\mathbb{Z}_2$-double cyclic code is permutation equivalent to a quasi-cyclic code of index 2 and even length (see [5]).

In recent times, $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have been studied (see [2], [3]). For $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, the set of coordinates is partitioned into two subsets, the first one of binary coordinates and the second one of quaternary coordinates.

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The simultaneous cyclic shift of the subsets of coordinates of a codeword has been first defined in [1], that studies $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes and these codes can be identified as $\mathbb{Z}_4[x]$-modules of a certain ring.

The aim of this paper is the study of the algebraic structure of $\mathbb{Z}_2$-double cyclic codes and their dual codes. It is organized as follows. In Section 2, we give the definition of $\mathbb{Z}_2$-double cyclic codes, we find the relation between some canonical projections of these codes and binary cyclic codes and we present the $\mathbb{Z}_2[x]$-module $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$, denoted by $R_{r,s}$. In Section 3, we discuss about the algebraic structure of a $\mathbb{Z}_2$-double cyclic code and we state some relations between its generators. In Section 4, we study the concept of duality and, given a $\mathbb{Z}_2$-double cyclic code, we determine the generators of the dual code in terms of the generators of the code. Finally, in Section 5, we give some conclusions and discuss about future research.

2 $\mathbb{Z}_2$-double cyclic codes

Let C be a binary code of length n. Let r and s be integers such that $n = r+s$. We consider a partition of the set of the n coordinates into two subsets of r and s coordinates, respectively, so that C is a subset of $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$.

Definition 1. *Let C be a binary linear code of length $n = r + s$. The code C is called* $\mathbb{Z}_2$-double cyclic *if*

$$(u_0, u_1, \ldots, u_{r-2}, u_{r-1} | u'_0, u'_1, \ldots, u'_{s-2}, u'_{s-1}) \in C$$

implies

$$(u_{r-1}, u_0, u_1, \ldots, u_{r-2} | u'_{s-1}, u'_0, u'_1, \ldots, u'_{s-2}) \in C.$$

Let $\mathbf{u} = (u_0, u_1, ..., u_{r-1} | u'_0, ..., u'_{s-1})$ be a codeword in C and i be an integer, then we denote by

$$\mathbf{u}^{(i)} = (u_{0+i}, u_{1+i}, ..., u_{r-1+i} | u'_{0+i}, ..., u'_{s-1+i})$$

the ith shift of $\mathbf{u}$, where the subscripts are read modulo r and s, respectively.

Let C_r be the canonical projection of C on the first r coordinates and C_s on the last s coordinates. The canonical projection is a linear map. Then, C_r and C_s are binary cyclic codes of length r and s, respectively. A code C is called *separable* if C is the direct product of C_r and C_s.

There is a bijective map between $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ and $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$ given by:

$$(u_0, u_1, \ldots, u_{r-1} | u'_0, \ldots, u'_{s-1}) \mapsto (u_0 + u_1 x + \cdots + u_{r-1} x^{r-1} | u'_0 + \cdots + u'_{s-1} x^{s-1}).$$

We denote the image of the vector $\mathbf{u}$ by $\mathbf{u}(x)$.

Definition 2. *Denote by $R_{r,s}$ the ring $\mathbb{Z}_2[x]/(x^r-1) \times \mathbb{Z}_2[x]/(x^s-1)$. We define the operation*

$$\star : \mathbb{Z}_2[x] \times R_{r,s} \to R_{r,s}$$

as

$$\lambda(x) \star (p(x)|q(x)) = (\lambda(x)p(x)|\lambda(x)q(x)),$$

where $\lambda(x) \in \mathbb{Z}_2[x]$ and $(p(x)|q(x)) \in R_{r,s}$.

The ring $R_{r,s}$ with the external operation $\star$ is a $\mathbb{Z}_2[x]$-module. Let $\mathbf{u}(x) = (u(x)|u'(x))$ be an element of $R_{r,s}$. Note that if we operate $\mathbf{u}(x)$ by x we get

$$\begin{aligned} x \star \mathbf{u}(x) &= x \star (u(x)|u'(x)) \\ &= (u_0x + \cdots + u_{r-2}x^{r-1} + u_{r-1}x^r | u'_0x + \cdots + u'_{s-2}x^{s-1} + u'_{s-1}x^s) \\ &= (u_{r-1} + u_0x + \cdots + u_{r-2}x^{r-1} | u'_{s-1} + u'_0x + \cdots + u'_{s-2}x^{s-1}). \end{aligned}$$

Note that $x \star \mathbf{u}(x)$ is the image of the vector $\mathbf{u}^{(1)}$. Thus, the operation of $\mathbf{u}(x)$ by x in $R_{r,s}$ corresponds to a shift of $\mathbf{u}$. In general, $x^i \star \mathbf{u}(x) = \mathbf{u}^{(i)}(x)$ for all i.

3 Algebraic structure and generators

In this section, we study submodules of $R_{r,s}$. We describe the generators of such submodules and state some properties. From now on, $\langle S \rangle$ will denote the submodule generated by a subset S of $R_{r,s}$.

Theorem 1. *The $\mathbb{Z}_2[x]$-module $R_{r,s}$ is a noetherian $\mathbb{Z}_2[x]$-module, and every submodule N of $R_{r,s}$ can be written as*

$$N = \langle (b(x)|0), (\ell(x)|a(x)) \rangle,$$

where $b(x), \ell(x) \in \mathbb{Z}_2[x]/(x^r-1)$ with $b(x)|(x^r-1)$ and $a(x) \in \mathbb{Z}_2[x]/(x^s-1)$ with $a(x)|(x^s-1)$.

From the previous results, it is clear that we can identify $\mathbb{Z}_2$-double cyclic codes in $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ as submodules of $R_{r,s}$. So, any submodule of $R_{r,s}$ is a $\mathbb{Z}_2$-double cyclic code.

Proposition 1. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then, C is permutation equivalent to a binary linear code with generator matrix of the form*

$$G = \left(\begin{array}{ccc|ccc} I_{r-\deg(b(x))} & A_1 & A_2 & 0 & 0 & 0 \\ 0 & B_\kappa & B & C_1 & I_\kappa & 0 \\ 0 & 0 & 0 & C_2 & R & I_{s-\deg(a(x))-\kappa} \end{array} \right),$$

where B_κ is a square matrix of full rank and $\kappa = \deg(b(x)) - \deg(\gcd(b(x), \ell(x)))$.

Corollary 1. *Let* $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code. Then,* C *is a binary linear code of dimension* $r + s - \deg(b(x)) - \deg(a(x))$.

Proposition 2. *Let* $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code. Then, we can assume that* $\deg(\ell(x)) < \deg(b(x))$.

Proposition 3. *Let* $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code. Then,* $b(x) \mid \frac{x^s-1}{a(x)} \ell(x)$.

Corollary 2. *Let* $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ *be a* $\mathbb{Z}_2$*-double cyclic code. Then,* $b(x) \mid \frac{x^s-1}{a(x)} \gcd(\ell(x), b(x))$.

Proposition 4. *Let* $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ *be a separable* $\mathbb{Z}_2$*-double cyclic code. Then,* $\ell(x) = 0$.

4 Duality

Let C be a $\mathbb{Z}_2$-double cyclic code and $C^\perp$ the dual code of C (see [4]). Taking a vector $\mathbf{v}$ of $C^\perp$, $\mathbf{u} \cdot \mathbf{v} = 0$ for all $\mathbf{u}$ in C. Since $\mathbf{u}$ belongs to C, we know that $\mathbf{u}^{(-1)}$ is also a codeword. So, $\mathbf{u}^{(-1)} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}^{(1)} = 0$ for all $\mathbf{u}$ from C, therefore $\mathbf{v}^{(1)}$ is in $C^\perp$ and $C^\perp$ is also a $\mathbb{Z}_2$-double cyclic code. Consequently, we obtain the following proposition.

Proposition 5. *Let* C *be a* $\mathbb{Z}_2$*-double cyclic code. Then the dual code of* C *is also a* $\mathbb{Z}_2$*-double cyclic code, we denote the generators by*

$$C^\perp = \langle (\bar{b}(x)|0), (\bar{\ell}(x)|\bar{a}(x)) \rangle,$$

where $\bar{b}(x), \bar{\ell}(x) \in \mathbb{Z}_2[x]/(x^r - 1)$ *with* $\bar{b}(x) \mid (x^r - 1)$ *and* $\bar{a}(x) \in \mathbb{Z}_2[x]/(x^s - 1)$ *with* $\bar{a}(x) \mid (x^s - 1)$.

The *reciprocal polynomial* of a polynomial $p(x)$ is $x^{\deg(p(x))} p(x^{-1})$ and is denoted by $p^*(x)$. As in the theory of binary cyclic codes, reciprocal polynomials have an important role (see [5]).

We denote the polynomial $\sum_{i=0}^{m-1} x^i$ by $\theta_m(x)$. Using this notation we have the following proposition.

Proposition 6. *Let* $n, m \in \mathbb{N}$. $x^{nm} - 1 = (x^n - 1)\theta_m(x^n)$.

From now on, m denotes the least common multiple of r and s.

Definition 3. *Let* $\boldsymbol{u}(x) = (u(x)|u'(x))$ *and* $\boldsymbol{v}(x) = (v(x)|v'(x))$ *be elements in* $R_{r,s}$. *We define the map*

$$\circ : R_{r,s} \times R_{r,s} \longrightarrow \mathbb{Z}_2[x]/(x^m - 1),$$

such that

$$\begin{aligned} \circ(\boldsymbol{u}(x), \boldsymbol{v}(x)) =& u(x)\theta_{\frac{m}{r}}(x^r) x^{m-1-\deg(v(x))} v^*(x) + \\ &+ u'(x)\theta_{\frac{m}{s}}(x^s) x^{m-1-\deg(v'(x))} v'^*(x) \bmod (x^m - 1). \end{aligned}$$

The map $\circ$ is linear in each of its arguments; i.e., if we fix the first entry of the map invariant, while letting the second entry vary, then the result is a linear map. Similarly, when fixing the second entry invariant. Then, the map $\circ$ is a bilinear map between $\mathbb{Z}_2[x]$-modules.

From now on, we denote $\circ(\mathbf{u}(x), \mathbf{v}(x))$ by $\mathbf{u}(x) \circ \mathbf{v}(x)$. Note that $\mathbf{u}(x) \circ \mathbf{v}(x)$ belongs to $\mathbb{Z}_2[x]/(x^m - 1)$.

Proposition 7. *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{Z}_2^r \times \mathbb{Z}_2^s$ with associated polynomials $\boldsymbol{u}(x) = (u(x)|u'(x))$ and $\boldsymbol{v}(x) = (v(x)|v'(x))$, respectively. Then, $\boldsymbol{u}$ is orthogonal to $\boldsymbol{v}$ and all its shifts if and only if*

$$\boldsymbol{u}(x) \circ \boldsymbol{v}(x) = 0 \bmod (x^m - 1).$$

Proposition 8. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then,*

$$|C_r| = 2^{r - \deg(b(x)) + \kappa}, |C_s| = 2^{s - \deg(a(x))},$$
$$|(C_r)^\perp| = 2^{\deg(b(x)) - \kappa}, |(C_s)^\perp| = 2^{\deg(a(x))},$$
$$|(C^\perp)_r| = 2^{\deg(b(x))}, |(C^\perp)_s| = 2^{\deg(a(x)) + \kappa},$$

where $\kappa = \deg(b(x)) - \deg(\gcd(b(x), \ell(x)))$.

Corollary 3. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code with dual code $C^\perp = \langle (\bar{b}(x)|0), (\bar{\ell}(x)|\bar{a}(x)) \rangle$. Then,*

$$\deg(\bar{a}(x)) = s - \deg(a(x)) - \deg(b(x)) + \deg(\gcd(b(x), \ell(x))).$$

Proposition 9. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Then, $\langle (0|\frac{x^s - 1}{a^*(x)}) \rangle \subseteq C^\perp$.*

Corollary 4. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code with $C^\perp = \langle (\bar{b}(x)|0), (\bar{\ell}(x)|\bar{a}(x)) \rangle$. Then, $\bar{a}(x)$ divides $\frac{x^s - 1}{a^*(x)}$.*

Corollary 5. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code. Let $T = \{(0|p(x)) \in R_{r,s} \mid (0|p(x)) \in C^\perp\}$. Then, T is generated by $\langle (0|\frac{x^s - 1}{a^*(x)}) \rangle$.*

The previous propositions and corollaries will be helpful to determine the relations between the generator polynomials of a $\mathbb{Z}_2$-double cyclic code and the generator polynomials of its dual code.

Proposition 10. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code and $C^\perp = \langle (\bar{b}(x)|0), (\bar{\ell}(x)|\bar{a}(x)) \rangle$. Then,*

$$\bar{b}(x) = \frac{x^r - 1}{\gcd(b(x), \ell(x))^*}.$$

Proposition 11. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a separable $\mathbb{Z}_2$-double cyclic code. Then, $C^\perp = \langle (\frac{x^r - 1}{b^*(x)}|0), (0|\frac{x^s - 1}{a^*(x)}) \rangle$.*

Corollary 6. *Let C be a separable $\mathbb{Z}_2$-double cyclic code. Then, $C^\perp$ is a separable $\mathbb{Z}_2$-double cyclic code.*

Proposition 12. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a non separable $\mathbb{Z}_2$-double cyclic code and $C^\perp = \langle (\bar{b}(x)|0), (\bar{\ell}(x)|\bar{a}(x)) \rangle$. Then,*

$$\bar{\ell}(x) = \frac{x^r - 1}{b^*(x)} \lambda(x),$$

for some $\lambda(x) \in \mathbb{Z}_2[x]$.

Corollary 7. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a non separable $\mathbb{Z}_2$-double cyclic code. Then,* $\deg(\lambda(x)) < \deg(b(x)) - \deg(\gcd(b(x), \ell(x)))$.

Proposition 13. *Let $C = \langle (b(x)|0), (\ell(x)|a(x)) \rangle$ be a $\mathbb{Z}_2$-double cyclic code and $C^\perp = \langle (\bar{b}(x)|0), (\bar{\ell}(x)|\bar{a}(x)) \rangle$. Then*

$$\bar{a}(x) = \frac{(x^s - 1)\gcd(b(x), \ell(x))^*}{a^*(x)b^*(x)}.$$

5 Conclusions and further research

We have introduced $\mathbb{Z}_2$-double cyclic codes. We have studied their algebraic description and the generators of $\mathbb{Z}_2$-double cyclic codes and we have determined its dual codes.

Given the fact that these codes are really related with binary cyclic codes, further research could be done on finding efficient encoding and decoding algorithms. Also self-duality could be studied.

References

[1] T. Abualrub, I. Siap, H. Aydin. $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes. *IEEE Trans. Info. Theory*, vol. 60, No. 3, pp. 1508-1514, Mar. 2014.

[2] J. Borges, C. Fernández-Córdoba, J. Pujol, J. Rifà and M. Villanueva. $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality. *Designs, Codes and Cryptography*, vol. 54, No. 2, pp. 167-179, 2010.

[3] C. Fernández-Córdoba, J. Pujol and M. Villanueva. $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: rank and kernel. *Designs, Codes and Cryptography*, vol. 56, pp. 43 - 59, 2010.

[4] W.C. Huffman, V. Pless. *Fundamentals of Error-Correcting Codes.* Cambridge University Press, Cambridge, 2003.

[5] F.J. MacWilliams, N.J.A. Sloane. *The Theory of Error-Correcting Codes.* North-Holland Publishing Company, Amsterdam, New York, Oxford, 1975.

Sobre los números de Ramsey $R(K_5 - e, K_5)$ y $R(K_6 - e, K_4)$

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Resumen. En este trabajo se mejoran en una unidad las cotas superiores de dos números de Ramsey obteniéndose que $R(K_5 - e, K_5)$ y $R(K_6 - e, K_4)$ son a lo más 33.

Palabras clave: Teoría extremal de grafos, números de Ramsey.

1 Introducción y preliminares

Sea G un grafo, se denotará como $\overline{G}$ al grafo complementario de G y como $n(G)$ y $e(G)$ a sus números de vértices y aristas respectivamente. $G \subseteq G'$ significará que G es un subgrafo de G'. Los conjuntos de vértices y aristas de G se denotarán como $V(G)$ y $E(G)$ respectivamente. $G[S]$ es G el subgrafo de G inducido por $S \subseteq V(G)$. La cardinalidad de un conjunto A se denotará por $|A|$. Si $v \in V(G)$ se denotará como $d_G(v)$ a la valencia de v en G. $\Delta(G)$ y $\delta(G)$ son las valencias máximas y mínimas de G respectivamente. Si $v \in V(G)$, denotateromos $N_G(v)$ al conjunto de vértices de G adyacentes a v y como $N^-(v)$ a $V(G) \setminus N_G(v) \setminus \{v\}$.

K_n es el grafo completo con n vértices y $J_n = K_n - e$ el obtenido al quitarle a K_n una arista.

Dados dos grafos F y H, $\mathcal{R}(F, H; n)$ es el conjunto de grafos G tales que $n(G) = n$, $F \nsubseteq G$ y $H \nsubseteq \overline{G}$. El número de Ramsey $R(F, H)$ es el menor entero n tal que $\mathcal{R}(F, H; n)$ es vacío. Claramente $R(H, F) = R(F, H)$ y si $F_1 \subseteq F_2$ y $H_1 \subseteq H_2$ entonces $R(F_1, H_1) \leq R(F_2, H_2)$.

El número de Ramsey $R(F, H)$ ha sido muy estudiado para grafos F y H con pocos vértices. Trivialmente se tiene que $R(K_1, H) = 1$ y $R(K_2, H) = n(H)$. $R(P_3, H)$ y $R(2K_2, H)$ han sido determinados en [11] para cualquier grafo H.

Cuando $n(F), n(H) \leq 5$, salvo dos casos, son conocidos los valores de $R(F, H)$ [1,2,6,9,10,11,12,18,9,24,25,31,36,13,45]. Los dos casos abiertos son $F \in \{J_5, K_5\}$ y $H = K_5$ (ver [39] para más detalles).

Las mejores cotas de estos casos son [15] $43 \leq R(K_5, K_5) \leq 49$ [37] y [16] $30 \leq R(J_5, K_5) \leq 34$ [17].

También son conocidos los valores exactos de $R(F, H)$ cuando $n(F) \leq 4$, $F \neq K_4$ y $n(H) = 6$ [3,19,27,28,33,34,38,42], por lo que tiene un especial interés la búsqueda de los números $R(K_4, H)$.

1.1 Números $R(J_m, K_n)$

El cálculo de $R(K_m, K_n)$ ha sido muy estudiado en la literatura desde 1935. A estos números de Ramsey se les denomina clásicos. $R(K_m, K_n)$, también denotado como $R(m, n)$, es el menor número de vértices que ha de tener un grafo completo para que en cualquier coloración de sus aristas con dos colores azul y rojo o bien hay m vértices unidos dos a dos por aristas azules o bien hay n vértices unidos dos a dos por aristas rojas.

Le siguen en interés $R(J_m, K_n)$ y $R(J_m, J_n)$. $R(J_m, K_n)$ es el menor número de vértices que ha de tener un grafo completo para que en cualquier coloración de sus aristas con dos colores azul y rojo o bien hay m vértices unidos dos a dos por aristas azules o bien hay n vértices entre los que hay a lo sumo una arista azul.

Se han estudiado principalmente estos números $R(K_m, K_n)$, $R(J_m, K_n)$ y $R(J_m, J_n)$ cuando m y n y son pequeños y también cuando m y n son iguales.

Además de los casos triviales $R(J_m, K_1) = 1$, $R(J_m, K_2) = m$, $R(J_2, K_n) =$ mín$\{2, n\}$ y $R(J_3, K_n) = 2n - 1$, se conocen unos pocos valores exactos de $R(J_m, K_n)$: cuando $n = 3$ y $4 \leq m \leq 10$ [12,11,19,21,16,35,40], cuando $m = 4$ y $4 \leq n \leq 6$ [2,11,38,43] y cuando $m = 5$ y $n = 4$ [18].

Además de [16] $30 \leq R(J_5, K_5) \leq 34$ [17], antes mencionado, las mejores cotas de los casos abiertos para los menores valores de m y n son [44] $42 \leq R(J_{11}, K_3) \leq 45$ [21], [5] $30 \leq R(J_6, K_4) \leq 34$ [5] y [32] $28 \leq R(J_4, K_7) \leq 30$ [7].

En este trabajo se mejorará en una unidad la cota superior de $R(J_5, K_5)$ y $R(J_6, K_4)$.

1.2 Evolución histórica de las cotas superiores de $R(J_5, K_5)$ y $R(J_6, K_4)$

Por tanto el estudio de $R(J_5, K_5)$ y $R(J_6, K_4)$ es interesante por varias razones, porque los números de Ramsey del tipo $R(J_m, K_n)$ son los más estudiados tras los clásicos, porque m y n son pequeños, porque para valores menores el número de Ramsey es conocido. Además $\{J_5, K_5\}$ es la pareja de grafos más pequeños en los que su número de Ramsey es desconocido y m y n son iguales.

- En 1935 a Erdös y Szekeres [14] a partir de la fórmula:
  $R(G_1, G_2) \leq R(G_1 - u, G_2) + R(G_1, G_2 - v)$, (1)

siendo la desigualdad estricta si $R(G_1 - u, G_2)$ y $R(G_1, G_2 - v)$ son pares, para todos $u \in V(G_1)$ y $v \in V(G_2)$, probaron la finitud del número de Ramsey de cualquier par de grafos finitos. Por inducción se obtiene que:

$$R(K_m, K_n) \leq \binom{m+n-2}{m-1}.$$

Consecuentemente, las primeras cotas superiores de los números $R(J_5, K_5)$ y $R(J_6, K_4)$ que se deducen son $R(J_5, K_5) \leq R(K_5, K_5) \leq 70$ y $R(J_6, K_4) \leq R(K_6, K_4) \leq 58$.

La Fórmula (1), aunque fue enunciada para grafos completos es fácilmente generalizable para cualquier par de grafos. A partir de ella la cota superior mejor conocida de $R(J_5, K_5)$ ha ido mejorando con el paso de los años. Aplicándo (1) y usando los resultados triviales se tiene que $R(J_4, K_3) \leq 9$, $R(J_5, K_3) \leq 14$, $R(J_6, K_3) \leq 19$, $R(J_4, K_4) \leq 16$, $R(J_5, K_4) \leq 29$ y por tanto $R(J_6, K_4) \leq 48$. Además $R(J_4, K_5) \leq 25$ y por tanto $R(J_5, K_5) \leq 54$.

- En 1972 Chvátal y Harary [11] probaron que $R(J_4, K_3) = 7$, por lo que $R(J_5, K_3) \leq 12$ y $R(J_6, K_3) \leq 17$. En ese mismo trabajo se probó que $R(J_4, K_4) = 11$, por lo que $R(J_5, K_4) \leq 23$ y por tanto $R(J_6, K_4) \leq 40$. Además $R(J_4, K_5) \leq 20$ y $R(J_5, K_5) \leq 43$.
- En 1977 Clancy [12] demostró que $R(J_5, K_3) = 11$, por lo que $R(J_5, K_4) \leq 22$ y por tanto $R(J_6, K_4) \leq 39$. Además $R(J_5, K_5) \leq 41$.
- En 1981 Bolze y Harborth [2] probaron que $R(J_4, K_5) = 16$, de lo que se deduce que $R(J_5, K_5) \leq 37$.
- En 1988 Exoo, Harborth y Mengersen [18] demostraron que $R(J_5, K_4) = 19$ y por tanto $R(J_6, K_4) \leq 36$ y $R(J_5, K_5) \leq 35$.
- En 1992 Exoo [17], probó $R(J_5, K_5) \leq 34$.
- Finalmente en 2013 [4] demostramos que $R(J_6, K_4) \leq 34$.

2 Resultados principales e idea de la demostración

Para probar los resultados principales, $R(J_5, K_5) \leq 33$ y $R(J_6, K_4) \leq 33$, procederemos por reducción al absurdo, es decir, supondremos que existe $G_1 \in \mathcal{R}(K_4, J_6; 33)$ ó $G_2 \in \mathcal{R}(J_5, K_5; 33)$.

Utilizaremos a partir de ahora V_i, E_i, d_i, N_i, N_i^-, Δ_i y δ_i en vez de $V(G_i)$, $E(G_i)$, d_{G_i}, N_{G_i}, $N_{G_i}^-$, Δ_{G_i} y δ_{G_i}, respectivamente, para $i = 1, 2$.

Para todo $v \in V_1$ se tiene que $G_1[N_1(v)] \in \mathcal{R}(K_3, J_6; d(v))$ y $G_1[N_1^-(v)] \in \mathcal{R}(K_4, J_5; 32 - d(v))$, por lo que $\Delta_1 \leq R(K_3, J_6) - 1 = 16$ y $\delta_1 \geq 33 - R(K_4, J_5) = 14$.

En la prueba de esta cota superior se utilizarán los conjuntos $\mathcal{R}(K_3, J_6; 14)$, $\mathcal{R}(K_3, J_6; 15)$, $\mathcal{R}(K_3, J_6; 16)$, $\mathcal{R}(K_4, J_5; 16)$, $\mathcal{R}(K_4, J_5; 17)$ y $\mathcal{R}(K_4, J_5; 18)$, los cuales también podían ser encontrados en [20] y tienen cardinales 4, 1, 1,

1494223, 3033 y 6, respectivamente. (Desgraciadamente esta página ya no está operativa.)

Análogamente, para todo $v \in V_2$ se tiene que $G_2[N_2(v)] \in \mathcal{R}(J_4, K_5; d(v))$ y $G[N_2^-(v)] \in \mathcal{R}(J_5, K_4; 32 - d(v))$, por lo que $\Delta_2 \leq R(J_4, K_5) - 1 = 15$ y $\delta_2 \geq 33 - R(J_5, K_4) = 14$.

Dado que en un grafo con un número impar de vértices no todos los vértices pueden tener valencia impar, se tiene que en G_2 existen vértices de valencia 14.

Los conjuntos $\mathcal{R}(J_4, K_5; 14)$ y $(J_5, K_4; 18)$ han sido mostrados en [20]. Se tiene que $|\mathcal{R}(J_4, K_5; 14)| = 856$ y $|\mathcal{R}(J_6, K_4; 18)| = 6$.

En ambos grafos, G_1 y G_2, se supone que v tiene valencia mínima y se deducen propiedades del grafo G_i. Finalmente se transforma el problema de demostrar la inexistencia de G_i en un problema de satisfacibilidad booleana de forma parecida a como se hace en [13,43]. Para ello asignamos a las aristas de K_{33} una variable booleana la cual deberá tomar el valor Verdadero si pertenece a E_i y el valor falso en otro caso.

Se construye, en cada uno de los dos casos, un conjunto de clausulas que han de ser satisfechas por las variables. Cada clausula representa un subgrafo de G_i ó $\overline{G_i}$ que ha de ser evitado.

Las simplificaciones obtenidas a partir de las propiedades encontradas de G_i permiten reducir el número de clausulas, el número total de variables y el número de variables dentro de cada clausula, que la no satisfacibilidad de ambos conjuntos de clausulas se puede comprobar en un tiempo razonable, del orden de horas. Para ello se utiliza el programa march rw [26], que fue el ganador de la medalla de oro de la 2011 International SAT Competition.

3 Otras nuevas cotas superiores

En la Tabla III de [39], que se actualiza periódicamente, se muestran las mejores cotas conocidas de $R(J_m, K_n)$ con $m \leq 11$ y $n \leq 8$. A partir de $R(J_5, K_5) \leq 33$ y $R(J_6, K_4) \leq 33$ y aplicando la Fórmula (1) y fórmulas publicadas en [29,30] se obtienen las siguientes cotas superiores:

- $R(J_6, K_5) \leq 66$.
- $R(J_6, K_6) \leq 110$.
- $R(J_6, K_7) \leq 192$.
- $R(J_6, K_8) \leq 300$.
- $R(J_7, K_5) \leq 110$.
- $R(J_7, K_7) \leq 388$.
- $R(J_7, K_8) \leq 657$.
- $R(J_8, K_4) \leq 74$.
- $R(J_8, K_5) \leq 180$.
- $R(J_8, K_6) \leq 371$.

- $R(J_8, K_7) \leq 746$.
- $R(J_8, K_8) \leq 1345$.
- $R(J_9, K_5) \leq 275$.
- $R(J_9, K_6) \leq 620$.
- $R(J_9, K_7) \leq 1325$.
- $R(J_9, K_8) \leq 2556$.
- $R(J_{10}, K_4) \leq 138$.
- $R(J_{10}, K_5) \leq 404$.
- $R(J_{10}, K_6) \leq 999$.
- $R(J_{10}, K_7) \leq 2281$.
- $R(J_{10}, K_8) \leq 4698$.
- $R(J_{11}, K_4) \leq 183$.
- $R(J_{11}, K_5) \leq 577$.
- $R(J_{11}, K_6) \leq 1538$.
- $R(J_{11}, K_7) \leq 3726$.
- $R(J_{11}, K_8) \leq 8177$.

Finalmente, con una pequeña modificación de la demostración de resultados de [8], tal como se hace en [9], éstos pueden ser generalizados y obtenerse también las siguientes cotas superiores:

- $R(J_5, K_6 - K_{1,2}) \leq \text{máx}\{R(J_5, K_5), 31\} \leq 33$.
- $R(J_6, K_5 - K_{1,2}) = R(J_6, K_5 - K_{1,3}) = R(J_6, K_5 - K_{1,4}) = R(J_6, K_4) \leq 33$.
- $R(J_6, K_6 - K_{1,4}) = R(J_6, K_5) \leq 66$.
- $R(J_6, K_6 - K_{1,3}) \leq \text{máx}\left\{R(J_6, K_5), \left\lceil \frac{5}{4} R(J_5, K_5) \right\rceil + 4\right\} \leq 66$. Por tanto, si $R(J_5, K_5) \leq 31$ entonces $R(J_6, K_6 - K_{1,3}) = R(J_6, K_5)$.
- $R(J_6, K_6 - K_{1,2}) \leq \text{máx}\left\{R(J_6, K_5), \left\lceil \frac{1}{3}(5R(J_5, K_6 - K_{1,2}) + 11) \right\rceil\right\} \leq 66$.

Referencias

[1] A. Babak, S.P. Radziszowski y Kung-Kuen Tse. Computation of the Ramsey Number $R(B_3, K_5)$. *Bulletin of the Institute of Combinatorics and its Applications*, 41:71–76, 2004.

[2] R. Bolze y H. Harborth. The Ramsey Number $r(K_4 - x, K_5)$, en *The Theory and Applications of Graphs*:109–116, (Kalamazoo, MI, 1980). John Wiley & Sons, New York, 1981.

[3] L. Boza. Corrections to The Ramsey Numbers for a Quadrilateral vs. All Graphs on Six Vertices. Por aparecer en *Journal of Combinatorial Mathematics and Combinatorial Computing*.

[4] L. Boza. Nuevas Cotas Superiores de Algunos Numeros de Ramsey del Tipo $r(K_m, K_n - e)$. *Proceedings de las VII Jornadas de Matemática Discreta y Algorítmica*, Castro Urdiales (Cantabria), 141–151, 2010.

[5] L. Boza. Sobre el Número de Ramsey $R(K_4, K_6 - e)$. VIII Encuentro Andaluz de Matemática Discreta, Sevilla, 2013.

[6] L. Boza. The Ramsey Number $r(K_5 - P_3, K_5)$. *Electronic Journal of Combinatorics*, 18:#P90, 10 páginas, 2011. Disponible en `http://www.combinatorics.org/Volume\_18/PDF/v18i1p90.pdf`.

[7] L. Boza y J.R. Portillo. Sobre el número de Ramsey $R(K_4 - e, K_7)$. *Proceedings of the VIII Jornadas de Matemática Discreta y Algorítmica*, Universidad de Almería, Almería, 193–200, 2012.

[8] S. A. Burr, P. Erdös, R. J. Faudree y R. H. Schelp. On the Difference between Consecutive Ramsey Numbers. *Utilitas Mathematica*, 35:115–118, 1989.

[9] J.A. Calvert, M.J. Schuster y S.P. Radziszowski. The Computation of $R(K_5 - P_3, K_5) = 25$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 82:131–140, 2012.

[10] V. Chvátal y F. Harary. Generalized Ramsey Theory for Graphs, II. Small Diagonal Numbers. *Proceedings of the American Mathematical Society*, 32:389–394, 1972.

[11] V. Chvátal y F. Harary. Generalized Ramsey Theory for Graphs, III. Small Off-Diagonal Numbers. *Pacific Journal of Mathematics*, 41:335–345, 1972.

[12] M. Clancy. Some small Ramsey numbers. *Journal of Graph Theory*, 1:89–91, 1977.

[13] S. Eliahou, J. M. Marín, M. P. Revuelta y M. I. Sanz. Weak Schur numbers and the search for G.W. Walker's lost partitions. *Computers & Mathematics with Applications*, 63:175–182, 2012.

[14] P. Erdös y G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[15] G. Exoo. A Lower Bound for $R(5, 5)$. *Journal of Graph Theory*, 13:97–98, 1989.

[16] G. Exoo. A Lower Bound for $r(K_5 - e, K_5)$. *Utilitas Mathematica*, 38:187–188, 1990.

[17] G. Exoo. Indiana State University. Comunicación personal a S.P. Radziszowski, 1992.

[18] G. Exoo, H. Harborth y I. Mengersen. The Ramsey Number of K_4 versus $K_5 - e$. *Ars Combinatoria*, 25A:277–286, 1988.

[19] R.J. Faudree, C.C. Rousseau y R.H. Schelp. All Triangle-Graph Ramsey Numbers for Connected Graphs of Order Six. *Journal of Graph Theory*, 4:293–300, 1980.

[20] R. Fidytek. Ramsey Graphs $R(K_n, K_m - e)$. `http://fidytek.inf.ug.edu.pl/ramsey`, 2010.

[21] J. Goedgebeur y S.P. Radziszowski. The Ramsey Number $R(3, K_{10} - e)$ and Computational Bounds for $R(3, G)$. *Electronic Journal of Combinatorics*, 20(4):#P19, 25 páginas, 2013. Disponible en `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i4p19/pdf`.

[22] R.E. Greenwood y A.M. Gleason. Combinatorial Relations and Chromatic Graphs. *Canadian Journal of Mathematics*, 7:1–7, 1955.

[23] U. Grenda y H. Harborth, The Ramsey Number $r(K_3, K_7 - e)$, *Journal of Combinatorics, Informations & System Sciences*, 7:166–169, 1982.

[24] G.R.T. Hendry. Ramsey Numbers for Graphs with Five Vertices. *Journal of Graph Theory*, 13:245–248, 1989.

[25] G.R.T. Hendry. The Ramsey Numbers $r(K_2+K_3, K_4)$ and $r(K_1+C_4, K_4)$. *Utilitas Mathematica*, 35:40–54, 1989. Apéndice en 36:25–32, 1989.

[26] M. Heule. March RW. SAT Competition 2011, 2011. `http://www.st.ewi.tudelft.nl/sat/`.

[27] M. Hoeth e I. Mengersen. Ramsey Numbers for Graphs of Order Four versus Connected Graphs of Order Six. *Utilitas Mathematica*, 57:3–19, 2000.

[28] C.J. Jayawardene y C.C. Rousseau. The Ramsey Numbers for a Quadrilateral vs. All Graphs on Six Vertices. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 35:71–87, 2000.

[29] Y. R. Huang, Y. Wang, W. Sheng, J. Yang, K. M. Zhang y J. Huang. New Upper Bound Formulas with Parameters for Ramsey Numbers. *Discrete Mathematics*, 307:760–763, 2007.

[30] Y.R. Huang y K.M. Zhang. New Upper Bounds for Ramsey Numbers. *European Journal of Combinatorics*, 19:391–394, 1998.

[31] J.G. Kalbfleisch. Construction of Special Edge-Chromatic Graphs. *Canadian Mathematical Bulletin*, 8:575–584, 1965.

[32] J.G. Kalbfleisch. Chromatic Graphs and Ramsey's Theorem. Tesis doctoral, University of Waterloo, 1966.

[33] G. Kéry. On a Theorem of Ramsey. *Matematikai Lapok*, 15:204–224, 1964.

[34] R. Lortz. A Note on the Ramsey Number of $K_{2,2}$ versus $K_{3,n}$. *Discrete Mathematics*, 306:2976–2982, 2006.

[35] B.D. McKay, K. Piwakowaski y S.P. Radziszowski. Ramsey Numbers for Triangles versus Almost-Complete Graphs. *Ars Combinatoria*, 73:205–214, 2004.

[36] B.D. McKay y S.P. Radziszowski. $R(4, 5) = 25$. *Journal of Graph Theory*, 19:309–322, 1995. Disponible en `http://www.cs.rit.edu/~spr/PUBL/paper32.pdf`.

[37] B.D. McKay y S.P. Radziszowski. Subgraph Counting Identities and Ramsey Numbers. *Journal of Combinatorial Theory, Series B*, 69:193–209, 1997. Disponible en `http://www.cs.rit.edu/~spr/PUBL/paper35.pdf`.

[38] J. McNamara, SUNY Brockport, resultados no publicados, 1995.

[39] S.P. Radziszowski. Small Ramsey numbers. *The Electronic Journal of Combinatoric*, Dinamic Survey 1, Revisión #14, 2014. Disponible en `http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1/pdf`

[40] S.P. Radziszowski. The Ramsey Numbers $R(K_3, K_8 - e)$ and $R(K_3, K_9 - e)$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 8:137–145, 1990.

[41] S.P. Radziszowski, J. Stinehour y Kung-Kuen Tse. Computation of the Ramsey Number $R(W_5, K_5)$. *Bulletin of the Institute of Combinatorics and its Applications*, 47:53–57, 2006. Disponible en `http://www.cs.rit.edu/~spr/PUBL/paper49.pdf`.

[42] C.C. Rousseau. Erratum to [28]. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 51:221, 2004.

[43] D. Shetler, M. Wurtz y S.P. Radziszowski. On Some Multicolor Ramsey Numbers Involving $K_3 + e$ and $K_4 - e$. *SIAM Journal on Discrete Mathematics*, 26:1256-1264, 2012. Disponible en `http://www.cs.rit.edu/~spr/PUBL/reu11.pdf`.

[44] Q.X. Wang, G.B. Wang y S.D. Yan. The Ramsey Numbers $R(K_3, K_q - e)$ (en chino). *Beijing Daxue Xuebao Ziran Kexue Ban*, 34:15–20, 1988.

[45] Yang Yuansheng y G.R.T. Hendry. The Ramsey Number $r(K_1 + C_4, K_5 - e)$. *Journal of Graph Theory*, 19:13–15, 1995.

Shortest paths in intersection graphs of unit disks*

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Abstract. Let G be a unit disk graph in the plane defined by n disks whose positions are known. We show that in G a shortest path tree from a given source can be computed in $\mathcal{O}(n \log n)$ time in the unweighted case and in $\mathcal{O}(n^{1+\varepsilon})$ time in the weighted case. Previous algorithms used $\mathcal{O}(n \log^6 n)$ expected time for the unweighted case and $\mathcal{O}(n^{4/3+\varepsilon})$ time for the weighted case.

Key words: Shortest paths, unit disk graph, Delaunay triangulation, bichromatic closest pair.

1 Introduction

Each set $\mathcal{S}$ of geometric objects in the plane defines its intersection graph in a natural way: the vertex set is $\mathcal{S}$ and there is an edge ss' in the graph, $s, s' \in S$, whenever $s \cap s' \neq \emptyset$. It is natural to seek faster algorithms when the input is constraint to geometric intersection graphs. Here we are interested in computing shortest path distances in unit disk graphs, that is, the intersection graph of equal sized disks.

A unit disk graph is uniquely defined by the centers of the disks. Thus, we will drop the use of disks and just refer to the graph $G(P)$ defined by a set P of n points in the plane. The vertex set of $G(P)$ is P. Each edge of $G(P)$ connects points p and p' from P whenever $\|p-p'\| \leq 1$, where $\|\cdot\|$ denotes the Euclidean norm. See Figure 1 for an example of such graph. Up to a scaling factor, $G(P)$ is isomorphic to a unit disk graph. In the *unweighted* case, each edge $pp' \in E(G(P))$ has unit weight, while in the *weighted* case, the weight of each edge $pp' \in E(G(P))$ is $\|p - p'\|$. In all our algorithms we assume that P is known. Thus, the input is P, as opposed to the abstract graph $G(P)$.

* Full version available as [3].

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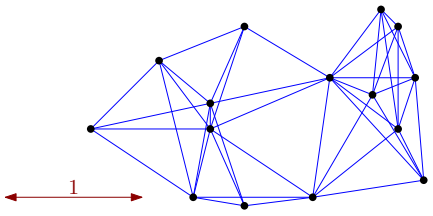

Fig. 1: Example of graph $G(P)$.

Exact computation of shortest paths in unit disks is considered by Roditty and Segal [11], under the name of *bounded leg* shortest path problem. They show that, for the weighted case, a shortest path tree can be computed in $\mathcal{O}(n^{4/3+\varepsilon})$ time. They also note that the dynamic data structure for nearest neighbors of Chan [5] imply that, in the unweighted case, shortest paths can be computed in $\mathcal{O}(n \log^6 n)$ expected time.

We show that in $G(P)$ a shortest path tree from a given source can be computed in $\mathcal{O}(n \log n)$ time in the unweighted case and in $\mathcal{O}(n^{1+\varepsilon})$ time in the weighted case. Thus, we provide a significant improvement over the results of Roditty and Segal. For the unweighted case we use the Delaunay triangulation to grow a search region and the algorithm is implementable. For the weighted case we use a modification of Dijkstra's algorithm combined with a data structure to dynamically maintain a bichromatic closest pair under an Euclidean weighted distance. Our algorithms are quite simple. Since the problem is very natural and we improve previous bounds, we think that simplicity is an added value.

Gao and Zhang [10] showed that the metric induced by a unit disk graph admits a compact well separated pair decomposition, extending the celebrated result of Callahan and Kosaraju [4] for Euclidean spaces. For making use of the well separated pair decomposition, Gao and Zhang [10] obtain a $(1+\varepsilon)$-approximation to shortest path distance in unit disk graphs in $\mathcal{O}(n \log n)$ time. Here we provide exact computation within comparable bounds.

Chan and Efrat [6] consider a graph defined on a point set but with more general weights in the edges. Namely, it is assumed that there is a function $\ell\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ such that the edge pp' gets weight $\ell(p,p')$. Moreover, it is assumed that the function $\ell(p,p')$ is increasing with $\|p-p'\|$. When $\ell(p,p') = \|p-p'\|^2 f(\|p-p'\|)$ for a monotone increasing function f, then a shortest path can be computed in $\mathcal{O}(n \log n)$ time. Otherwise, if ℓ has constant size description, a shortest path can be computed in roughly $\mathcal{O}(n^{4/3})$ time.

Shortest path trees can be computed for unit disk graphs in polynomial time. One can just construct $G(P)$ explicitly and run a standard algorithm for shortest paths. The main objective here is to obtain a faster algorithm that avoids the explicit construction of $G(P)$ and exploits the geometry of P. There are several problems that can be solved in polynomial time, but

faster algorithms are known for geometric settings. A classical example is the computation of the minimum spanning tree of a set of points in the Euclidean plane. Using the Delaunay triangulation, the number of relevant edges is reduced from quadratic to linear. For more advanced examples see Vaidya [12], Efrat, Itai and Katz [7], Eppstein [9], or Agarwal, Overmars and Sharir [2].

Proofs omitted can be found in the full version [3].

2 Unweighted shortest paths

In this section we consider the unweighted version of $G(P)$ and compute a shortest path tree from a given point $s \in P$. Pseudocode for the eventual algorithm is provided in Figure 2. Before moving into the details, we provide the main ideas employed in the algorithm.

As it is usually done for shortest path algorithms we use tables distan[$\cdot$] and $\pi[\cdot]$ indexed by the points of P to record, for each point $p \in P$, the distance $d(s,p)$ and the ancestor of p in a shortest (s,p)-path. We start by computing the Delaunay triangulation $DT(P)$ of P. We then proceed in rounds for increasing values of i, where at round i we find the set W_i of points at distance exactly i from the source s. We start with $W_0 = \{s\}$. At round i, we use $DT(P)$ to grow a neighbourhood around the points of W_{i-1} that contains W_i. More precisely, we consider the points adjacent to W_{i-1} in $DT(P)$ as candidate points for W_i. For each candidate point that is found to lie in W_i, we also take its adjacent vertices in $DT(P)$ as new candidates to be included in W_i. For checking whether a candidate point p lies in W_i we use a data structure to find the nearest neighbour of p in W_{i-1}, denoted by $: NN \longrightarrow (W_{i-1}, p)$. Such data structure is just a point location data structure in the Voronoi diagram of W_{i-1}. Similarly, the shortest path tree is constructed by connecting each point of W_i to its nearest neighbour in W_{i-1}. See Figure 2 for the eventual algorithm UnweightedShortestPath. In Figure 3 we show an example of what edges of the shortest path tree are computed in one iteration of the main loop.

We would like to emphasize a careful point that we employ to achieve the running time $\mathcal{O}(n \log n)$. In lines 16 and 17 of the algorithm, we check whether p is at distance at most 1 from *some* point in W_{i-1}, namely its nearest neighbour in W_{i-1}. Checking whether p is at distance at most 1 from $\pi(q)$ (or q when $q \in W_{i-1}$) would lead to a potentially larger running time. Thus, we do not grow each disk $D(w,1)$ independently for each $w \in W_{i-1}$, but we grow the whole region $\bigcup_{w \in W_{i-1}} D(w,1)$ at once. Growing each disk $D(w,1)$ separately would force as to check the same edge qp of $DT(P)$ several times, once for each $w \in W_{i-1}$ such that $q \in D(w,1)$. The main property used to prove correctness is summarized in the following statement.

```
UnweightedShortestPath(P, s)
 1  for p ∈ P
 2      distan[p] = ∞
 3      π[p] = nil
 4  distan[s] = 0
 5  build the Delaunay triangulation DT(P)
 6  W_0 = {s}
 7  i = 1
 8  while W_{i-1} ≠ ∅
 9      build data structure for nearest neighbour queries in W_{i-1}
10      Q = W_{i-1}      // candidate points
11      W_i = ∅
12      while Q ≠ ∅
13          q an arbitrary point of Q
14          remove q from Q
15          for qp edge in DT(P)
16              w =: NN ⟶ (W_{i-1}, p)
17              if distan[p] = ∞ and ‖p − w‖ ≤ 1
18                  distan[p] = i
19                  π[p] = w
20                  add p to Q
21                  add p to W_i
22      i = i + 1
23  return distan[·] and π[·]
```

Fig. 2: Algorithm to compute a shortest path tree in the unweighted case.

Lemma 1. *Let p be a point from $P \setminus \{s\}$ such that $d(s,p) < \infty$. There exists a point w in P and a path π in $DT(P) \cap G(P)$ from w to p such that $d(s,w)+1 = d(s,p)$ and each internal vertex p_j of π satisfies $d(s,p_j) = d(s,p)$.*

Theorem 1. *Let P be a set of n points in the plane and let s be a point from P. In time $\mathcal{O}(n \log n)$ we can compute a shortest path tree from s in the unweighted graph $G(P)$.*

3 Weighted shortest paths

In this section we consider the `SSSP` problem on the weighted version of $G(P)$: points p and q have an edge between them iff $\|p-q\| \leq 1$ and the weight of that edge is $\|p-q\|$. Our algorithm uses a dynamic data structure for bichromatic closest pairs. We first review the precise data structure that we will employ. We then describe the algorithm and discuss its properties.

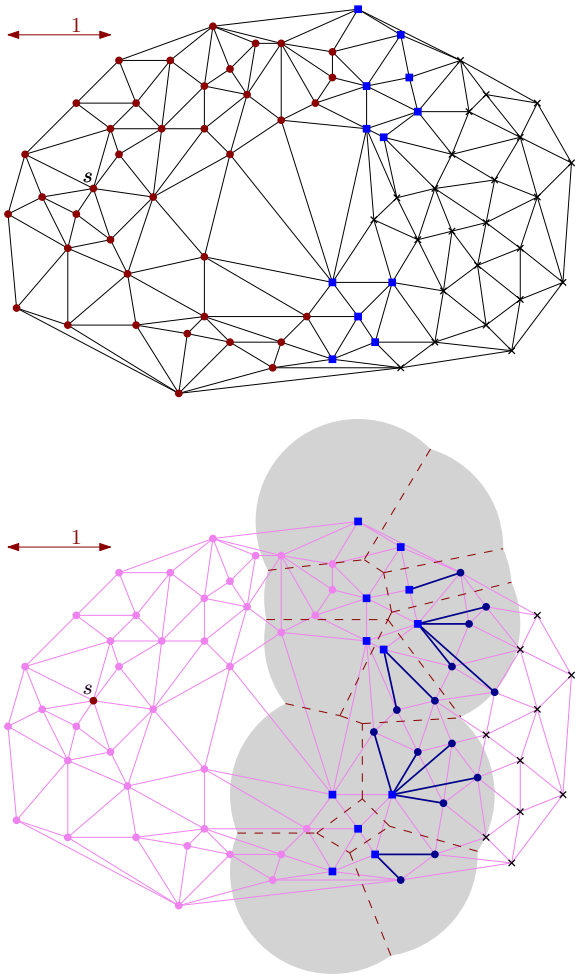

Fig. 3: Top: A point set with its Delaunay triangulation. The source is marked as s. Points p such that $d(s,p) \leq 3$ are marked with red dots. Points from W_4 are marked with blue boxes. Bottom: The new edges added to the tree at iteration 5 and the new vertices are shown. The light grey region is $\bigcup_{p \in W_4} D(p,1)$, where the Voronoi diagram of W_4 is superimposed.

3.1 Bichromatic closest pair

Let P be a set of n points in the plane and let each point $p \in P$ have a weight $w_p \geq 0$. We call a function $\delta\colon \mathbb{R}^2 \times P \longrightarrow \mathbb{R}_+$ a *(additive) weighted Euclidean metric*, if it is of the form

$$\delta(q,p) = w_p + \|q - p\|,$$

where $\|\cdot\|$ denotes the Euclidean distance.

Let $\varepsilon > 0$ denote an arbitrary constant. Agarwal, Efrat and Sharir [1] showed that for any P and δ as above, P can be preprocessed in $\mathcal{O}(n^{1+\varepsilon})$ time

into a data structure of size $\mathcal{O}(n^{1+\varepsilon})$ so that points can be inserted into or deleted from P in $\mathcal{O}(n^{\varepsilon})$ amortized time per update, and a nearest-neighbour query can be answered in $\mathcal{O}(\log n)$ time. Eppstein [8] had already shown that if such a dynamic data structure existed, then a bichromatic closest pair (BCP) under δ of red and blue points in the plane could be maintained, adding only a polylogarithmic factor to the update time. Combining these two results gives

Theorem 2 (Agarwal, Efrat, Sharir [1]). *Let R and B be two sets of points in the plane with a total of n points. We can store $R \cup B$ in a dynamic data structure of size $\mathcal{O}(n^{1+\varepsilon})$ that maintains a bichromatic closest pair in $R \times B$, under any weighted Euclidean metric, in $\mathcal{O}(n^{\varepsilon})$ amortized time per insertion or deletion.* □

3.2 Algorithm

We will use a variant of Dijkstra's algorithm. As before, we maintain tables distan[·] and π[·] containing distances from the source and parents of points in the shortest path tree. As in Dijkstra's algorithm we will maintain a set S (containing the source s) of points for which the correct distance from s has already been computed, and a set $P \setminus S$ of points for which the distance has yet to be computed. For the points of S, distan[·] stores the true distance from the source.

In our approach we split S into sets B and D, called "blue" and "dead" points, respectively. We call the points in $R = P \setminus S$ "red" points. The reason for the introduction of the "dead" points D is that, as it will be proved, at no stage of the algorithm there is an edge of $G(P)$ between a point in D and a point in R. Thus, the points of D are not relevant to find the last edge in a shortest path to points of R.

We store $R \cup B$ in the dynamic data structure from Theorem 2 that maintains the bichromatic closest pair (BCP) in $R \times B$ under the weighted Euclidean metric

$$\delta(r, b) := \text{distan}[b] + \|r - b\|.$$

At each iteration of the main **while** loop, we query the data structure for a BCP pair (b^*, r^*). If b^*r^* is not an edge in our underlying graph $G(P)$, meaning $\|b^* - r^*\| > 1$, then b^* will never be the last vertex to any point in R, and therefore we will move it from B to D. If b^*r^* is an edge of $G(P)$, then, as it happens with Dijkstra's algorithm, we have completed a shortest path to r^*. The algorithm is given in Figure 4. Figure 5 shows sets D, B, and R in the middle of a run of the algorithm.

Let us explain the actual bottleneck of our approach to reduce the time from $\mathcal{O}(n^{1+\varepsilon})$ to $\mathcal{O}(n \operatorname{polylog} n)$. The inner workings of the data structure of Theorem 2 is based on two dynamic data structures. One of them has to compute $\min_{b \in B} \delta(r_0, b)$ for a given $r_0 \in R$. The other has to compute

```
WeightedShortestPaths(P, s)
 1  for p ∈ P
 2        distan[p] = ∞
 3        π[p] = NIL
 4  distan[s] = 0
 5  B = {s}
 6  D = ∅
 7  R = P \ {s}
 8  store R ∪ B in the BCP dynamic DS of Theorem 2 wrt δ(r, b)
 9  while R ≠ ∅
10        if B = ∅
11              return distan[·] and π[·]          // G(P) is not connected
12        else (b*, r*) =: BCP ⟶ (B, R)
13        if ‖b* − r*‖ > 1
14              : delete ⟶ (B, b*)
15              D = D ∪ {b*}
16        else distan[r*] = distan[b*] + ‖b* − r*‖
17              π[r*] = b*
18              : delete ⟶ (R, r*)
19              : insert ⟶ (B, r*)
20  return distan[·] and π[·]
```

Fig. 4: Algorithm for `SSSP` in the weighted case.

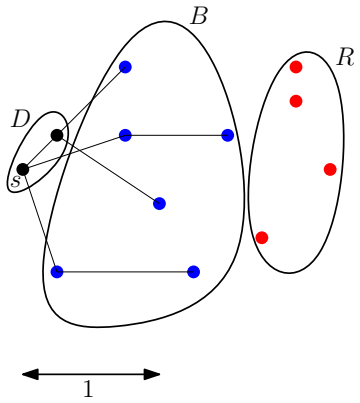

Fig. 5: Sets D, B, and R after a couple of iterations of the **while** loop in WeightedShortestPaths.

$\min_{r \in R} \delta(r, b_0)$ for a given $b_0 \in B$. For the latter data structure we could use the dynamic nearest neighbour data structure by Chan [5], yielding polylogarithmic update and query times. However, for the former we need a dynamic weighted Voronoi diagram, and for this we only have the data structure devel-

oped by Agarwal, Efrat and Sharir [1]. A dynamic data structure for dynamic weighted Voronoi diagrams with updates and queries in polylogarithmic time readily would lead to $\mathcal{O}(n \operatorname{polylog} n)$.

Note that in the algorithm a point can only go from red to blue and from blue to dead. Dead points stay dead. The main property to prove correctness is the following statement.

Lemma 2. *Once a point b^* is moved from B to D, it no longer has any edges to points in R.*

Theorem 3. *Let P be a set of n points in the plane, $s \in P$, and $\varepsilon > 0$ an arbitrary constant. The algorithm* WeightedShortestPaths(P, s) *returns the correct distances from the source in the graph $G(P)$ in $\mathcal{O}(n^{1+\varepsilon})$ time.*

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References

[1] P. K. Agarwal, A. Efrat, and M. Sharir. Vertical Decomposition of Shallow Levels in 3-Dimensional Arrangements and Its Applications. *SIAM J. Comput.*, 29(3):912–953, 1999.

[2] P. K. Agarwal, M. H. Overmars, and M. Sharir. Computing Maximally Separated Sets in the Plane. *SIAM J. Comput.*, 36(3):815–834, 2006.

[3] S. Cabello and M. Jejcic. Shortest paths in intersection graphs of unit disks. *CoRR*, abs/1402.4855, 2014.

[4] P. B. Callahan and S. R. Kosaraju. A Decomposition of Multidimensional Point Sets with Applications to k-Nearest-Neighbors and n-Body Potential Fields. *J. ACM*, 42(1):67–90, 1995.

[5] T. M. Chan. A dynamic data structure for 3-D convex hulls and 2-D nearest neighbor queries. *J. ACM*, 57(3), 2010.

[6] T. M. Chan and A. Efrat. Fly Cheaply: On the Minimum Fuel Consumption Problem. *J. Algorithms*, 41(2):330–337, 2001.

[7] A. Efrat, A. Itai, and M. J. Katz. Geometry Helps in Bottleneck Matching and Related Problems. *Algorithmica*, 31(1):1–28, 2001.

[8] D. Eppstein. Dynamic Euclidean Minimum Spanning Trees and Extrema of Binary Functions. *Discrete & Computational Geometry*, 13:111–122, 1995.

[9] D. Eppstein. Testing bipartiteness of geometric intersection graphs. *ACM Transactions on Algorithms*, 5(2), 2009.

[10] J. Gao and L. Zhang. Well-Separated Pair Decomposition for the Unit-Disk Graph Metric and Its Applications. *SIAM J. Comput.*, 35(1):151–169, 2005.

[11] L. Roditty and M. Segal. On Bounded Leg Shortest Paths Problems. *Algorithmica*, 59(4):583–600, 2011.

[12] P. M. Vaidya. Geometry Helps in Matching. *SIAM J. Comput.*, 18(6):1201–1225, 1989.

Two graph families with maximum quasiperfect domination number *

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Abstract. A subset $S \subseteq V$ in a graph $G = (V, E)$ is a quasiperfect domination set if every vertex $v \in V \setminus S$ is adjacent to at least one but no more than two vertices in S. Those sets were studied in [1] as a generalization of perfect domination. In that paper, the authors highlighted the difficulty in finding examples where the minimum quasiperfect domination set was the whole set of vertices, and in this work we provide two infinite families verifying such property.

Key words: Domination, $[1, 2]$-sets, perfect domination, quasiperfect domination, split graphs, Kneser graphs.

1 Introduction

Let $G = (V, E)$ be a graph of *order* $n = |V|$ and *size* $m = |E|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u|uv \in E\}$ of vertices adjacent to v. Each vertex in $N(v)$ is called a *neighbor* of v. The *degree* of a vertex v is $deg(v) = |N(v)|$ and the maximum degree of a vertex in G is denoted by $\Delta(G)$. The *closed neighborhood* of a vertex $v \in V$ is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood of a set* $S \subseteq V$ of vertices is $N(S) = \bigcup_{v \in S} N(v)$, while the *closed neighborhood of a set* S is the set $N[S] = \bigcup_{v \in S} N[v]$.

A set S is a *dominating set* of a graph G if $N[S] = V$, that is, for every $v \in V$, either $v \in S$ or $v \in N(u)$ for some vertex $u \in S$. The minimum cardinality of a dominating set in a graph G is called the *domination number*, and it is denoted $\gamma(G)$ (see [4] for a comprehensive introduction to the topic). A dominating set S is *perfect dominating* if $|N(v) \cap S| = 1$ for each $v \in V \setminus S$. Perfect dominating sets were introduced in [9] and its study is motivated by the need to have dominating sets, for example, acting as servers in a computing network, or acting as sets of monitoring devices in situations requiring surveillance, but with the need to establish such sets as efficiently

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or as cost effectively as possible, that is, without creating any redundancy. The *perfect domination number* $\gamma_{[1,1]}(G)$ is the minimum cardinality of a perfect dominating set of G. A dominating set of cardinality $\gamma_{[1,1]}(G)$ is called a $\gamma_{[1,1]}(G)$-*code*.

In [2], it is introduced the following generalization of perfect dominating sets: A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a *quasiperfect dominating set* if for every vertex $v \in V \setminus S$ we have that $1 \leq |N(v) \cap S| \leq 2$. The cardinality of a minimum quasiperfect dominating set in a graph G is called its *quasiperfect dominating number* and denoted as $\gamma_{[1,2]}(G)$. Such a minimum set is a $[1, 2]$-*set* of the graph. From the above definitions, it follows immediately that:

$$\gamma(G) \leq \gamma_{[1,2]}(G) \leq \gamma_{[1,1]}(G) \leq n$$

From the above inequality, the first questions that come into mind have to do with the extremal values of $\gamma_{[1,2]}(G)$. In [1], the authors obtained several sufficient conditions to ensure for a graph to have $\gamma(G) = \gamma_{[1,2]}(G)$. For instance, if the graph G is claw-free, a cograph, a caterpillar or $\Delta(G) \leq 2$ then $\gamma(G) = \gamma_{[1,2]}(G)$. However and regarding the other extreme bound, they wrote that "Does there exist a connect graph G or order $n \geq 2$ with $\gamma_{[1,2]}(G) = n$? Such graphs seem to be difficult to characterize".

In this paper, we obtain two families of graphs having $\gamma_{[1,2]}(G) = n$. In the next section, we show the conditions that should verify a split graph to have maximum quasiperfect dominating number, and in Section 3 we prove that the only $[1, 2]$-sets for Kneser graphs $K_{n:k}$ such that $n \geq 5k - 1$ are the whole set of vertices.

2 Split graphs

Our first family is a subset of split graphs. A *split graph* is a graph where the vertices can be partitioned into a clique and an independent set. Those graphs were first studied by Földes and Hammer [3] (in a slightly different definition), and independently by Tyshkevich and Chernyak [7].

Split graphs can be characterized as the graphs in which no induced subgraph is a cycle on four or five vertices, or a pair of disjoint edges. They are chordal graphs where its complement is also chordal. Consequently, split graphs are perfect. Moreover, split graphs are the intersection graphs of distinct substars in a star graph.

The following result provides the conditions under a split graph G verifies $\gamma_{[1,2]}(G) = |V(G)|$.

Theorem 1. *Let G be a connected split graph and let $V_1 \cup V_2$ be a partition of V such that V_1 is a maximal independent set and $< V_2 >$ is a maximal clique of size at least 3. $\gamma_{[1,2]}(G) = n$ if and only if the following conditions hold:*

1. *$\delta(a) \geq 3$ for any vertex $a \in V_1$.*
2. *There exists a vertex $u \in V_2$ such that $|N(u) \cap V_1| \geq 3$ or $N(u) \cap V_1 = \emptyset$.*
3. *For any vertex of $u \in V_2$ exists a pair of vertices $a, b \in V_1$ such that $N(a) \cap N(b) \neq \emptyset$, $u \notin N(a) \cup N(b)$.*
4. *For any pair of vertices $u, v \in V_2$, there exists a vertex $a \in V_1$ such that $a \notin N(u) \cup N(v)$.*

Proof. Let G be a connected split graph with $\gamma_{[1,2]}(G) = n$ and let us see that the conditions hold:

1. If there exists a vertex $a \in V_1$ with $\delta(a) = 2$ then $V \setminus \{a\}$ is a quasiperfect dominating set, which is a contradiction (see Observation 12 in [1]).
2. If any vertex $u \in V_2$ satisfies $1 \leq |N(u) \cap V_1| \leq 2$, then V_1 is a quasiperfect dominating set.
3. On the contrary, suppose that exists $u \in V_2$ such that either there not exist $a, b \in V_1$ with $N(a) \cap N(b) \neq \emptyset$ or for any pair of vertices $a, b \in V_1$ with $N(a) \cap N(b) \neq \emptyset$ satisfies $u \in N(a) \cup N(b)$.
   Suppose that for any $a, b \in V_1$ with $a, b \in V_1$ with $N(a) \cap N(b) \neq \emptyset$ exists a vertex $u \in N(a) \cup N(b)$ and suppose $u \in N(a)$. The we can define $S = \{u\} \cup (V_1 \setminus N(u))$ and we claim that S is a quasiperfect dominating set. First S dominates the vertex set of G. Secondly, let be $v \in V_2$, if there exist $a, b \in S \setminus \{u\}$ such that $v \in N(a) \cap N(b)$ then as $u \in N(a) \cup N(b)$, a can not be a vertex of S which is a contradiction. Thus v has at most two neighbors in S and S is a quasiperfect dominating set with $|S| < n$ contradicting the fact that $\gamma_{[1,2]}(G) = n$.
   Suppose now that there not exist $a, b \in V_1$ with $N(a) \cap N(b) \neq \emptyset$. Then we can define $S = \{u\} \cup (V_1 \setminus N(u))$. S dominates the vertex set of G because u dominates V_2 and $V_1 \cap N(u)$. On top, given a vertex $v \in V_2$ different to u, v has at the most a neighbor in $V_1 \setminus N(u)$ because any $a, b \in V_1$ satisfy $N(a) \cap N(b) = \emptyset$. So $v \in V_2$ has at most two neighbors in S therefore S is a quasiperfect dominating set and $|S| < n$ contradicting the fact that $\gamma_{[1,2]}(G) = n$.
4. On the contrary, suppose that condition 4 does not hold. So there exist $u, v \in V_2$ such that $V_1 \subseteq N(u) \cup N(v)$ and the set $\{u, v\}$ is a quasiperfect dominating set, contradicting the fact that $\gamma_{[1,2]}(G) = n$.

Conversely, suppose a graph G satisfying the previous conditions and let D be a quasiperfect dominating set. We will check all the possible cases depending on the cardinality of $V_2 \cap D$:

If $V_2 \cap D = \emptyset$ it is clear that $V_1 = D$. As condition 2 holds, there exists a vertex $u \in V_2$ such that either $|N(u) \cap V_1| \geq 3$ or $N(u) \cap V_1 = \emptyset$. If u has no neighbor in V_1, then D can not be a dominating set. If v has three neighbors in V_1, u must belong to D. In both cases, this leads a contradiction.

Suppose that $V_2 \cap D = \{u\}$. By condition 3, there exist $a, b \in V_1$ such that $N(a) \cap N(b) \neq \emptyset$ and $u \notin N(a) \cap N(b)$, thus $a, b \in D$. Let $v \in N(a) \cap N(b)$

so v is dominated by three vertices from D which contradicts the assumption that D is a quasiperfect dominating set.

In the case of $V_2 \cap D = \{u, v\}$, by condition 4 there exists a vertex $a \in V_1$ such that $a \notin N(u) \cap N(v)$, so $a \in D$. Since G is connected, there exits $w \in V_2 \cap N(a)$, then w is dominated by three vertices in D (a, u and v) so $w \in D$, which is not possible.

Finally, if $|V_2 \cap D| \geq 3$ then $V_2 \subseteq D$, and by condition 1 $\delta(a) \geq 3$, then $V_1 \subseteq D$ and $D = V(G)$. Therefore $\gamma_{[1,2]}(G) = n$ □ ∎

3 Kneser graphs

The *Kneser graph* $K_{n:k}$ has the k-subsets of the n-set $[n] = \{1, 2, \ldots, n\}$ as vertices, and edges connecting disjoint sets. As an example, the graph $K_{5:2}$ is the Petersen graph. Kneser graphs were introduced by Lovász in [6] in order to prove a Kneser's conjecture, which is equivalent to claim that the chromatic number of $K_{n:k}$ is $n - 2k + 2$. From now onwards, we will deal only with connected Kneser graphs, i.e., $n > 2k$. Kneser graphs are vertex transitive and edge transitive, each vertex has exactly $\binom{n-k}{k}$ neighbors, and its diameter is $\lceil \frac{k-1}{n-2k} \rceil + 1$ (see [8]). In [5], the authors proved the following result:

Theorem 2. *[5] The domination number of the Kneser graph $K_{n:2}$ for each $n \geq 3$ is equal to 3.*

As a consequence, it can be stated that the domination number of any connected Kneser graph with $n \geq 3$ is greater or equal to 3. We will use that fact, to prove our result.

Theorem 3. *Let $K_{n:k}$ be a Kneser graph where $n \geq 5k-1$. Then $\gamma_{[1,2]}(K_{n:k}) = \binom{n}{k}$.*

Proof.

Let $\mathcal{S} \subseteq V(K_{n:k})$ be a quasiperfect dominating set. In particular, $\mathcal{S}$ is a dominating set hence $|\mathcal{S}| \geq 3$.Let A, B and C be vertices in $\mathcal{S}$ and let $R_1, \ldots, R_m$ be all vertices in $K_{n:k}$ such that $A \cap R_i = B \cap R_i = C \cap R_i = \emptyset$. As each R_i has three neighbors in $\mathcal{S}$, it is clear that $R_i \in \mathcal{S} \quad \forall i \in \{1, \ldots, m\}$. Let X be a vertex in $K_{n:k} \setminus \{A, B, C, R_1, \ldots, R_m\}$. We will show that $X \in \mathcal{S}$.

Case 1: A, B and C are pairwise disjoint. Then $|A \cup B \cup C| = 3k$ and suppose w.l.o.g. that $A \cup B \cup C = \{1, 2, \ldots, 3k\}$.
Suppose that X intersects exactly a vertex among A, B or C, and assume that $X \cap A \neq \emptyset$ and $X \cap B = X \cap C = \emptyset$. Then X has at most $k-1$ elements in $\{3k + 1, \ldots, 5k - 1\}$. Assume that $X \subseteq \{1, 2, \ldots, 3k, 3k + 1, \ldots, 3k + (k - 1) = 4k - 1\}$. Therefore $R_j = \{4k, 4k + 1, \ldots, 4k + (k - 1) = 5k - 1\}$

which is a vertex of $\mathcal{S}$ and verifies $X \cap R_j = \emptyset$. Thus X has three neighbors in $\mathcal{S}$ and $X \in \mathcal{S}$.

Finally, suppose that X intersects at leats two vertices among A, B and C, and assume that $X \cap A \neq \emptyset$ and $X \cap B \neq \emptyset$. Then $X \subseteq \{1, 2, \ldots, 3k, 3k+1, \ldots, 3k+(k-2) = 4k-2\}$. Let us consider the following vertices:

$$R_1 = \{4k-1, 4k, \ldots, 4k+(k-2) = 5k-2\}$$
$$R_2 = \{4k-1, 4k, \ldots, 4k+(k-3) = 5k-3, 4k+(k-1) = 5k-1\}$$
$$R_3 = \{4k, \ldots, 4k+(k-1) = 5k-1\}$$

Then $R_1, R_2, R_3 \in \mathcal{S}$ and $X \cap R_1 = X \cap R_2 = X \cap R_3 = \emptyset$, hence X has three neighbors in S and $X \in \mathcal{S}$.

Case 2: A, B and C are not pairwise disjoint. Then $|A \cup B \cup C| \leq 3k$ and suppose w.l.o.g. that $A \cup B \cup C \subseteq \{1, 2, \ldots, 3k-1\}$. So $X \subseteq \{1, 2, \ldots, 3k-1, 3k, 3k+1, \ldots, 3k+(k-2) = 4k-2\}$ and we can reasoning analogously as in as in Case 1.

□ ∎

References

[1] M. Chellali, T.W. Haynes, S.T. Hedetniemi and A. McRae. [1,2]-sets in graphs. *Discrete Applied Mathematics*, 161:2885–2893, 2013.

[2] I.J. Dejter. Quasiperfect domination in triangular lattices. *Discuss. Math. Graph Theory*, 29(1):179–198, 2009.

[3] S. Földes and P.L. Hammer. Split graphs. *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977)*, *Congressus Numerantium* no. XIX:311–315. Utilitas Math, Winnipeg, MB, 1977.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater. *Fundamentals of domination in graphs*. Marcel Dekker Inc., New York, New York, 1998.

[5] J. Ivanco and B. Zelinka. Domination in Kneser graphs. *Math. Bohemica*, 118(2):147–152, 1993.

[6] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. of Comb. Theory, Series A*, 25(3):319–324, 1978.

[7] R.I. Tyshkevich and A.A. Chernyak. Canonical partition of a graph defined by the degrees of its vertices. *Isv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk (in Russian)*, 5:14–26, 1979.

[8] M. Valencia-Pabon and J.C. Vera. On the diameter of Knese graphs. *Discrete Mathematics*, 305(1-3):383–385, 2005.

[9] P.M. Weichsel. Dominating sets in n-cubes. *J. Graph Theory*, 18(5):479-488, 1994.

The graph distance game and some graph operations *

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Abstract. In the graph distance game, two players alternate in constructing a maximal path. The objective function is the distance between the two endpoints of the path, which one player tries to maximize and the other tries to minimize. In this paper we examine the distance game for various graph operations: the join, the corona and the lexicographic product of graphs. We provide general bounds and exact results for special graphs.

Key words: Distance game, graph operations.

1 Introduction

Combinatorial games have been widely studied and are constantly developed. Many such games are modifications of some previous ones [17], other are entirely new. Combinatorial games remain an active field of new and interesting research. For an extensive bibliography on combinatorial games and related topics see [12].

Notice that, as shown in [14], in many combinatorial games, the winner is determined by who moves last, as studied for example in [3]. In others, the players compete to construct a desired goal, by taking one element at a time from the universe, as studied for example in [2]. Another class of games consists of those where the players compete to maximize or minimize some quantity, such as the game chromatic number introduced in [11], the competition chromatic number introduced in [21] , graph competition independence introduced in [18] or the domination game introduced in [4]. We consider here a game that falls in the latter category. (See [19] for more on such competitive games.)

Some of this games remained more or less unnoticed for many years, as for example, the game chromatic number. However, in the last several years

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various authors focus their attention in this topics. Many of this combinatorial games are specially studied in some families of graphs, such as trees, forests, outerplanar graphs, cactuses or wheels [5,6,15,20,16]. Others authors infer interrelationship between this games and some operations in graphs [9,1,22,10].

In [14], the graph distance game is introduced as follows. Given a graph G, two players alternate in constructing a path. The first player picks a vertex, the second player picks a neighbor of the first vertex, the first player picks a neighbor of the second vertex that has not yet been picked, and so on. This is continued until the path cannot be extended. One player tries to maximize the final distance from the start, and the other player tries to minimize this distance. That is, the value at the end of the game is the distance between the start and the finish, regardless of the path taken. We call this the distance game.

2 Basics and known results of graph distance game

In [14], the authors explore the distance game for various graphs and provide general bounds and exact results for simple graphs. They also show that the parameter can be calculated in a tree in linear time. Further, the values for small grids are determined.

There are two versions, depending on which of the Minimizer or Maximizer moves first. We let $S_m(G)$ denote the value of the game on graph G when the minimizer chooses the first vertex, and $S_M(G)$ the value when the maximizer chooses the first vertex. We call the first vertex of the path the *source*. Clearly, if the graph is vertex transitive, all sources are equivalent. If it does not matter who goes first, then we drop the subscript and write $S(G)$.

For a trivial example, the value of this parameters is always 1 in the complete graph. Table 1 shows the values of these parameters for paths, cycles, complete bipartite graphs and wheels.

Table 1: Distance game parameters of some basic graphs.

G	P_n	C_n	K_n	$K_{p,q}$	W_n
$S_m(G)$	$\lfloor \frac{n}{2} \rfloor$	1	1	1 (if $p = q$)	1
$S_M(G)$	$n-1$	1	1	1, 2 (if $p \neq q$)	1

The radius and the diameter of the graph give bounds of these parameters.

Proposition 1. *Let G be a graph of order n, radius $rad(G) = r$ and diameter $diam(G) = d$. Then, $1 \leq S_m(G) \leq r$ and $1 \leq S_M(G) \leq d$.*

In Figure 1, we show the distance game pairs (x, y) of alls vertices of P_6. In these terms, S_m is the minimum of the x-numbers and S_M is the maximum of the y-numbers. In this example, the Maximizer choose as first vertex one of the leaves and Minimizer select one central vertex as source.

Fig. 1: Distance game pairs of all vertices of P_6. Notice that $S_m(P_6) = 3$ and $S_M(P_6) = 5$.

In other graphs, the same vertex could be a good source for the two players, the Maximizer and the Minimizer.

3 Operations

In this paper we examine the distance game for various graph operations. We follow the notation of [7].

In [14], the authors explore the distance game for small grids. Specifically, the following results were obtained.

Theorem 1. *i)* $S_M(P_2 \square P_m) = 1$ *and* $S_m(P_2 \square P_m) = 2\lfloor m/4 \rfloor + 1$,

$$ii) S_M(P_3 \square P_m) = \begin{cases} m+1, \textit{ for odd } m, \\ 1, \qquad \textit{for even } m. \end{cases}$$

$$S_m(P_3 \square P_m) = \begin{cases} 2, \textit{ for odd } m \geq 3, \\ 1, \textit{ for even } m. \end{cases}$$

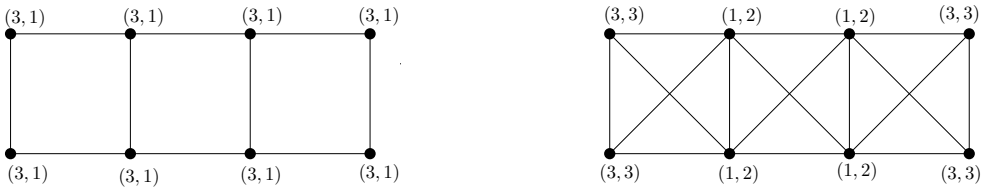

Fig. 2: Distance game pairs of vertices of the cartesian and the strong product of P_2 and P_4.

We focus our attention in the study of the distance game for join, corona and lexicographic product of graphs.

3.1 Join

The *join* $G = G_1 \vee G_2$ is the graph such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Remark 1. Notice that $1 \leq S_m(G \vee H) \leq 2$ and $1 \leq S_M(G \vee H) \leq 2$, since $1 \leq diam(G \vee H) \leq 2$.

Theorem 2. *Let G, H be graphs. Then, $S_m(G \vee H) = 1$.*

Remark 2. Notice that $S_M(P_2 \vee P_3) = 1$ and $S_M(P_2 \vee P_4) = 2$.

3.2 Corona product

Let G and H be two graphs and let n be the order of G. The *corona product* of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G and n copies of H, and then joining by an edge the i^{th} vertex of G to every vertex in the i^{th} copy of H. The corona product is neither associative nor commutative.

Theorem 3. *Let G, H be graphs. Then,*
$1 \leq S_m(G \odot H) \leq 2$ *and* $1 \leq S_M(G \odot H) \leq 3$.

Remark 3. For both parameters all values are possible:

- If $H \cong (K_2 + K_2) \vee K_1$, then $S_m(G \odot H) = 1$.
- If $H \cong P_3$ (or $H \cong K_q$) and $G \ncong K_1$, then $S_m(G \odot H) = 2$.

And

- If $H \cong K_p$ and $G \cong K_1$, then $S_M(G \odot H) = 1$.
- If $H \cong P_3 \odot K_1$, then $S_M(G \odot H) = 2$.
- If $H \cong K_q$ and $G \ncong K_1$, then $S_M(G \odot H) = 3$.

Remark 4. Notice that

- If $H \cong P_n$ and $n \geq 5$, then $S_M(G \odot H) \leq 3 < n - 1 = S_M(H)$.
- If $H \cong K_r$ and $G \ncong K_1$, then $S_M(G \odot H) = 3 > 1 = S_M(H)$.
- If $H \cong P_3 \odot K_1$ and $G \cong K_r$, then $S_M(G \odot H) = 2 > 1 = S_M(G)$.
- If $G \cong P_n$ with $n \geq 5$ and $H \cong K_r$, then $S_M(G \odot H) = 3 < n-1 = S_M(G)$.

3.3 Lexicographic product

The *lexicographic product* of graphs G and H is the graph $G \circ H$ on vertex set $V(G) \times V(H)$ in which vertices (g_1, h_2) and (g_2, h_2) are adjacent if and only if either $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$. This graph operation is also known as the graph *composition* and denoted by $G[H]$. The graph $G \circ H$ is called nontrivial if both factors are graphs on at least two vertices. Next, we show a basic list of properties of this graph operation, whose proofs are direct consequences of the definition.

Proposition 2. *Let G, H be graphs.*

1. *The graph $G \circ H$ is connected if and only if G is connected.*
2. *The lexicographic product is associative but not commutative.*
3. *If G is connected, then*
   - $d_{G\circ H}((g,h),(g',h')) = d_G(g,g')$ *if* $g \neq g'$,
   - $d_{G\circ H}((g,h),(g,h')) = 2$ *if* $hh' \in E(H)$,
   - $d_{G\circ H}((g,h),(g,h')) = 1$ *if* $hh' \in E(H)$.

We have obtained the following result.

Theorem 4. *Let G, H be graphs. Then,*

$$1 \leq S_m(G \circ H) \leq S_m(G) \quad \textit{and} \quad S_M(G) \leq S_M(G \circ H).$$

Conjecture 1. Let G be a graph. Then,

$$1 \leq S_m(G \circ K_q) = S_m(G) \quad \text{and} \quad S_M(G) = S_M(G \circ K_q).$$

References

[1] T. Bartnicki, B. Bresar, J. Grytczuk, M. Kovse, Z. Miechowicz, I. Peterin, Game chromatic number of Cartesian product graphs, *Electron. J. Combin.* 15, R72, 13 pp., 2008.

[2] J. Beck, Combinatorial Games: Tic-tac-toe theory, Cambridge University Press, 2008.

[3] E. R. Berlekamp, J. H. Conway, and R. K. Guy, Winning Ways for your Mathematical Plays. Vol. 1. A K Peters Ltd., Natick, MA, 2nd Ed., 2001.

[4] B. Bresar, S. Klavzar, D. F. Rall, Domination game and an imagination strategy, *SIAM J. Discrete Math.*, 24, 979–991, 2010.

[5] B. Bresar, S. Klavzar, D. F. Rall, Domination game played on trees and spanning subgraphs, *Discrete Math.*, 313 (8), 915–923, 2013.

[6] C. BujtÃ¡s, Domination game on forests, 2014arXiv1404.1382B, 2014.

[7] G. Chartrand, L. Lesniak, P. Zhang. Graphs and Digraphs, 5th Ed., CRC Press, Boca Raton, 2011.

[8] T. Dinski, X. Zhu, Game chromatic number of graphs, *Discrete Math.* 196, 109–115, 1999.

[9] P. Dorbec, G. Kosmrlj, G. Renault, The domination game played on unions of graphs, manuscript, 2013.

[10] L. Esperet, X. Zhu, Game colouring of the square of graphs, *Discrete Math.* 309, 4514–4521, 2009.

[11] U. Faigle, U. Kern, H. Kierstead, and W. T. Trotter. On the game chromatic number of some classes of graphs. *Ars Combin.*, 35, 143–150, 1993.

[12] A. Fraenkel, Combinatorial games: selected bibliography with a succinct gourmet introduction, *Electron. J. Combin.* DS2, 88 pp., 2009.

[13] M. Gardner, Mathematical games, *Scientic American* 244, 18–26, 1981.

[14] W. Goddard, A. Sinko, P. Slater and H. Xu, The graph distance game, *AKCE Int. J. Graphs Comb.*, 8 (1), 85–96, 2011.

[15] D.J. Guan, X. Zhu, Game chromatic number of outerplanar graphs, *J. Graph Theory*, 30, 67–70, 1999

[16] J. Y. Kim, The incidence game chromatic number of paths and subgraphs of wheels, *Discrete Appl. Math.*, 159 (8), 683–694, 2011.

[17] J. Nesetril, E. Sopena, On the oriented game chromatic number, *Electron. J. Combin.* 8 R14, 13pp., 2001.

[18] J.B. Phillips and P.J. Slater. Graph competition independence and enclaveless parameters. *Congr. Numer.*, 154, 79–100, 2002.

[19] S.J. Seo and P. Slater, Competition parameters of a graph, *AKCE Int. J. Graphs Comb.*, 4 (2), 183–190, 2007.

[20] E. Sidorowicz, The game chromatic number and the game colouring number of cactuses Information Processing Letters, 102 (4), 147–151, 2007

[21] M. Walsh. The competition chromatic numbers of a graph. *Congr. Numer.*, 164,153–158, 2003.

[22] X. Zhu, Game coloring the Cartesian product of graphs, *J. Graph Theory* 59, 261–278, 2008.

On perfect and quasiperfect domination in graphs *

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Abstract. Given a graph G, a set $D \subset V(G)$ is a dominating set of G if every vertex not in D is adjacent to at least one vertex of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

If moreover, every vertex not in D is adjacent to exactly one vertex of D, then D is called a perfect dominating set of G. The perfect domination number $\gamma_{11}(G)$ is the minimum cardinality of a perfect dominating set of G. In general, for every integer $k \geq 1$, a dominating set D is called a k-quasiperfect dominating set if every vertex not in D is adjacent to at most k vertices of D. The k-quasiperfect domination number $\gamma_{1k}(G)$ is the minimum cardinality of a k-quasiperfect dominating set of G. These parameters are related in the following general way (Δ the maximum degree of G and by n the number of vertices): $\gamma(G) = \gamma_{1\Delta}(G) \leq \cdots \leq \gamma_{12}(G) \leq \gamma_{11}(G) \leq n$.

In this work we study the perfect domination number, with the help of this decreasing chain of domination parameters, in the following graph families: graphs with extremal maximum degree, that is, graphs with $\Delta \geq n-3$ or $\Delta = 3$, and also in cographs, claw-free graphs and trees. We also study the behavior of these parameters under some usual product operations.

Key words: Perfect domination, quasiperfect domination, claw-free graphs, cographs.

1 Introduction

All the graphs considered are finite, undirected, simple, and connected. Given a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* $deg(v)$ of a vertex $v \in V(G)$ is the number of neighbors of v, i.e., $deg(v) = |N(v)|$. The *maximum degree* of G, denoted by $\Delta(G)$, is the largest degree among all vertices of G. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [3].

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Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* of G if every vertex v not in D is adjacent to at least one vertex of D, i.e., if $N(v) \cap D \neq \emptyset$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a γ-*code*[5].

If moreover, every vertex not in D is adjacent to exactly one vertex of D, then D is called a *perfect dominating set* of G [1,7]. The *perfect domination number* $\gamma_{11}(G)$ is the minimum cardinality of a perfect dominating set of G. A dominating set of cardinality $\gamma_{11}(G)$ is called a γ_{11}-*code*. This definition can be generalized in the following way.

Definition 1 ([4]). *For $k \geq 1$, we define a dominating subset $S \subseteq V$ in a graph $G = (V, E)$ to be a* k-quasiperfect dominating set *if every vertex not in D is adjacent to at most k vertices of D.*

Definition 2 ([4]). *For $k \geq 1$, The* k-quasiperfect domination number *$\gamma_{1k}(G)$ is the minimum cardinality of a k-quasiperfect dominating set of G. A dominating set of cardinality $\gamma_{1k}(G)$ is called a* γ_{1k}-code.

Certainly, 1-quasiperfect dominating sets and Δ-quasiperfect dominating sets are precisely the perfect dominating sets and dominating sets, respectively. There is an obvious relationship among these domination paramenters. If G is a graph of order n and maximum degree Δ, then

$$\gamma(G) = \gamma_{1\Delta}(G) \leq \ldots \gamma_{12}(G) \leq \gamma_{11}(G) \leq n$$

In this work we study this decreasing chain of domination parameters. We present our main contributions when restricting ourselves to the following graph families:

- Graphs with maximum degree $\Delta \geq n - 3$ or $\Delta = 3$.
- Cographs.
- Claw-free graphs.
- Trees.

We also study the behavior of these parameters under product operations.

2 Results

Theorem 1 ([4]). *If G is a graph of order n that satisfies some of the following conditions, then $\gamma(G) = \gamma_{12}(G)$:*

- $\Delta(G) \geq n - 3$.
- $\Delta(G) \leq 2$.
- *G is a P_4-free graph (cograph).*
- *G is a $K_{1,3}$-free graph (claw-free graph).*
- *Every vertex of G is either a support vertex or has degree at most 2.*

As a result of Theorem above, in graphs that satisfy some of its conditions the chain of quasiperfect domination parameters is shorter that in the general case: $\gamma(G) = \gamma_{12}(G) \leq \gamma_{11}(G) \leq n$, and it is interesting to consider what happen with the parameter γ_{11}. We have obtained the following results.

2.1 Graphs with maximum degree $\Delta(G) \geq n - 3$

In this case we have obtained realization results for the parameter γ_{11}, that show that it can achieve all values in the interval between 2 and n, with a small number of exceptions.

Theorem 2. *Let k, n be integers such that $n \geq 4$, $2 \leq k \leq n$ and $(n, k) \notin \{(5,5), (5,4), (4,4), (4,3)\}$. Then, there exists a graph $G = (V, E)$ of order n such that $\Delta(G) = n - 2$ and $\gamma_{11}(G) = k$.*

Theorem 3. *Let k, n be positive integers such that $n \geq 8$ and $2 \leq k \leq n$. Then, there exists a graph G of order n such that $\Delta(G) = n - 3$ that satisfies $\gamma_{11}(G) = k$.*

2.2 Graphs with small maximum degree

The family of connected graphs with maximum degree $\Delta = 2$ contains just paths and cycles, and in both cases parameter γ_{11} is completely determined: $\gamma_{11}(P_n) = \lceil \frac{n}{3} \rceil$ and $\gamma_{11}(C_n) = \lceil \frac{2n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$. So we focus on graphs with maximum degree $\Delta = 3$ and we have obtained the following result that provide an upper bound for γ_{11}.

Theorem 4. *If $\Delta(G) = 3$ and G is other than the bull graph, then $\gamma_{11}(G) \leq n - 3$. Note also that the bull graph H has 5 vertices and $\gamma_{11}(H) = 3 = n - 2$*

2.3 Cographs

In the family of P_4-free graphs, we have calculated the exact values of γ_{11}, depending on the value of de domination parameter γ.

Theorem 5. *Let G be a cograph of order n. Then:*

- *If $\gamma(G) = 2$, then $\gamma_{11}(G) \in \{2, n\}$.*
- *Cographs such that $\gamma(G) = \gamma_{11}(G) = 2$ are completely characterized.*
- *If $\gamma(G) \geq 3$, then $\gamma_{11}(G) = n$.*

2.4 Claw-free graphs

In this family of graphs, we have also studied the values of γ_{11} in relationship with the values of γ. But in contrast with the case above, the family of cographs, in this occasion a wider range of values can be achieved.

Theorem 6. *Let h, k, n be integers such that $2 \leq h \leq k < n$ and $h + k \leq n$. Then, there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = k$.*

Proposition 1. *Let n be an integer such that $n \geq 6$. Then,*

- *there exists a claw-free graph G of order n and such that $\gamma(G) = 2$ and $\gamma_{11}(G) = n - 1$,*
- *there exists a claw-free graph G of order n and such that $\gamma(G) = 2$ and $\gamma_{11}(G) = n$.*

Proposition 2. *Let h, n be integers such that $n \geq 7$, $2 \leq h \leq \lfloor \frac{n-1}{3} \rfloor$. Then, there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = n$.*

3 Trees

The following result about trees is known.

Theorem 7 ([2]).
Let T be a tree of order $n \geq 3$ with k leaves. Then,

- *Every $[1, 1]$-set contains all its strong support vertices.*
- $\gamma_{11}(T) \leq \frac{n}{2}$.
- $\gamma_{11}(T) = \frac{n}{2}$ *if and only if* $T = T' \odot K_1$, *for some tree* T'.
- $\gamma_{11}(T) \leq n - k$.
- $\gamma_{11}(T) = n - k$ *if and only if T contains a $[1, 1]$-code D such that $V \setminus D$ induces a coclique.*

So we focus our attention on the relationship between γ and γ_{11}. We have obtained a complete result in the particular case of caterpillars and a general inequality between both parameters that is satisfied for any tree.

Proposition 3. *Let T be a caterpillar. Then*

$$\gamma(T) = \gamma_{12}(T) \leq \gamma_{11}(T) < 2\gamma(T)$$

Proposition 4. *Let $\{h, k, n\}$ be integers with $1 \leq h \leq k \leq \frac{n}{2}$ and $h < 2k$. Then there exists a caterpillar T of order n such that $\gamma_{12}(T) = h$, $\gamma_{11}(T) = k$.*

Theorem 8. *For every tree T, $\gamma(T) \leq \gamma_{11}(T) \leq 2\gamma(T) - 1$. Moreover, both bounds are tight.*

4 Product graphs

Finally we present some results on the behavior of the quasiperfect domination parameters with standard product operations.

We begin with the *cartesian product* [6] of two connected graphs G and H, denoted by $G\Box H$, which is the graph with the vertex set $V(G) \times V(H)$ in which vertices (g, h) and (g', h') are adjacent whenever $gg' \in E(G)$ and $h = h' \in E(H)$ or $g = g' \in E(G)$ and $hh' \in E(H)$. The following result is known.

Proposition 5 ([4]). *For every grid graph $G = P_h \Box P_k$, $\gamma_{13}(G) = \gamma(G)$.*

We have obtained a general upper bound for this product-type operation.

Theorem 9. *Let G and H be two graphs and let r be an integer. Then, $\gamma_{1r}(G\Box H) \leq \min\{\gamma_{1r}(G)|V(H)|, |V(G)|\gamma_{1r}(H)\}$. Moreover, this bound is tight.*

On the other hand, the *strong product* [6] of graphs two connected G and H, denoted by $G \boxtimes H$, is the graph such that $V(G \boxtimes H) = (V(G) \times V(H)$ and $E(G \boxtimes H) = E(G \times H) \cup E(G\Box H)$. In this case, the following result is proved.

Proposition 6. *Let G be a graph and let k be an integer such that $\gamma_{1k}(G) = |V(G)|$. Then, $\gamma_{1k}(G \boxtimes H) = |V(G \boxtimes H)|$, for any graph H.*

Finally we have calculated exact values of parameters γ_{11} and γ_{12} for strong product of paths, cycles and complete graphs.

Proposition 7. $\gamma_{11}(P_r \boxtimes P_s) = \gamma(P_r \boxtimes P_s) = \gamma(P_r) \cdot \gamma(P_s)$

Proposition 8.

- $\gamma_{12}(C_r \boxtimes C_s) = \gamma(C_r \boxtimes C_s) = \gamma(C_r)\gamma(C_s) = \lceil \frac{r}{3} \rceil \lceil \frac{s}{3} \rceil$.
- $\gamma_{11}(C_r \boxtimes C_s) = \gamma(C_r \boxtimes C_s)$, *if* $r = 3a$ *and* $s = 3b$.
- $\gamma_{11}(C_r \boxtimes C_s) = rs = n$, *if* $r \neq 3a$ *or* $s \neq 3b$.

Proposition 9. $\gamma_{11}(K_r \boxtimes P_s) = \gamma(K_r \boxtimes P_s) = \lceil \frac{s}{3} \rceil$

Proposition 10.

- $\gamma_{12}(K_r \boxtimes C_s) = \gamma(K_r \boxtimes C_s) = \gamma(C_s) = \lceil \frac{s}{3} \rceil$.
- $\gamma_{11}(K_r \boxtimes C_s) = \gamma(K_r \boxtimes C_s)$, *if* $s = 3a$.
- $\gamma_{11}(K_r \boxtimes C_s) = rs = n$, *if* $s \neq 3a$.

Proposition 11.

- $\gamma_{12}(C_r \boxtimes P_s) = \gamma(C_r \boxtimes P_s) = \gamma(C_r)\gamma(P_s) = \lceil \frac{r}{3} \rceil \lceil \frac{s}{3} \rceil$.
- $\gamma_{11}(C_r \boxtimes P_s) = \gamma(C_r \boxtimes P_s)$, *if* $r = 3a$.
- $\gamma_{11}(C_r \boxtimes P_s) = rs = n$, *if* $r \neq 3a$.

References

[1] N. Biggs, Perfect codes in graphs, *J. Comb. Theory (B)*, 15:289-296, 1973.

[2] Y. Caro, A. Hansberg, M. Henning, Fair domination in graphs, *Discrete Mathematics*, 312(19):2905–2914, 2012.

[3] G. Chartrand, L. Lesniak, P. Zhang, *Graphs and Digraphs, (5th edition).* CRC Press, Boca Raton, Florida, 2011.

[4] M. Chellali, T.W. Haynes, S.T. Hedetniemi, A. McRae, [1, 2]-sets in graphs, *Discrete Applied Mathematics*, 161(18):2885–2893, 2013.

[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of domination in graphs.* Marcel Dekker, New York, 1998.

[6] W. Imrich, S. Klavzar, *Product graphs. Structure and recognition.* John Wiley & Sons, Inc., New York, 2000.

[7] M. Livingston, Q.F. Stouty, Perfect dominating sets, *Congressus Numerantium*, 79:187-203, 1990.

Rings of graphs

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Abstract. We construct all possible rings taking graphs as elements, the union or sum of graphs as the binary operation of the group and graph product as the second operation. Furthermore we show that every ring formed with union operation is isomorphic to some ring with sum operation. In every case, we give an homomorphism from the ring to the integers polynomial. (This is an extended abstract of [1])

Key words: Ring, Graph products, clique polynomial, union and sum of graphs

1 Preliminaries

For two graphs G, H, the union $\cup$ is defined as $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. In the set of graphs with the union (or sum) operation, we define a monoid. The first part of this work is to construct the Grothendieck group $K_{\cup}(G)$ over this monoid by defining the equivalence relation $\sim$ as $(G, G') \sim (H, H')$ if and only if $G \cup H'$ is isomorphic to $G' \cup H$, and forming the quotient group $\frac{\mathcal{G}\times\mathcal{G}}{\sim}$, where $\mathcal{G}$ is the space of graphs. (see [2]). It is easy to proof that this group is abelian.

Then, we would like to add another operation to $K_{\cup}(G)$ that make up a ring structure, more precisely, we would like to see which of the graphs products can be added to form a ring. Since $K_{\cup}(G)$ is an abelian group, the graph products we are looking must be associative and distributive over the union operation (left and right).

2 Graph products

The graph products have been studied since 1912 (see [5]). A *graph product* $\wedge$ is an operation between two graphs G, H such that the vertex set of $G \wedge H$ is the cartesian product of the vertex sets of G and H, and $(g_1, h_1)(g_2, h_2) \in G \wedge H$

depending if g_1 and g_2 (h_1, h_2 resp.) are adjacent, not adjacent or the same vertex in G (H resp.).

We use the matrix representation defined in [4]. For a graph G, call E to the set of the pairs of adjacent vertices, C the set of the pairs of non-adjacent vertices and $\triangle$ to set set of the pair of equal vertices. So, for two graphs G, H and a graph product $\wedge$, the adjacency between two vertices $(g,h),(g'.h')$ totally depends of the contention of $(g,g'),(h,h')$ in one of the three mentioned sets of G and H respectively. We can resume the conditions of adjacencies by taking the matrix:

$$\begin{array}{c} \\ E \\ \triangle \\ C \end{array} \begin{array}{c} \begin{array}{ccc} E & \triangle & C \end{array} \\ \begin{pmatrix} E^2 & E\triangle & EC \\ \triangle E & \triangle & \triangle C \\ CE & C\triangle & C^2 \end{pmatrix} \end{array}$$

where the column represents the sets in G and the rows represents the sets in H and fill the entry by E if the condition in the entry of the matrix allows a edge in the product and by C in the other case.

We will give an example of this representation. The first products ever studied were the *direct product (tensor product)* and the *lexicographic product* (see [5]). In the direct product of graphs G, H, we have that two vertices $(g,h)(g',h')$ are adjacent if and only if $(g,g') \in E(G)$ and $(h,h') \in E(H)$, so we can write the product :

$$\begin{array}{c} \\ E \\ \triangle \\ C \end{array} \begin{array}{c} \begin{array}{ccc} E & \triangle & C \end{array} \\ \begin{pmatrix} E & C & C \\ C & \triangle & C \\ C & C & C \end{pmatrix} \end{array}$$

Since every entry must be filled by E or C, there are $2^8 = 256$ possible ways to define a graph product.

3 Rings of graphs

We are interested in determining which are associative and distributive over union. In [4], the authors characterized the associative graph products and shows that there are 39. In order to see which are the distributive products over union, we will used the two following lemmas.

Lemma 1. *A graph product $\wedge$ is distributive on left with respect to the union, if and only if, its associated matrix has the following form:*

$$\begin{pmatrix} \_ & \_ & C \\ \_ & \triangle & C \\ \_ & \_ & C \end{pmatrix}$$

Lemma 2. *A graph product $\wedge$ is distributive on right with respect to the union $\cup$, if and only if, its associated matrix has the following form:*

$$\begin{pmatrix} \_ & \_ & \_ \\ \_ & \triangle & \_ \\ C & C & C \end{pmatrix}$$

The two lemmas above imply the following theorem.

Theorem 1. *A graph product $\wedge$ is distributive with respect to the union $\cup$, if and only if, its associated matrix has the following form:*

$$\begin{pmatrix} \_ & \_ & C \\ \_ & \triangle & C \\ C & C & C \end{pmatrix}$$

So there are $2^3 = 8$ distributive products, and by using the results of [4], we found that 6 of them can be add to $K_{\cup}(G)$ to form a ring. We will denote those rings by $(\mathcal{G}, \cup, \wedge)$. Now we set the following lemma and we resume it in a theorem:

Lemma 3. *If $\wedge$ is an associative graph product (distributive resp.), then it is associative (distributive resp.) on $K(\mathcal{G})$.*

Theorem 2. $(\mathcal{G}, \cup, \wedge)$ *is a ring of graphs, if and only if, $\wedge$ is one of the following products:*

$$\begin{pmatrix} C & C & C \\ C & \triangle & C \\ C & C & C \end{pmatrix}. \begin{pmatrix} C & E & C \\ C & \triangle & C \\ C & C & C \end{pmatrix}. \begin{pmatrix} C & C & C \\ E & \triangle & C \\ C & C & C \end{pmatrix}. \begin{pmatrix} C & E & C \\ E & \triangle & C \\ C & C & C \end{pmatrix}. \begin{pmatrix} E & C & C \\ C & \triangle & C \\ C & C & C \end{pmatrix}. \begin{pmatrix} E & E & C \\ E & \triangle & C \\ C & C & C \end{pmatrix}.$$

In order, the empty product, the manifold product*, the* co-manifold product*, the cartesian product, the categorical product, the strong product*

For two graph G, H, the *sum* of graph $+$ is defined as $V(G + H) = V(G) \cup V(H)$ ans $E(G + H) = E(G) \cup E(H) \cup \{gh | g \in G, h \in H\}$. As we perform an analogous analysis for the graph sum as we have done for the union , we find six products $\vee$ for which $(\mathcal{G}, +, \vee)$ is a ring of graphs. Those

rings are isomorphic to the rings of graphs constructed with respect to the union. We use another definition from [4] to set the next two theorem. For a graph product $\wedge$, we define its *complementary product* $\vee$ as $G \vee H = \overline{\overline{G} \wedge \overline{H}}$, where $\overline{G}$ denotes the complement of the graph G.

Theorem 3. *[1] If $(\mathcal{G}, \cup, \wedge)$ is a ring then $(\mathcal{G}, +, \vee)$ is a ring of graph, where $\vee$ is the complementary product of $\wedge$*

Theorem 4. *[1] There exists an isomorphism Ψ between the rings of graphs $(\mathcal{G}, \cup, \wedge)$ and $(\mathcal{G}, +, \vee)$.*

4 Clique Polynomials

A k-clique of a graph G is a complete subgraph of G with k vertices i.e a subgraph with all possible edges. An independent subset G' of G is a vertex subset of $V(G)$ such that $gg' \notin E(G)$ for every $g, g' \in G'$, i.e, a set without any edge. The clique polynomial of a graph G is defined as $P_G(X) = \sum a_i x^i$ where the the coefficient a_i is the number of complete subgraph of size i and $a_0 = 1$, in the independence polynomial, the coefficients a_i represent the number of independent set of size i.

In an attempt to relate these graphs rings with some algebraic space, we give an homomorphism between the graphs rings and the integer polynomials. There exists al lot of different polynomials defined on the graph space, however we choose to work with the clique polynomial and the independence polynomial (see [3]). Our first result on the graph polynomial is the following theorem:

Theorem 5. *[1] For every ring of graphs $(\mathcal{G}, \cup, \wedge)$, there exists a product $\wedge'$ in $\mathbb{Z}_0[x]$ (the integers polynomials with $p(0) = 0$) and an homomorphism $\phi_\wedge$ from $(\mathcal{G}, \cup, \wedge)$ to $(\mathbb{Z}_0[x], +, \wedge')$*

Then, we give the formulae of the induced clique polynomials of each of the 6 products in $K_\cup(G)$ and of the induced independence polynomials of the 6 products in $K_+(G)$.

For graphs G, H with clique polynomials $P_G = \sum_{i=0}^{n} g_i x^i$ and $P_H = \sum_{i=0}^{m} h_i x^i$ with $a_0 = 1, b_0 = 1$. In order for the empty product, the manifold product, the co-manifold product, the cartesian product, the categorical product, the strong product, the formulaes are:

$$P_\emptyset = |G||H|x + 1$$

$$P_{G \sqcup H} = |H|P_G - |H| + 1$$

$$P_{G \sqcap H} = |G|P_H - |G| + 1$$

$$P_{G\square H} = |H|P_G + |G|P_H - (|H| + |G| + |G||H|x) + 1$$

$$P_{G\times H} = \sum_{i=0}^{min(n,m)} i!g_i h_i x^i$$

$$P_{G\boxtimes H} = \sum_{i=0}^{n}\sum_{j=0}^{m} \tilde{g}_i\tilde{h}_j(K_{ij})$$

where $\tilde{g}_i = \sum_{s=i}^{n} g_i\binom{s}{i}(-1)^{i+s}$ and $\tilde{h}_i = \sum_{s=i}^{m} h_i\binom{s}{i}(-1)^{i+s}$

We ending this note by making the two following remarks:

Remark 1. Since $G \wedge H = \overline{\overline{G} \vee \overline{H}}$ we have that $P_{G\wedge H} = P_{\overline{\overline{G}\vee\overline{H}}} = I_{\overline{G}\vee\overline{H}}$ so the independence polynomials of the complementary products are easily computed from our previous computations.

Remark 2. The 6 rings of the form $(\mathcal{G}, \cup, \wedge)$ are associative $\mathbb{Z} - algebras$. Furthermore, for $\wedge = \begin{pmatrix} C & C & C \\ C & \triangle & C \\ C & C & C \end{pmatrix}$ or $\begin{pmatrix} E & C & C \\ C & \triangle & C \\ C & C & C \end{pmatrix}$, the $\mathbb{Z}$-algebras are commutative but not unital. If $\wedge = \begin{pmatrix} C & E & C \\ C & \triangle & C \\ C & C & C \end{pmatrix}$ or $\begin{pmatrix} C & C & C \\ E & \triangle & C \\ C & C & C \end{pmatrix}$, then they are not commutative and not unital $\mathbb{Z}$-algebras (either they have right and left unite resp.). Finally, if $\wedge = \begin{pmatrix} E & E & C \\ E & \triangle & C \\ C & C & C \end{pmatrix}$ or $\begin{pmatrix} C & E & C \\ E & \triangle & C \\ C & C & C \end{pmatrix}$, the $\mathbb{Z}$-algebras are commutative and unital, with identity element K_1.

References

[1] Campanelli Nicolas, Martín Eduardo Frías Armenta, Jose L. Martinez-Morales. Ring of graphs, Submited to boletin de la sociedad mexicana de matemáticas.

[2] Martín-Eduardo Frías Armenta, Jose L. Martinez-Morales. The Grothendieck group of isomorfism classes of graphs, Submited to boletin de la sociedad mexicana de matemáticas.

[3] Cornelis Hoede and Xueliang Li, Clique polynomials and independent set polynomials of graph; Discrete Mathematics 125 (1994) 219ï¿½228.

[4] Wilfried Imrich, Leoben and Herbert Izbicki, Wien. Monatsh. Associative products of Graphs, Math., 80, 4, Page(s): 277–281, 1975.

[5] A. N. Whitehead and B. Russell. Principia Mathematica, volume 2. Cambridge University Press, Cambridge, 1912.

Isoperimetric inequalities in graphs and surfaces

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Abstract. Let $\mathfrak{M}$ be the set of metric spaces that are either graphs with bounded degree or Riemannian manifolds with bounded geometry. Kanai proved the quasi-isometric stability of several geometric properties (in particular, of isoperimetric inequalities) for the spaces in $\mathfrak{M}$. Kanai proves directly these results for graphs with bounded degree; in order to prove the general case, he uses a graph (an ε-net) associated to a Riemannian manifold with bounded geometry. This paper studies the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces (endowed with their Poincaré metrics). The present work proves the stability of the linear isoperimetric inequality for planar surfaces (genus zero surfaces) without the condition on bounded geometry. It is also shown the stability of any non-linear isoperimetric inequality.

Key words: Isoperimetric inequality; linear isoperimetric inequality; quasi-isometry; infinite graphs; Riemann surfaces.

1 Introduction and main results

An interesting problem in the study of geometric properties of graphs and surfaces is to consider their stability under appropriate deformations. Let $\mathfrak{M}$ be the set of metric spaces that are either graphs with bounded degree or Riemannian manifolds with bounded geometry. In the 1985, in [16] M. Kanai proved the quasi-isometric stability (see the definition of quasi-isometry after Theorem 1) of several geometric properties (in particular, of isoperimetric inequalities) for the spaces in $\mathfrak{M}$.

We shall be interested not only in his results but in the ideas behind the proofs. Concretely, those relating the manifold with a particular graph (an ε-net of the manifold) in order to study the stability of isoperimetric inequalities by quasi-isometries. Several authors have followed Kanai in studying the stability of some other property, or in proving the equivalence of a manifold

with a different associated graph (see, e.g., [1], [6], [12], [15], [17], [18], [21], [22], [23], [24], [25], [26], [14]).

Quasi-isometries play a central role in the theory of Gromov hyperbolic spaces for they preserve hyperbolicity of geodesic metric spaces (see, e.g., [16], [18]).

A *non-exceptional* Riemann surface S will mean a two-dimensional manifold with a complete conformal metric of constant negative curvature -1. In this case, the universal covering space of S is the unit disk $\mathbb{D}$ endowed with its Poincaré metric. The only exceptional Riemann surfaces are the sphere, the plane, the punctured plane and the tori.

A Riemann surface S satisfies the *α-isoperimetric inequality* $(1/2 \leq \alpha \leq 1)$ if there exists a constant $c_\alpha(S)$ such that

$$A_S(\Omega)^\alpha \leq c_\alpha(S) L_S(\partial\Omega) \tag{1}$$

for every relatively compact domain $\Omega \subset S$. Throughout, A_S, L_S and d_S refer to Poincaré area, length and distance of S and LII refers to the 1-isoperimetric inequality also known as the *linear isoperimetric inequality*.

The isoperimetric inequality on a graph G with bounded degree can be defined as follows. For a subset T of $V(G)$, define its *boundary* as

$$\partial T := \{q \in V(G) \setminus T : d_G(q, T) = 1\}.$$

It is said that G satisfies the *α-isoperimetric inequality* if there exists a constant $c_\alpha(G)$ so that

$$\big(\#T\big)^\alpha \leq c_\alpha(G)\, \#\partial T$$

for any non-empty finite subset T of $V(G)$, where $\#$ denotes the cardinal.

There are close connections between LII and some conformal invariants of Riemann surfaces, namely the bottom of the spectrum of the Laplace-Beltrami operator, the exponent of convergence, and the Hausdorff dimensions of the sets of both bounded geodesics and escaping geodesics in the surface (see [3], [4, p.228], [8], [9], [10], [11], [19], [27, p.333]). Isoperimetric inequalities are of interest in pure and applied mathematics (see, e.g., [7], [20]).

The *injectivity radius $\iota(p)$ of $p \in S$* is defined as the supremum of those $r > 0$ such that $B_S(p, r)$ is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at p. The *injectivity radius $\iota(S)$ of S* is the infimum over $p \in S$ of $\iota(p)$.

In this work we consider the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces. This stability was proved by Kanai in [16] in the very general setting of graphs and Riemannian manifolds in $\mathfrak{M}$ (bounded geometry in a Riemannian manifold M means a lower bound for the Ricci curvature and $\iota(M) > 0$). We have an example showing that the stability fails, even for Riemann surfaces, without the hypothesis $\iota(S) > 0$. Since this example involves non-zero genus surfaces, it is natural to wonder if the stability holds for planar surfaces.

The main result in this paper is the following.

Theorem 1. *Let S and S' be quasi-isometric non-exceptional genus zero Riemann surfaces. Then S' satisfies the linear isoperimetric inequality if and only if S satisfies the linear isoperimetric inequality. Furthermore, if $f : S \longrightarrow S'$ is a c-full (a, b)-quasi-isometry, and $c_1(S') < \infty$ then $c_1(S) \leq C$, where C is a universal constant which just depends on a, b, c and $c_1(S')$.*

A function between two metric spaces $f : X \longrightarrow Y$ is said to be an (a, b)-*quasi-isometric embedding* with constants $a \geq 1$, $b \geq 0$, if

$$\frac{1}{a}\, d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a\, d_X(x_1, x_2) + b\,,$$

for every $x_1, x_2 \in X$. Such a quasi-isometric embedding f is a *quasi-isometry* if, furthermore, there exists a constant $c \geq 0$ such that f is *c-full*, i.e., if for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq c$.

Two metric spaces X and Y are *quasi-isometric* if there exists a quasi-isometry between them. It is easy to check that to be quasi-isometric is an equivalence relation on the set of metric spaces.

For surfaces of positive finite genus, the following result shows that the first conclusion of Theorem 1 holds:

Theorem 2. *Let S and S' be quasi-isometric non-exceptional Riemann surfaces with finite genus. Then S' satisfies the LII if and only if S satisfies the LII.*

However, we have an example showing that the second conclusion of Theorem 1 fails in this case of positive finite genus.

The idea behind the proof of Theorem 1 is simple: each surface is split into a thin part (with small injectivity radius) and a thick part; a slight modification of the proof of Kanai's theorem applied to the thick part, together with some new arguments to show that the thin part is "essentially" preserved under the quasi-isometry give the theorem. A difficulty is the following: two quasi-isometric surfaces have a similar shape at a large scale (if viewed from sufficiently far), but they can look very different at a small scale (by definition a quasi-isometry may not be continuous). In particular, the image of a continuous loop by a quasi-isometry need not be a continuous curve, and thus the injectivity radii can be very different in two quasi-isometric surfaces. Theorem 3 deals with this situation and states that a quasi-isometry between planar surfaces maps points with small injectivity radius to points with small injectivity radius (in a precise quant itative way).

Theorem 3. *Let S and S' be non-exceptional genus zero Riemann surfaces and let $f : S \longrightarrow S'$ be a c-full (a, b)-quasi-isometry. For each $\varepsilon' > 0$ there exists $\varepsilon > 0$ which just depends on ε', a, b, c, such that if $\iota(z) < \varepsilon$ then $\iota(f(z)) < \varepsilon'$.*

We show that a very different situation appears when dealing with the α-isoperimetric inequality, $1/2 \leq \alpha < 1$.

Theorem 4. *Let S and S' be quasi-isometric non-exceptional Riemann surfaces with $\iota(S) > 0$, and $1/2 \leq \alpha < 1$. Then S' satisfies the α-isoperimetric inequality if and only if S satisfies the α-isoperimetric inequality and $\iota(S') > 0$.*

Note that here we have no hypothesis on genus.

Hence, the behavior of the α-isoperimetric inequality in Riemann surfaces under quasi-isometries is very different in the cases $\alpha = 1$ and $\alpha < 1$.

One of the main ingredients in the proofs is the relation between a surface S and a graph (an ε-net) associated to S.

2 Sketch of the proof of Theorem 1

This section is devoted to present the main ideas in the proof of Theorem 1, which follows Kanai's approach. See [5] for details.

In Kanai's results it is essential that both $\iota(S)$ and $\iota(S')$ are positive; these conditions will be avoided due to Theorem 3 and the thick-thin decomposition of Riemann surfaces given by Margulis Lemma (see, e.g., [2, p.107]). Concretely, for any $\varepsilon < \sinh^{-1} 1$ ($\sinh^{-1}$ denotes the inverse function of sinh) any Riemann surface, S, can be partitioned into a thick part, $S_\varepsilon := \{z \in S : \iota(z) > \varepsilon\}$, and a thin part, $S \setminus S_\varepsilon$, whose connected components have a simple structure (the fundamental group of each connected component of $S \setminus S_\varepsilon$ is generated by a single element).

In order to prove Theorem 1, it will be shown that it suffices to consider the thick parts of S and S' for some particular choices of ε and ε', so that Kanai's insight can be brought to S_ε and $S'_{\varepsilon'}$ if we avoid the (possible) contribution to the LII given by ∂S_ε and $\partial S'_{\varepsilon'}$.

Let us consider $H > 0$, a metric space X, and a subset $Y \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$ is called the *H-neighborhood* of Y in X.

We will need the following technical results.

Lemma 1. *Let S and S' be non-exceptional genus zero Riemann surfaces, and $f : S \longrightarrow S'$ be a c-full (a, b)-quasi-isometry. Then, given $0 < \varepsilon, \varepsilon_1 < \sinh^{-1} 1$, there exist $0 < \varepsilon', \tilde{\varepsilon} < \varepsilon_1$, which just depend on $\varepsilon, \varepsilon_1, a, b, c$, so that*

$$f(S_\varepsilon) \subset S'_{\varepsilon'} \subset V_c(f(S_{\tilde{\varepsilon}})).$$

As a first goal it is going to be proved the LII intrinsic to a bordered surface, S_ε contained in S; note that S_ε is not necessarily connected. To this end, we define below the "thick" boundary of a subset of S as its intrinsic boundary in S_ε, and the "intrinsic" LII that will refered to as LII_ε.

Definition 1. *Given a non-exceptional Riemann surface S, $\varepsilon > 0$ and a domain Ω in S_ε, define*

$$\partial_\varepsilon \Omega := \partial\Omega \cap S_\varepsilon = \partial\Omega \setminus \partial S_\varepsilon.$$

Definition 2. *S_ε is said to satisfy the ε-linear isoperimetric inequality, LII_ε, if there exists a positive constant c, such that if Ω is a relatively compact domain in S_ε with smooth boundary, then*

$$A_S(\Omega) \leq c\, L_S(\partial_\varepsilon \Omega). \tag{2}$$

A reduction is that it suffices to prove LII_ε for intrinsic geodesic domains in S_ε. A domain $\Omega \subset S$ is said to be a *geodesic domain* if $\partial\Omega$ is a finite number of simple closed geodesics, and $A_S(\Omega)$ is finite. An *intrinsic geodesic domain* is a geodesic domain intrinsic to S_ε, i.e., the intersection of a geodesic domain in S with S_ε.

Let us denote by $c_1(S_\varepsilon)$ the sharp ε-linear isoperimetric constant of S_ε and by $c_{1,g}(S_\varepsilon)$ the sharp ε-linear isoperimetric constant of S_ε for intrinsic geodesic domains.

Lemma 2. *Let S be a non-exceptional Riemann surface and $\varepsilon \geq 0$ so that $\varepsilon < \sinh^{-1} 1$. Then,*

$$S_\varepsilon \text{ has } LII_\varepsilon \iff S_\varepsilon \text{ has } LII_\varepsilon \text{ for intrinsic geodesic domains in } S_\varepsilon.$$

In fact, $c_{1,g}(S_\varepsilon) \leq c_1(S_\varepsilon) \leq c_{1,g}(S_\varepsilon) + 2$.

Following Kanai's procedure, the LII will be transferred from bordered surfaces to nets and viceversa. To this end, a subset G of S is said to be δ-separated for $\delta > 0$, if $d_S(p, q) > \delta$ whenever p and q are distinct points of G. It is called *maximal* if it is maximal with respect to the order relation of inclusion.

Consider the distance d_G in G induced by the distance d_S of S. Concretely, given $p_1, p_2 \in G$, $d_G(p_1, p_2) = M$ if and only if $M \geq 0$ is the only natural number such that

$$\delta M \leq d_S(p_1, p_2) < \delta(M + 1). \tag{3}$$

The set of neighbors of p in G is defined as $N(p) = \{q \in G : d_G(p, q) = 1\}$ and this gives a graph structure to the set G. Such graph will be referred to as δ-net.

Let S be a Riemann surface and $0 < \varepsilon < \sinh^{-1} 1$. We have that $\iota\left(V_\varepsilon(S_\varepsilon)\right) \geq c(\varepsilon)$, where $c(\varepsilon) := \sinh^{-1}\left(e^{-\varepsilon} \sinh \varepsilon\right)$. The pair (G, δ) will denote a δ-net *associated* to the pair (S, ε) as follows: Set $\delta \leq \frac{1}{2}\, \iota(V_\varepsilon(S_\varepsilon))$, and choose a maximal δ-net G on S_ε so that

$$A_S(S_\varepsilon \cap B_S(p, \delta)) > \frac{1}{2} A_S(B_S(p, \delta)), \tag{4}$$

for all $p \in G$; such choice of G is possible due to Collar Lemma. Note also that G does not need to be connected.

The strategy of the proof of Theorem 1 is as follows: Consider S and S' Riemann surfaces and $f : S \longrightarrow S'$ a quasi-isometry, (G, δ) and (G', δ') nets in (S, ε) and (S', ε'). It will be assumed that S' satisfies the LII that will be transferred to the net (G', δ'). Then it will be shown that (G, δ) and (G', δ') are quasi-isometric and so (G, δ) also satisfies the LII. Finally, this LII will be transferred to S. The next two results deal with transferring the LII between surfaces and nets. A direct application of [16, Lemma 4.5] is the following result:

Lemma 3. *There exists a universal constant ε_0 with the following property: Let S' be any non-exceptional Riemann surface satisfying LII and $0 < \varepsilon' < \min\left\{\varepsilon_0, (12c_1(S'))^{-1}\right\}$. Let (G', δ') be a δ'-net associated to (S', ε'). Then,*

$$(G', \delta') \text{ also satisfies the } LII \text{ and } c_1(G') \leq \frac{12 \sinh \delta'}{\cosh(\delta'/2) - 1}\, c_1(S').$$

Lemma 4. *Let (G, δ) be a δ-net associated to (S, ε). Then*

$$(G, \delta) \text{ has } LII \Longrightarrow S_\varepsilon \text{ has } LII_\varepsilon. \tag{5}$$

Moreover, $c_1(S_\varepsilon) \leq 2\, m\, c_{1,l}(S_\varepsilon) \max\left\{1,\, 2\, c_1(G) \left(\dfrac{\sinh(9\delta/4)}{\sinh(\delta/4)}\right)^2\right\} + 2$, where $c_{1,l}(S_\varepsilon)$ is the constant in the local LII and $m =: \sup_{z \in S} \#\{p \in G : z \in B_S(p, \delta)\} < \infty$.

As a last step, it will be constructed a quasi-isometry between the two nets (G, δ) and (G', δ') associated to (S, ε) and (S, ε') respectively with $0 < \varepsilon < \sinh^{-1} 1$ and $0 < \varepsilon', \tilde{\varepsilon} < \varepsilon$ given by Lemma 1.

Proposition 1. *The nets (G, δ) and (G', δ') are quasi-isometric. More precisely, there is a C'-full (A, B)-quasi-isometry $g : G \longrightarrow G'$, with $A = a \max\left\{\dfrac{\delta'}{\delta}, \dfrac{\delta}{\delta'}\right\}$, $B = 5 + \dfrac{a\delta}{\delta'} + \dfrac{b}{\delta'}$ and $C' = 2 + \dfrac{a(2\delta + C(\varepsilon, \tilde{\varepsilon})) + 2b + c}{\delta'}$ where $C(\varepsilon, \tilde{\varepsilon})$ is the maximum diameter of the connected components of $S_{\tilde{\varepsilon}} \setminus S_\varepsilon$ where $\tilde{\varepsilon}$ is given by Lemma 1.*

Moreover, for any $X \subset G$, $\#X \leq \mu \#g(X)$ where $\mu \leq 13^{\frac{a(2\delta' + b)}{\delta}}$.

In [16, Lemma 4.2] Kanai proves that the LII on graphs is preserved by quasi-isometries; thus an immediate consequence is:

Corollary 1. *For (G, δ) and (G', δ') as above,*

$$(G, \delta) \text{ satisfies the } LII \iff (G', \delta') \text{ satisfies the } LII.$$

Moreover, $c_1(G) \leq \mu 12^{A(B+2C-1)+C-2} c_1(G')$, with μ as in Proposition 1.

Finally, the combination of all previous results will give the proof of Theorem 1.

Proof of Theorem 1. Assume that S' has LII. If ε_0 is the constant in Lemma 3, let us fix $0 < \varepsilon < \varepsilon_0$ and let $0 < \varepsilon', \tilde{\varepsilon} < \min\left\{\varepsilon_0, (12c_1(S'))^{-1}\right\}$ given by Lemma 1. Let (G', δ') be a net associated to (S', ε'). Since S' has LII, by Lemma 3, G' has LII. If (G, δ) is a net associated to (S, ε), then Proposition 1 gives that (G, δ) and (G', δ') are quasi-isometric, and Corollary 1 concludes that (G, δ) has LII. Lemma 4 states that S_ε has LII_ε and, since $0 < \varepsilon < \varepsilon_0$, S has LII.

Moreover, the isoperimetric constant obtained $c_1(S) < \infty$ depends just on $\varepsilon, a, b, c, c_1(S')$. In order to avoid the dependence on ε, it suffices to take $\varepsilon = \varepsilon_0/2$, since ε_0 is a universal constant. □

References

[1] Alvarez, V., Rodríguez, J.M., Yakubovich, V.A., Subadditivity of p-harmonic "measure" on graphs, *Michigan Mathematical Journal* **49** (2001), 47-64.

[2] Ballman, W., Gromov, M., Schroeder, V., Manifolds of Non-positive Curvature, Birkhauser, Boston, 1985.

[3] Bishop, C. J., Jones, P. W., Hausdorff dimension and Kleinian groups, *Acta Math.* **179** (1997), 1-39.

[4] Buser, P., Geometry and Spectra of Compact Riemann Surfaces. Birkhäuser, Boston, 1992.

[5] Cantón, A., Granados, A., Portilla, A., Rodríguez, J. M., Quasi-isometries and isoperimetric inequalities in planar domains. To appear in *J. Math. Soc. Japan.*

[6] Cantón, A., Fernández, J. L., Pestana, D., Rodríguez, J. M., On harmonic functions on trees, *Potential Analysis* **15** (2001), 199-244.

[7] Chavel, I., Isoperimetric inequalities: differential geometric and analytic perspectives. Cambridge University Press, Cambridge, 2001.

[8] Cheeger, J., A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in Analysis*, Princeton University Press, Princeton (1970), 195-199.

[9] Fernández, J. L., Melián, M. V., Bounded geodesics of Riemann surfaces and hyperbolic manifolds, *Trans. Amer. Math. Soc.* **347** (1995), 3533-3549.

[10] Fernández, J. L., Melián, M. V., Escaping geodesics of Riemannian surfaces, *Acta Math.* **187** (2001), 213-236.

[11] Fernández, J. L., Rodríguez, J. M., The exponent of convergence of Riemann surfaces. Bass Riemann surfaces. *Annales Acad. Sci. Fenn. A. I.* **15** (1990), 165-183.

[12] Fernández, J. L., Rodríguez, J. M., Area growth and Green's function of Riemann surfaces, *Arkiv för matematik* **30** (1992), 83-92.

[13] Ghys, E., de la Harpe, P., *Sur les Groupes Hyperboliques d'après Mikhael Gromov.* Progress in Mathematics, Volume 83, Birkhäuser, 1990.

[14] Gromov, M., Hyperbolic groups, in "Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[15] Holopainen, I., Soardi, P. M., p-harmonic functions on graphs and manifolds, *Manuscripta Math.* **94** (1997), 95-110.

[16] Kanai, M., Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Japan* **37** (1985), 391-413.

[17] Kanai, M., Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Japan* **38** (1986), 227-238.

[18] Kanai, M., Analytic inequalities and rough isometries between non-compact Riemannian manifolds. Curvature and Topology of Riemannian manifolds (Katata, 1985). Lecture Notes in Math. **1201**. Springer (1986), 122-137.

[19] Melián, M. V., Rodríguez, J. M., Tourís, E., Escaping geodesics in Riemannian surfaces with pinched variable negative curvature. Submitted.

[20] Pólya, G., Isoperimetric inequalities in mathematical physics. Princeton University Press, 1951.

[21] Portilla, A., Rodríguez, J. M., Tourís, E., Gromov hyperbolicity through decomposition of metric spaces II, *J. Geom. Anal.* **14** (2004), 123-149.

[22] Portilla, A., Tourís, E., A characterization of Gromov hyperbolicity of surfaces with variable negative curvature, *Publ. Mat.* **53** (2009), 83-110.

[23] Rodríguez, J. M., Isoperimetric inequalities and Dirichlet functions of Riemann surfaces, *Publ. Mat.* **38** (1994), 243-253.

[24] Rodríguez, J. M., Two remarks on Riemann surfaces, *Publ. Mat.* **38** (1994), 463-477.

[25] Rodríguez, J. M., Tourís, E., Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica* **23** (2007), 209-228.

[26] Soardi, P. M., Rough isometries and Dirichlet finite harmonic functions on graphs, *Proc. Amer. Math. Soc.* **119** (1993), 1239-1248.

[27] Sullivan, D., Related aspects of positivity in Riemannian geometry, *J. Diff. Geom.* **25** (1987), 327-351.

[28] Tourís, E., Graphs and Gromov hyperbolicity of non-constant negatively curved surfaces, *J. Math. Anal. Appl.* **380** (2011), 865-881.

Characterization of the hyperbolicity in the lexicographic product

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Abstract. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. In this paper we characterize the lexicographic product of two graphs $G_1 \circ G_2$ which are hyperbolic, in terms of G_1 and G_2: the lexicographic product graph $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic, unless if G_1 is a trivial graph (the graph with a single vertex); if G_1 is trivial, then $G_1 \circ G_2$ is hyperbolic if and only if G_2 is hyperbolic. In particular, we obtain that $\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$ if G_1 is not a trivial graph, and we find families of graphs for which the inequalities are attained.

Key words: Lexicographic product graphs; geodesics; Gromov hyperbolicity; infinite graphs.

1 Introduction

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1,16,18]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1,16,18]).

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [20]

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that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension; the same holds for many complex networks, see [15]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [6,7]). Another important application of these spaces is the study of the spread of viruses through on the internet (see [20]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [11,20]).

If X is a metric space we say that the curve $\mathsf{g} : [a, b] \longrightarrow X$ is a *geodesic* if we have $L(\mathsf{g}|_{[t,s]}) = d(\mathsf{g}(t), \mathsf{g}(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$.

In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. Throughout this paper, $G = (V, E)$ denotes a simple connected graph such that every edge has length 1. These properties guarantee that any graph is a geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality, since [5, Theorems 8 and 10] reduce the problem of compute the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

Consider a polygon $J = \{J_1, J_2, \ldots, J_n\}$ with sides $J_j \subseteq X$ in a geodesic metric space X. We say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. Let us denote by $\delta(J)$ the sharp thin constant of J, *i.e.*, $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If x_1, x_2, x_3 are three points in X, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. We say that X is δ-*hyperbolic* if every geodesic triangle in X is δ-thin, and we denote by $\delta(X)$ the sharp hyperbolicity constant of X, *i.e.*, $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. A geodesic *bigon* is a geodesic triangle $\{x_1, x_2, x_3\}$ with $x_2 = x_3$. Therefore, every bigon in a δ-hyperbolic geodesic metric space is δ-thin.

Trivially, any bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. A normed linear space is hyperbolic if and only if it has dimension one. A geodesic space is 0-hyperbolic if and only if it is a metric tree. If a complete Riemannian

manifold is simply connected and their sectional curvatures satisfy $K \leq c$ for some negative constant c, then it is hyperbolic. See the classical references [1,16] in order to find more background and further results.

A main problem in the theory is to characterize in a simple way the hyperbolic graphs. Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic. However, for a general graph deciding whether or not a space is hyperbolic seems an unabordable problem. Therefore, it is interesting to study the hyperbolicity of particular classes of graphs. The papers [3,5,26,30] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, strong product graphs, corona and join of graphs, line graphs, Cartesian product graphs, cubic graphs, tessellation graphs, short graphs, median graphs and k-chordal graphs. In [3,5,26] the authors characterize the hyperbolic product graphs (for strong product, corona and join of graphs, and Cartesian product) in terms of properties of the factor graphs.

The study of lexicographic product graphs is a subject of increasing interest (see, *e.g.*, [20] and the references therein). In this paper we characterize the hyperbolic lexicographic product of two graphs $G_1 \circ G_2$, in terms of G_1 and G_2: if G_1 has at least two vertices, then $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic; besides, if G_1 has a single vertex, then $G_1 \circ G_2$ is hyperbolic if and only if G_2 is hyperbolic (see Theorem 8 and Remark 6). We also prove some sharp inequalities involving $\delta(G_1 \circ G_2)$ and $\delta(G_1)$, see Theorems 1 and 6 (finding families of graphs for which the inequalities are attained, see Theorem 10). Furthermore, we obtain the precise value of the hyperbolicity constant for many lexicographic products (see Theorem 10).

We state now the main results in this work and the lemmas used in their proofs, see [4] for the detailed proofs.

2 Distances in lexicographic products

In order to estimate the hyperbolicity constant of the lexicographic product of two graphs G_1 and G_2, we must obtain bounds on the distances between any two arbitrary points in $G_1 \circ G_2$. Besides, we study the geodesics in $G_1 \circ G_2$, relating them with the geodesics in G_1. The lemmas of this section provide these results.

Definition 1. *Let $G_1 = (V(G_1), E(G_1))$ and $G_2 = (V(G_2), E(G_2))$ be two graphs. The lexicographic product $G_1 \circ G_2$ of G_1 and G_2 has $V(G_1) \times V(G_2)$ as vertex set, so that two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \circ G_2$ are adjacent if either $[u_1, u_2] \in E(G_1)$, or $u_1 = u_2$ and $[v_1, v_2] \in E(G_2)$.*

Remark 1. The Cartesian and the strong product of two graphs are subgraphs of the lexicographic product of two graphs, i.e., $G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \circ G_2$.

Along this work by *trivial graph* we mean a graph having just a single vertex, and we denote it by E_1. An *isomorphism of graphs* G_1 and G_2 is a bijection between the vertex sets of G_1 and G_2, $f : V(G_1) \to V(G_2)$ such that any two vertices u and v of G_1 are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. If an isomorphism exists between G_1 and G_2, then the graphs are called *isomorphic* and we write $G_1 \simeq G_2$.

Remark 2. Let G be any graph. Then $G \circ E_1 \simeq G$ and $E_1 \circ G \simeq G$.

In what follows we denote by π the projection $\pi : G_1 \circ G_2 \to G_1$. The following result allows to compute the distance between any two vertices of $G_1 \circ G_2$.

Lemma 1. *Let G_1 be a non-trivial graph and G_2 any graph and (u, v), (u', v') two vertices in $G_1 \circ G_2$. Then*

$$d_{G_1 \circ G_2}((u, v), (u', v')) = \begin{cases} \min\{2, d_{G_2}(v, v')\}, & \text{if } u = u', \\ d_{G_1}(u, u'), & \text{if } u \neq u'. \end{cases}$$

Let X be a metric space, Y a non-empty subset of X and ε a positive number. We call *ε-neighborhood* of Y in X, denoted by $\mathcal{V}_\varepsilon(Y)$ to the set $\{x \in X : d_X(x, Y) \leq \varepsilon\}$.

Lemma 2. *Let G_1 be a non-trivial graph and G_2 any graph. Then $G_1 \circ G_2 \subseteq \mathcal{V}_{3/2}(G_1 \circ \{v\})$ for every $v \in V(G_2)$.*

Lemma 3. *Let y_1, y_2 be any points in G_2 with $d_{G_2}(y_1, y_2) \leq 5/2$ and x_0 a fixed vertex in G_1. Then $\mathsf{g} := \{x_0\} \times [y_1 y_2]$ is a geodesic in $G_1 \circ G_2$ joining the points (x_0, y_1) and (x_0, y_2).*

Corollary 1. *Let G_1 be a non-trivial graph and G_2 any graph, y_1, y_2 any points in G_2 with $d_{G_2}(y_1, y_2) > 3$ and x_0 a fixed vertex in G_1. Then $\{x_0\} \times [y_1 y_2]$ is not a geodesic in $G_1 \circ G_2$.*

Remark 3. Let y_1, y_2 be two midpoints in any graph G_2 with $d_{G_2}(y_1, y_2) = 3$ and x_0 a fixed vertex in any graph G_1. Then $\{x_0\} \times [y_1 y_2]$ is a geodesic in $G_1 \circ G_2$ joining (x_0, y_1) and (x_0, y_2).

Lemma 4. *Let G_1 be a non-trivial graph and G_2 be any graph. If g is a geodesic in $G_1 \circ G_2$ joining x and y with $L(\mathsf{g}) > 3$, then $\pi(\mathsf{g})$ contains at least three vertices in G_1.*

Furthermore, if σ is a path in $G_1 \circ G_2$ joining x and y, then $\pi(\sigma)$ contains at least three vertices in G_1.

Lemma 5. *Let G_1 be a non-trivial graph and G_2 be any graph. Consider a geodesic g in $G_1 \circ G_2$ joining x and y. If $L(\mathsf{g}) > 3$, then $\pi(\mathsf{g})$ is a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$. Besides, if $L(\mathsf{g}) = 3$ then $\pi(\mathsf{g})$ contains a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$.*

Remark 4. Let $\mathtt{g}$ be a geodesic in $G_1 \circ G_2$ joining x and y. If $L(\mathtt{g}) = 3$ and $\pi(\mathtt{g})$ is not a geodesic in G_1 joining $\pi(x)$ and $\pi(y)$, then x, y are midpoints of edges, $\pi(x) = \pi(y) \in V(G_1)$ and $\operatorname{diam}(\pi(\mathtt{g})) = 1$.

Corollary 2. *Let* $\mathtt{g}$ *be a geodesic in* $G_1 \circ G_2$ *joining* x *and* y. *If* $\pi(\mathtt{g})$ *is not a geodesic in* G_1 *joining* $\pi(x)$ *and* $\pi(y)$, *then* $\operatorname{diam}\big(\pi(\mathtt{g})\big) < 3$.

3 Hiperbolicity in lexicographic products

Some bounds for the hyperbolicity constant of the lexicographic product of two graphs are studied in this section. These bounds allow to prove Theorem 8, which characterizes the hyperbolic lexicographic products of two graphs.

We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$. The following result which appears in [31, Lemma 5] will be useful.

Lemma 6. *If* Γ *is an isometric subgraph of* G, *then* $\delta(\Gamma) \leq \delta(G)$.

The next theorem shows an important qualitative result: if G_1 is not hyperbolic then $G_1 \circ G_2$ is not hyperbolic.

Theorem 1. *Let* G_1 *and* G_2 *two graphs, then* $\delta(G_1) \leq \delta(G_1 \circ G_2)$.

In order to prove Theorem 1 it suffices to show that G_1 is an isometric subgraph of $G_1 \circ G_2$ and to apply Lemma 6.

Proposition 1. *Let* G_1 *be a non-trivial graph and* G_2 *any graph. Consider isometric subgraphs* Γ_1, Γ_2 *of* G_1, G_2, *respectively, with* Γ_1 *non-trivial. Then* $\Gamma_1 \circ \Gamma_2$ *is an isometric subgraph to* $G_1 \circ G_2$.

Note that taking Γ_1 as a trivial graph, $\Gamma_1 \circ \Gamma_2$ is not an isometric subgraph to $G_1 \circ G_2$ if $\operatorname{diam} V(\Gamma_2) \geq 3$.

In order to prove Proposition 1 it suffices to show that $d_{\Gamma_1 \circ \Gamma_2}(x, y) \leq d_{G_1 \circ G_2}(x, y)$ for every $x, y \in \Gamma_1 \circ \Gamma_2$.

Theorem 2. *Let* G_1 *be a non-trivial graph and* G_2 *any graph. Then*

$$\delta(G_1 \circ G_2) = \max\{\delta(\Gamma_1 \circ \Gamma_2) : \Gamma_i \text{ is isometric to } G_i \text{ for } i = 1, 2 \text{ and } \Gamma_1 \text{ non-trivial}\}.$$

Theorem 3. *If* G_1 *and* G_2 *are non-trivial graphs, then* $\delta(G_1 \circ G_2) \geq 1$.

In order to prove Theorem 3 it suffices to show that $P_2 \circ P_2$ is an isometric subgraph of $G_1 \circ G_2$ and to apply Theorem 2.

Theorem 4. *Let* G_2 *be any non-trivial graph and* G_1 *any graph. If* $\operatorname{diam} V(G_1) = 2$, *then* $\delta(G_1 \circ G_2) \geq 5/4$. *If* $\operatorname{diam} V(G_1) \geq 3$, *then* $\delta(G_1 \circ G_2) \geq 3/2$.

In order to prove Theorem 4 it suffices to show that $P_3 \circ P_2$ and $P_4 \circ P_2$ are isometric subgraphs of $G_1 \circ G_2$ when $\operatorname{diam} V(G_1) = 2$ and $\operatorname{diam} V(G_1) \geq 3$, respectively. Then Theorem 2 gives the result.

Theorem 5. *If G_1 is any non-trivial graph and G_2 is any graph with* $\operatorname{diam} G_2 > 2$*, then $\delta(G_1 \circ G_2) \geq 5/4$.*

In order to prove Theorem 5 it suffices to show that $P_2 \circ P_4$ is an isometric subgraph of $G_1 \circ G_2$ and to apply Theorem 2.

Theorem 6. *Let G_1 be a non-trivial graph and G_2 any graph. Then we have $\delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2$.*

Theorem 10 shows that the equality in Theorem 6 is attained.

Remark 5. Let G_1 be any hyperbolic graph which is not a tree and let G_2 be any graph. The argument in the proof of Theorem 6 gives that if $\delta(G_1 \circ G_2) = \delta(G_1) + 3/2$ then there is a geodesic triangle $T = \{x, y, z\}$ with $x, y, z \in J(G_1 \circ G_2)$ and a midpoint $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = \delta(G_1) + 3/2$. Besides, $d_{G_1 \circ \{w\}}(V_p, [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]) = \delta(G_1)$ and the distance is attained in a vertex $\alpha \in [\pi(x)\pi(z)] \cup [\pi(z)\pi(y)]$.

We obtain the following consequences of Theorem 1 and Theorem 6.

Theorem 7. *Let G_1 be a non-trivial graph and G_2 any graph. Then*

$$\delta(G_1) \leq \delta(G_1 \circ G_2) \leq \delta(G_1) + 3/2.$$

Corollary 3. *If G_1 is any infinite tree and G_2 is any non-trivial graph, then $\delta(G_1 \circ G_2) = 3/2$.*

Theorem 8. *Let G_1 be a non-trivial graph and G_2 be any graph. The lexicographic product $G_1 \circ G_2$ is hyperbolic if and only if G_1 is hyperbolic.*

Remark 6. For any graph G and the trivial graph E_1, the lexicographic product graph $E_1 \circ G$ is hyperbolic if and only if G is hyperbolic, since $\delta(E_1 \circ G) = \delta(G)$. This trivial result completes the characterization of hyperbolic lexicographic products.

The following result allows to attain the bound in Theorem 6.

Theorem 9. *Let G_1 be any hyperbolic graph and let G_2 be any graph. If $\delta(G_1 \circ G_2) = \delta(G_1) + 3/2$, then G_1 is a tree, G_2 is a non-trivial graph and $\delta(G_1 \circ G_2) = 3/2$.*

Theorem 10 below is a converse of Theorem 9; furthermore, it provides the exact value of the hyperbolicity constant of the lexicographic product of many trees and graphs. We need some lemmas.

Lemma 7. *If G_1 is a tree with $1 \leq \operatorname{diam} G_1 \leq 2$, G_2 is any graph and $\delta(G_1 \circ G_2) = 3/2$, then there is a geodesic triangle $T = \{x, y, z\}$ in $G_1 \circ G_2$ that is a cycle contained in $\{v_0\} \circ G_2$ for some $v_0 \in V(G_1)$ with $x, y, z \in J(\{v_0\} \circ G_2)$ and a vertex $p \in [xy]$ such that $d_{G_1 \circ G_2}(p, [xz] \cup [zy]) = d_{G_1 \circ G_2}(p, x) = d_{G_1 \circ G_2}(p, y) = 3/2$.*

In the following results we use the definition of the class of graphs $\mathcal{F}$ given in [4].

Lemma 8. *Let G be any graph. Assume that there is a geodesic triangle $T = \{x, y, z\}$ in G that is a cycle with x, y midpoints, $L([xy]), L([yz]), L([zx]) \leq 3$ and $\delta(T) = 3/2 = d_G(p, [yz] \cup [zx])$ for some $p \in [xy]$. Then $G \in \mathcal{F}$.*

Theorem 10. *Let G_1 be a tree and G_2 any non-trivial graph.*

(1) *If $\operatorname{diam} G_1 \geq 3$, then $\delta(G_1 \circ G_2) = 3/2$.*
(2) *If $1 \leq \operatorname{diam} G_1 \leq 2$, then $\delta(G_1 \circ G_2) = 3/2$ if and only if $G_2 \in \mathcal{F}$.*
(3) *If G_1 is trivial, then $\delta(G_1 \circ G_2) = 3/2$ if and only if $\delta(G_2) = 3/2$.*

This result allows to compute, in a simple way, the hyperbolicity constant of the lexicographic product of any tree and any graph.

Theorem 11. *Let G_1 be any tree and G_2 any graph. Then*

$$\delta(G_1 \circ G_2) = \begin{cases} \delta(G_2), \text{ if } G_1 \simeq E_1, \\ 0, \quad \text{if } G_2 \simeq E_1, \\ 1, \quad \text{if } \operatorname{diam} G_1 = 1 \text{ and } 1 \leq \operatorname{diam} G_2 \leq 2, \\ 5/4, \quad \text{if } \operatorname{diam} G_1 = 1 \text{ and } \operatorname{diam} G_2 > 2 \text{ and } G_2 \notin \mathcal{F}, \\ 5/4, \quad \text{if } \operatorname{diam} G_1 = 2 \text{ and } \operatorname{diam} G_2 \geq 1 \text{ and } G_2 \notin \mathcal{F}, \\ 3/2, \quad \text{if } 1 \leq \operatorname{diam} G_1 \leq 2 \text{ and } G_2 \in \mathcal{F}, \\ 3/2, \quad \text{if } \operatorname{diam} G_1 \geq 3 \text{ and } \operatorname{diam} G_2 \geq 1. \end{cases}$$

References

[1] Alonso, J., Brady, T., Cooper, D., Delzant, T., Ferlini, V., Lustig, M., Mihalik, M., Shapiro, M. and Short, H., *Notes on word hyperbolic groups, in: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), Group Theory from a Geometrical Viewpoint, World Scientific, Singapore,1992.*

[2] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M., Gromov hyperbolic graphs, *Discr. Math.* **313** (2013), 1575-1585.

[3] Carballosa, W., Casablanca, R. M., de la Cruz, A. and Rodríguez, J. M., Gromov hyperbolicity in strong product graphs, *Electr. J. Comb.* **20**(3) (2013), P2.

[4] Carballosa, W., de la Cruz, A. and Rodríguez, J. M., Gromov hyperbolicity in lexicographic product of graphs. Submitted. *Preprint in http://gama.uc3m.es/index.php/jomaro.html*

[5] Carballosa, W., Rodríguez, J. M. and Sigarreta, J. M., Hyperbolicity in the corona and join of graphs. Submitted. *Preprint in http://gama.uc3m.es/index.php/jomaro.html*

[6] Chen, B., Yau, S.-T. and Yeh, Y.-N., Graph homotopy and Graham homotopy, *Discrete Math.* **241** (2001), 153-170.

[7] Chepoi, V. and Estellon, B., Packing and covering δ-hyperbolicspaces by balls, *APPROX-RANDOM* 2007 pp. 59-73.

[8] Eppstein, D., Squarepants in a tree: sum of subtree clustering and hyperbolic pants decomposition, SODA'2007, *Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, pp. 29-38.

[9] Ghys, E. and de la Harpe, P., Sur les Groupes Hyperboliques d'après Mikhael Gromov. *Progress in Mathematics 83, Birkhäuser Boston Inc., Boston, MA, 1990.*

[10] Gromov, M., Hyperbolic groups, in "Essays in group theory". *Edited by S. M. Gersten, M. S. R. I. Publ.* **8**. Springer, 1987, 75-263.

[11] Imrich, W. and Klavzar, S., Product Graphs: Structure and Recognition, *Wiley Series in Discrete Mathematics and Optimization*, (2000).

[12] Jonckheere, E. and Lohsoonthorn, P., A hyperbolic geometry approach to multipath routing, Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002), *Lisbon, Portugal, July 2002. FA5-1.*

[13] Jonckheere, E. A., Contrôle du traffic sur les réseaux à géométrie hyperbolique– Vers une théorie géométrique de la sécurité l'acheminement de l'information, *J. Europ. Syst. Autom.* **8** (2002), 45-60.

[14] Krioukov, D., Papadopoulos, F., Kitsak, M., Vahdat, A. and Boguñá, M., Hyperbolic geometry of complex networks, *Physical Review E* **82**, 036106 (2010).

[15] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M., Gromov hyperbolicity in cartesian product graphs, *Proc. Indian Acad. Sci. Math. Sci.* **120** (2010), 1-17.

[16] Michel, J., Rodríguez, J. M., Sigarreta, J.M., and Villeta, M., Hyperbolicity and parameters of graphs. *Ars Combinatoria* **100** (2011), 43-63.

[17] Oshika, K., Discrete groups, *AMS Bookstore, 2002.*

[18] Rodríguez, J. M., Characterization of Gromov hyperbolic short graphs, *Acta Math. Sinica* **30** (2014), 197-212.

[19] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M. and Villeta, M., On the hyperbolicity constant in graphs, *Discr. Math.* **311** (2011), 211-219.

[20] Saputro, S. W., Simanjuntak, R., Uttunggadewa, S., Assiyatun, H., Baskoro, E. T., Salman, A. N. M. and Baca, M., The metric dimension of the lexicographic product of graphs, *Discrete Math.* **313**(9) (2013), 1045-1051.

[21] Shavitt, Y., Tankel, T., On internet embedding in hyperbolic spaces for overlay construction and distance estimation, *INFOCOM 2004*.

Distortion of the hyperbolicity constant in minor graphs

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Abstract. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. In the context of graphs, to remove and to contract an edge of a graph are natural transformations. The main aim in this work is to obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ (respectively, G/e) obtained from the graph G by deleting (respectively, contracting) an arbitrary edge e from it. A graph H is a minor of a graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices. Since we deal with connected graphs, our work provides information about the hyperbolicity constant of minor graphs.

Key words: Gromov hyperbolicity, edge contraction, removing an edge, minor graphs, infinite graphs, geodesics.

1 Introduction

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [16,18]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [16,18]). The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [18]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, e.g., [18]); indeed, hyperbolic groups are strongly geodesically automatic, i.e., there is an automatic structure on the group [5].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [20]

that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension; the same holds for many complex networks, see [15]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [6,7,8,14]). Another important application of these spaces is the study of the spread of viruses through on the internet (see [20,21]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [11,20,21,17]).

We say that the curve γ in a metric space X is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u, v]$.

In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. Throughout this paper, $G = (V, E)$ denotes a simple graph such that every edge has length 1.

If X is a geodesic metric space and x, y, z are in X, a geodesic triangle $T = \{x, y, z\}$ is the union of the three geodesics $[xy]$, $[yz]$ and $[zx]$ in X. The space X is δ-hyperbolic (in the Gromov sense) if any side of T is contained in the δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \inf\{\delta : X \text{ is } \delta\text{-hyperbolic}\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$.

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices. Minor graphs is an interesting class of graphs. This topic started with one well-known result on planar graphs, the Wagner's theorem, which says that a graph is planar if and only if it does not include as a minor the complete graph K_5 nor the complete bipartite graph $K_{3,3}$, see, e.g., [21].

In the context of graphs, to remove and to contract an edge of a graph are natural transformations. The main aim in this work is to obtain quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ (respectively, G/e) obtained from the graph G by deleting (respectively, contracting) an arbitrary edge e from it. Since we deal with connected

graphs, our work provides information about the hyperbolicity constant of minor graphs.

Frequently, natural transformations preserve many properties. For instance, in [3] the authors obtain a monotony property for the hyperbolicity constant under a non-trivial transformation (the line graph $\mathcal{L}(G)$ of a graph G), i.e., $\delta(G) \leq \delta\big(\mathcal{L}(G)\big)$. We show here that there is not a monotony property for the hyperbolicity constant under these natural operations of graphs, i.e., the hyperbolicity constant of $G\backslash e$ (respectively, G/e) can be smaller or larger than $\delta(G)$, see Examples 1 and 3.

2 Hyperbolicity and edge contraction

In this section we study the distortion of the hyperbolicity constant by contraction of one edge in any graph. If G is a graph and $e := [A, B] \in E(G)$, we denote by G/e the graph obtained by contracting the edge e from it.

Denote by V_e the vertex in G/e obtained by identifying A and B in G. Note that any vertex $V \in V(G) \setminus \{A, B\}$ can be seen as a vertex in $V(G/e)$. Also we can write any edge in $E(G/e)$ in terms of its endpoints, but we write V_e instead of A or B.

We define the map $h : G \to G/e$ in the following way: if x belongs to the edge e, then $h(x) := V_e$; if $x \in G$ does not belong to e, then $h(x)$ is the "natural inclusion map". Clearly h is onto, i.e., $h(G) = G/e$. Besides, the restriction of h to the complement of e in G is an injective map.

Lemma 1. *Let G be any graph and $e \in E(G)$. Then*

$$d_{G/e}\big(h(x), h(y)\big) \leq d_G(x, y) \leq d_{G/e}\big(h(x), h(y)\big) + 3/2, \quad \forall x, y \in G. \tag{1}$$

Furthermore, if x, y are not contained in an induced cycle C_3 of G containing e, then

$$d_{G/e}\big(h(x), h(y)\big) \leq d_G(x, y) \leq d_{G/e}\big(h(x), h(y)\big) + 1. \tag{2}$$

Proof. Proof (Sketch) Fix $x, y \in G$. Let $[xy]_G$ be a geodesic in G joining x and y. Clearly, $h([xy]_G)$ is a path joining $h(x)$ and $h(y)$ with length at most $L([xy]_G)$, thus, we obtain $d_{G/e}\big(h(x), h(y)\big) \leq d_G(x, y)$. The rest of the proof is straightforward analyzing separately the cases of $x, y \in G$. ∎

This lemma has the following consequence about the continuity of h.

Proposition 1. *The map h is an 1-Lipschitz continuous function.*

Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is said to be an (α, β)-*quasi-isometric embedding*, with constants $\alpha \geq 1,\ \beta \geq 0$ if for every $x, y \in X$:

$$\alpha^{-1}d_X(x,y)-\beta\leq d_Y(f(x),f(y))\leq \alpha d_X(x,y)+\beta.$$

The function f is ε-*full* if for each $y\in Y$ there exists $x\in X$ with $d_Y(f(x),y)\leq \varepsilon$.

A map $f:X\longrightarrow Y$ is said to be a *quasi-isometry*, if there exist constants $\alpha\geq 1,\ \beta,\varepsilon\geq 0$ such that f is an ε-full (α,β)-quasi-isometric embedding.

A fundamental property of hyperbolic spaces is the following:

Theorem 1 (Invariance of hyperbolicity). *Let $f:X\longrightarrow Y$ be an (α,β)-quasi-isometric embedding between the geodesic metric spaces X and Y. If Y is hyperbolic, then X is hyperbolic. Furthermore, if Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which just depends on α,β,δ.*

Besides, if f is ϵ-full for some $\epsilon\geq 0$ (a quasi-isometry), then X is hyperbolic if and only if Y is hyperbolic. Furthermore, if X is δ-hyperbolic, then Y is δ'-hyperbolic, where δ' is a constant which just depends on $\alpha,\beta,\delta,\epsilon$.

Using the *Invariance of hyperbolicity* (see Theorem 1), we can obtain the main qualitative aim in this section.

Theorem 2. *Let G be any graph and e one of its edges. Then G is hyperbolic if and only if G/e is hyperbolic. Furthermore, if G (respectively, G/e) is δ-hyperbolic, then G/e (respectively, G) is δ'-hyperbolic, where δ' is a constant which just depends on δ.*

Proof. Lemma 1 gives that h is a 0-full $(1,3/2)$-quasi-isometry from G on G/e and, consequently, by Theorem 1 we obtain the result. ∎

Intuitively, it is expected that the edge contraction is an operation which preserves the hyperbolicity monotonically, ie, the hyperbolicity constant always decrease by edge contraction. However, the following example provides some family of graphs in which hyperbolicity constant increases by contracting certain edge. The *girth* of a graph G is the infimum of the lengths of the cycles in G.

Example 1. In [31, Theorem 11], authors obtain the precise value of hyperbolicity constant of wheel graph with n vertices W_n, i.e., $\delta(W_4)=\delta(W_5)=1$, $\delta(W_n)=3/2$ for every $7\leq n\leq 10$, and $\delta(W_n)=5/4$ for $n=6$ and for every $n\geq 11$. Note that we can obtain W_n from W_{n+1} by edge contraction, so, we have that $\delta(W_{11})=5/4$ and $\delta(W_{10})=3/2$.

Other aim in this work is to obtain quantitative relations between $\delta(G/e)$ and $\delta(G)$. Since the proofs of these inequalities are long, in order to make the arguments more transparent, we collect some results in technical lemmas. Denote by $J(G)$ the set of vertices and midpoints of edges in G.

Lemma 2. *Let G be any graph and $e = [A, B] \in E(G)$. Suppose that for some $x, y \in G \setminus \{e\}$ there are two geodesic γ_G and $\gamma_{G/e}$ in G and G/e, respectively, joining x, y and $h(x), h(y)$ such that $L(\gamma_G) = L(\gamma_{G/e})$, $L(h(\gamma_G)) = L(\gamma_G)$, $V_e \in \gamma_{G/e}$ and $h^{-1}\big(\gamma_{G/e} \setminus \{V_e\}\big)$ is separated in two disjoint curves at distance 1 (i.e., if split the vertex V_e to e then it is not a path). Then we have*

$$d_{G/e}(h(\alpha'), \gamma_{G/e}) \leq \delta(G) \quad \forall\ \alpha' \in \gamma_G \tag{3}$$

and

$$d_{G/e}\big(\alpha, h(\gamma_G)\big) \leq 2\delta(G) \quad \forall\ \alpha \in \gamma_{G/e}. \tag{4}$$

Lemma 3. *Let G be any graph and $e \in E(G)$. Let $[xy]_G$ be a geodesic in G joining $x, y \in J(G)$. Suppose that $h([xy]_G)$ is not a geodesic in G/e, besides, suppose that if e is contained in an induced C_3 of G then $h([xy]_G)$ contain at most one edge of C_3 (i.e., $L([xy]_G \cap C_3) \leq 1$). Consider $[h(x)h(y)]_{G/e}$ a geodesic in G/e. Then we have*

$$d_G(h^{-1}(\alpha'), [xy]_G) \leq \delta(G/e), \quad \forall\ \alpha' \in [h(x)h(y)]_{G/e} \tag{5}$$

and

$$d_G\big(\alpha, h^{-1}([h(x)h(y)]_{G/e})\big) \leq 2\delta(G/e), \quad \forall\ \alpha \in [xy]_G. \tag{6}$$

For the proof of Lemma 2 (Lemma 3, respectively) we use an auxiliar geodesic triangle in G (G/e, respectively). We use known technics on geodesic metric spaces in order to obtain the Hausdorff's distance between a geodesic and a quasigeodesic, for proof details see [4].

Remark 1. Let G be any graph and $e \in E(G)$. Let $T_G = \{\gamma_1, \gamma_2, \gamma_3\}$ be a geodesic triangle in G. Then at least one of $h(\gamma_1), h(\gamma_2), h(\gamma_3)$ is a geodesic in G/e, otherwise there are three different points in G with geodesics between them containing one common edge (the edge e).

The following result in [1] will be useful. As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

Theorem 3. *[1, Theorem 2.7] For any hyperbolic graph G, there exists a geodesic triangle $T = \{x, y, z\}$ that is a cycle with $x, y, z \in J(G)$ and $\delta(T) = \delta(G)$.*

Previous results allow to obtain the quantitative version of Theorem 2.

Theorem 4. *Let G be any graph and e one of its edges. Then*

$$\frac{1}{3}\delta(G/e) \leq \delta(G) \leq 4\delta(G/e) + 1. \tag{7}$$

Proof. Proof (Sketch) The proof follows using strongly Lemmas 2 and 3 making casuistic analysis on the possible structure of an arbitrary geodesic triangle of G by Remark 1. It is straightforward despite being long. The details of the proof are in [4]. ∎

Bounds in Theorem 4 are sharp, the following example provides it.

Example 2. Let G be a diamond graph, i.e., the complete graph with 4 vertices K_4 without one edge. Contract the edge e which is a cordal in G, so, we obtain G/e as isomorphic graph to P_3 (path graph with 3 vertices). Clearly, we have $\delta(G) = 1$ and $\delta(G/e) = 0$. This fact allows to obtain many graphs G which attain the upper bound in Theorem 4. On the other hand, if T any tree and e any edge of T then $\delta(T) = \delta(T/e) = 0$, and so, the lower bound in Theorem 4 is attained.

3 Removing an edge

In this section we deal with one of the main problems in this work to obtain quantitative relations between $\delta(G \setminus e)$ and $\delta(G)$, where e is any edge of G. As usual, we define the graph $G \setminus e$ as the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.

The following example provides some graphs in which hyperbolicity constant increases and decreases respectively by removing certain edge.

Example 3. Consider G as a cycle graph C_n with $n \geq 3$ vertices. Clearly, $G \setminus e$ is isomorphic to a path graph P_{n-1}. Hence, we have $\delta(C_n) = n/4$ and $\delta(P_{n-1}) = 0$. So, it is a decrease example which is non-uniform bounded. Let $C_{a,b,c}$ be the graph with two vertices and three disjoint paths joining them with lengths $a \leq b \leq c$. Consider an edge e of $C_{a,b,c}$ contained in a path with length a. Then, it is known that $\delta(C_{a,b,c} \setminus e) = (c + b)/4$ and $\delta(C_{a,b,c}) = (c + \min\{b, 3a\})/4$, see [25, Lemma 19]. Thus, choice $3a < b$ provides an increase example.

Since the proofs of these inequalities are long and technical, in order to make the arguments more transparent, we collect some results we need along the proof in technical lemmas. For proof details of the following lemmas see [10].

Lemma 4. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u \in \Gamma_{G\setminus e}\ ,\ \exists\ u' \in \Gamma_G \setminus e\ :\ d_{G\setminus e}(u, u') \leq 2\delta(G). \tag{8}$$

Lemma 5. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u' \in \Gamma_G\ ,\ \exists\ u \in \Gamma_{G\setminus e}\ :\ d_G(u', u) \leq \delta(G). \tag{9}$$

Furthermore,

$$\forall\ u' \in \Gamma_G \setminus e\ ,\ \exists\ u \in \Gamma_{G\setminus e}\ :\ d_{G\setminus e}(u', u) \leq 2\delta(G). \tag{10}$$

This previous lemmas allow to obtain the quantitative information about the distortion of the hyperbolicity constant of the graph $G \setminus e$ obtained from the graph G by removing an arbitrary edge e from it.

Theorem 5. *Let G be any graph and $e \in E(G)$ with $G \setminus e$ connected. The following inequality holds*

$$\delta(G \setminus e) \leq 5\delta(G). \tag{11}$$

Proof. Proof (Sketch) The proof follows using strongly Lemmas 4 and 5 making casuistic analysis on the possible structure of an arbitrary geodesic triangle of $G \setminus e$. The details of the proof are in [4]. ∎

In order to obtain the next theorem we collect some results we need along the proof in technical lemmas. The details of the proof are in [10].

Lemma 6. *Let G be any graph and $e = [A, B] \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u \in \Gamma_{G\setminus e}\ ,\ \exists\ u' \in \Gamma_G \setminus e\ :\ d_G(u, u') \leq 2\delta(G \setminus e) + \frac{1}{2} d_{G\setminus e}(A, B). \tag{12}$$

Lemma 7. *Let G be any graph and $e = [A, B] \in E(G)$ with $G \setminus e$ connected. For all $x, y \in G \setminus e$, if $\Gamma_G = [xy]_G$ is a geodesic in G containing e and $\Gamma_{G\setminus e} = [xy]_{G\setminus e}$ is a geodesic in $G \setminus e$, then*

$$\forall\ u' \in \Gamma_G\ ,\ \exists\ u \in \Gamma_{G\setminus e}\ :\ d_G(u', u) \leq 5\delta(G \setminus e) + d_{G\setminus e}(A, B). \tag{13}$$

Finally, we can prove a kind of converse of Theorem 5.

Theorem 6. *Let G be any graph and $e = [A, B] \in E(G)$ with $G \setminus e$ connected. Then*

$$\max\left\{\frac{1}{5}\delta(G \setminus e), \frac{1}{4}\big(d_{G\setminus e}(A, B) + 1\big)\right\} \leq \delta(G) \leq 6\delta(G \setminus e) + d_{G\setminus e}(A, B). \tag{14}$$

Proof. Proof (Sketch) The proof follows using strongly Lemmas 6 and 7 making casuistic analysis on the possible structure of an arbitrary geodesic triangle of G. The details of the proof are in [4]. ∎

Finally, Theorems 4 and 6 give the following result.

Theorem 7. *Let G be any graph and G' a minor graph of G obtained by a finite operations (remove or contract edge). Then G is hyperbolic if and only if G' is hyperbolic.*

References

[1] Bermudo, S., J. M. Rodríguez, and J. M. Sigarreta, *Computing the hyperbolicity constant*, Comput. Math. Appl. **62** (2011), 4592-4595.

[2] Carballosa, W., D. Pestana, J. M. Rodríguez, and J. M. Sigarreta, *Distortion of the hyperbolicity constant of a graph*, Electr. J. Comb. **19** (2012), P67.

[3] Carballosa, W., J. M. Rodríguez, and J. M. Sigarreta, *New inequalities on the hyperbolity constant of line graphs*, to appear in Ars Combinatoria. Preprint in http://gama.uc3m.es/index.php/jomaro.html

[4] Carballosa, W., J. M. Rodríguez, and J. M. Sigarreta, *Gromov hyperbolicity of minor graphs*, Submitted. Preprint in http://gama.uc3m.es/index.php/jomaro.html

[5] Charney, R., *Artin groups of finite type are biautomatic*, Math. Ann. **292** (1992), 671-683.

[6] Chepoi, V. and B. Estellon, *Packing and covering δ-hyperbolic spaces by balls*, APPROX-RANDOM 2007 pp. 59-73.

[7] Eppstein, D., *Squarepants in a tree: sum of subtree clustering and hyperbolic pants decomposition*, SODA'2007, Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms, pp. 29-38.

[8] Gavoille, C. and O. Ly, *Distance labeling in hyperbolic graphs*, In ISAAC 2005, pp. 171-179.

[9] Ghys, E. and P. de la Harpe, "Sur les Groupes Hyperboliques d'après Mikhael Gromov". Progress in Mathematics 83, Birkhäuser Boston Inc., Boston, MA, 1990.

[10] Gromov, M., "Hyperbolic groups, in Essays in group theory". Edited by S. M. Gersten, M. S. R. I. Publ. **8**. Springer, 1987, 75-263.

[11] Jonckheere, E. and P. Lohsoonthorn, *A hyperbolic geometry approach to multipath routing*, Proceedings of the 10th Mediterranean Conference on Control and Automation (MED 2002), Lisbon, Portugal, July 2002. FA5-1.

[12] Jonckheere, E. A., *Contrôle du traffic sur les réseaux à géométrie hyperbolique– Vers une théorie géométrique de la sécurité l'acheminement de l'information*, J. Europ. Syst. Autom. **8** (2002), 45-60.

[13] Jonckheere, E. A. and P. Lohsoonthorn, *Geometry of network security*, Amer. Control Conf. **ACC** (2004), 111-151.

[14] Krauthgamer, R. and J. R. Lee, *Algorithms on negatively curved spaces*, FOCS 2006.

[15] Krioukov, D., F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Boguñá, *Hyperbolic geometry of complex networks*, Physical Review E **82**, 036106 (2010).

[16] Michel, J., J. M. Rodríguez, J. M. Sigarreta, and M. Villeta, *Hyperbolicity and parameters of graphs*, Ars Comb. **100** (2011), 43-63.

[17] Narayan, O. and I. Saniee, *Large-scale curvature of networks*, Physical Review E **84**, 066108 (2011).

[18] Oshika, K., "Discrete groups", AMS Bookstore, 2002.

[19] Rodríguez, J. M., J. M. Sigarreta, J.-M. Vilaire, and M. Villeta, *On the hyperbolicity constant in graphs*, Discr. Math. **311** (2011), 211-219.

[20] Shavitt, Y.,T. Tankel, *On internet embedding in hyperbolic spaces for overlay construction and distance estimation*, INFOCOM 2004.

[21] Wagner, K., *Über eine Eigenschaft der ebenen Komplexe*, Mathematische Annalen **114**(1) (1937), 570-590.

Distinctive power of the alliance polynomial for regular graphs

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Abstract. The alliance polynomial of a graph G with order n and maximum degree Δ is the polynomial $A(G;x) = \sum_{k=-\Delta}^{\Delta} A_k(G)\, x^{n+k}$, where $A_k(G)$ is the number of exact defensive k-alliances in G. We obtain some properties of $A(G;x)$ and its coefficients for regular graphs. In particular, we characterize the degree of regular graphs by the number of non-zero coefficients of their alliance polynomial. Besides, we prove that the family of alliance polynomials of Δ-regular graphs with small degree is a very special one, since it does not contain alliance polynomials of graphs which are not Δ-regular.

Key words: Finite Graphs; Regular Graphs; Defensive Alliances; Alliance Polynomials.

1 Introduction

Some parameters of a graph allow to define polynomials on the graph, for instance, the parameters associated to matching sets [8,9], independent sets [4,16,11], domination sets [1,3,2], chromatic numbers [30,14] and many others. In [6], the authors use the exact index of alliance (introduced in [5]) in order to define the exact alliance polynomial of a graph. The main aim of this work is to obtain further results about the alliance polynomial of regular graphs, since they are a very interesting class of graphs.

We begin by stating the terminology. Throughout this paper, $G = (V, E)$ denotes a simple graph (no necessarily conneted) of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices u and v by $u \sim v$. For a nonempty set $X \subseteq V$ and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors v has in X: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of v in X will be denoted by $\delta_X(v) = |N_X(v)|$. We denote by δ and Δ the minimum and maximum degree of G, respectively. If G is a regular graph with degree Δ we say that G is

Δ-regular. The subgraph induced by $S \subset V$ will be denoted by $\langle S \rangle$ and the complement of the set $S \in V$ will be denoted by $\overline{S}$.

A nonempty set $S \subseteq V$ with $\langle S \rangle$ connected is a *defensive k-alliance* in $G = (V, E)$, $k \in [-\Delta, \Delta] \cap \mathbb{Z}$, if for every $v \in S$,

$$\delta_S(v) \geq \delta_{\bar{S}}(v) + k. \tag{1}$$

We consider the value of k in the set of integers $\mathcal{K} := [-\Delta, \Delta] \cap \mathbb{Z}$. For some graphs, there are some values of $k \in \mathcal{K}$ without defensive k-alliances. For instance, for $k \geq 2$ do not exist defensive k-alliances in the star graph S_n with n vertices. Besides, $V(G)$ is a defensive δ-alliance in G if G is connected. Note that for any S with $\langle S \rangle$ connected there exists some $k \in \mathcal{K}$ such that it is a defensive k-alliance in G.

We denote by $k_S := \max\{k \in \mathcal{K} : S \text{ is a defensive } k\text{-alliance}\}$. We say that k_S is the *exact index of alliance* of S, or also, S is an *exact defensive k_S-alliance* in G.

Remark 1. The exact index of alliance of S in G is $k_S = \min_{v \in S}\{\delta_S(v) - \delta_{\overline{S}}(v)\}$.

We define the *alliance polynomial* of a graph G with variable x as follows: $A(G; x) = \sum_{k \in \mathcal{K}} A_k(G) x^{n+k}$, with $A_k(G)$ the number of exact defensive k-alliances in G. In [6] the authors introduce this alliance polynomial and [13] deals with the alliance polynomials of cubic graphs. In this paper we study the alliance polynomials of regular graphs and their coefficients, see Section 2. In Section 3 we focus on the alliance polynomials of connected regular graphs; besides, we prove that the family of alliance polynomials of connected Δ-regular graphs with small degree is a very special one, since it does not contain alliance polynomials of graphs which are not connected Δ-regular.

2 Alliance polynomials for regular graphs

Below, a quick reminder of some previous results for general graphs (not necessarily regular) that will be useful, see [6]. We denote by $\mathrm{Deg}(p)$ the degree of the polynomial p.

Theorem 1. *Let G be any graph. Then $A(G; x)$ satisfies the following properties:*

1. *$A(G; x)$ does not have zeros in the interval $(0, \infty)$.*
2. *$A(G; 1) < 2^n$, and it is the number of connected induced subgraphs in G.*
3. *If G has at least an edge and its degree sequence has exactly r different values, then $A(G; x)$ has at least $r + 1$ terms.*
4. *$A(G; x)$ is a symmetric polynomial (i.e., an even or an odd function of x) if and only if the degree sequence of G has either all values odd or all even.*

5. $A_{-\Delta}(G)$ *and* $A_{-\Delta+1}(G)$ *are the number of vertices in* G *with degree* Δ *and* $\Delta - 1$*, respectively.*
6. $A_{\Delta}(G)$ *is equal to the number of connected components in* G *which are* Δ*-regular.*
7. $n + \delta \leq \mathrm{Deg}(A(G;x)) \leq n + \Delta$.

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex. The following lemma is a well known result of graph theory.

Lemma 1. *If* $r \geq 2$ *is a natural number and* G *is any graph with* $\delta(v) \geq r$ *for every* $v \in V(G)$*, then there exists a cycle* η *in* G *with* $L(\eta) \geq r + 1$.

We show now some results about the alliance polynomial of regular graphs and their coefficients, for details see [7]. If G is a graph and $v \in V(G)$, we denote by $G \setminus \{v\}$ the subgraph obtained by removing from G the vertex v and the edges incident to v. We say that v is a *cut vertex* if $G \setminus \{v\}$ has more connected components than G. Besides, if p is a polynomial we denote by $\mathrm{Deg}_{min}(p)$ the minimum degree of their non-zero coefficients.

Theorem 2. *For any* Δ*-regular graph* G*, its alliance polynomial* $A(G;x)$ *satisfies the following properties:*

1. $A_{-\Delta+2i}(G)$ *is the number of connected induced subgraphs of* G *with minimum degree* i $(0 \leq i \leq \Delta)$.
2. $\mathrm{Deg}_{\min}\big(A(G;x)\big) = n - \Delta$ *and* $A_{-\Delta}(G) = n$.
3. $\mathrm{Deg}\big(A(G;x)\big) = n + \Delta$*. Besides,* $2n = \mathrm{Deg}_{\min}\big(A(G;x)\big) + \mathrm{Deg}\big(A(G;x)\big)$ *and* $8m = \mathrm{Deg}^2\big(A(G;x)\big) - \mathrm{Deg}^2_{\min}\big(A(G;x)\big)$.
4. $1 \leq A_{\Delta}(G) \leq n/(\Delta+1)$*. Furthermore,* G *is a connected graph if and only if* $A_{\Delta}(G) = 1$.
5. *If* $\Delta > 0$*, then* $A_{-\Delta+2}(G) \geq m$ *and* $A_{\Delta-2}(G) \geq n+n_0$ *with* n_0 *the number of cut vertices; in particular,* $A_{\Delta-2}(G) \geq n$.
6. $A(G;x)$ *is either an even or an odd function of* x*. Furthermore,* $A(G;x)$ *is an even function of* x *if and only if* $n + \Delta$ *is even.*
7. *The unique real zero of* $A(G;x)$ *is* $x = 0$*, and its multiplicity is* $n - \Delta$.

Theorem 3. *Let* G *be any connected graph. Then* G *is regular if and only if* $A_{\Delta}(G) = 1$.

Proof. If G is regular, then by Theorem 1 vi) we obtain $A_{\Delta}(G) = 1$. Besides, if $A_{\Delta}(G) = 1$, then there is an exact defensive Δ-alliance S in G with $\delta_S(v) \geq \delta_{\bar{S}}(v) + \Delta \geq \Delta$ (i.e., $\delta_S(v) = \Delta$ and $\delta_{\overline{S}}(v) = 0$) for every $v \in S$. So, the connectivity of G gives that G is a Δ-regular graph. ∎

Theorem 4. *Let G_1, G_2 be two regular graphs. If $A(G_1;x) = A(G_2;x)$, then G_1 and G_2 have the same order, size, degree and number of connected components.*

Proof. The proof is direct using Theorem 2 ii-iii) and Theorem 1 vi). ∎

The next theorem characterizes the degree of any regular graph by the number of non-zero coefficients of its alliance polynomial.

Theorem 5. *Let G be any Δ-regular graph with order n. Then $A(G;x)$ has $\Delta+1$ non-zero coefficients. Furthermore, $A(G;x) = \sum_{i=0}^{\Delta} A_{\Delta-2i}(G)\ x^{n+\Delta-2i}$, with $A_{-\Delta}(G) = n$, $A_{\Delta}(G) \geq 1$, and $A_{\Delta-2i}(G) \geq \frac{n\binom{\Delta}{i}}{\min\{\Delta, n-i\}}$ for $1 \leq i \leq \Delta - 1$ if $\Delta > 0$.*

Proof. By Theorem 2 we obtain the structure of $A(G;x)$ with $A_{-\Delta}(G) = n$ and $A_{\Delta}(G) \geq 1$. Hence, we build an exact defensive $(\Delta - 2i)$-alliance for $1 \leq i \leq \Delta - 1$ and we obtain a lower bound of the number of exact defensive $(\Delta - 2i)$-alliance with the given structure. For details see [7]. ∎

The following results in this section are interesting properties of the coefficients of $A(G;x)$ that will be useful along this work.

Theorem 6. *Let G be any Δ-regular graph with order $n < 2\Delta$. Then we have $A_{\Delta-2}(G) = n$.*

Proof. Note that G is a Hamiltonian graph with $\Delta \geq 2$, hence, we have that $V \setminus \{u\}$ is an exact defensive $(\Delta - 2)$-alliance in G for every $u \in V$ and so $A_{\Delta-2}(G) \geq n$. Besides, we prove that there is not an exact defensive $(\Delta-2)$-alliance $S \subset V$ with $|S| \leq n - 2$. In order to complete the proof we proceed by contradiction using the double counting technic. For details see [7]. ∎

Lemma 2. *Let G be any Δ-regular graph with order n, $\Delta \geq 3$ and $2\Delta \leq n \leq 2\Delta + 1$. If G contains two cliques of cardinality Δ, then these cliques are disjoint. In particular, G contains at most two cliques of cardinality Δ.*

Proof. Seeking for a contradiction, assume that there exist cliques $S_1, S_2 \subset V$ of cardinality Δ with $S_1 \cap S_2 \neq \emptyset$. We compute $\delta_{S_1 \cup S_2}(u)$ for every $u \in S_1 \cap S_2$ and we obtain $|S_1 \cap S_2| = \Delta - 1$. In fact, $\Delta = 3$ and $n = 6$. Therefore, G is a graph isomorphic to either $K_{3,3}$ or the Cartesian product $P_2 \square K_3$. This finishes the proof since it is impossible to have three disjoint cliques of cardinality Δ contained in G. ∎

Remark 2. Let G be any Δ-regular graph with order n and $\Delta \geq 1$ such that G has two disjoint cliques of cardinality Δ. Then:

1. If $n = 2\Delta$, then G is isomorphic to the Cartesian product graph $P_2 \Box K_\Delta$.
2. If $n = 2\Delta + 1$, then Δ is even (since $n\Delta = 2m$) and G can be obtained from $P_2 \Box K_\Delta$ by removing $\Delta/2$ copy edges of P_2 and connecting the Δ vertices with degree $\Delta - 1$ with a new vertex.

Theorem 7. *Let G be any Δ-regular graph with order n, size m, $\Delta \geq 3$ and $2\Delta \leq n \leq 2\Delta + 1$. Then $n \leq A_{\Delta-2}(G) \leq n + m + 2$.*

Theorem 8. *Let G be a Δ-regular connected graph with order n and let G^* be a graph with order n_1 and, minimum and maximum degrees δ_1 and Δ_1, respectively. If $A(G^*; x) = A(G; x)$, then G^* is a connected graph with exactly n vertices of degree $\Delta_1 = \Delta + n_1 - n$, $n_1 \geq n$, $\Delta_1 \geq \Delta$ and $\delta_1 \equiv \Delta_1 (mod\ 2)$.*

Furthermore, if $n_1 > n$, then $\frac{\Delta_1 + \delta_1 + 2}{2} \leq \Delta$.

The proofs of Theorems 7 and 8 are long and they use some previous results with combinatorial arguments, for details see [7].

3 Alliance polynomials of regular graphs with small degree

The theorems in this section can be seen as a natural continuation of [6,13] in the sense of showing the distinctive power of the alliance polynomial of a graph. In particular, we show that the family of alliance polynomials of Δ-regular graphs with small degree Δ is a special family of alliance polynomials since there not exists a non Δ-regular graph with alliance polynomial equal to one of their members, see Theorems 9 and 10.

Theorem 9. *Let G be a Δ-regular graph with $0 \leq \Delta \leq 3$ and G^* another graph. If $A(G^*; x) = A(G; x)$, then G^* is a Δ-regular graph with the same order, size and number of connected components of G.*

Proof. In [6, Theorem 2.7] the authors obtain the uniqueness of the alliance polynomials of 0-regular graphs (the empty graphs). The result for $1 \leq \Delta \leq 2$ follows from [6, Theorems 3.3] and Theorems 1 and 5. Finally, [13, Theorem 2.6] gives the result for $\Delta = 3$. For details see [7]. ∎

Now we prove a similar result for Δ-regular graphs with $\Delta > 3$. First, we prove some technical results that will be useful.

Lemma 3. *Let G_1 be a graph with minimum and maximum degree δ_1 and Δ_1, respectively, and let $n \geq 3$ be a fixed natural number. Assume that G_1 has order $n_1 > n$ with exactly n vertices of degree Δ_1, and such that its alliance polynomial $A(G_1; x)$ is symmetric. The following statements hold:*

1. *If $\delta_1 = 1$, then $A(G_1; x)$ is not a monic polynomial of degree $2n - n_1 + \Delta_1$.*
2. *If $\delta_1 = 2$, then we have $2n_1 < 2\Delta_1 + n$ or $A(G_1; x)$ is not a monic polynomial of degree $2n - n_1 + \Delta_1$.*

Proof. We proceed by contradiction, i.e., we assume that $A(G_1; x)$ is a monic polynomial with degree $2n - n_1 + \Delta_1$. The argument in the proof of Theorem 8 gives that G_1 is a connected graph with a determined structure since there is an unique exact defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance in G_1. Using Theorem 8 obtain $1 \leq \delta_1 \leq 2$, so, working separately in each case we obtain contradictions; this provides the result. For details see [7]. ∎

Lemma 4. *Let G_1 be a graph with minimum and maximum degree 2 and Δ_1, respectively, and let $n \geq 3$ be a fixed natural number. Assume that G_1 has order $n_1 > n$ with exactly n vertices of degree Δ_1, and such that its alliance polinomial $A(G_1; x)$ is symmetric. If $n < 2[\Delta_1 - (n_1 - n)]$ and $A(G_1; x)$ is a monic polynomial of degree $2n - n_1 + \Delta_1$, then $A_{2(n-n_1)+\Delta_1-2}(G_1) > n$.*

Proof. As in the proof of Lemma 3 we have that G_1 is a connected graph with a determined structure since there is an unique exact defensive $\big[\Delta_1 - 2(n_1 - n)\big]$-alliance in G_1. We continue in this way obtaining $n + 1$ exact defensive $\big[\Delta_1 - 2(n_1 - n) - 2\big]$-alliances in G_1, and this finishes the proof. For details see [7]. ∎

Theorem 10. *Let G be a connected Δ-regular graph with $\Delta \leq 5$ and G^* another graph. If $A(G^*; x) = A(G; x)$, then G^* is a connected Δ-regular graph with the same order and size of G.*

Proof. If $0 \leq \Delta \leq 3$, then the result follows from Theorem 9. Assume that $4 \leq \Delta \leq 6$. Let n, n_1 be the orders of G, G^*, respectively, and let δ_1, Δ_1 be the minimum and maximum degree of G^*, respectively. By Theorem 8, G^* is a connected graph and $n_1 \geq n$. Seeking for a contradiction assume that $n_1 > n$. By Theorem 8 and Lemma 3 we can obtain a contradiction if $\Delta = 4$. If $\Delta = 5$ then by Theorem 8 we have $\delta_1 = 1$ and $\Delta_1 = 7$, or $\delta_1 = 2$ and $\Delta_1 = 6$. Hence, by Lemma 3 we have directly a contradiction in the first case. In the second case, since $n_1 = n + 1$, Lemma 3 gives that $n < 10$. Thus, Lemma 4 and Theorem 6 gives $n < A_2(G^*) = A_3(G) = n$. For details see [7]. ∎

References

[1] S. Akbari, S. Alikhani and Y-H. Peng, Characterization of graphs using domination polynomials. *Europ. J. Combin.* **31**(7) (2010), 1714-1724.

[2] S. Alikhani and Y-H. Peng, Introduction to Domination Polynomial of a Graph, to appear in *Ars Comb.*

[3] J. L. Arocha and B. Llano, Mean value for the matching and dominating polynomial, *Discuss. Math. Graph Theory* **20**(1), (2000) 57-69.

[4] J. I. Brown, K. Dilcher, R. J. Nowakowski, Roots of Independence Polynomials of Well Covered Graphs, *J. Alg. Comb.* **11**(3), (2000) 197-210.

[5] W. Carballosa, Exact Defensive Alliances in Graphs, *Appl. Math. Sci.* **7**(74) (2013), 3673-3679.

[6] W. Carballosa, J. M. Rodríguez, J. M. Sigarreta and Y. Torres, Computing the alliance polynomial of a graph. Submitted.

[7] W. Carballosa, J. M. Rodríguez, J. M. Sigarreta and Y. Torres, Computing the alliance polynomial of regular graphs. Submitted.

[8] E. J. Farrell, An introduction to matching polynomials, *J. Comb. Theory Ser. B* **27** (1979), 75-86.

[9] C. D. Godsil and I. Gutman, On the theory of the matching polynomial, *J. Graph Theory* **5** (1981), 137-144.

[10] I. Gutman and F. Harary, Generalizations of the matching polynomial, *Utilitas Math.* **24** (1983), 97-106.

[11] C. Hoede and X. Li, Clique polynomials and independent set polynomials of graphs, *Discrete Math.* **125** (1994), 219-228.

[12] R. C. Read, An introduction to chromatic polynomials, *J. Comb. Theory* **4**(1) (1968), 52-71.

[13] J. M. Rodríguez, J. M. Sigarreta and Y. Torres, Alliance polynomial of cubic graphs. Submitted.

[14] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canad. J. Math* **6**(80-91) (1954), 3-4.

Laplacian matrix of a weighted graph with new pendant vertices *

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Abstract. The Laplacian matrix of a simple graph has been widely studied, as a consequence of its applications. However the Laplacian matrix of a weighted graph is still a challenge. In this work we provide the Moore–Penrose inverse of the Laplacian matrix of the graph obtained adding new pendant vertices to an initial graph, in terms of the Moore–Penrose inverse of the Laplacian matrix of the original graph. As an application we can compute the effective resistances and the Kirchhoff index of the new network.

Key words: Laplacian matrix, Moore–Penrose inverse, Kirchhoff index, Green matrix, pendant vertex.

1 Introduction

One of the main problems in network theory is to obtain global information of complex networks from more simple networks where this information either is known or, at least, it is easily computable. Understanding complex networks as a perturbation of more simple networks, our aim is to compute the Green function and then the Kirchhoff index of the perturbed network in terms of the same parameters of the original network and the values of the perturbation itself. The case when the perturbation consist on adding new edges and/or to increase the conductance of the actual edges, has been studied. Then, the resulting expressions became from a more o less explicit adaptation of the well–known Sherman–Morrison–Woodbury formulas. For instance, Y. Yang and D.J.Klein [8], use the Sherman–Morrison formula to obtain a recursive

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procedure for the computation of effective resistances in the perturbed network in terms of the effective resistances of the original network by adding a new edge in each step. In [2], some of the authors analyzed the effect of a perturbation of the network by computing the effective resistance of perturbed networks through Sherman–Morrison–Woodbury like–formulas, instead of using the Sherman–Morrison formula recursively. In fact, since adding edges to a network does not modify the space of functions on the vertex set of the network, this class of perturbation was placed into the general framework of perturbations of discrete elliptic operators. Specifically, we show that this problem corresponds with the superposition of rank one perturbations, that are orthogonal to the eigenfunction associated with the lowest eigenvalue of the elliptic operator.

The scenario changes when the perturbation consists on adding new vertices to the network, and only few works have been developed, see for instance [3], the effect of adding a new vertex. In this work we consider perturbations that consist on adding pendant vertices to a network. After some well–known operations on the Laplacian of the perturbed network, that involves the inverse of the Schur complement of the block corresponding to the added vertices, we show that this Schur complement can be seen as a perturbation of the laplacian of the original network, understood as a discrete elliptic operator, that is a superposition of perturbations with rank one that, this time, are not orthogonal to the eigenfunction associated with the lowest eigenvalue of the elliptic operator. Therefore, we can apply the general theory developed in [2] for this kind of perturbations.

2 Definitions and notation

A network $\Gamma = (V, E, c)$ or weighted graph is composed by a set of elements V called vertices, a set of pairs of vertices E called edges, and a symmetric map $c : V \times V \to [0, \infty)$ named conductance, associated to the edges. The order of the network, n, is the number of vertices.

We denote by $\mathcal{C}(V)$ the space of real valued functions on V. The scalar product on $\mathcal{C}(V)$ is $\langle u, v\rangle = \sum_{x\in V} u_x v_x$ for each $u, v \in \mathcal{C}(V)$. It is well–known that there exists a one to one correspondence between matrices $\mathcal{M}_{n\times n}$ and operators defined in $\mathcal{C}(V)$. That is, any matrix $K \in \mathcal{M}_{n\times n}$ can be considered as a function $K : V \times V \to \mathbb{R}$ and determines an endomorphim of $\mathcal{C}(V)$ by assigning to any $u \in \mathcal{C}(V)$ the real function $\mathcal{K}(u) = \sum_{y\in V} K(\cdot, y)u(y)$. Conversely, each endomorphism of $\mathcal{C}(V)$ is determined by the kernel $K(x, y) = \langle \mathcal{K}(\varepsilon_y), \varepsilon_x\rangle$ for any x, $y \in V$, where $\varepsilon_x \in \mathcal{C}(V)$, denotes the Dirac function at x.

A unitary and positive function ω is called a weight and $\Omega(V)$ denote the set of weights. Given σ, $\tau \in \mathcal{C}(V)$, we denote by $\boldsymbol{P}_{\sigma,\tau}$ the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\boldsymbol{P}_{\sigma,\tau}(u) = \langle \tau, u\rangle\sigma$, and it is

called a projector, as it assigns to any $u \in \mathcal{C}(V)$ its projection on σ along τ. Observe that the corresponding kernel is the matrix $(P)_{x,y} = (\sigma \otimes \tau)(x,y) = \sigma(x)\tau(y)$. In particular, when $\omega \neq 0$ the projector $P_{\omega,\omega}$ is denoted simply by P_ω.

If $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$, it is self–adjoint when $\langle \mathcal{K}(u), v\rangle = \langle u, \mathcal{K}(v)\rangle$ for any $u \in \mathcal{C}(V)$. Moreover, $\mathcal{K}$ is positive semi–definite when $\langle \mathcal{K}(u), u\rangle \geq 0$ for any $u \in \mathcal{C}(V)$. A self–adjoint operator $\mathcal{K}$ is elliptic if it is positive semi–definite and its lowest eigenvalue λ is simple. Moreover, there exists a unique unitary function $\omega \in \mathcal{C}(V)$, up to sign, satisfying $\mathcal{K}(\omega) = \lambda\omega$, so $\mathcal{K}$ is called (λ,ω)–elliptic. It is straighforward that (λ,ω)–elliptic is singular iff $\lambda = 0$.

Given $\lambda \geq 0$, $\omega \in \Omega(V)$ and a (λ,ω)–elliptic operator $\mathcal{F}$, we will be concerned with the so–called Poisson equation for $\mathcal{F}$ on V: for a given $f \in \mathcal{C}(V)$ find $u \in \mathcal{C}(V)$ such that $\mathcal{F}(u) = f$. The inverse of a (λ,ω)–elliptic operator $\mathcal{F}$ on $\omega^\perp$ is called orthogonal Green operator and it is denoted by Γ. This operator on $\omega^\perp$ can be extended to $\mathcal{C}(V)$ by assigning to any $f \in \mathcal{C}(V)$ the unique solution of the Poisson equation $\mathcal{F}(u) = f - \boldsymbol{P}_\omega(f)$. Moreover, the orthogonal Green operator Γ of $\mathcal{F}$ satisties

$$\mathcal{F} \circ \Gamma = \Gamma \circ \mathcal{F} = \mathcal{I} - \boldsymbol{P}_\omega.$$

And when $\mathcal{F}$ is non–singular $\mathcal{F}^{-1} = \Gamma + \lambda^{-1}\boldsymbol{P}_\omega$, (see [2]).

Now consider a non–null function $\sigma \in \mathcal{C}(V)$, the associated self–adjoint projection $\boldsymbol{P}_\sigma$ and the operator

$$\mathcal{H}_\sigma = \mathcal{F} + \boldsymbol{P}_\sigma,$$

called the perturbation of $\mathcal{F}$ by σ. The relation between Green operators of both $\mathcal{H}$ and $\mathcal{F}$ can be found in [2, Corollary 3.4].

Corollary 1. *Consider $\sigma \in \mathcal{C}(V)$ and the operator of the perturbation of $\mathcal{F}$ by σ: $\mathcal{H}_\sigma = \mathcal{F} + \boldsymbol{P}_\sigma$, then when $\langle \sigma, \omega\rangle = 0$*

$$\Gamma^{\mathcal{H}} = \Gamma - \frac{1}{1 + \langle \Gamma(\sigma), \sigma\rangle}\boldsymbol{P}_{\Gamma(\sigma)},$$

whereas, when $\langle \sigma, \omega\rangle \neq 0$

$$\mathcal{H}^{-1} = \Gamma - \frac{1}{\beta}\Big[\lambda \boldsymbol{P}_{\Gamma(\sigma)} - \langle \sigma, \omega\rangle \left(\boldsymbol{P}_{\Gamma(\sigma),\omega} - \boldsymbol{P}_{\omega,\Gamma(\sigma)}\right) - (1 + \langle \sigma, \omega\rangle)\, \boldsymbol{P}_\omega\Big],$$

where $\beta = \lambda(1 + \langle \Gamma(\sigma), \sigma\rangle) + \langle \sigma, \omega\rangle^2$.

Moreover, if we consider several $\sigma_i \in \mathcal{C}(V)$ for any $i = 1, \ldots, m$, the operator $\mathcal{H} = \mathcal{F} + \sum_{i=1}^m \boldsymbol{P}_{\sigma_i}$ is a perturbed operator. The relation between the corresponding inverse operators is given in following theorem.

Theorem 1. *If $\sigma_i \in \mathcal{C}(V)$, $i = 1, \ldots, m$, then $\mathcal{H}$ is positive definite and*

$$\mathcal{H}^{-1} = \Gamma + \alpha \boldsymbol{P}_\omega + \sum_{i=1}^{m} \beta_i [\boldsymbol{P}_{\Gamma(\sigma_i),\omega} - \boldsymbol{P}_{\omega,\Gamma(\sigma_i)}] - \sum_{i,j=1}^{m} \gamma_{ij} \boldsymbol{P}_{\Gamma(\sigma_i),\Gamma(\sigma_j)},$$

where $(b_{ij}) = (I + \langle \Gamma(\sigma_j), \sigma_i \rangle)^{-1}$, and

$$\begin{aligned}
\alpha &= \left(\lambda + \sum_{r,s} b_{r,s} \langle \Gamma(\sigma_j), \sigma_i \rangle \langle \sigma_s, \omega \rangle \right)^{-1}, \\
\beta_i &= \alpha \sum_{r=1}^{m} b_{ir} \langle \sigma_r, \omega \rangle, \\
\gamma_{ij} &= b_{ij} - \alpha \left(\sum_{r=1}^{m} b_{ir} \langle \sigma_r, \omega \rangle \right) \left(\sum_{r=1}^{m} b_{jr} \langle \sigma_r, \omega \rangle \right).
\end{aligned}$$

On the other hand the following classic result provide us a fundamental tool for the former results.

Lemma 1. *If $A \in \mathcal{M}_{n\times n}$, $B \in \mathcal{M}_{n\times m}$ and $D \in \mathcal{M}_{m\times m}$ are such that D and $\begin{pmatrix} A & B \\ B^\intercal & D \end{pmatrix}$ are invertible, then*

$$\begin{pmatrix} A & B \\ B^\intercal & D \end{pmatrix}^{-1} = \begin{pmatrix} H^{-1} & -H^{-1}BD^{-1} \\ -D^{-1}B^\intercal H^{-1} & D^{-1} + D^{-1}B^\intercal H^{-1} BD^{-1} \end{pmatrix},$$

where $H = A - BD^{-1}B^\intercal$ is the Schur complement of D in $\begin{pmatrix} A & B \\ B^\intercal & D \end{pmatrix}$.

3 Inverse Laplacian of networks with new pendant vertices

The combinatorial Laplacian, $\mathcal{L}$, is a singular elliptic operator on $\mathcal{C}(V)$ and moreover $\mathcal{L}(u) = 0$ iff u is a constant function.

Given $q \in \mathcal{C}(V)$ the Schrödinger operator on Γ with potential q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}_q(u)(x) = \mathcal{L}(u)(x) + q(x)u(x) = \sum_{y\in V} c(x,y)(u(x) - u(y)) + q(x)u(x).$$

Given a weight $\omega \in \Omega(V)$, the function $q_\omega = -\frac{1}{\omega}\mathcal{L}(\omega)$ is the potential determined by ω. If there exist $\lambda \geq 0$ and $\omega \in \Omega(V)$ such that $q = q_\omega + \lambda$, the $\mathcal{L}_q$ is (λ, ω)–elliptic, see [2]. Observe that if $\omega = \frac{1}{\sqrt{n}}\mathrm{j}$, where j is the all–ones vector, then it is verified that $q_\omega = -\frac{1}{\omega}\mathcal{L}(\omega) = 0$ and therefore $q = q_\omega + \lambda = \lambda$. In

this case, we denote by g the orthogonal Green operator and by G_λ its corresponding kernel. It is also verified that $G_\lambda = L_\lambda^{-1} - \frac{1}{n\lambda}J_n$, where $L_\lambda = L + \lambda I_n$ and J_n is the all–ones matrix.

Firstly we consider the simplest case of the addition of a new vertex to the graph, which is the addition of a pendant vertex. Consider the network $\Gamma' = (V', E', c')$ obtained by connecting a new vertex x' to one of the vertices $x \in V(\Gamma)$. Therefore $V' = V \cup \{x'\}$, $E' = E \cup \{e_{x,x'}\}$, the conductance of the pendant vertex is $c' = c'(x, x') > 0$ and the weight of the new network is $\omega = \frac{1}{\sqrt{n+1}}\mathsf{j} \in \Omega(V')$. As a consequence, the Schrödinger matrix of Γ' is the Laplacian matrix of the weighted network, $L'_q = L' + \lambda I_{n+1} = L'_\lambda$, and the relation between the Schrödinger matrices of the initial and the new graph is given by

$$L'_\lambda = \begin{pmatrix} L_1 & -c'\varepsilon_x \\ -c'\varepsilon_x & c' + \lambda \end{pmatrix},$$

where $L_1 = L_\lambda + c'P_{\varepsilon_x}$. Therefore we can apply the Lemma 1 to obtain the Green matrix G'_λ in terms of the Green matrix G_λ of the initial graph. We compute the matrix $H = L_\lambda + \frac{c'\lambda}{c'+\lambda}P_{\varepsilon_x}$, which can be considered as the matrix of the perturbed operator $\mathcal{L}^H = \mathcal{L} + \boldsymbol{P}_{\sigma_{x,x'}}$, taking as a perturbation $\sigma_{x,x'} = \sqrt{\frac{c'\lambda}{c'+\lambda}}\varepsilon_x$, and thus we can apply the Corollary 1 to compute its inverse.

Therefore we can apply Lemma 1 to obtain the Green matrix G'_λ in terms of the Green matrix G_λ of the initial graph.

Proposition 1. *It holds*

$$(L'_\lambda)^{-1} = \begin{pmatrix} H^{-1} & -\frac{1}{c'+\lambda}H^{-1}\varepsilon_x \\ -\frac{1}{c'+\lambda}\varepsilon_x^\intercal H^{-1} & \frac{1}{c'+\lambda} - \frac{1}{(c'+\lambda)^2}(H^{-1})_{xx} \end{pmatrix}, \tag{1}$$

where H^{-1} *is*

$$H^{-1} = G_\lambda - \frac{1}{\lambda(1+\sigma_{x,x'}G_\lambda\sigma_{x,x'}^\intercal) + \frac{\lambda}{n(c'+\lambda)}}\left[\lambda P_{G_\lambda\sigma_{x,x'}^\intercal} - \left(1 + \sqrt{\frac{c'\lambda}{n(c'+\lambda)}}\right)\frac{1}{n}J_n\right].$$

Proof. The proof is straightforward by applying Corollary 1 and Lemma 1. ∎

Now we compute the Green matrix of the new graph as

$$G'_\lambda = (L'_\lambda)^{-1} - \frac{1}{(n+1)\lambda}J_{n+1}.$$

Moreover, as the Kirchhoff index is the trace of the Green matrix, we have the following relation between both Kirchhoff indexes.

Corollary 2.

$$k'(\lambda) = \sum_{x\in V}(G'_\lambda)_{xx} = k_p(\lambda) + \frac{1}{c'+\lambda} - \frac{1}{(c'+\lambda)^2}\varepsilon_x^\intercal H^{-1}\varepsilon_x - \frac{1}{\lambda},$$

where $k_p(\lambda)$ is the Kirchhoff index of the perturbed matrix L_1.

Secondly consider a more general case, when we add m pendant vertices x'_i to some vertices $x_i \in V(\Gamma)$, for $i = 1,\dots,m-n$, and consider the network $\Gamma' = (V', E', c')$ such that $V' = V \cup \{x'_1,\dots,x'_{m-n}\}$, $E' = E \cup \{e_{x_1,x'_1},\dots,e_{x_{m-n},x'_{m-n}}\}$, the new conductance $c'_i = c'(x_i,x'_i) > 0$ and new weight $\omega' = \frac{1}{\sqrt{m}}\mathsf{j} \in \Omega(V')$. Thus consider the Schrödinger matrix of Γ', $L'_q = L' + \lambda I_m = L'_\lambda$, and observe that the relation between both Schrödinger matrices of Γ and Γ' is given by

$$L'_\lambda = \begin{pmatrix} L_1 & B \\ B^\intercal & D_{q'} \end{pmatrix},$$

where $L_1 = L_\lambda + \sum_{i=1}^m c'_i P_{\varepsilon_{x_i}}$, $B = (-c'_1\varepsilon_{x_1},\dots,-c'_m\varepsilon_{x_m})$, and where $D_{q'}$ is an invertible diagonal matrix having in the diagonal the potential of the new vertices $q'_{x'_i} = c'_i + \lambda \neq 0$. Following the same steps of the previous case, we compute the inverse of L_1 by applying Lemma 1, and thus we first compute the matrix

$$H = L_1 - BD^{-1}B^\intercal = L_1 - \sum_{i=1}^m \frac{(c'_i)^2}{c'_i+\lambda}P_{\varepsilon_{x_i}} = L_\lambda + \sum_{i=1}^m \left(c'_i - \frac{(c'_i)^2}{c'_i+\lambda}\right)P_{\varepsilon_{x_i}}.$$

Now, for obtaining its inverse observe that L_1 can be considered as the kernel of the perturbed operator $\mathcal{L}_1 = \mathcal{L}_\lambda + \sum_{i=1}^m \boldsymbol{P}_{\sigma_i}$, with $\sigma_i = \sqrt{\frac{c'_i\lambda}{c'_i+\lambda}}\,\varepsilon_{x_i}$, for any $1 \le i \le m$. Consequently, we apply Theorem 1 to obtain the following result.

Proposition 2. *It is verified*

$$(L'_\lambda)^{-1} = \begin{pmatrix} H^{-1} & -H^{-1}BD_{q'}^{-1} \\ -D_{q'}^{-1}B^\intercal H^{-1} & D_{q'}^{-1} + D_{q'}^{-1}B^\intercal H^{-1}BD_{q'}^{-1} \end{pmatrix}, \tag{2}$$

where H^{-1} is

$$H^{-1} = G_\lambda + \frac{\alpha}{n}J_n - \sum_{i,j=1}^m \gamma_{ij}\boldsymbol{P}_{G_\lambda\sigma_i, G_\lambda\sigma_j},$$

where

$$(b_{ij}) = (I + \sigma_i G\sigma_j^\intercal)^{-1},$$

$$\alpha = \left(\lambda + \frac{1}{m}\sum_{r,s}\sqrt{\frac{c'_s\lambda}{c'_s+\lambda}}b_{r,s}\sigma_i G\sigma_j^\intercal\right)^{-1},$$

$$\gamma_{ij} = b_{ij} - \frac{\alpha}{m^2}\left(\sum_{r=1}^m b_{ir}\sqrt{\frac{c'_r\lambda}{c'_r+\lambda}}\right)\left(\sum_{r=1}^m b_{jr}\sqrt{\frac{c'_r\lambda}{c'_r+\lambda}}\right).$$

We point out that this last result can be obtained by applying the previous one step by step.

4 Example

In this section we apply the results obtained in previous section to derive a formula for computing the Green matrix of a simple star graph S_n, by adding $n-2$ pendant vertices at once to an edge.

Consider a simple edge between two vertices x_0, x_1 with an edge e_{x_0,x_1} with conductance 1. We denote by $L_{\lambda,2}$ the Schrödinger matrix of S_2. Observe that

$$L_{\lambda,2} = \begin{pmatrix} 1+\lambda & -1 \\ -1 & 1+\lambda \end{pmatrix}.$$

If we add $n-2$ pendant vertices of conductance 1 to x_0, by applying Proposition 2 observe that the matrix $D_{q'}^{-1} = (\lambda+1)^{-1} I_{n-2}$,

$$H = \begin{pmatrix} \frac{\lambda^2+n\lambda+1}{\lambda+1} & -1 \\ -1 & 1+\lambda \end{pmatrix} \Rightarrow H^{-1} = \begin{pmatrix} \frac{1+\lambda}{\lambda^2+n\lambda} & \frac{1}{\lambda^2+n\lambda} \\ \frac{1}{\lambda^2+n\lambda} & \frac{\lambda^2+n\lambda+1}{(\lambda+1)(\lambda^2+n\lambda)} \end{pmatrix}.$$

Observe that $B = \begin{pmatrix} -j_{n-2} \\ 0_{n-2} \end{pmatrix} \in \mathcal{M}_{2,n-2}$, and thus

$$-H^{-1}BD_{q'}^{-1} = \begin{pmatrix} \frac{1}{\lambda^2+n\lambda} j_{n-2} \\ \frac{1}{(1+\lambda)(\lambda^2+n\lambda)} j_{n-2} \end{pmatrix},$$

and $D_{q'}^{-1} + D_{q'}^{-1} B^{\intercal} H^{-1} B D_{q'}^{-1} = \dfrac{1}{\lambda+1} I_{n-2} + \dfrac{1}{(\lambda+1)(\lambda^2+n\lambda)} J_{n-2}$.

Finally, we obtain

$$L_{\lambda,2}^{-1} = \begin{pmatrix} \frac{1+\lambda}{\lambda^2+n\lambda} & \frac{1}{\lambda^2+n\lambda} & \frac{1}{\lambda^2+n\lambda} & \cdots & \frac{1}{\lambda^2+n\lambda} \\ \frac{1}{\lambda^2+n\lambda} & \frac{\lambda^2+n\lambda+1}{(\lambda+1)(\lambda^2+n\lambda)} & \frac{1}{(1+\lambda)(\lambda^2+n\lambda)} & \cdots & \frac{1}{(1+\lambda)(\lambda^2+n\lambda)} \\ \frac{1}{\lambda^2+n\lambda} & \frac{1}{(1+\lambda)(\lambda^2+n\lambda)} & \frac{\lambda^2+n\lambda+1}{(\lambda+1)(\lambda^2+n\lambda)} & \cdots & \frac{1}{(\lambda+1)(\lambda^2+n\lambda)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda^2+n\lambda} & \frac{1}{(1+\lambda)(\lambda^2+n\lambda)} & \frac{1}{(\lambda+1)(\lambda^2+n\lambda)} & \cdots & \frac{\lambda^2+n\lambda+1}{(\lambda+1)(\lambda^2+n\lambda)} \end{pmatrix}.$$

We point out that our method allow us to compute the inverse matrix with an easy method that the usual. Furthermore, the Green matrix is

$$G_\lambda = L_{\lambda,2}^{-1} - \frac{1}{n\lambda} J_n.$$

And the Kirchhoff index is the trace of the former matrix, so

$$k'(\Gamma) = (n-1)\frac{\lambda+(n-1)}{(\lambda+1)(\lambda+n)}.$$

References

[1] E. Bendito, A. Carmona, A.M. Encinas. The Kirchoff indices of join networks, *Discrete Appl. Math.*, vol. 160:24–37. Springer, 2012.

[2] A. Carmona, A.M. Encinas, M. Mitjana Discrete elliptic operators and their Green operators. *Linear Algebra Appl.*, vol. 442:115–134. Springer, 2014.

[3] Z. Cinkir. Deletion and contraction identities for the resistance values and the Kirchhoff index. *Int. J. Quantum Quem.*, vol. 111:4030–4041. Springer, 2011.

[4] F. R. K. Chung, Spectral Graph Theory, *CBMS Regional Conference Series in Mathematics*, 92. American Mathematical Society, 1997.

[5] R. Merris, Laplacian matrices of graphs: a survey, *Linear Algebra Appl.*, vol. 197-198:143–176. Springer, 1994.

[6] Bojan Mohar, Some applications of Laplace eigenvalues of graphs, *Graph Symmetry: Algebraic Methods and Applications*, vol. 497:225–275. Springer, 1997.

[7] W. Wang, D. Yang, Y. Luo. The Laplacian polynomial and Kirchhoff index of graphs derived from regular graphs. *Discrete Appl. Math.*, vol. 161:3063–3071. Springer, 2013.

[8] Y. Yang, D.J. Klein. A recursion formula of resistance distances and its applications. *Discrete Appl. Math.*, vol. 161: 2702–2715, 2013.

[9] Q. Zhu, Y. Wang. The Kirchhoff Index of Weighted Graphs *Int. J. Algebra*, vol. 7: 267–280. Hikari, 2013.

Potential Theory on Finite Networks

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Abstract. In this work we survey some techniques from Potential Theory and show their use in electrical networks. We are focused on the linear aspect of the theory; that is, on the analysis of singular an positive semidefinite Schrödinger operators and we show the relation between this class of operators and singular M–matrices, paying attention on the properties of superharmonic functions. We also apply these methods to establish some global results on the associated Markov chains.

Key words: Discrete Potential Theory, Schrödinger operators, Green functions, singular M–matrices, electrical networks, Markov chains.

1 Introduction

In its very early stages, *Potential Theory* was referred to the study of the properties of harmonic functions; that is, functions that satisfy the Laplace equation. This terminology was coined in 19th–century physics, when it was realized that the fundamental forces of nature could be modeled using potentials which satisfy Laplace equation and hence, potential theory was the study of functions that could serve as potentials. A fruitful approach to the study of harmonic functions is the consideration of inequalities satisfied by they, being the most basic the *maximum principle*, that allows us to analyze the level sets of superharmonic functions.

Modern potential theory can be defined in more axiomatic terms, see [10], as either the search of *kernels* satisfying analogous properties to those satisfy by the newtonian one, this is called the *linear aspect*, or as the search of (semi)–norms satisfying analogous properties to those satisfy by the classical Dirichlet norm, this is called the *quadratic aspect*, and involves the so–called *Dirichet forms*. Modern potential theory is also intimately connected with probability

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and the theory of Markov chains. In the finite state space case, this connection can be understood by introducing an electrical network on the state space, with resistance between points inversely proportional to transition probabilities and densities proportional to potentials. In this framework, potentials and kernels can be seen as vectors and matrices of order the number of states. Moreover, the *Kirchhoff's problem* is formulated as a linear system whose coefficient matrix is the *Combinatorial Laplacian* of the network, that represents the finite analogue of the Laplacian in classical potential theory. Combinatorial laplacians or, more generally, positive (semi)–definite Schrödinger operators, or matrices, are known as M–matrices in the ambit of matrix theory. In view of their numerous applications, M–matrices have deserved a great attention and many of their properties has been studied. An important problem related with M–matrices is the so–called *inverse M–matrix problem*, that consists in characterizing all nonnegative matrices whose inverses are M–matrices. This is a longstanding and difficult problem that has generated a big amount of literature and has been partially solved, mainly using methods from Potential Theory. Some of the authors characterized in [5] the symmetric invertible M–matrices as *resistive inverses* associated with networks, introducing an adequate extension of the effective resistance of a network. Unlike the previous case, in this work we survey some techniques from Potential Theory and show their use in network theory when we consider singular an positive semidefinite Schrödinger operators. For the origins and motivations of this relation see [1] and references therein. Since the quadratic aspects of the theory have been studied in [3], we only treat here the linear aspects.

Let $\mathit{\Gamma} = (V, E, c)$ be a finite connected network, that is, a finite connected graph without loops nor multiple edges, with vertex set V and edge set E, in which each edge (x, y) has been assigned a *conductance* $c(x, y) > 0$. Therefore, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. We say that x is adjacent to y, $x \sim y$, if $(x, y) \in E$. A subset $F \subset V$ is said to be connected if each pair of vertices of F is joined by a path entirely contained in F. Given $x, y \in V$, $d(x, y)$ denotes the minimum length between the paths joining x and y. It its well known that d determines a distance on V.

The set of real functions on V, denoted by $\mathcal{C}(V)$, and the set of non–negative functions on V, $\mathcal{C}^+(V)$, are naturally identified with $\mathbb{R}^{|V|}$ and the positive cone of $\mathbb{R}^{|V|}$, respectively. Moreover, if $u \in \mathcal{C}(V)$ and $F \subset V$ the notation $u > 0$ on F means that $u(x) > 0$ for any $x \in F$. If $u \in \boldsymbol{C}(V)$, its *support* is $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$. Moreover, if $F \subset V$, then $\mathcal{C}(F) = \{u \in \mathcal{C}(V) : \mathsf{supp}(u) \subset F\}$, $\mathcal{C}^+(F) = \mathcal{C}(F) \cap \mathcal{C}^+(V)$. For any $x \in V$, ε_x is the Dirac function at x. The identity $\langle u, v\rangle = \sum_{z \in V} u(z)v(z)$ determines an inner product on $\boldsymbol{C}(V)$. If $\omega > 0$ on V and $\langle \omega, \omega\rangle = 1$ then it is called a *weight*. The set of weights is denoted by $\mathit{\Omega}(V)$.

Given $F \subset V$, we denote by F^c its complementary in V and we call *interior, boundary* and *closure of* F the subsets $\overset{\circ}{F} = \{x \in F : y \in F \text{ for all } y \sim x\}$, $\delta(F) = \{x \in F^c : d(x,F) = 1\}$ and $\bar{F} = F \cup \delta(F)$, respectively. Observe that $F = \delta(F^c) \cup \overset{\circ}{F}$, $\delta(F^c) \cap \overset{\circ}{F} = \emptyset$ and $\delta(\overset{\circ}{F}) \subset \delta(F^c)$. When F is a proper set, $r(F) = \max_{x \in F}\{d(x, \delta(F))\}$ is called the *radius* of F and for any $i = 0, \ldots, r(F)$ we consider $D_i(F) = \left\{x \in \bar{F} : d(x, \delta(F)) = i\right\}$. Observe that $D_0(F) = \delta(F)$ whereas $D_1(F) = \delta(F^c)$.

2 Schrödinger operators on finite networks

The *combinatorial Laplacian* of Γ or simply the *Laplacian*, is the linear operator $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \boldsymbol{C}(V)$ the function $\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big)$. The laplacian is self–adjoint; that is $\langle v, \mathcal{L}(u)\rangle = \langle u, \mathcal{L}(v)\rangle$ for any $u, v \in \boldsymbol{C}(V)$, and positive semi–definite. Moreover $\sum_{x \in V} \mathcal{L}(u)(x) = 0$ for any $u \in \mathcal{C}(V)$ and $\mathcal{L}(u) = 0$ iff u is a constant function.

A *Schrödinger operator* on Γ is a linear operator $\mathcal{L}_q : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \boldsymbol{C}(V)$ the function $\mathcal{L}_q(u)(x) = \mathcal{L}(u)(x) + q(x)u(x)$, where $q \in \mathcal{C}(V)$ is called the *potential*. Clearly $\mathcal{L}_q$ is self–adjoint and a function $u \in \boldsymbol{C}(V)$ is called q–harmonic, q–subharmonic or q–superharmonic on F iff $\mathcal{L}_q(u) = 0$, $\mathcal{L}_q(u) \leq 0$ or $\mathcal{L}_q(u) \geq 0$ on F, respectively. When the above inequalities are strict, then u is called strictly sub or superharmonic, respectively. The bilinear form $\boldsymbol{E}_q(u,v) = \langle u, \mathcal{L}_q(v)\rangle$ is called the *energy of* $\mathcal{L}_q$.

We associate a potential to any $\omega \in \Omega(V)$ by defining $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$. Since $\langle \omega, q_\omega\rangle = 0$, q_ω takes positive and negative values, except when ω is constant in which case $q_\omega = 0$ and the corresponding Schrödinger operator coincides with the Laplacian. It is satisfied, see [3], that $\langle \omega, \mathcal{L}_{q_\omega}(u)\rangle = 0$ for any $u \in \boldsymbol{C}(V)$ and moreover, $\mathcal{L}_{q_\omega}(u) = 0$ iff u is a multiple of ω. Therefore, the only q–sub or superharmonic functions on V are q–harmonic and are all multiple of ω. In [3, Proposition 3.3] it was proved that the energy is positive semi–definite iff $q = q_\omega$ for some weight ω.

For any $u \in \boldsymbol{C}(V)$, we can interpret $\mathcal{L}_{q_\omega}(u)(x)$ as the (signed) charge associated with the potential u at vertex x. One of the main problems in potential theory ask for the potential determined by a given charge. Specifically, given $f \in \boldsymbol{C}(V)$ we wonder under which conditions the so–called *Poisson equation* $\mathcal{L}_{q_\omega}(u) = f$ has solution and when non–negative or positive charges determine non–negative potentials. This question can have no solution since the above equation has solution only if the total charge is null and then the potential is unique up to a multiple of ω and moreover, takes positive and negative values. However, the situation changes if we demand that the charge is concentrated in a proper set $F \subset V$. Specifically, given $f \in \boldsymbol{C}(F)$ and $g \in \boldsymbol{C}(\delta(F))$ we set

the so–called *Dirichlet problem* $\mathcal{L}_{q_\omega}(u) = f$ on F and $u = g$ on $\delta(F)$. In this case the Dirichlet principle [3, Proposition 3.5] establishes that the problem has a unique solution. We tackle below the question about the sign of the charges.

3 Potential Theory for singular elliptic Schrödinger operators

In the sequel $F \subset V$ always denotes a proper connected subset of V and $q = q_\omega$ where $\omega \in \Omega(V)$. Moreover, we refer to q–harmonic, q–sub or q–superharmonic functions simply as harmonic, sub or superharmonic functions, respectively.

We start with a generalization of a known property, see [2, Theorem 3.3].

Theorem 1 (General Minimun principle). *If $u \in \mathcal{C}(\bar{F})$ is superharmonic on F then $\min_{x \in D_i(F)} \left\{\frac{u(x)}{\omega(x)}\right\} \leq \min_{x \in D_{i+1}(F)} \left\{\frac{u(x)}{\omega(x)}\right\}$ for any $i = 0, \ldots, r(F) - 1$. Moreover, if the equality holds then u is a multiple of ω on $\bigcup_{j=i}^{r(F)} D_j(F)$.*

The following property is a refined consequence of the above one and sometimes is named as *strong minimum principle*, see [3, Proposition 4.10].

Proposition 1 (Monotonicity Principle). *Given u a superharmonic function on F such that $u \geq 0$ on $\delta(F)$, then either $u > 0$ on F or $u = 0$ on $\bar{F}$.*

An important potential–theoretic problem in an electrical network, that is the discrete counterpart of the very same question in the continuous case, is the following: given a potential function $u \in \boldsymbol{C}(V)$ on V is it possible to sweep out the positive charges associated with u located outside F onto F, so that the potential function $v \in \boldsymbol{C}(V)$ associated with the new distribution of charges preserves the same values on F? This question is answered in the following result, see [1, Theorem 2.2.11].

Corollary 1 (Balayage (Sweeping–out) Principle). *If $u \in \boldsymbol{C}(V)$ is superharmonic on F^c, then there exists a unique $v \in \boldsymbol{C}(V)$ harmonic on F^c such that $v \leq u$ and $v = u$ on F. In addition, the total charge of u in F^c has been swept–out onto $\delta(F^c)$ since $\sum_{x \in (\overset{\circ}{F})^c} \omega(x)\mathcal{L}_q(u)(x) = \sum_{x \in \delta(F^c)} \omega(x)\mathcal{L}_q(v)(x)$.*

In the study of general electrical networks, the existence of a non–negative function with an associated point charge is very helpful. However, on a finite network such a function cannot be defined on V since $\langle \omega, \mathcal{L}_q(u) \rangle = 0$ for any $u \in \boldsymbol{C}(V)$. However, this function can be defined on any proper subset.

Theorem 2 (Green Function of a proper set). *For any $y \in F$ there exists a unique function $G_y^F \in \boldsymbol{C}^+(F)$ such that $\mathcal{L}_q(G_y^F) = \varepsilon_y$ on F. In addition, it satisfies that $0 < \omega(y)G_y^F(x) < \omega(x)G_y^F(y)$ for any $x \in F$.*

The kernel $G\colon F \times F \longrightarrow \mathbb{R}^+$ defined as $G^F(x,y) = G^F_y(x)$ for any $x, y \in F$ is called *Green function of* F, whereas $\Gamma^F\colon \boldsymbol{C}(F) \longrightarrow \boldsymbol{C}(F)$ defined for any $f \in \boldsymbol{C}(F)$ as $\Gamma^F(f)(x) = \sum_{y\in F} G^F(x,y)f(y)$, $x \in F$ is the *Green operator of* F. Then for any $f \in \boldsymbol{C}(F)$, the function $u = \Gamma^F(f) \in \boldsymbol{C}(F)$ satisfies that $\mathcal{L}_q(u) = f$ on F. The self–adjointness of Γ^F implies that G^F is symmetric.

As we explain above, a positive Green function defined on V has no sense. However, as $\mathcal{L}_q$ determines an isomorphism of $\omega^\perp = \{u \in \boldsymbol{C}(V) : \langle \omega, u\rangle = 0\}$, we can define the *Green operator of* Γ as the inverse of $\mathcal{L}_q$ on $\omega^\perp$. Clearly, this operator can be extended to $\boldsymbol{C}(V)$ by defining $\Gamma(f) = \Gamma(f - \omega\langle\omega, f\rangle)$ for any $f \in \boldsymbol{C}(V)$. If $G\colon V \times V \longrightarrow \mathbb{R}$ is the kernel of Γ; that is, given $x \in V$, $\Gamma(f)(x) = \sum_{y\in V} G(x,y)f(y)$, then for any $y \in V$ the function $G_y \in \boldsymbol{C}(V)$ is the unique function in $\omega^\perp$ such that $\mathcal{L}_q(G_y) = \varepsilon_y - \omega(y)\omega$ and hence is the potential corresponding to a signed charge that is positive at y, negative otherwise and with null total charge. Kernel G is called the *Green function of* V and it is symmetric, diagonally positive with some negative off–diagonal entries, since $\langle \omega, G_y\rangle = 0$. In addition $\Gamma \circ \mathcal{L}_q = \mathcal{L}_q \circ \Gamma = \boldsymbol{I} - \boldsymbol{P}_\omega$, where $\boldsymbol{I}$ is the Identity and $\boldsymbol{P}_\omega$ is the projection onto the subspace generated by ω.

From the matrix point of view, the above results say that for any proper connected subset F, the block of the Schrödinger matrix corresponding to F is an invertible M–matrix. In addition, the matrix corresponding to the Green function of V is the *Moore–Penrose inverse* of the matrix corresponding to the Schrödinger operator $\mathcal{L}_q$.

The following principle is closely related to the theory of Dirichlet Forms.

Proposition 2 (Condenser Principle). *Let $\{A, B\}$ a partition of $\delta(F)$, where $A, B \neq \emptyset$ and two real numbers $b < a$. If $u \in \boldsymbol{C}(\bar{F})$ is the unique solution of the boundary value problem $\mathcal{L}_q(u) = 0$ on F; $u = a\,\omega$ on A and $u = b\,\omega$ on B, then $a\,\omega < u < b\,\sigma$ on F, $\mathcal{L}_q(u) > 0$ on A and $\mathcal{L}_q(u) < 0$ on B.*

Notice that, if we allow $a = b$ then $u = a\,\omega$ and hence $\mathcal{L}_q(u) = 0$ on $\bar{F}$.

Under the hypotheses of the above proposition, $\bar{F}$ is called *condenser* with *positive and negative plates* A and B, respectively and the boundary value problem is called *the condenser problem corresponding to* $\bar{F}$.

Next, we introduce a concept that is closely related with the condenser problem, namely the *Effective Resistance* between two non empty subsets. Consider A, B two disjoint nonempty subsets of V such that $F = V \setminus (A \cup B)$ is connected and moreover $\delta(F) = A \cup B$. Therefore, V is a condenser with plates A and B. If, in addition, we choose the values $a = 1$ and $b = 0$, then the positive plate, say A, is called the *source* and the negative plate, B, is called the *sink*. If $u \in \boldsymbol{C}(V)$ is the solution of the corresponding condenser problem, then $\boldsymbol{E}_q(u,u) > 0$, otherwise u is a multiple of ω and hence, u can not verify $u = 0$ on B and $u = \omega$ on A, simultaneously.

The *effective resistance between* A, B*, with respect to* ω, is defined as the value $\mathsf{R}_\omega(A,B) = \mathcal{E}_q(u,u)^{-1}$. Clearly, the effective resistance, is a symmetric set function; that is, $\mathsf{R}_\omega(A,B) = \mathsf{R}_\omega(B,A)$ since $\mathcal{E}_q(u,u) = \mathcal{E}_q(\omega - u, \omega - u)$. So, it is irrelevant which plate acts as the source and which acts as the sink.

In the special case when $A = \{x\}$ and $B = \{y\}$ if u is the unique solution of corresponding condenser problem, then denoting $R_\omega(x,y) = R_\omega(\{x\},\{y\})$

$$R_\omega(x,y)^{-1} = \boldsymbol{E}_q(u,u) = \langle u(z), \mathcal{L}_q(u)\rangle = \omega(x)\ \mathcal{L}_q(u)(x) = -\omega(y)\ \mathcal{L}_q(u)(y),$$

and hence, $R_\omega(x,y)\,\mathcal{L}_q(u) = \omega(x)^{-1}\varepsilon_x - \omega(y)^{-1}\varepsilon_y$ on V. Therefore, if v is a solution of the Poisson equation $\mathcal{L}_q(v) = \omega(x)^{-1}\varepsilon_x - \omega(y)^{-1}\varepsilon_y$, then $R_\omega(x,y) = \mathcal{E}_q(v,v) = \omega(x)^{-1}v(x) - \omega(y)^{-1}v(y)$.

Since the last Poisson equation has sense for any $x, y \in V$ we can define the effective resistance between any pair of vertices from the above identity. In particular $R_\omega(x,y) = 0$ iff $x = y$ and moreover, R_ω determines a distance on V. In addition, given $x, y, z \in V$ the equality $R_\omega(x,y) + R_\omega(y,z) = R_\omega(x,z)$ iff y is a cut vertex on Γ, see [5, Corollary 4.4].

Now, we go back to Dirichlet problems, including a property of the support of the solution and solving the question on if non–negative charges determine positive potentials.

Proposition 3 (Generalized Equilibrium Principle). *For each* $f \in \boldsymbol{C}(F)$ *there exists a unique* $u \in \boldsymbol{C}(F)$ *such that* $\mathcal{L}_q(u) = f$ *on* F*. In addition, if* $f \geq 0$ *on* F*, then* $u \geq 0$ *on* F *and* $\mathsf{supp}(f) \subset \mathsf{supp}(u)$*.*

The above principle assures the existence of strictly superharmonic functions on F supported by F. In particular, we obtain the following result.

Corollary 2 (Equilibrium Principle). *There exists a unique* $\nu^F \in \boldsymbol{C}(F)$ *such that* $\mathcal{L}_q(u) = \omega$ *on* F*. Moreover* $\nu^F > 0$ *on* F*.*

Function ν^F is called *equilibrium measure of* F and the value $\mathsf{cap}(F) = \langle \omega, \nu^F\rangle = \boldsymbol{E}_q(\nu^F, \nu^F)$ is called the *capacity of* F. In particular, for any $x \in V$, the equilibrium measure and the capacity of $F = V \setminus \{x\}$ are denoted by ν^x and $\mathsf{cap}(x)$, respectively.

It seems reasonable to expect that equilibrium measures can be computed easily when F or V have many symmetries. The authors have computed the equilibrium measures of symmetric networks as stars, paths, circles and distance–regular graphs to name only a few, see for instance [6,7].

The role of equilibrium measures is crucial both in classical and modern potential theory. For finite networks, the equilibrium measures are useful to obtain the effective resistance and the Green functions. Specifically, see [3, Theorem 2.3] and [6, Theorem 3.6], we have obtained that

$$G^F(x,y) = \frac{\nu^F(x)}{\mathsf{cap}(F) - \mathsf{cap}(F\setminus\{y\})}\Big(\nu^F(x) - \nu^{F\setminus\{y\}}(x)\Big), \quad x,y\in V,$$
$$G(x,y) = \omega(x)\Big(\mathsf{cap}(x)\omega(y) - \nu^x(y)\Big), \qquad x,y\in V,$$
$$R_\omega(x,y) = \frac{\nu^x(y)}{\omega(y)} + \frac{\nu^y(x)}{\omega(x)}, \qquad x,y\in V.$$

The *Kirchhoff Index, with respect to* ω, also called total resistance of Γ, see [5,6] is the value $\mathsf{k}(\omega) = \frac{1}{2}\sum\limits_{x,y\in V} R_\omega(x,y)\omega^2(x)\omega^2(y)$. This is a global parameter of the network that was introduced in Organic Chemistry as a better alternative to other parameters used for discriminating among different molecules with similar shapes and structures; see [11]. A considerable amount of production has been developed and the Kirchhoff Index has been computed for some classes of graphs; see for instance [4,13] and the references therein.

The global character to the Kirchhoff index can be understood from the following identities $\mathsf{k}(\omega) = \sum\limits_{x\in V} G(x,x) = \sum\limits_{x\in V}\omega(x)^2\mathsf{cap}(x)$, where $\mathsf{k}(\omega)$ appears as the trace of the Green function and as the mean capacity of the network.

The effective resistance is related with the escape probability for a reversible Markov chain and the Kirchhoff index parameter is named the *Kemeny constant* of the chain. Therefore, the effective resistance with respect to ω correspond to a generalization of the escape probability. In turns, the equilibrium measure with respect to ω can be seen as a generalization of the *hitting time*. The probability laws governing the evolution of the chain are given by the (one step) *transition probability kernel* with respect to ω, that is defined for any states $x,y\in V$ as $\mathsf{P}_\omega(x,y) = \dfrac{c(x,y)\omega(y)}{\sum\limits_{z\in V} c(x,y)\omega(y)}$. Then, P is markovian, reversible and its *stationay distribution* is $\pi_\omega(x) = \omega(x)\sum\limits_{z\in V} c(x,y)\omega(y)$. If for any $k\in\mathbb{N}$, we consider $\mathsf{tr}_\omega(k) = \sum\limits_{x\in V}\mathsf{P}^k_\omega(x,x)$, the trace of the k–step transition probability kernel, then $\mathsf{tr}_\omega(0) = n$, whereas $\mathsf{tr}_\omega(1) = 0$. The following key result was proved in [6, Theorem 4.2].

Theorem 3 (Foster's Formula). *Given $k\in\mathbb{N}^*$, the following identity holds:*

$$\frac{1}{2}\sum_{x,y\in V} R_\omega(x,y)\tau_\omega(x)\mathsf{P}^k(x,y) = \sum_{j=0}^{k-1}\mathsf{tr}_\omega(j) - k.$$

We end back to the inverse M–matrix problem, that in the singular case ask when the Moore–Penrose inverse of a M–matrix inverse is also a M–matrix. When the combinatorial laplacian of Γ has this property, we say that Γ satisfies the M–property. Since the Green function of $\mathcal{L}_q$ is symmetric and positive semidefinite, it determines a M–matrix iff its off–diagonal entries are non–positive. From the expression of the Green function and applying the minimum

principle, we obtain that $\mathcal{L}_q$ has the M–property iff $\mathsf{cap}(x) \leq \omega(y)^{-1}\nu^x(y)$ for any $x \sim y$. In [8] this result was applied to tridiagonal matrix, obtaining that for any order there exists infinite singular, irreducible and tridiagonal matrices such that both, they and their Moore–Penrose inverse, are M–matrices. In [9] the authors have characterized which distance-regular graphs have the M–property. Roughly speaking, all distance–regular graphs with diameter one; about half of the strongly regular graphs; only some imprimitive distance–regular graphs with diameter three, and no distance–regular graphs with diameter greater than three, have the M–property. In addition, the conjecture that no primitive distance–regular graph with diameter three has the M–property has been almost solved in [12].

References

[1] V. Anandam. *Harmonic Functions and Potentials on Finite or Infinite Networks*. Lecture Notes of the Unione Matematica Italiana, 12. Springer, 2011.

[2] C. Araúz, A. Carmona, A.M. Encinas. Discrete Serrin's Problem. *Linear Algebra Appl.*, to appear.

[3] E. Bendito, A. Carmona, A.M. Encinas. Potential Theory for Schrödinger operators on finite networks. *Rev. Mat. Iberoam.*, 21: 771–818, 2005.

[4] E. Bendito, A. Carmona, A.M. Encinas, J.M. Gesto. A formula for the Kirchhoff index. *Int. J. Quantum Chem.*, 108: 1200–1206, 2008.

[5] E. Bendito, A. Carmona, A.M. Encinas, J.M. Gesto. Characterization of symmetric M–matrices as resistive inverses. *Linear Algebra Appl.*, 430: 1336–1349, 2009.

[6] E. Bendito, A. Carmona, A.M. Encinas, J.M. Gesto and M. Mitjana. Kirchhoff Indexes of a network. *Linear Algebra Appl.*, 432: 2278–2292, 2010.

[7] E. Bendito, A. Carmona, A.M. Encinas, and M. Mitjana. Generalized inverses of symmetric M–matrices. *Linear Algebra Appl.*, 432: 2438–2454, 2010.

[8] E. Bendito, A. Carmona, A.M. Encinas, and M. Mitjana. The M—matrix inverse problem for singular and symmetric Jacobi matrices. *Linear Algebra Appl.*, 436: 1090–1098, 2012.

[9] E. Bendito, A. Carmona, A.M. Encinas, and M. Mitjana. Distance–regular graphs having the M–property. *Linear Multilinear Algebra*, 60: 225–240, 2012.

[10] J. Deny. Les deux aspects of the théorie du potentiel. *Séminaire Bourbaki 1956–1958*, exp. n^o 148, 151–168, 1957.

[11] D.J. Klein, M. Randić. Resistance distance. *J. Math. Chem.*, 12, 81–95, 1993.

[12] J. Koolen, J. Park. A note on distance–regular graphs with a small number of vertices compared to the valency. *European J. Combin.*, 34, 935–940, 2013.

[13] Y. Yang, Y.H. Zhang. Kirchhoff index of linear hexagonal chains. *Int. J. Quantum Chem.*, 108, 503–512, 2008.

The Kirchhoff index of unicycle weighted chains

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Abstract. We consider a class of generalized linear chains, namely, the unicycle chains as a perturbation of a $2n$–path by adding one weighted edge between two specific vertices. This class of chains includes, in particular, the weighted cycle. In this work, we obtain the Green function, the effective resistance and the Kirchhoff index of the unicycle chains as a function of the Green function, the effective resistance and the Kirchhoff index of a path, respectively.

Key words: Unicycle network, Weighted cycle, Kirchhoff index, Green function, Effective Resistance.

1 Introduction

In the context of the study of the effect of a perturbation of a discrete elliptic operator on its orthogonal Green operator, here we consider the addition of a self–adjoint and positive semi–definite operator. That is, for a given base network we consider the addition of one specific edge. In [3], the authors analyzed the effect of a perturbation of a network by computing the effective resistance of perturbed networks through Sherman–Morrison–Woodbury like–formulas, instead of using the Sherman–Morrison formula recursively. We apply the obtained results to the study of the Green function and the Kirchhoff index of a weighted unicycle networks, and in particular for the weighted cycle.

Let $\Gamma = (V, E, c)$ be a connected finite network on the vertex set $V = \{1, 2, \ldots, n\}$ and edge set E, where each edge (i, j) has been assigned a *conductance* $c_{ij} > 0$. The *(weighted) degree* of vertex i is defined as $\delta_i = \sum_{j=1}^{n} c_{ij}$. The space of real valued functions on V is denoted by $\mathcal{C}(V)$ and for any $i \in V$, $\varepsilon_i \in \mathcal{C}(V)$ stands for the Dirac function at i. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle \cdot, \cdot \rangle$; that is, $\langle u, v \rangle = \sum_{i \in V} u(i)\, v(i)$ for each $u, v \in \mathcal{C}(V)$.

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The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{L}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(i) = \sum_{j \in V} c_{ij} \big(u(i) - u(j)\big), \quad i \in V. \tag{1}$$

The Laplacian is a singular, self–adjoint and positive semi–definite operator on $\mathcal{C}(V)$ and moreover, $\mathcal{L}(u) = 0$ iff u is a constant function.

Therefore, $\mathcal{L}$ is an isomorphism on the subspace orthogonal to constant functions, whose inverse extended to $\mathcal{C}(V)$ is denoted by $\mathcal{G}$ and called *Green operator* and whose kernel G is called *Green function*.

For any pair $i, j \in V$, the *dipole between i and j* is the function $\tau_{ij} = \varepsilon_i - \varepsilon_j$. Clearly, $\tau_{ij} = -\tau_{ji}$ and moreover, $\tau_{ij} = 0$ iff $i = j$.

Given $i, j \in V$, the *effective resistance between i and j* is defined as the value

$$R_{ij} = \langle \mathcal{G}(\tau_{ij}), \tau_{ij} \rangle = G(i,i) + G(j,j) - 2G(i,j). \tag{2}$$

It is known, see for instance [2], that the kernel R determines a metric on V. Moreover, for any $i, j, k \in V$ the triangular inequality $R_{ij} \leq R_{ik} + R_{kj}$ is an equality iff k separates i and j.

The *Kirchhoff index* of the network Γ is the value

$$\mathsf{k} = \frac{1}{2} \sum_{i,j \in V} R_{ij} = |V| \sum_{i \in V} G(i,i), \tag{3}$$

see [1,2].

For a given path P on $2n$ vertices, labelled as $V = \{1, \ldots, 2n\}$, the class of *generalized linear chains* supported by the path P, consists of all connected networks whose conductance satisfies $c_i := c_{i\,i+1} > 0$ for $i = 1, \ldots, 2n-1$, $a_i := c_{i\,2n+1-i} \geq 0$ for any $i = 1, \ldots, n-1$ and $c_{ij} = 0$ otherwise, see Figure 1.

A *unicycle chain* is a generalized linear chain such that there exists a unique h, $1 \leq h \leq n-1$, with $a_h > 0$. In particular, the $2n$–cycle corresponds to the case $h = 1$, i.e. $a_1 > 0$ and $a_j = 0$, $j = 2, \ldots, n-1$.

Let G and R be the Green function and the effective resistance of the path P. Since each interior vertex in a path is a cut vertex, the triangular equality holds

$$R_{ij} = R_{\min\{k,i\}\,\min\{k,j\}} + R_{\max\{k,i\}\,\max\{k,j\}}, \quad i, j, k = 1, \ldots, 2n. \tag{4}$$

Moreover, the authors proved in [8] that for any $i, j = 1, \ldots, 2n$, the Green function of a weighted path is

$$G_{ij} = \frac{1}{4n^2} \left[\sum_{k=1}^{\min\{i,j\}-1} \frac{k^2}{c_k} + \sum_{k=\max\{i,j\}}^{2n-1} \frac{(2n-k)^2}{c_k} - \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1} \frac{k(2n-k)}{c_k} \right],$$

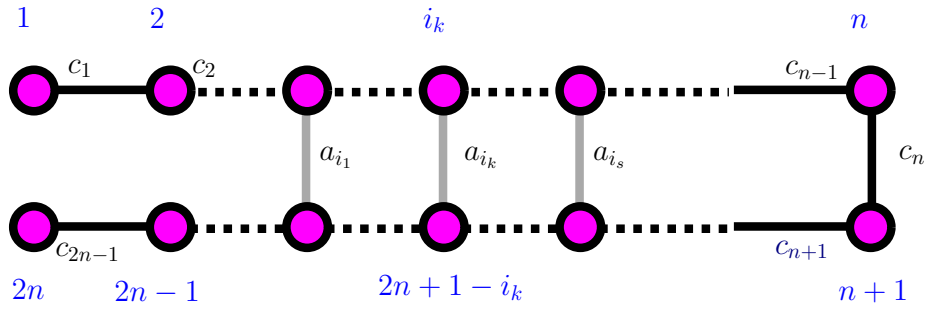

Fig. 1: A Generalized Linear Chain

where we use the usual convention that empty sums are defined as zero. Therefore, the effective resistance and the Kirchhoff index of the path are,

$$R_{ij} = \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1} \frac{1}{c_k}, \quad i,j = 1,\dots,2n \quad \text{and} \quad \mathsf{k} = \sum_{k=1}^{2n-1} \frac{k(2n-k)}{c_k}.$$

2 Unicycle chain network

The combinatorial Laplacian of a unicycle chain appears as the combinatorial Laplacian of the weighted path perturbed by the addition of exactly one edge with conductance a_h, between vertices h and $2n+1-h$, $1 \le h \le n-1$. Since we interpret a unicycle chain as a perturbation of the path by adding a single weighted edge, we use [3, Theorem 2.1] to obtain the Green function, G^{Γ}, the effective resistance, R^{Γ}, and the Kirchhoff index, k^{Γ}, of such a chain. In order to apply the result, we need to consider some previous tools. We define the coefficient M as

$$\mathsf{M} = \left(1 + a_h R_{h\,2n+1-h}\right)^{-1} = \frac{1}{a_h}\left[\frac{1}{a_h} + \sum_{j=h}^{2n-h} \frac{1}{c_j}\right]^{-1}.$$

Moreover, for any $j = 1,\dots,2n$, we consider $\mathsf{v}_j = \dfrac{\sqrt{a_h}}{2}\left[R_{2n+1-h\,j} - R_{h\,j}\right]$ and $\mathsf{u}_j = \mathsf{M}\mathsf{v}_j$, and also $\mathsf{r} = \dfrac{1}{2n}\sum_{j=1}^{2n} \mathsf{v}_j$.

According to the mentioned results and the previous notation, the Green function, the effective resistance and the Kirchhoff index of a unicycle chain are given, respectively, by the following expressions.

Theorem 1. *For any $i,j = 1,\dots,2n$, we get*

$$G^{\Gamma}_{ij} = G_{ij} - \mathsf{M}(\mathsf{r}-\mathsf{v}_i)(\mathsf{r}-\mathsf{v}_j) \quad \textit{and} \quad R^{\Gamma}_{ij} = R_{ij} - (\mathsf{u}_i - \mathsf{u}_j)(\mathsf{v}_i - \mathsf{v}_j).$$

In particular, the Kirchhoff index of the unicycle chain is given by

$$\mathsf{k}^{\Gamma} = \mathsf{k} + 4n^2\mathsf{Mr}^2 - 2n\sum_{j=1}^{2n}\langle \mathsf{u}_j, \mathsf{v}_j\rangle.$$

We introduce the following function $\phi_h\colon \{1,\dots,2n\} \longrightarrow \{h,\dots,2n+1-h\}$ for any $h = 1,\dots,n-1$,

$$\phi_h(j) = \begin{cases} h, & 1 \le j \le h, \\ j, & h \le j \le 2n+1-h, \\ 2n+1-h, & 2n+1-h \le j \le 2n. \end{cases}$$

Clearly, ϕ_h is nondecreasing and moreover, for any $i,j = 1,\dots,2n$, we have a useful version of Identity (4),

$$R_{ij} = R_{\min\{h,i\}\,\min\{h,j\}} + R_{\max\{2n+1-h,i\}\,\max\{2n+1-h,j\}} + R_{\phi_h(i)\,\phi_h(j)}. \tag{5}$$

Lemma 1. *The following equalities hold,*

$$\begin{aligned} \mathsf{v}_j &= \frac{\sqrt{a_h}}{2}\Big[R_{h\,2n+1-h} - 2R_{h\,\phi_h(j)}\Big], && 1 \le j \le n, \\ \mathsf{v}_j &= \frac{\sqrt{a_h}}{2}\Big[2R_{\phi_h(j)\,2n+1-h} - R_{h\,2n+1-h}\Big], && n+1 \le j \le 2n, \\ \mathsf{v}_i - \mathsf{v}_j &= \frac{\sqrt{a_h}}{2}R_{\phi_h(i)\,\phi_h(j)}, && 1 \le i \le j \le 2n. \end{aligned}$$

In addition, $-\mathsf{v}_{2n+1-h} = \mathsf{v}_h = \dfrac{\sqrt{a_h}}{2}R_{h\,2n+1-h}$, $\mathsf{v}_j = \mathsf{v}_h$ *and* $\mathsf{v}_{2n+1-j} = \mathsf{v}_{2n+1-h}$, *for any* $1 \le j \le h$.

Moreover, $\mathsf{r} = \dfrac{\sqrt{a_h}}{2n}\displaystyle\sum_{j=h}^{2n-h}\frac{(j-n)}{c_j}$.

Next, we can derive the expressions of the Green function, the effective resistance and the Kirchhoff index for a unicycle chain.

Proposition 1. *For any* $i,j = 1,\dots,2n$, *we get that*

$$\begin{aligned} G^{\Gamma}_{ij} &= \frac{1}{4n^2}\left[\sum_{k=1}^{\min\{i,j\}-1}\frac{k^2}{c_k} + \sum_{k=\max\{i,j\}}^{2n-1}\frac{(2n-k)^2}{c_k} - \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{k(2n-k)}{c_k}\right] \\ &\quad - \frac{1}{4n^2}\left[\frac{1}{c_{2n}} + \sum_{k=h}^{2n-h}\frac{1}{c_k}\right]^{-1}\left[\sum_{k=h}^{2n-h}\frac{k}{c_k} - 2n\sum_{k=\phi_h(i)}^{2n-h}\frac{1}{c_k}\right]\left[\sum_{k=h}^{2n-h}\frac{k}{c_k} - 2n\sum_{k=\phi_h(j)}^{2n-h}\frac{1}{c_k}\right] \\ R^{\Gamma}_{ij} &= \sum_{k=\min\{h,i,j\}}^{\min\{h,\max\{i,j\}\}-1}\frac{1}{c_k} + \sum_{k=\max\{2n+1-h,\min\{i,j\}\}}^{\max\{2n+1-h,i,j\}-1}\frac{1}{c_k} \\ &\quad + \left[\frac{1}{c_{2n}} + \sum_{k=h}^{2n-h}\frac{1}{c_k}\right]^{-1}\left[\sum_{k=\phi_h(\min\{i,j\})}^{\phi_h(\max\{i,j\})-1}\frac{1}{c_k}\right]\left[\frac{1}{c_{2n}} + \sum_{k=h}^{\phi_h(\min\{i,j\})-1}\frac{1}{c_k} + \sum_{k=\phi_h(\max\{i,j\})}^{2n-h}\frac{1}{c_k}\right]. \end{aligned}$$

In particular,

$$\mathsf{k}^{\Gamma} = \sum_{k=1}^{2n} \frac{k(2n-k)}{c_k} + \left[\frac{1}{c_{2n}} + \sum_{k=h}^{2n-h} \frac{1}{c_k}\right]^{-1} \left[\left[\sum_{k=h}^{2n-h} \frac{k}{c_k}\right]^2 - 2n \sum_{k=h}^{2n-h} \left[\sum_{m=k}^{2n-h} \frac{1}{c_m}\right]^2 - 2n(h-1)\left[\sum_{k=h}^{2n-h} \frac{1}{c_k}\right]^2\right].$$

Proof. For any $i, j = 1, \ldots, 2n$, we have

$$G_{ij}^{\Gamma} = G_{ij} - \mathsf{M}\,(\mathsf{r} - \mathsf{v}_i)(\mathsf{r} - \mathsf{v}_j) \text{ and } R_{ij}^{\Gamma} = R_{ij} - \mathsf{M}(\mathsf{v}_i - \mathsf{v}_j)^2.$$

The expression for the Green function is a consequence of the last identity in Lemma 1, whereas the expression for the effective resistance appears as a consequence of the mentioned Lemma and Identity (5).

Finally, since $\mathsf{k}^{\Gamma} = 2n \sum_{j=1}^{2n} G_{jj}^{\Gamma}$, we have

$$\mathsf{k}^{\Gamma} = \mathsf{k} - \frac{1}{2n}\left[\frac{1}{c_{2n}} + \sum_{k=h}^{2n-h} \frac{1}{c_k}\right]^{-1} \sum_{j=1}^{2n} \left[\sum_{m=h}^{2n-h} \frac{m}{c_m} - 2n \sum_{m=\phi_h(j)}^{2n-h} \frac{1}{c_m}\right]^2.$$

On the other hand,

$$\begin{aligned}
&\sum_{j=1}^{2n} \left[\sum_{m=h}^{2n-h} \frac{m}{c_m} - 2n \sum_{m=\phi_h(j)}^{2n-h} \frac{1}{c_m}\right]^2 = \\
&\quad = 2n \left[\sum_{m=h}^{2n-h} \frac{m}{c_m}\right]^2 - 4n \left[\sum_{m=h}^{2n-h} \frac{m}{c_m}\right] \left[\sum_{j=1}^{2n} \sum_{m=\phi_h(j)}^{2n-h} \frac{1}{c_m}\right] + 4n^2 \sum_{j=1}^{2n} \left[\sum_{m=\phi_h(j)}^{2n-h} \frac{1}{c_m}\right]^2 \\
&\quad = 4n^2(h-1)\left[\sum_{m=h}^{2n-h} \frac{1}{c_m}\right]^2 + 4n^2 \sum_{j=h}^{2n-h} \left[\sum_{m=j}^{2n-h} \frac{1}{c_m}\right]^2 - 2n \left[\sum_{m=h}^{2n-h} \frac{m}{c_m}\right]^2,
\end{aligned}$$

since

$$\sum_{j=1}^{2n} \sum_{m=\phi_h(j)}^{2n-h} \frac{1}{c_m} = h \sum_{m=h}^{2n-h} \frac{1}{c_m} + \sum_{j=h+1}^{2n-h} \sum_{m=j}^{2n-h} \frac{1}{c_m} = \sum_{m=h}^{2n-h} \frac{m}{c_m},$$

$$\sum_{j=1}^{2n} \left[\sum_{m=\phi_h(j)}^{2n-h} \frac{1}{c_m}\right]^2 = (h-1)\left[\sum_{m=h}^{2n-h} \frac{1}{c_m}\right]^2 + \sum_{j=h}^{2n-h} \left[\sum_{m=j}^{2n-h} \frac{1}{c_m}\right]^2.$$

and hence, the expression for the Kirchhoff index follows. □

In particular, if $c_k = c$ for all $k = 1, \ldots, 2n-1$ and $c_h = a$ the Green function, for any $i, j = 1, \ldots, 2n$, of this particular unicycle is

$$G^{\Gamma}_{ij} = \frac{1}{12nc}\Big[(2n+1)(4n+1) + 3\Big(i(i-2n-1) + j(j-2n-1) - 2n|i-j|\Big)\Big] -$$
$$- \frac{a(2\phi_h(i)-2n-1)(2\phi_h(j)-2n-1)}{4c\big(c+a+2a(n-h)\big)}$$

Moreover, the Kirchooff index of the unicycle chain is

$$\mathsf{k}^{\Gamma} = \frac{n\left[(4n^2-1)c + a(2(n-h)+1)(n(2(n-h)+1)+4h(h-1)-1)\right]}{3c(c+a(2(n-h)+1)}.$$

3 The weighted cycle

Next, we particularize the above theorem to $h = 1$ that corresponds to the $2n$–cycle. Although the case of cycles with constant weight and conductances is well–known, see for instance [4], as far as authors' knowledge, this is the first time that the orthogonal Green function for a weighted cycle is obtained.

Corollary 1. *The Green function and the effective resistance for the weighted $2n$–cycle are given by*

$$G^{\Gamma}_{ij} = \frac{1}{4n^2}\left[\sum_{k=1}^{\min\{i,j\}-1}\frac{k^2}{c_k} + \sum_{k=\max\{i,j\}}^{2n}\frac{(2n-k)^2}{c_k} - \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{k(2n-k)}{c_k}\right]$$
$$\frac{1}{4n^2}\left[\sum_{k=1}^{2n}\frac{1}{c_k}\right]^{-1}\left[\sum_{k=1}^{2n}\frac{k}{c_k} - 2n\sum_{k=i}^{2n}\frac{1}{c_k}\right]\left[\sum_{k=1}^{2n}\frac{k}{c_k} - 2n\sum_{k=j}^{2n}\frac{1}{c_k}\right],$$
$$R^{\Gamma}_{ij} = \left[\sum_{k=1}^{2n}\frac{1}{c_k}\right]^{-1}\left[\sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{1}{c_k}\right]\left[\sum_{k=1}^{\min\{i,j\}-1}\frac{1}{c_k} + \sum_{k=\max\{i,j\}}^{2n}\frac{1}{c_k}\right].$$

Moreover, the Kirchhoff index is given by

$$\mathsf{k}^{\Gamma} = \sum_{k=1}^{2n}\frac{k(2n-k)}{c_k} + \left[\sum_{k=1}^{2n}\frac{1}{c_k}\right]^{-1}\left[\left[\sum_{k=1}^{2n}\frac{k}{c_k}\right]^2 - 2n\sum_{j=1}^{2n}\left[\sum_{k=j}^{2n}\frac{1}{c_k}\right]^2\right].$$

In particular, if $c_k = c$ for all $k = 1, \ldots, 2n-1$ and $c_{2n} = a$, we get

$$G^{\Gamma}_{ij} = \frac{1}{12nc}\Big[(2n+1)(4n+1) + 3\Big(i(i-2n-1) + j(j-2n-1) - 2n|i-j|\Big)\Big]$$
$$\frac{a}{4c\big[c+a(2n-1)\big]}\big[2(n-i)+1\big]\big[2(n-j)+1\big],$$
$$R^{\Gamma}_{ij} = \frac{|i-j|}{c\big[c+a(2n-1)\big]}\Big(c + a\big(2n-1-|i-j|\big)\Big),$$

and hence,

$$\mathsf{k}^{\Gamma} = \frac{n(4n^2-1)\big(c+a(n-1)\big)}{3c\big(c+a(2n-1)\big)}.$$

The last expression, when $a = c$ coincides, as expected, with the one obtained in [4].

References

[1] E. Bendito, A. Carmona, A. M. Encinas, and J. M. Gesto. Characterization of symmetric M–matrices as resistive inverses. *Linear Algebra Appl.*, 430(4):1336–1349, 2009.

[2] E. Bendito, A. Carmona, A. M. Encinas, J. M. Gesto, and M. Mitjana. Kirchoff indexes of a network. *Linear Algebra Appl.*, 432(9):2278–2292, 2010.

[3] A. Carmona, A.M. Encinas, and M. Mitjana. Green matrices associated with generalized linear polyominoes. *Linear Algebra and its Applications*, to appear.

[4] Robert B. Ellis. *Chip–firing games with Dirichlet eigenvalues and discrete Green's functions.* PhD thesis, University of California, San Diego, 2002.

Estructuras combinatorias asociadas a algunas álgebras no asociativas de dimensión finita

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Resumen. En el presente trabajo, mostramos métodos algorítmicos para asociar estructuras combinatorias tanto a las álgebras de Lie como a las de Leibniz de dimensión finita. Comparamos ambos procedimientos y estudiamos las estructuras asociadas a estas álgebras para dimensión 3.

Palabras clave: Digrafo, pseudodigrafo, estructura combinatoria, álgebra de Leibniz, álgebra de Lie.

1 Introducción

Las aplicaciones de las álgebras no asociativas abarcan múltiples campos de conocimiento, que van desde la Ingeniería y la Física a la Matemática Aplicada, siendo muy amplia la investigación realizada hasta la fecha sobre dichas aplicaciones. Entre las álgebras más relevantes a este respecto, caben destacar las álgebras de Lie y las de Leibniz. Estas últimas fueron introducidas a principios de la década de 1990 por J.L. Loday [6] y son una generalización no conmutativa de las primeras. Por este motivo, muchos de los resultados conocidos para álgebras de Lie pueden ser generalizados y aplicados a las álgebras de Leibniz. No obstante, quedan diversos aspectos por estudiar en referencia a ambos tipos de álgebras, como puede ser la obtención de una clasificación completa de las álgebras resolubles y nilpotentes en ambos casos.

Por su parte, la Teoría de Grafos se ha vuelto esencial para la resolución de múltiples problemas que van surgiendo en diferentes áreas de investigación. Por este motivo, creemos que esta teoría podría permitir un mejor estudio de las álgebras no asociativas y, llegado el caso, resolver problemas aún abiertos, como puede ser el de clasificación antes indicado. Por tanto, nuestro principal objetivo es continuar estudiando la relación entre las estructuras combinatorias (grafos y complejos simpliciales) y las álgebras no asociativas (de Lie y de

Leibniz), siguiendo la línea comenzada en [1] y profundizada posteriormente en [2,3,4] para el caso de las álgebras de Lie. En esta ocasión, generalizamos el método visto en [1] para asociar estructuras combinatorias sobre álgebras de Lie pero para el caso de las álgebras de Leibniz, haciendo una primera comparativa entre ambos métodos.

2 Preliminares

A continuación, presentamos algunos conceptos preliminares sobre álgebras de Lie y de Leibniz, pudiéndose consultar [6,7] para revisión general del tema. En este trabajo, sólo consideramos álgebras de dimensión finita sobre el cuerpo de los números complejos.

Definición 1. *Un* álgebra de Leibniz $\mathcal{L}$ *es un espacio vectorial con una segunda ley interna bilineal* ($[\cdot,\cdot]$)*, llamada producto corchete, que verifica la llamada identidad de Leibniz:*

$$L(X,Y,Z) = [[X,Y],Z] - [[X,Z],Y] - [X,[Y,Z]] = 0,\ \forall\, X,Y,Z \in \mathcal{L}.$$

Si además $[X,X] = 0$ *para todo* $X \in \mathcal{L}$*, entonces* $\mathcal{L}$ *es un* álgebra de Lie*. En este caso, se verifica* $[X,Y] = -[Y,X]$ *para todo* $X,Y \in \mathcal{L}$ *y la identidad de Leibniz es equivalente a la identidad de Jacobi.*

Una base $\{e_i\}_{i=1}^n$ *de* $\mathcal{L}$ *se caracteriza por sus constantes de estructura o de Maurer-Cartan:* $[e_i,e_j] = \sum c_{i,j}^h e_h$*, para* $1 \le i,j \le n$*.*

Definición 2. *El centro de un álgebra de Leibniz* $\mathcal{L}$ *se define como sigue:*

$$Z(\mathcal{L}) = \{X \in \mathcal{L} \mid [X,Y] = 0 = [Y,X],\ \forall\, Y \in \mathcal{L}\}.$$

Definición 3. *Dada un álgebra de Leibniz* $\mathcal{L}$*, su* sucesión de resolubilidad *es:*

$$\mathcal{L}_1 = \mathcal{L},\ \mathcal{L}_2 = [\mathcal{L},\mathcal{L}],\ \ldots,\ \mathcal{L}_k = [\mathcal{L}_{k-1},\mathcal{L}_{k-1}],\ \ldots$$

Diremos que $\mathcal{L}$ *es* resoluble *si existe* $m \in \mathbb{N}$ *tal que* $\mathcal{L}_m = \{0\}$*. Además, si* $\mathcal{L}_{m-1} \neq \{0\}$*, entonces se dice que* $\mathcal{L}$ *es resoluble de paso* $(m-1)$ *o que* $\mathcal{L}$ *tiene índice de resolubilidad* m*.*

Definición 4. *Dada un álgebra de Leibniz* $\mathcal{L}$*, su* sucesión de nilpotencia *es:*

$$\mathcal{L}^1 = \mathcal{L},\ \mathcal{L}^2 = [\mathcal{L},\mathcal{L}],\ \ldots,\ \mathcal{L}^k = [\mathcal{L}^{k-1},\mathcal{L}],\ \ldots$$

Diremos que $\mathcal{L}$ *es* nilpotente *si existe* $m \in \mathbb{N}$ *tal que* $\mathcal{L}^m = \{0\}$*. Además, si* $\mathcal{L}^{m-1} \neq \{0\}$*, entonces* $\mathcal{L}$ *se dice nilpotente de paso* $(m-1)$ *o con índice de nilpotencia* m*.*

Observación 1. Toda álgebra nilpotente es también resoluble, pues $\mathcal{L}_i \subseteq \mathcal{L}^i$ para todo $i \in \mathbb{N}$.

Los conceptos de Teoría de Grafos empleados son ampliamente conocidos y pueden consultarse en [5].

3 Asociando estructuras combinatorias a álgebras de Lie

Dado un álgebra de Lie $\mathfrak{g}$ de dimensión n, con base $\mathcal{B} = \{e_i\}_{i=1}^n$ y ley dada por $[e_i, e_j] = \sum_{k=1}^n c_{i,j}^k e_k$, podemos asociarle al par $(\mathfrak{g}, \mathcal{B})$ una estructura combinatoria siguiendo el método introducido en [1]:

a) Dibuja un vertice i, para cada vector $e_i \in \mathcal{B}$.
b) Dados tres vértices $i < j < k$, dibuja un triángulo lleno $\{i, j, k\}$ si y sólo si $(c_{i,j}^k, c_{j,k}^i, c_{i,k}^j) \neq (0, 0, 0)$. Las aristas $\{i, j\}$, $\{j, k\}$ e $\{i, k\}$ tienen pesos $c_{i,j}^k$, $c_{j,k}^i$, y $c_{i,k}^j$, respectivamente (véase Figura 1):
   b1) Usa una línea discontinua (denominada *arista fantasma*) para las aristas con peso cero.
   b2) Si dos triángulos $\{i, j, k\}$ e $\{i, j, l\}$, con $1 \leq i < j < k < l \leq n$, satisfacen $c_{i,j}^k = c_{i,j}^l$, entonces dibuja sólo una arista $\{i, j\}$, que será compartida por ambos triángulos (véase Figura 2).

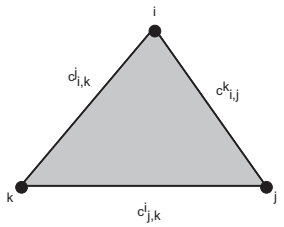

Figura 1: Triángulo lleno.

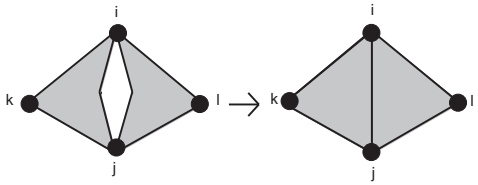

Figura 2: Compartiendo una arista.

c) Dados dos vértices i y j con $1 \leq i < j \leq n$ y tal que $c_{i,j}^i \neq 0$ (resp. $c_{i,j}^j \neq 0$), dibuja una arista dirigida de j a i (resp. de i a j); como ilustra Figura 3.

Obsérvese que estas estructuras combinatorias pueden no corresponder a digrafos, sino que pueden ser 2-complejos simpliciales. En caso de que un vértice v no sea incidente a triángulos llenos, el vértice v será un *sumidero* (resp. una *fuente*) si todas las aristas incidentes con v están orientadas hacia v (resp. desde v). Estas nociones están representadas en la Figura 4.

Figura 3: Aristas dirigidas.

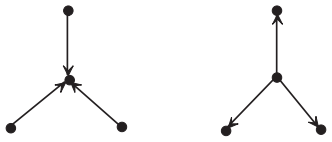

Figura 4: Vértices sumidero y fuente.

4 Asociando estructuras combinatorias a álgebras de Leibniz

Sea $\mathcal{L}$ un álgebra de Leibniz de dimensión n con base $\mathcal{B} = \{e_i\}_{i=1}^n$ y ley $[e_i, e_j] = \sum_{h=1}^n c_{i,j}^h e_h$. El par $(\mathcal{L}, \mathcal{B})$ se puede asociar a la estructura combinatoria obtenida con el siguiente procedimiento:

a) Dibuja un vértice i para cada vector $e_i \in \mathcal{B}$.
b) Dibuja un lazo sobre cada vértice i que satisfaga $[e_i, e_i] \neq 0$. Como puede verse en Figura 5, el peso del lazo viene dado por el vector $(c_{i,i}^1, c_{i,i}^2, \ldots, c_{i,i}^n)$.

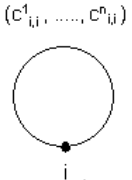

Figura 5: Lazo sobre el vértice i.

c) Dados tres vértices $i < j < k$ tales que $(c_{i,j}^k, c_{j,i}^k, c_{j,k}^i, c_{k,j}^i, c_{i,k}^j, c_{k,i}^j) \neq (0,0,0,0,0,0)$, dibuja un triángulo lleno $\{i,j,k\}$ en el que las aristas $\{i,j\}$, $\{j,k\}$ e $\{i,k\}$ tienen pesos $(c_{i,j}^k, c_{j,i}^k)$, $(c_{j,k}^i, c_{k,j}^i)$ y $(c_{i,k}^j, c_{k,i}^j)$, respectivamente (véase Figura 6). Además:
   c1) Usa una línea discontinua (llamada *arista fantasma*) para aristas con peso $(0,0)$.
   c2) Si dos triángulos $\{i,j,k\}$ e $\{i,j,l\}$ verifican $(c_{i,j}^k, c_{j,i}^k) = (c_{i,j}^l, c_{j,i}^l)$, dibuja sólo una arista $\{i,j\}$, que será compartida por ambos triángulos (véase Figura 7).

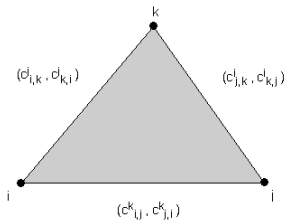

Figura 6: Triángulo lleno.

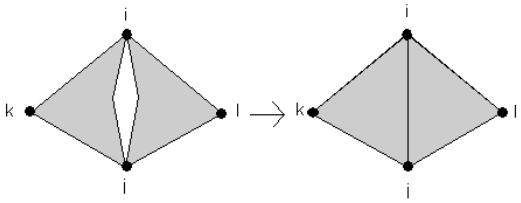

Figura 7: Compartiendo una arista.

d) Dados dos vértices $i < j$ verificando $(c_{i,j}^j, c_{j,i}^j) \neq (0,0)$, dibuja un arista dirigida de i a j cuyo peso está dado por el par $(c_{i,j}^j, c_{j,i}^j)$. Si $(c_{i,j}^i, c_{j,i}^i) \neq (0,0)$, dibuja otra arista dirigida, de j a i cuyo peso es $(c_{i,j}^i, c_{j,i}^i)$.

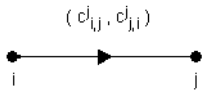

Figura 8: Arista dirigida.

Por tanto, fijada una base, todo álgebra de Leibniz puede asociarse a una estructura combinatoria, consistente en un complejo simplicial de dimensión menor que 3. Nótese que un vértice aislado en dicha estructura correspondería a un vector de la base del álgebra perteneciente al centro de la misma.

Este método para álgebras de Leibniz generaliza al procedimiento visto en la Sección 3 para álgebras de Lie ya que, como muestra el siguiente resultado, ni existirían lazos ni sería necesario usar vectores para los pesos de las aristas.

Proposición 1. *La estructura combinatoria de un álgebra de Leibniz que también sea un álgebra de Lie verifica las siguientes condiciones:*

1. *No hay lazos.*
2. *El peso de las aristas del vértice i al vértice j viene dado por $(c^j_{i,j}, -c^j_{i,j})$.*
3. *El peso de las aristas en un triángulo lleno $\{i,j,k\}$ viene dado por $(c^k_{i,j}, -c^k_{i,j})$, $(c^i_{j,k}, -c^i_{j,k})$ y $(c^j_{i,k}, -c^j_{i,k})$.*

5 (Pseudo)digrafos conexos asociados a álgebras de Lie y de Leibniz de dimensión 3

En primer lugar, recordamos el lema que permite determinar las configuraciones que no pueden estar contenidas como subdigrafos en los digrafos asociados a álgebras de Lie en virtud de lo visto en [1]:

Lema 1. *[1, Lemma 3.1] Si $\mathfrak{g}$ es un álgebra de Lie asociada a un digrafo G, entonces las configuraciones mostradas en Figura 9 no pueden ser subdigrafos de G; i.e. están prohibidas para cualesquiera tres vértices i, j, k (independientemente del peso de las aristas).*

Es más, este resultado puede generalizarse para el caso de las álgebras de Leibniz como sigue:

Lema 2. *Todo (pseudo)digrafo que contenga una de las configuraciones de la Figura 9 como subdigrafo, no puede estar asociado a un álgebra de Leibniz.*

En virtud del Lemma 1, solo existen cuatro digrafos de 3 vértices que pueden estar asociados a álgebras de Lie (véase Figura 10) según lo probado en [1]. Este resultado es también generalizable al caso de las álgebras de Leibniz y a estos cuatro digrafos de 3 vértices construidos a partir del método para

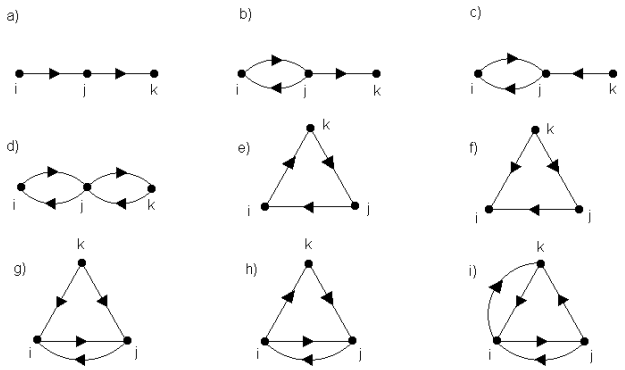

Figura 9: Configuraciones prohibidas para álgebras de Lie y de Leibniz.

las álgebras de Leibniz. La única diferencia significativa es que para las tres primeras configuraciones dependiendo de los coeficientes de estructuras, el álgebra de Leibniz de dimensión 3 puede ser de Lie o no; mientras que la cuarta estructura cuando está asociada a un álgebra de Leibniz, ésta tiene que ser de Lie.

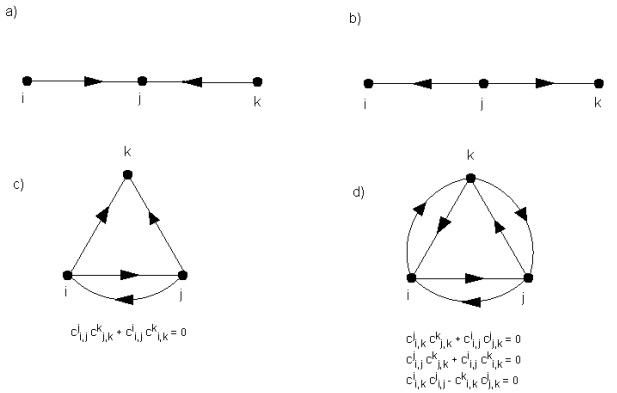

Figura 10: Digrafos conexos asociados a álgebras de Lie y Leibniz de dimensión 3.

Si a estas cuatro estructuras (que son las únicas que no están prohibidas como subdigrafos de un pseudodigrafo asociado a un álgebra de Leibniz) le añadimos lazos en los vértices, podemos obtener los pseudodigrafos conexos asociados a álgebras de Leibniz de dimensión 3. En este caso, no están permitidas todas las posibles configuraciones obtenidas al añadir lazos como subpseudodigrafos del pseudodigrafo de partida. Esto se puede resumir en la siguiente:

Proposición 2. *Si G es un pseudodigrafo conexo de* 3 *vértices y asociado a un álgebra de Leibniz, entonces G debe ser isomorfo a alguna de las configuraciones mostradas en la Figura 11.*

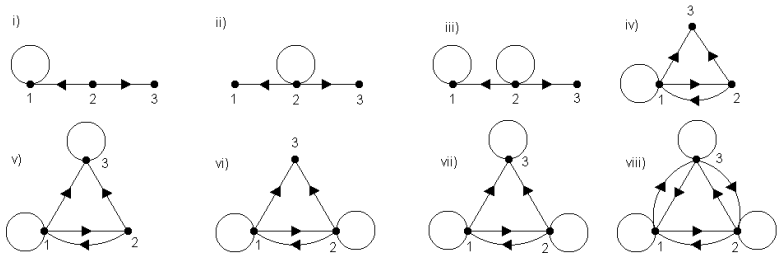

Figura 11: Pseudodigrafos conexos de 3 vértices asociados a álgebras de Leibniz.

Corolario 1. *Los pseudodigrafos conexos de 3 vértices mostrados en la Figura 12 no pueden estar contenidos como subpseudodigrafos de un pseudodigrafo asociado a un álgebra de Leibniz (i.e. son configuraciones prohibidas).*

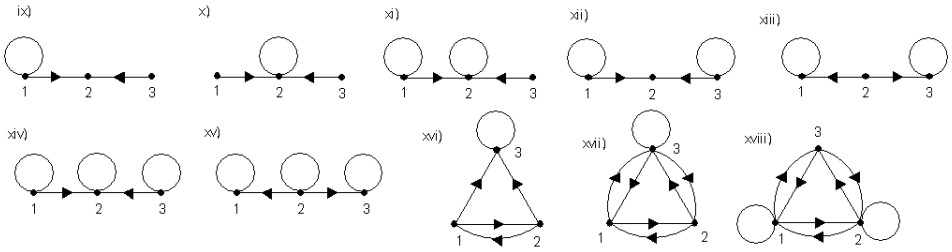

Figura 12: Pseudodigrafos conexos de 3 vértices no asociados a álgebras de Leibniz.

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Referencias

[1] A. Carriazo, L. M. Fernández and J. Núñez. Combinatorial structures associated with Lie algebras of finite dimension. *Linear Algebra Appl.*, 389: 43–61, 2004.

[2] J. Cáceres, M. Ceballos, J. Núñez, M.L. Puertas and A.F. Tenorio. Combinatorial structures of three vertices and Lie algebras. *Int. J. Comput. Math.*, 89: 1879–1900, 2012.

[3] M. Ceballos, J. Núñez and A.F. Tenorio. Complete triangular structures and Lie algebras. *Int. J. Comput. Math.*, 88: 1839–1851, 2011.

[4] M. Ceballos, J. Núñez and A.F. Tenorio. Study of Lie algebras by using combinatorial structures. *Linear Algebra Appl.*, 436(2): 349–363, 2012.

[5] F. Haray. Graph Theory. *Addison-Wesley*, Reading, 1969.

[6] J.L. Loday. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign. Math. (2)*, 239: 269–293, 1993.

[7] V.S. Varadarajan. *Lie Groups, Lie Algebras and Their Representations.* Springer, Reading, New York, 1984.

Connected covering numbers *

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Abstract. A connected covering is a design system in which the corresponding block graph is connected. The minimum size of such coverings are called connected coverings numbers. In this work, we present various formulas and bounds for several parameter settings for these numbers. We also investigate results in connection with Turán systems. Finally, a new general upper bound, improving an earlier result, is given. The latter is used to improve upper bounds on a question concerning oriented matroid due to Las Vergnas

Key words: Covering design, Turán-system, uniform oriented matroid

1 Introduction

Let n, k, r be positive integers such that $n \geq k \geq r \geq 1$. A (n, k, r)-*covering* is a family $\mathcal{B}$ of k-subsets of $\{1, \ldots, n\}$, called *blocks*, such that each r-subset of $\{1, \ldots, n\}$ is contained in at least one of the blocks. The number of blocks is the covering's *size*. The minimum size of such a covering is called the *covering number* and is denoted by $C(n, k, r)$. Given a (n, k, r)-covering $\mathcal{B}$, its graph $G(\mathcal{B})$ has as vertices $\mathcal{B}$ and two vertices are joined if they have one r-subset in common. We say that a (n, k, r)-covering is *connected* if the graph $G(\mathcal{B})$ is connected. The minimum size of a connected (n, k, r)-covering is called the *connected covering number* and is denoted by $CC(n, k, r)$.

The graph corresponding to a connected $(7, 4, 3)$-covering is illustrated in Figure 1. In this work, we mainly focus our attention to coverings when $k = r + 1$ and thus, we will denote $C(n, r + 1, r)$ (resp. $CC(n, r + 1, r)$) by $C(n, r)$ (resp. by $CC(n, r)$) for short. The original motivation to study $CC(n, r)$ comes from the following question posed by Michel Las Vergnas.

Question 1. Let $U_{r,n}$ be the rank r uniform matroid on n elements. What is the smallest number $s(n, r)$ of circuits of $U_{r,n}$, that uniquely determines all

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orientations of $U_{r,n}$? This is, whenever two uniform oriented matroids coincide on these circuits they must be equal.

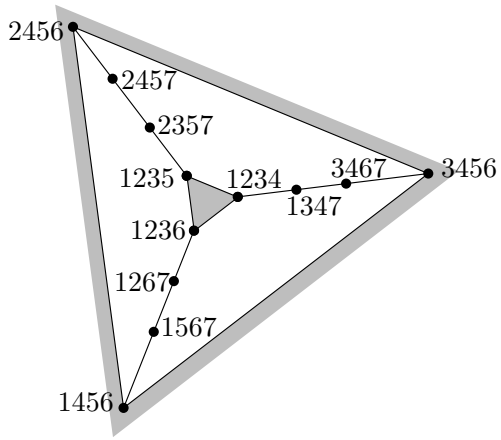

Fig. 1: A connected $(7, 4, 3)$-covering with 12 blocks.

In [2], Forge and Ramírez Alfonsín introduced the notion of connected coverings and proved that

$$s(n, r) \leq CC(n, r). \tag{1}$$

The latter was then used to improve the best upper bound, $s(n, r) \leq \binom{n-1}{r}$, known at that time due to Hamidoune and Las Vergnas [6]; see also [3] for related results.

It turns out that $s(n, r)$ is also closely related to $C(n, r)$.

Proposition 1. *Let n and r be positive integers such that $n \geq r + 1$. Then,*

$$C(n, r) \leq s(n, r).$$

Although the study of covering designs have already a long history of research (see [5] for many upper bounds and [13] for a survey in the dual setting of Turán-systems), Proposition 1 motivates even further the study of $C(n, r)$ and specially, in view of (1), its behavior in relation to $CC(n, r)$.

This relationship was already remarked in [2] where it was proved that

$$CC(n, r) \leq 2C(n, r). \tag{2}$$

That was done by observing that the graph G associated to a covering with $C(n, r)$ blocks (and thus with $|V(G)| = C(n, r)$) can be make it connected by adding at most $C(n, r)$ extra vertices (blocks), obtaining a graph corresponding to a $(n, r + 1, r)$-connected covering with at most $2C(n, r)$ blocks.

Many interesting variants of Question 1 can be investigated. For instance, for non-uniform (oriented) matroids (graphic, representable, etc.) and by varying the notion of what *determine* means (up to orientations, bijections, etc.). These (and other) variants are treated in another paper (see [9]).

2 Basic results

Let n, m, p be positive integers such that $n \geq m \geq p$. A (n, m, p)-*Turán-system* is a family $\mathcal{D}$ of p-subsets of $\{1, \ldots, n\}$, called *blocks*, such that each m-subset of $\{1, \ldots, n\}$ contains at least one of the blocks. The number of blocks is the *size* of the Turán-system. The minimum size of such a covering is called the *Turán Number* and is denoted by $T(n, m, p)$. Given a (n, m, p)-Turán-system $\mathcal{D}$, with $0 \leq 2p - m \leq p$, its graph $G(\mathcal{D})$ has as vertices $\mathcal{D}$ and two vertices are joined if they have one $2p - m$-subset in common. We say that a (n, m, p)-Turán-system with $0 \leq 2p - m \leq p$ is *connected* if $G(\mathcal{D})$ is connected.

The minimum size of a connected (n, m, p)-Turán-system is the *connected Turán Number* and is denoted by $CT(n, m, p)$. By applying set complement to blocks, it can be obtained that

$$C(n, k, r) = T(n, n - r, n - k). \tag{3}$$

Moreover, if $0 \leq n - 2k + r \leq n - k$ then

$$CC(n, k, r) = CT(n, n - r, n - k). \tag{4}$$

Note that the precondition for (4) is fulfilled if $k = r + 1$.

Most of the papers on coverings consider n large compared with k and r, while for Turán numbers it has frequently been considered n large compared with m and p, and often focusing on the quantity $\lim_{n\to\infty} T(n, m, p)/\binom{n}{p}$ for fixed m and p. Thus, for Turán-type problems, the value $C(n, k, r)$ has usually been studied in the case when k and r are not too far from n.

Forge and Ramírez Alfonsín [2] proved that

$$CC(n, r) \geq \frac{\binom{n}{r} - 1}{r} =: CC_1^*(n, r). \tag{5}$$

Moreover, Sidorenko [14] proved that $T(n, r + 1, r) \geq \left(\frac{n-r}{n-r+1}\right) \frac{\binom{n}{r}}{r}$, which by (3), we obtain that

$$CC(n, r) \geq C(n, r) = T(n, n-r, n-r-1) \geq \left(\frac{r+1}{r+2}\right) \frac{\binom{n}{r+1}}{n-r-1} =: CC_2^*(n, r). \tag{6}$$

Combining (5) and (6), together with a straight forward computation we have

$$CC(n,r) \geq \max\{CC_1^*(n,r), CC_2^*(n,r)\}, \tag{7}$$

where the maximum is attained by the second term if and only if $r \geq \frac{2}{3}(n-1)$.

The following recursive lower bound for covering numbers was obtained by Schönheim [12] and, independently, by Katona, Nemetz and Simonovits [8]

$$C(n,r) \geq \left\lceil \frac{n}{r+1} C(n-1,r-1) \right\rceil \tag{8}$$

which can be iterated yielding to

$$C(n,r) \geq \left\lceil \frac{n}{r+1} \left\lceil \frac{n-1}{r} \left\lceil \dots \left\lceil \frac{n-r+1}{2} \right\rceil \dots \right\rceil \right\rceil \right\rceil =: L(n,r). \tag{9}$$

The following recursive upper bound for $CC(n,r)$ due to Forge and Ramírez Alfonsín [2] will be used later.

$$CC(n,r) \leq CC(n-1,r) + C(n-1,r-1). \tag{10}$$

3 Results for small and large r

In this section, we investigate connected covering numbers for *small* and *large* r, that is, when r is very close to either 1 or n. Let us start with the following observations.

Remark 1.
a) $CC(n,0) = 1$ since any 1-element set contains the empty set.
b) $CC(n,1) = n-1$ by taking the edges of a spanning tree of K_n.
c) $CC(n,n-2) = n-1$ by taking all but one $(n-1)$-sets.
d) $CC(n,n-1) = 1$ by taking the entire set.

All these values coincide with the corresponding covering numbers except in the case $r = 1$, where $C(n,1) = \lceil \frac{n}{2} \rceil$.

3.1 Results when r is small

For ordinary covering numbers, Fort and Hedlund [4] have shown that $C(n,2) := \lceil \frac{n}{3} \lceil \frac{n}{2} \rceil \rceil$ that coincides with the lower bounds given in (9) when the case $r = 2$. We could obtain the precise value for the connected case when $r = 2$.

Theorem 1. *Let n be a positive integer with $n \geq 3$. Then, we have*

$$CC(n,2) = \left\lceil \frac{\binom{n}{2} - 1}{2} \right\rceil.$$

The precise value of $C(n,3)$ remains unknown only for finitely many n, see [10,11,7]. The situation for connected coverings is worse.

Theorem 2. *Let n be a positive integer with $4 \leq n \leq 12$. Then, we have*

$$CC(n,3) = \left\lceil \frac{\binom{n}{3} - 1}{3} \right\rceil.$$

Theorem 2 supports the following

Conjecture 1. For every positive integer $n \geq 4$, we have

$$CC(n,3) = CC_1^*(n,3).$$

Even, more ambitious,

Question 2. Let n and r be two positive integers such that $n \geq r+1 \geq 4$. Is it true that if $CC(n,r) = CC_1^*(n,r)$ then $CC(n',r) = CC_1^*(n',r)$ for every integer $n' \geq n$?

3.2 Results when r is large

We could obtain the precise value for $CC(n, n-3)$.

Theorem 3. *Let n be a positive integer with $n \geq 3$. Then, we have*

$$CC(n,n-3) = \binom{\lceil \frac{n}{2} \rceil}{2} + \binom{\lfloor \frac{n}{2} \rfloor}{2} + 1.$$

For the connected covering $CC(n,n-4)$, we could obtain the following upper bound.

Theorem 4. *Let $n \neq 5,6,8,9$ be a positive integer with $n \geq 4$. Then, we have*

$$CC(n,n-4) \leq \begin{cases} m(m-1)(2m-1) & \text{if } n = 3m, \\ m^2(2m-1) & \text{if } n = 3m+1, \\ m^2(2m+1) & \text{if } n = 3m+2. \end{cases}$$

If $n = 5,6,9$ the value of $CC(n,n-4)$ is one larger than claimed in the formula. Further, $CC(8,4) \in \{20,21\}$, i.e., it remains open if the above formula has to be increased by one or not in order to give the precise value.

A famous conjecture of Turán [15] states that the bounds in Theorem 4 are best possible for $C(n,n-4)$. By combining (1) and Proposition 3, we have

$$C(n,n-4) \leq CC(n,n-4) \leq \begin{cases} m(m-1)(2m-1) & \text{if } n = 3m, \\ m^2(2m-1) & \text{if } n = 3m+1, \\ m^2(2m+1) & \text{if } n = 3m+2. \end{cases} \quad (11)$$

Turán's conjecture has been verified [14] for all $n \leq 13$ and so, by (11), the connected covering number can also be determined for these same values.

Towards proving Turán's conjecture, it would be of interest to investigate the following.

Question 3. Is it true that one of the inequalities in (11) is actually an equality?

Bounds and precise values for all $CC(n, r)$ with $n \leq 14$ are given in Table 1. All the exact values previously given in [2] for the same range have been improved by using our above results.

$r \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1		1	2	3	4	5	6	7	8	9	10	11	12	13
2			1	3	$5^{e,t}$	7^{e}	10^{e}	14^{e}	18^{e}	22^{e}	27^{e}	33^{e}	39^{e}	45^{e}
3				1	4	$7^{p,t}$	$12^{p,u}$	19^{p}	28^{p}	40^{p}	55^{p}	73^{p}	$[95^{l}, 97^{r}]$	$[121^{l}, 123^{r}]$
4					1	5	10^{t}	$[20, 21^{u}]$	$[32^{l}, 34^{r}]$	$[53^{l}, 59^{r}]$	$[83^{l}, 89^{r}]$	$[124^{l}, 136^{r}]$	$[179^{l}, 193^{r}]$	$[250^{l}, 271^{r}]$
5						1	6	13^{t}	31^{u}	$[51^{l}, 60^{r}]$	$[96^{a}, 111^{r}]$	$[159^{l}, 177^{r}]$	$[258^{l}, 290^{r}]$	$[401^{l}, 447^{r}]$
6							1	7	17^{t}	45^{u}	$[84^{a}, 95^{r}]$	$[165^{a}, 195^{r}]$	$[286^{l}, 327^{r}]$	$[501^{l}, 572^{r}]$
7								1	8	21^{t}	63^{u}	$[126^{a}, 147^{r}]$	$[269^{a}, 323^{r}]$	$[491^{l}, 587^{r}]$
8									1	9	26^{t}	84^{u}	$[185^{a}, 210^{r}]$	$[419^{a}, 505^{r}]$
9										1	10	31^{t}	112^{u}	$[259^{s}, 297^{r}]$
10											1	11	37^{t}	$[143^{s}, 144^{u}]$
11												1	12	43^{t}
12													1	13
13														1

Table 1: Bounds and values of $CC(n, r)$ for $n \leq 14$.

Key of Table 1 :

- r — Upper bound for $CC(n, r)$(from (10))
- e — Exact values for $CC(n, 2)$ (Theorem 1)
- t — Exact values for $CC(n, n-3)$ (Theorem 3)
- l — Lower bound $CC_1^*(n, r)$
- p — Some exact values for $CC(n, 3)$ (Theorem 2)
- u — Upper bound for $CC(n, n-4)$ (Proposition 4)
- s — Lower bound for $C(n, r)$ (from (8))
- a — Lower bounds for $C(n, r)$ (from [1])

4 A general upper bound

Let n and r be positive integers such that $n \geq r+1 \geq 3$. Forge and Ramírez Alfonsín [2] obtained the following general upper bound

$$CC(n,r) \leq \sum_{i=1}^{\lfloor \frac{n-r+1}{2} \rfloor} \binom{n-2i}{r-1} + \left\lfloor \frac{n-r}{2} \right\rfloor =: S(n,r). \tag{12}$$

Let us notice that the upper bounds obtained by applying the recursive equation (10), that were used in Table 1, are better than the one given by (12). Moreover, by iterating (10) it can be obtained

$$CC(n,r) \leq \sum_{i=r}^{n-1} C(i,r-1). \tag{13}$$

Although (13) might be used to get an *explicit* upper bound for $s(n,r)$, it is not clear how good it would be since that would depend on the known exact values and the upper bounds of $C(n,r)$ used in the recurrence (and thus intrinsically difficult to compute). On the contrary, in [2] was used (12) to give the best known (to our knowledge) explicit upper bound for $s(n,r)$.

We could construct a connected $(n, r+1, r)$-covering giving an upper bound for $CC(n,r)$ better than $S(n,r)$ and so, yielding to a better upper bound for $s(n,r)$ than that given in [2].

Theorem 5. *Let n and r be positive integers such that $n \geq r+1 \geq 3$. Then,*

$$CC(n,r) \leq N(n,r),$$

with

$$N(n,r) := \sum_{i=0}^{\lceil \frac{n-r}{2} \rceil - 1} (n-r-2i)\binom{r-2+2i}{r-2} + \left\lceil \frac{n-r}{2} \right\rceil - 1 + \delta_0 C(n-2, r-2), \tag{14}$$

where δ_0 is the parity function of $n-r$, that is,

$$\delta_0 = \begin{cases} 0 & \text{if } n-r \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Let us illustrate the construction given in the above theorem.

Example 1. $N(7,4) = 10$. We consider

$$\mathcal{N}_0 = \{12345, 12346, 12347\} \text{ and } \mathcal{N}_1 = \{12567, 13567, 14567, 23567, 24567, 34567\}.$$

It can be checked that $\mathcal{N}_0 \cup \mathcal{N}_1$ is a $(7,5,4)$-covering and $G(\mathcal{N}_0)$ and $G(\mathcal{N}_1)$ are connected. Now, by taking $C_0 = 12456$, it follows that $G(\mathcal{N}_0 \cup C_0 \cup \mathcal{N}_1)$ is connected.

Finally, we could show that $S(n,r) > N(n,r)$.

Theorem 6. *Let r and n be positive integers such that $n-r$ is an even number. Then,*

$$S(n,r) \geq N(n,r) + \sum_{i=0}^{\frac{n-r}{2}-2} \left(\frac{n-r}{2} - i - 1\right) \binom{r-2+2i}{r-3}.$$

References

[1] D. Applegate, E.M. Rains, and N.J.A. Sloane. On asymmetric coverings and covering numbers. *J. Combin. Des.*, 11(3):218–228, 2003.

[2] D. Forge and J.L. Ramírez Alfonsín. Connected coverings and an application to oriented matroids. *Discrete Math.*, 187(1-3):109–121, 1998.

[3] D. Forge, J.L. Ramírez Alfonsín, and H. Yeun. Disconnected coverings for oriented matroids via simultaneous mutations. *Discrete Math.*, 258(1-3):353–359, 2002.

[4] M.K. Fort, Jr. and G.A. Hedlund. Minimal coverings of pairs by triples. *Pacific J. Math.*, 8:709–719, 1958.

[5] D.M. Gordon, O. Patashnik, and G. Kuperberg. New constructions for covering designs. *J. Combin. Des.*, 3(4):269–284, 1995.

[6] Y.O. Hamidoune and M. Las Vergnas. Directed switching games on graphs and matroids. *J. Combin. Theory Ser. B.* 40(3):237–269, 1986.

[7] L. Ji. An improvement on covering triples by quadruples. *J. Combin. Des.*, 16(3):231–243, 2008.

[8] G. Katona, T. Nemetz, and M. Simonovits. On a problem of Turán in the theory of graphs. *Mat. Lapok.* 15:228–238, 1964.

[9] K. Knauer, L.P. Montejano, and J.L. Ramírez Alfonsín. How many circuits determine a (oriented) matroid? (in preparation).

[10] W.H. Mills. On the covering of triples by quadruples. *Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing* (Florida Atlantic Univ., Boca Raton, Fla., 1974), pages 563–581. Congressus Numerantium, No. X, Winnipeg, Man., 1974. *Utilitas Math.*

[11] W.H. Mills. A covering of triples by quadruples. *Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing.* Vol. II (Baton Rouge, La., 1981), volume 33, pages 253–260, 1981.

[12] J. Schönheim. On coverings. *Pacific J. Math.*, 14:1405–1411, 1964.

[13] A. Sidorenko. What we know and what we do not know about Turán numbers. *Graphs Combin.*, 11(2):179–199, 1995.

[14] A.F. Sidorenko. The method of quadratic forms in a combinatorial problem of Turán. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (1):3–6, 76, 1982.

[15] P. Turán. Research problems. *Magyar Tud. Akad. Mat. Kut. Int. Közl.*, 6:417–423, 1961.

Combinatorial proof for a stability property of plethysm coefficients *

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Abstract. Plethysm coefficients are important structural constants in the representation theory of the symmetric groups and general linear groups. Remarkably, some sequences of plethysm coefficients stabilize (they are ultimately constants). In this paper we give a new proof of such a stability property, proved by Brion with geometric representation theory techniques. Our new proof is purely combinatorial: we decompose plethysm coefficients as a alternating sum of terms counting integer points in polytopes, and exhibit bijections between these sets of integer points.

Key words: Combinatorial representation theory, symmetric functions, plethysm.

1 Introduction

Representation theory of groups is a fundamental tool in geometry and mathematical physics. But even one of the simplest cases, the representation theory of the general linear groups $GL_n(\mathbb{C})$, still raises unsolved problems. It is known that any (finite-dimensional, complex, analytic) linear representation V of $GL_n(\mathbb{C})$ decomposes as a direct sum of irreducible representations:

$$V \approx \bigoplus_{\lambda} m_\lambda S_\lambda(\mathbb{C}^n).$$

Here the m_λ are nonnegative integers (the *multiplicities* of the irreducible representations in V). The irreducible representations $S_\lambda(\mathbb{C}^n)$, indexed by the integer partitions λ of length at most n (an integer partition of length k is a weakly decreasing sequence of k positive integers), can be built explicitly by means of explicit combinatorial constructions [7,6]. These constructions can be applied actually to any representations: the S_λ are actually endofunctors of the category of representation of $GL_n(\mathbb{C})$ (the *Schur functors*) that comprise as particular cases the *symmetric powers* and the *exterior powers*.

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Three important, non–trivial, constructions of new representations from old ones are: tensor products; restrictions to a subgroup; plethysms (composition of functors S_λ). They define three important families of structural constants. Firstly, the *Littlewood–Richardson coefficients* $c^\lambda_{\mu,\nu}$ are the multiplicities arising when decomposing a tensor product of irreducible representations $S_\mu(\mathbb{C}^n) \otimes S_\nu(\mathbb{C}^n)$. Next, the *Kronecker coefficients* $g^{\mu,\nu}_\lambda$ are the multiplicities arising when considering an irreducible representation of $GL_{mn}(\mathbb{C})$ as a representation of $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ by means of the Kronecker product of matrices, and decomposing it into irreducibles. Finally, the *plethysm coefficients* $a^\lambda_{\pi,\nu}$ are the multiplicities obtained when applying a Schur functor S_π to an irreducible representation $S_\nu(\mathbb{C}^n)$ and decomposing the resulting representation in irreducibles.

The Littlewood–Richardson coefficients are quite well understood: they have many combinatorial interpretations (they count *Littlewood–Richardson Young tableaux*, or alternatively the integral points in the *hive polytopes*) that are as many efficient tools for proofs and as well for computations. Finding similar general interpretations for the Kronecker coefficients and the plethysm coefficients are major open problems in combinatorial representation theory.

Murnaghan [10] and Littlewood [8] observed some remarkable stability properties of the Kronecker coefficients: some sequences of Kronecker coefficients are ultimately constants. Proofs for these stability properties were provided by Thibon and his his collaborators by means of vertex operators on symmetric functions [11] and by Brion by means of tools from geometric representation theory [1]. Very recently a new proof were provided by Church, Farb and Ellenberg by means of explicit constructions of sequences of representations of symmetric groups in the setting of their theory of FI–modules [4], at the same time they observed the ubiquity of the phenomenon of stability of multiplicities.

The plethysm coefficients also exhibit several stability properties, some observed by Foulkes [5], that were proved in the 1990's essentially by means of combinatorial arguments (vertex operators and symmetric functions) by Carré and Thibon [2] on the one hand, and by tools from geometric representation theory by Brion on the other hand [1] in a more general setting (algebraic groups in general rather than just general linear groups). Some of the stability properties proved by Brion lack combinatorial proofs. In this paper we consider one of them.

Theorem 1 (Brion [1]). *For any partitions π, λ and μ such that $|\lambda| \cdot |\pi| = |\mu|$, the sequence with general term $u_n = a^{\mu+n\lambda}_{\pi+(n),\,\lambda}$ stabilizes.*

Here for any partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, the notation $|\alpha|$ stands for the sum of its parts,

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k,$$

and we treat partitions as vectors: we can addi them, and multiply them by scalars (*e.g.* in $\mu + n\lambda$), adding trailing zeros whenever necessary (*e.g* adding two partitions of different lengths).

Our contribution is a new, simple, combinatorial proof of Theorem 1, that we sketch now. We set the plethysm coefficients $a^{\mu}_{\pi,\lambda}$ in the framework of symmetric functions and express them as alternated sums of coefficients $b^{\mu+\omega'}_{\pi+\omega,\,\lambda}$. The stability of the sequence coefficients $a^{\mu+n\lambda}_{\pi+(n),\lambda}$ will therefore be established as soon as all sequences of coefficients $b^{\mu+n\lambda\omega'}_{\pi+\omega+(n),\,\lambda}$ are shown to be stable.

We show that the coefficient $b^{\mu}_{\pi,\,\lambda}$ counts the integer points in a polytope $Q(\pi, \lambda, \mu)$. Let $E(n)$ stand for $Q(\pi+(n), \lambda, \mu+n\lambda)$. We build explicit injections $E(n) \hookrightarrow E(n+1)$. Last we show that these injections are surjective for n big enough.

2 From plethysm coefficients to integer points in polytopes

The (complex, analytic, finite-dimensional) linear representations of $GL(n, \mathbb{C})$ are completely described, up to isomorphism, by their character, which is a symmetric polynomial [7,6]. This allows to set computations of plethysms in the framework of *symmetric functions*. Remember (see for instance [9,12,6]) that the ring of symmetric functions is a graded ring endowed with a scalar product, that admits several importat linear basis. Among them: the *Schur functions* s_λ, the monomial functions m_λ, and the products of complete functions $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$. All these bases are indexed by the integer partitions λ (the complete sums h_k are indexed by the nonnegative integers and as a result the products of complete sums are also indexed by integer partitions). With respect to the scalar product, the Schur functions are an orthonormal basis. The monomial functions m_λ and the complete sums h_λ are dual bases.

The plethysm of representations induces an operation $(f, g) \mapsto f[g]$ on the ring of symmetric functions, called *plethysm of symmetric functions* (see [9], I.8). This operation is associative but non–commutative, and not even bilinear. It is, nevertheless, linear in the first argument. In this framework, the plethysm coefficient $a^{\lambda}_{\mu,\,\nu}$ is the coefficient of s_λ in the expansion in the Schur basis of the plethysm of Schur functions $s_\mu[s_\nu]$. Alternatively, this coefficient is extracted by means of a scalar product:

$$a^{\lambda}_{\mu,\,\nu} = \langle s_\mu[s_\nu] \mid s_\lambda \rangle \tag{1}$$

We will expand, in this expression, s_μ and s_λ in the h–basis. This is done by means of the *Jacobi–Trudi identity* that we recall here.

Lemma 1 (Jacobi–Trudi identity, [9] I. (3.4)).
Let λ be a partition with length at most N. Then

$$s_\lambda = \det\left(h_{\lambda_j+i-j}\right)_{1\le i,j\le N}$$

with $h_0 = 1$ and $h_r = 0$ if $r < 0$, and λ is completed with trailing zeros if necessary.

This expansion writes explicitly as a sum over the permutations σ in the symmetric group $\mathfrak{S}_N$ ([9] I. (3.4'))

$$s_\lambda = \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) h_{\lambda + \omega(\sigma)}$$

where $\omega(\sigma)_j = \sigma(j) - j$ for all j between 1 and N.

We now perform this Jacobi–Trudi expansion for s_μ and s_ν in (1). We get the following expansion for the plethysm coefficients.

Lemma 2. *Let N and N' be positive integers. Let λ, μ and ν be partitions, such that μ has length at most N and λ has length at most N'. Then*

$$a^\lambda_{\mu,\,\nu} = \sum_{\sigma,\tau} \varepsilon(\sigma)\varepsilon(\tau) \left\langle h_{\mu+\omega(\sigma)}[s_\nu] \mid h_{\lambda+\omega(\tau)} \right\rangle$$

where the sum is carried over all permutations $\sigma \in \mathfrak{S}_N$ and $\tau \in \mathfrak{S}_{N'}$.

The scalar products that appear on the right–hand side are interesting. For any partition ν and any finite sequences μ and λ of integers we set:

$$b^\lambda_{\mu,\,\nu} = \langle h_\mu[s_\nu] \mid h_\lambda \rangle .$$

It turns out that these coefficients count the nonnegative solutions of systems of linear Diophantine equations whose constant terms depend linearly on the parts of μ and λ. In particular, they count integer points in polytopes with a nice description.

To state this precisely we introduce some notations. For any partition ν and any positive integer N let $t(\nu; N)$ be the set of semi–standard Young tableaux (see [12] 7.10) of shape ν with entries between 1 and N. Let $\mathcal{P}_{\nu;\,N} = (p_{T,j})_{T,j}$ be the matrix whose rows are indexed by the tableaux $T \in t(\nu; N)$, whose columns are indexed by the integers j between 1 and N, such that $p_{T,j}$ is the number of occurrences of j in T (so that row T of $\mathcal{P}_{\nu;\,N}$ is the *weight* of the tableau T).

Proposition 1. *Let μ, ν and λ be partitions. Let $\ell(\mu)$ be the length of μ and N be at least the length of λ.*

The coefficient $b^\lambda_{\mu,\,\nu}$ is the cardinal of the set $Q(\mu; \nu; \lambda; N)$ of matrices $\mathcal{M} = (m_{i,T})$ with nonnegative integer entries whose rows are indexed by the integers i between 1 and $\ell(\mu)$ and whose columns are indexed by the tableaux $T \in t(\nu; N)$ such that:

- *The sum of the entries in row i of $\mathcal{M}$ is μ_i.*
- *The sum of the entries in column j of $\mathcal{M} \cdot \mathcal{P}_{\nu;\,N}$ is λ_j.*

Proof. [Sketch] Let x_1, x_2, ... be the underlying variables of the symmetric functions and, for any finite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, let $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$.

Since the scalar product with h_λ extracts the coefficient of m_λ in the expansion in basis of monomial functions, the constant $b^\lambda_{\mu,\,\nu}$ can be interpreted as the coefficient as the monomial x^λ in $h_\mu[s_\nu]$. Instead of working with symmetric functions (with infinitely many variables) we can work with symmetric polynomials in N variables, provided N is at least the length of λ. We now use the expansion of the Schur polynomial $s_\nu(x_1, x_2, \ldots, x_N)$ in monomials ([6]):

$$s_\nu(x_1, x_2, \ldots, x_N) = \sum_{T \in t(\nu;N)} x_1^{p_{1,T}} x_2^{p_{2,T}} \cdots x_N^{p_{N,T}}.$$

We also use that a plethysm $f[g]$ when g is a sum of monomials $g = \sum_i x^{u_i}$ is just the evaluation $f(x^{u_1}, x^{u_2}, \ldots)$ (see [12] 7.8). We use finally that the complete sum h_j is the sum of all monomials of degree j. These three facts and a short combinatorial reasoning provide the expansion:

$$h_\mu[s_\nu(x_1, \ldots, x_N)] = \sum_\lambda \mathrm{Card}\left(Q(\mu;\nu;\lambda;N)\right)\, x^\lambda.$$

This proves the proposition. ∎

3 Stability

We now consider a positive integer N and three partitions λ, π and μ, where λ and μ have length at most N and $|\lambda| \cdot |\pi| = |\mu|$. Applying Lemma 2 to the plethysm coefficients $u_n = a^{\mu+n\lambda}_{\pi+(n),\,\lambda}$ we get

$$u_n = \sum_{\sigma \in \mathfrak{S}_{\ell(\pi)}, \tau \in \mathfrak{S}_N} \varepsilon(\sigma)\varepsilon(\tau)\, v_n(\sigma, \tau)$$

where $v_n(\sigma, \tau) = b^{\mu+n\lambda+\omega(\tau)}_{\pi+\omega(\sigma)+(n),\,\lambda}$.

Therefore, in order to establish Theorem 1 (the stability of the sequence with general term u_n), it is enough to show that each of the sequences $v_n(\sigma, \tau)$ stabilizes. Each of these sequences is of the type $b^{\mu'+n\lambda}_{\pi'+(n),\,\lambda}$ for some sequences π' and μ'.

Theorem 2. *Let λ be a partition and π and μ be finite sequences of integers. Then, the sequence with general term $v_n = b^{\mu+n\lambda}_{\pi+(n),\,\lambda}$ stabilizes.*

Proof. [Sketch] Let N be an integer bigger than or equal to the lengths of λ and μ. After Proposition 1,

$$v_n = \mathrm{Card}\left(Q(\pi + (n);\ \lambda;\ \mu + n\lambda)\right).$$

Set $E(n)$ for $Q(\pi + (n);\ \lambda;\ \mu + n\lambda)$.

Let T_0 be the tableau in $t(\lambda; N)$ whose row number i is filled with occurrences of i, for each i. Consider the injection $\iota(n) : E(n) \hookrightarrow E(n+1)$ that maps any matrix $\mathcal{M} \in E(n)$ to the matrix $\mathcal{M}'$ where the coefficient m_{1,T_0} is incremented by one, and all other coefficients are unchanged.

We contend that $\iota(n)$ is also surjective for n big enough. It is not difficult to check that $\iota(n)$ is surjective if and only if for all $\mathcal{M}' \in E(n+1)$, the entry m_{1,T_0} is non–zero. Thus proving the surjectivity of $\iota(n)$ amounts to showing that $m_{1,T_0} > 0$.

Let $\mathcal{M}' \in E(n+1)$. Observe that among all tableaux in $t(\lambda; N)$, the tableau T_0 is the unique one with maximum weight for the *dominance ordering* ([9] I.1). Denote $||\alpha||$ for $\sum_{k=1}^{N}\sum_{j=1}^{k}\alpha_j$ and p_T for row T of $\mathcal{P}_{\lambda;\,N}$. Then for all $T \neq T_0$,

$$||p_T|| \leq ||\lambda|| - 1,$$

and for $T = T_0$,

$$||p_{T_0}|| = ||\lambda||.$$

Using now the row sums conditions on the matrix $\mathcal{M}'$, a few more elementary operations lead to the inequality

$$m_{1,T_0} \geq ||\mu|| + \pi_1 - |\pi| \cdot ||\lambda|| + (n+1)$$

that proves that $m_{1,T_0} > 0$ as soon as $n \geq |\pi| \cdot ||\lambda|| - ||\mu|| - \pi_1$. ∎

4 Conclusion

We have provided with rather elementary tools (the combinatorics of symmetric functions) a new proof of a stability property for plethysm coefficients, otherwise proved using the more eleborate machinery of geometric representation theory. Our approach enhanced the importance of other constants (the $b^{\nu}_{\lambda,\,\mu}$ in our paper) that seem interesting by themselves. In particular, since they count integer points in polytopes, it should be possible to evaluate them efficiently by means of Barvinok's algorithm. Does this lead to more efficient algorithms for computing the plethysm coefficients? A similar approach was followed successfully for the Kronecker coefficients [3].

References

[1] Michel Brion. Stable properties of plethysm: on two conjectures of Foulkes. *Manuscripta Math.*, 80(4):347–371, 1993.

[2] Christophe Carré and Jean-Yves Thibon. Plethysm and vertex operators. *Adv. in Appl. Math.*, 13(4):390–403, 1992.

[3] M. Christandl, B. Doran, and M. Walter. Computing multiplicities of Lie group representations. In *Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on*, pages 639–648, Oct 2012.

[4] Thomas Church, Jordan Ellenberg, and Benson Farb. Fi–modules: a new approach to stability for S_n-representations. arXiv:1204.4533[math.RT], april 2012.

[5] H. O. Foulkes. Concomitants of the quintic and sextic up to degree four in the coefficients of the ground form. *J. London Math. Soc.*, 25:205–209, 1950.

[6] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[7] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[8] D. E. Littlewood. The Kronecker product of symmetric group representations. *J. London Math. Soc.*, 31:89–93, 1956.

[9] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[10] Francis D. Murnaghan. The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group. *Amer. J. Math.*, 60(3):761–784, 1938.

[11] Thomas Scharf, Jean-Yves Thibon, and Brian G. Wybourne. Reduced notation, inner plethysms and the symmetric group. *J. Phys. A*, 26(24):7461–7478, 1993.

[12] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

The degree/diameter problem in maximal planar bipartite graphs

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Abstract. The (Δ, D) (degree/diameter) problem consists of finding the largest possible number of vertices n among all the graphs with maximum degree Δ and diameter D. We consider the (Δ, D) problem for maximal planar bipartite graphs, that are simple planar graphs in which every face is a quadrangle. We obtain that for the $(\Delta, 2)$ problem, the number of vertices is $n = \Delta + 2$; and for the $(\Delta, 3)$ problem, $n = 3\Delta - 1$ if Δ is odd and $n = 3\Delta - 2$ if Δ is even. Then, we study the general case (Δ, D) and obtain that an upper bound on n is approximately $3(2D+1)(\Delta-2)^{\lfloor D/2 \rfloor}$, and another one is $C(\Delta - 2)^{\lfloor D/2 \rfloor}$ if $\Delta \geq D$ and C is a sufficiently large constant. Our upper bounds improve for our kind of graphs the one given by Fellows, Hell and Seyffarth for general planar graphs. We also give a lower bound on n for maximal planar bipartite graphs, which is approximately $(\Delta - 2)^k$ if $D = 2k$, and $3(\Delta - 3)^k$ if $D = 2k + 1$, for Δ and D sufficiently large in both cases.

Key words: (Δ, D) problem, maximal planar bipartite graphs.

1 Introduction

We consider simple graphs $G = G(V, E)$ that are bipartite, planar and with the maximum possible number of edges. In a *bipartite* graph, each cycle has an even length. If a graph can be drawn on the plane without any crossing of its edges, then the graph is called *planar*. A planar bipartite graph is maximal if when we add a new edge, the graph obtained is no longer planar or bipartite. A maximal planar bipartite graph divides the plane only into quadrangles (see Ringel [12]). From the Euler characteristic $|V| - |E| + |F| = 2$, which

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relates the numbers of vertices $n = |V|$, edges $|E|$ and faces $|F|$ in a planar embedding of G, and the fact that each face is incident to four edges, one obtains the well-known relations $|E| = 2n - 4$ and $|F| = n - 2$.

The (Δ, D) problem consists of finding the maximum possible number of vertices $n = |V|$ in a graph G with maximum degree Δ and diameter D. This is a prominent topic in graph theory, with results obtained for many cases. Information about this problem for graphs in general can be found in the comprehensive survey by Miller and Sirán [9] and for planar graphs also on the web page by Loz, Pérez-Rosés, and Pineda-Villavicencio [7].

For simple maximal planar graphs (that is, triangulations), the (Δ, D) problem with diameter $D = 2$ and $\Delta \geq 8$ was solved by Seyffarth in [13]. She proved that, in this case, the number of vertices is $n \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$ if $\Delta \geq 8$, and that this bound is best possible. Later, Hell and Seyffarth [8] showed that this result also holds for the larger class of all simple planar graphs. Yang, Lin, and Dai [16] solved the remaining case $\Delta < 8$ for $D = 2$, for both graph classes. Fellows, Hell and Seyffarth [2] found that an upper bound on the number of vertices for planar graphs is $8\Delta + 12$ with diameter $D = 3$, and $3(2D+1)(2\Delta^{\lfloor D/2 \rfloor} + 1)$ for any diameter.

Regarding lower bounds on the maximum number of vertices of planar graphs, given a fixed diameter D and a maximum degree Δ, Fellows, Hell and Seyffarth [3] proved that $n \geq \frac{9}{2}\Delta^{(D-1)/2} - o(\Delta^{(D-1)/2})$ for odd values of D, and $n \geq \frac{3}{2}\Delta^{D/2} - o(\Delta^{D/2})$ for even values. Later, Feria-Puron and Pineda-Villavicencio [4] presented a lower bound on the maximum number of vertices for planar graphs of maximum degree $\Delta \geq 6$ and odd diameter $D \geq 5$, which is $\left(\lfloor \frac{9\Delta}{2} \rfloor - 12\right) \frac{\Delta(\Delta-1)^{(D-3)/2}-2}{\Delta-2} + 9$. For planar graphs with even diameter $D = 2k$ and maximum degree Δ, Tishchenko [14] obtained the lower bound $\left\lfloor \frac{3\Delta}{2} \frac{(\Delta-1)^k-1}{\Delta-2} \right\rfloor + 1$, and proved that this is also an upper bound for cases with large Δ, concretely for $\Delta \geq 6(12k+1)$.

Related bounds on the (Δ, D) problem for sparse graph classes and for graphs embedded on surfaces can be found in Nevo, Pineda-Villavicencio and Wood [10], and Pineda-Villavicencio and Wood [11].

We study the (Δ, D) problem for maximal planar bipartite graphs. In this text, we omit most of the proofs because of lack of space. We show in Section 2, that in the $(\Delta, 2)$ problem $n = \Delta + 2$ and that only the complete bipartite graph $K_{2,\Delta}$ satisfies this equation. Moreover, we solve the $(\Delta, 3)$ problem and prove that $n = 3\Delta - 1$ if Δ is odd, and $n = 3\Delta - 2$ if Δ is even. In Section 3, we study the general case (Δ, D) and obtain that n is bounded from above by approximately $3(2D+1)(\Delta-2)^{\lfloor D/2 \rfloor}$. For the case $\Delta \geq D$ we also obtain the upper bound $n \leq C(\Delta-2)^{\lfloor D/2 \rfloor}$, for some constant C. Our upper bounds improve for our kind of graphs the one given by Fellows, Hell and Seyffarth for general planar graphs. We also give a lower bound on n for maximal planar bipartite graphs, which is approximately $(\Delta-2)^k$ if $D = 2k$, and $3(\Delta-3)^k$ if

$D = 2k + 1$, for Δ and D sufficiently large in both cases. The precise bounds are given in that section.

2 The $(\Delta, 2)$ and $(\Delta, 3)$ problems in maximal planar bipartite graphs

For maximal planar bipartite graphs with diameter $D = 2$, we solve the $(\Delta, 2)$ problem with the following result.

Proposition 1. *Consider a maximal planar bipartite graph G with diameter $D = 2$, maximum degree Δ and maximum number of vertices n, then $n = \Delta + 2$. The only graph that satisfies this equation is the complete bipartite graph $K_{2,\Delta}$.*

Proof. For $\Delta = 2$, the only maximal planar bipartite graph is the cycle on four vertices C_4, which satisfies $n = \Delta+2$. Therefore, let us assume that $\Delta > 2$. Let v be a vertex of degree Δ in G. Suppose, for the sake of contradiction, that there is more than one vertex at distance two from v in G. Denote two such vertices by w and z. Then, w and z are not both adjacent to all neighbors of v; otherwise G would contain the complete bipartite graph $K_{3,3}$ as a subgraph, contradicting the planarity of G. Hence, there is a neighbor u of v incident to at most one of w and z, say u is not adjacent to z. The vertices u and z belong to different partite classes, which implies that the distance between them is odd. It follows that the distance between u and z is at least three, contradicting $D = 2$. Thus, $n = \Delta + 2$ and only the complete bipartite graph $K_{2,\Delta}$ attains this bound. ∎

For maximal planar bipartite graphs with diameter $D = 3$, our main result is the following.

Theorem 1. *Consider a maximal planar bipartite graph G with diameter $D = 3$, maximum degree Δ and maximum number of vertices n, then*

$$n = \begin{cases} 3\Delta - 1 \text{ if } \Delta \text{ is odd,} \\ 3\Delta - 2 \text{ if } \Delta \text{ is even.} \end{cases}$$

3 The (Δ, D) problem in maximal planar bipartite graphs

3.1 An upper bound

Fellows, Hell and Seyffarth [2] obtained bounds on the (Δ, D) problem for planar graphs applying the following theorem by Lipton and Tarjan [6].

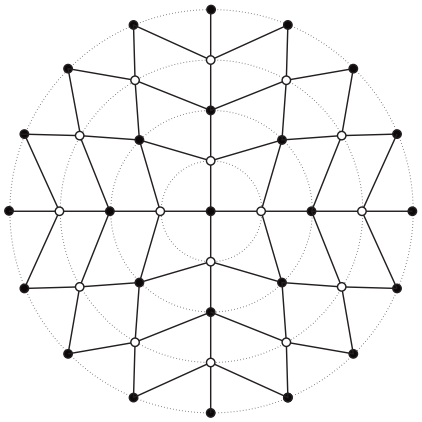

Fig. 1: An almost maximal subgraph for $\Delta = 4$

Theorem 2 ([6]). *Let G be a planar graph on n vertices containing a spanning tree of radius r. Then $V(G)$ can be partitioned into sets A, B and C such that no edges join vertices in A with vertices in B, $|A| \leq \frac{2}{3}n$, $|B| \leq \frac{2}{3}n$, and $|C| \leq 2r + 1$.*

Clearly, this theorem also holds for maximal planar bipartite graphs. We give an upper bound on the number of vertices for this kind of graphs. The cases $D = 2$ and $D = 3$ are studied in Section 2. No maximal planar bipartite graphs with $\Delta = 3$ has more than $n = 8$ vertices. Therefore, we assume that $D \geq 4$ and $\Delta \geq 4$.

Our aim is to give an upper bound on n, computing from each vertex of C the maximum possible number of vertices at distance at most $\lfloor D/2 \rfloor$. We build a subgraph adding vertices at distance i from a given (root) vertex of C in step i ($0 \leq i \leq \lfloor D/2 \rfloor$), to obtain a planar bipartite graph (which is almost maximal, meaning that all its interior faces are quadrangles), as shown in Fig. 1. This is done in a similar way to the tree built to find the Moore bound (see Miller and Sirán [9]).

Let n_i be the number of vertices at distance i (for $0 \leq i \leq \lfloor D/2 \rfloor$). For $i \geq 3$, n_i follows the recurrence

$$n_i = (\Delta - 2)n_{i-1} - n_{i-2}.$$

Solving this recurrence equation with a generating function, we obtain for $\Delta > 4$

$$n_i = \frac{\Delta}{\sqrt{\Delta(\Delta-4)}} \left[\left(\frac{\Delta - 2 + \sqrt{\Delta(\Delta-4)}}{2} \right)^i - \left(\frac{\Delta - 2 - \sqrt{\Delta(\Delta-4)}}{2} \right)^i \right].$$

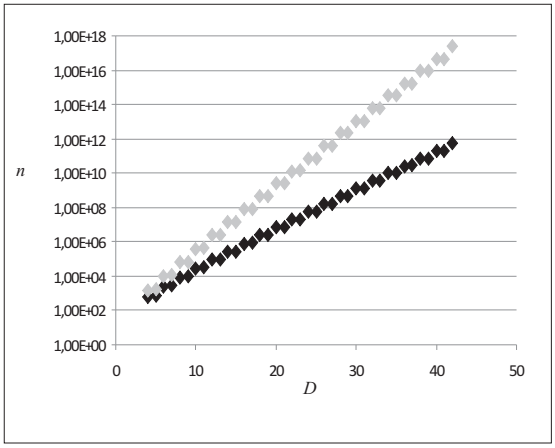

Fig. 2: Plot of the log (base 10) of the number of vertices n with respect to the diameter D according to our bound given by Eq. (1) (black points) and the one by Fellows, Hell and Seyffarth given by Eq. (2) (grey points), for $\Delta = 5$ and $4 \leq D \leq 42$

Thus, the total number of vertices $n = \sum_{i=0}^{\lfloor D/2 \rfloor} n_i$ is obtained as the difference of two geometric series. With this result, and with a similar approach as in Fellows, Hell, and Seyffarth [2], that is using Theorem 2, we obtain the following theorem.

Theorem 3. *Let G be a maximal planar bipartite graph on n vertices with maximum degree $\Delta \geq 4$ and diameter $D \geq 4$. Then,*

(a) If $\Delta = 4$: $n \leq 6(2D+1)\left(\left\lfloor \frac{D}{2} \right\rfloor^2 + \left\lfloor \frac{D}{2} \right\rfloor + 1\right)$.

(b) If $\Delta > 4$:

$$n \leq 3(2D+1)\left[\frac{\sqrt{\Delta(\Delta-4)}}{2(\Delta-4)^2}\left[(\Delta-4+\sqrt{\Delta(\Delta-4)}\,)\left(\frac{\Delta-2-\sqrt{\Delta(\Delta-4)}}{2}\right)^{\lfloor D/2\rfloor+1}\right.\right.$$
$$-2\sqrt{\Delta(\Delta-4)}$$
$$\left.\left.+\ (4-\Delta+\sqrt{\Delta(\Delta-4)}\,)\left(\frac{\Delta-2+\sqrt{\Delta(\Delta-4)}}{2}\right)^{\lfloor D/2\rfloor+1}\right]+2\right], \qquad (1)$$

which is approximately $3(2D+1)\left[(\Delta-2)^{\lfloor D/2\rfloor}+1\right]$ if Δ is sufficiently large.

The upper bound given by Fellows, Hell and Seyffarth [2] for planar graphs is

$$n \leq 3(2D+1)(2\Delta^{\lfloor D/2 \rfloor}+1). \tag{2}$$

As our graphs are planar, this bound also applies to maximal planar bipartite graphs, but our bound is much better for this kind of graphs. See an example for $\Delta = 5$ in Fig. 2, with the values of our bound given by Theorem 3 and the one by Fellows, Hell and Seyffarth.

We also give an alternative upper bound for the (Δ, D) problem of the form $n < C(\Delta-2)^{\lceil D/2 \rceil}$, for some constant C, which improves the bound of Theorem 3 by a factor D, when D is even and sufficiently large. However, it remains for further research to determine the smallest value of C for which this bound holds. It is based on the following theorem of Chepoi, Estellon, and Vaxès [1]. The ball of center $v \in G$ and radius k consists of all vertices of G at distance at most k from v.

Theorem 4 ([1]). *There exists a constant C such that any planar graph G of diameter $D \leq 2k$ can be covered with at most C balls of radius k.*

As for a lower bound for Theorem 4, Gavoille Gavoille, Peleg, Raspaud, and Sopena in [5] presented a family of planar graphs which requires $C \geq 4$.

Corollary 1. *There exists a constant C such that each maximal planar bipartite graph G with maximum degree Δ and diameter D has at most $n \leq C(\Delta-2)^{\lceil D/2 \rceil}$ vertices.*

We further strengthen the bound for the (Δ, D) problem given in Corollary 1 to $C(\Delta-2)^{\lfloor D/2 \rfloor}$, for the case D odd and $\Delta \geq D$. We use the N-separator theorem by Tishchenko [14,15] to obtain the following result.

Theorem 5. *There exists a constant C such that each maximal planar bipartite graph G with maximum degree Δ and diameter D, for $\Delta \geq D$, has at most $n \leq C(\Delta-2)^{\lfloor D/2 \rfloor}$ vertices.*

3.2 A lower bound

In this section we present maximal planar bipartite graphs $G_{\Delta,D}$, with given maximum degree Δ and diameter D, which have a large number $n = n(G_{\Delta,D})$ of vertices.

The graph which attains the bound in the following Theorem 6 (a) is based on the one depicted in Fig. 1; from a given root vertex we build a planar bipartite graph adding the maximum number of vertices at distance i from the root in step i (for $0 \leq i \leq D/2$). We draw this graph on a sphere, with the root placed on the north pole and the vertices at distance $D/2$ from the root are placed on the equator. Then we add a copy of this graph on the lower hemisphere, with the root on the south pole, and such that the vertices at distance $D/2$ from the south pole are identified with those already

placed on the equator. For the bounds (b) and (c) of Theorem 6, we use an iterative construction: We start with the graph on the left of Fig. 3 and then we substitute the dotted edges by complete bipartite subgraphs $K_{2,t}$ (where the value of t differs according to the subcases of Theorem 6). The generic step of this construction is shown on the right of Fig 3. In each step of this construction, the diameter of the graph increases by two units.

Theorem 6. (a) *For any diameter $D = 2k$ $(k \geq 1)$ and maximum degree Δ $(\Delta \geq 5)$, there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is*

$$\frac{\Delta\left(\Delta - 2 + \sqrt{\Delta(\Delta-4)}\right)^k + \Delta\left(\Delta - 2 - \sqrt{\Delta(\Delta-4)}\right)^k}{(\Delta-4)2^k} - \frac{8}{\Delta-4},$$

which is approximately $(\Delta-2)^k$, for Δ and D sufficiently large.

(b) *For any diameter $D = 2k+1$ $(k \geq 1)$ and odd maximum degree Δ $(\Delta \geq 9)$, there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is*

$$\begin{aligned}
&n(G_{\Delta,3}) = 3\Delta - 1 && \text{for } D = 3,\\
&n(G_{\Delta,5}) = 3\Delta^2 - 21\Delta + 26 && \text{for } D = 5,\\
&n(G_{\Delta,2k+1}) = 3\Delta^2 - 21\Delta + 26 + \tfrac{3(\Delta-7)(\Delta-2)^2((\Delta-3)^{k-2}-1)}{(\Delta-4)} && \text{for } D = 2k+1 \\
& && \text{and } k > 2,
\end{aligned}$$

which is approximately $3(\Delta-3)^k$, for Δ and D sufficiently large.

(c) *For any diameter $D = 2k+1$ $(k \geq 1)$ and even maximum degree Δ $(\Delta \geq 10)$, there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is*

$$\begin{aligned}
&n(G_{\Delta,3}) = 3\Delta - 2 && \text{for } D = 3,\\
&n(G_{\Delta,5}) = 3\Delta^2 - 22\Delta + 26 && \text{for } D = 5,\\
&n(G_{\Delta,2k+1}) = 3\Delta^2 - 22\Delta + 26 + \tfrac{(3\Delta-22)(\Delta-2)^2((\Delta-3)^{k-2}-1)}{(\Delta-4)} && \text{for } D = 2k+1 \\
& && \text{and } k > 2,
\end{aligned}$$

which is approximately $3(\Delta-3)^k$, for Δ and D sufficiently large.

References

[1] V. Chepoi, B. Estellon, Y. Vaxès. On covering planar graphs with a fixed number of balls. *Discrete Comput. Geom.* 37:237–244, 2007.

[2] M. Fellows, P. Hell, K. Seyffarth. Large planar graphs with given diameter and maximum degree. *Discrete Appl. Math.* 61:133–153, 1995.

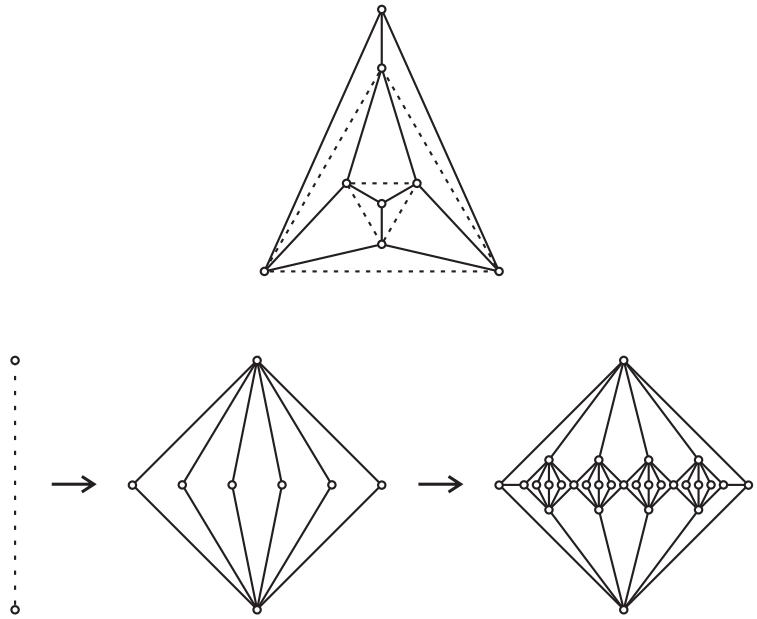

Fig. 3: The iterative construction

[3] M. Fellows, P. Hell, K. Seyffarth. Constructions of large planar networks with given degree and diameter. *Networks* 32:275–281, 1998.

[4] R. Feria-Puron, G. Pineda-Villavicencio. Constructions of large graphs on surfaces. *Graph. Combinator.*, DOI 10.1007/s00373-013-1323-y, 2013.

[5] C. Gavoille, D. Peleg, A. Raspaud, E. Sopena. Small k-dominating sets in planar graphs with applications. 27th International Workshop on Graph-Theoretic Concepts in Computer Science 2001, *Lect. Notes Comput. Sc.* 2204:201–216, Springer-Verlag, Berlin, 2001.

[6] R.J. Lipton, R.E. Tarjan. A separator theorem for planar graphs. *SIAM J. Appl. Math.* 36:177–189, 1979.

[7] E. Loz, H. Pérez-Rosés, G. Pineda-Villavicencio. The degree/diameter problem for planar graphs. http://combinatoricswiki.org/wiki/The_Degree_Diameter_Problem_for_Planar_Graphs, 2008, accessed on 3 March 2014.

[8] P. Hell, K. Seyffarth. Largest planar graphs of diameter two and fixed maximum degree. *Discrete Math.* 111:313–322, 1993.

[9] M. Miller, J. Sirán. Moore graphs and beyond: A survey of the degree/diameter problem. *Electron. J. Combin.* 20(2), #DS14v2, 2013.

[10] E. Nevo, G. Pineda-Villavicencio, D.R. Wood. On the maximum order of graphs embedded in surfaces. arXiv:1312.1753, 2013.

[11] G. Pineda-Villavicencio, D.R. Wood. The degree-diameter problem for sparse graph classes. arXiv:1307.4456, 2013.

[12] G. Ringel. Two trees in maximal planar bipartite graphs. *J. Graph Theory* 17:755–758, 1993.

[13] K. Seyffarth. Maximal planar graphs of diameter two. *J. Graph Theory* 13:619–648, 1989.

[14] S.A. Tishchenko. Maximum size of a planar graph with given degree and even diameter. *Eur. J. Combin.* 33:380–396, 2012.

[15] S.A. Tishchenko. N-separators in planar graphs. *Eur. J. Combin.* 33:397–407, 2012.

[16] Y. Yang, J. Lin, Y. Dai. Largest planar graphs and largest maximal planar graphs of diameter two. *J. Comput. Appl. Math.* 144:349–358, 2002.

An Approximate Algorithm for the Chromatic Number of Graphs *

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Abstract. We have designed a novel polynomial-time approximate algorithm for the graph vertex colouring problem. Contrary to the common focus top-down for solving the coloring graph problem, we propose a novel algorithm with a paradigm down-top for coloring graphs. Given an input graph G, we discard the nodes in G not involved in odd cycles forming so a subgraph $G_0 \subseteq G$. G_0 will be coloured at the end of the colouring proccess with the two last colours.

On each iteration of our procedure a maximal indpendent set (MIS) K_i is formed and discarded from G_i, building a new subgraph $G_{i+1} = (G_i - K_i)$. This iterative process continues until achieving a polynomial-time 2-colourable subgraph. We build each MIS K_i of G_i in such way that the number of edges in G_i covered by the nodes of K_i is at least, the number of current nodes minus 1, that is, $|E_{G_i}(K_i)| \leq |V(G_i)|$. That lower bound allow us to design an iterative procedure such that, if each remained subgraph $G_{i+1} = (G_i - K_i)$ is connected, then we obtain an upper bound to approximate the coloring of the input graph, since our heuristic establishes an average number of $\lceil \delta(G)/2 \rceil$ +2 colours to approximate the chromatic number of G, where $\delta(G)$ is the average degree of G.

Key words: Graph Coloring, Approximate Algorithm, Chromatic Number.

1 Introduction

Graph vertex colouring problem is an active field of research, with many interesting subproblems [4,5,6]. This problem has many applications in areas like scheduling, frequency allocation, planning, etc [2].

The graph colouring problem consists in colouring properly the vertices of a graph with the smallest possible number of colours, so that no two adjacent vertices receive the same colour. If a colouring with k colours exists, then the graph is k-colourable. The chromatic number of a graph G, denoted as $\chi(G)$, represents the minimum number of colours for proper colouring G.

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The interesting part of the chromatic number $\chi(G)$ of a graph G is that it appears in a variety of applications. Furthermore, it was one of the first twenty two NP-complete problems found.

The chromatic number $\chi(G)$ is polynomial computable when $\chi(G) \leq 2$, but when $\chi(G) \geq 3$ the problem becomes NP-complete, even for graphs G with degree $\Delta(G) \geq 3$. As a consequence, there are many unanswered questions related to the colouring of a graph[5].

Following the line of exact algorithms and using maximal independent sets to compute the chromatic number, Eppstein established an $O(2.4151^n)$ time algorithm [1]. Subsequently, Byskov provided an $O(2.4023^n)$ time algorithm [2]. All of those algorithms have an orientation top-down and they use a combination of improved upper bounds on the number of maximal independent sets of size at most k, as well as some changes in the way the table of subsolutions is fulled, using a dynamic programming approach.

We have designed a novel heuristic for coloring graphs with an orientation down-top. Given an input graph G, first we form a subgraph $G_0 \subseteq G$ with the nodes in G not involved in odd cycles. After, the procedure iterate building in each iteration a MIS K_i and discarding it from G_i, forming so a subgraph $G_{i+1} = (G_i - K_i)$, until achieving a polynomial-time 2-colourable subgraph.

The knowledge of lower bounds for the independence number of the graph ($\alpha(G)$) has been a relevant measure to determine combinatorial properties of a graph. For example, for determining upper bounds to $\chi(G)$. However, we show that $\alpha(G)$ is not the unique useful measure for computing $\chi(G)$.

Related to $\alpha(G)$ is also the maximum number of edges of G that the nodes of a MIS of G can cover. To the best of our knowledge, the relation between the nodes in a MIS and the number of edges in the original graph covered by those nodes has not been addressed to approximate $\chi(G)$. For a connected graph G and a MIS K of G, we establish here the first lower bound on the maximum number of edges in $E(G)$ incident to the nodes of K, and we show how that lower bound is helpful to establish a new upper bound for the chromatic number of a graph.

We build a MIS K_i for each subgraph G_i holding that the number of edges of G_i incident to nodes of K_i, is at least the number of current nodes minus 1, i.e. $|E_{G_i}(K_i)| \leq |V(G_i)| - 1$. That lower bound for each K_i of G_i allows us to design an iterative procedure such that, if each remained subgraph $G_{i+1} = (G_i - K_i)$ is connected, then our procedure establishes an average number of $\lceil \delta(G)/2 \rceil$ +2 colours to approximate the chromatic number of G, where $\delta(G)$ is the average degree of G.

2 Preliminaries

Let $G = (V, E)$ be an undirected simple graph (i.e. finite, loop-less and without multiple edges) with vertex set V and set of edges E. $E(G)$ and $V(G)$

emphasize that these are the edges and vertex sets of a particular graph G. Two vertices v and w are called *adjacent* if there is an edge $\{v, w\} \in E$, joining them. The *neighbourhood* of $x \in V$ is $N(x) = \{y \in V : \{x, y\} \in E\}$ and its *closed neighbourhood* is $N(x) \cup \{x\}$ which is denoted by $N[x]$. Note that v is not in $N(v)$, but is in $N[v]$.

We denote the cardinality of a set A, by $|A|$. Given a graph $G = (V, E)$, the degree of a vertex $x \in V$, denoted by $\delta(x)$, is $|N(x)|$. If A is a set of vertices from a graph G, $N(A)$ is the set of neighbour vertices from any vertex of A, that is, $N(A) = \cup_{x \in A} N(x)$, while $N[A] = N(A) \cup A$.

The maximum degree of G or just the degree of G is $\Delta(G) = max\{\delta(x) : x \in V\}$, while we denote with $\delta_{min}(G) = min\{\delta(x) : x \in V\}$ and with $\delta(G) = (2 \cdot |E|)/|V|$ the average degree of the graph.

Given a subset of vertices $S \subseteq V(G)$ the subgraph of G denoted by $G|S$ has vertex set S and a set of edges $E(G|S) = \{\{u, v\} \in E : u, v \in S\}$. $G|S$ is called the *subgraph of G induced by S*. We write $G - S$ to denote the graph $G|(V - S)$. The subgraph induced by $N(v)$ is denoted as $H(v) = G|N(v)$ which has to $N(v)$ as the set of nodes and all edges upon them.

Given a subgraph $H \subseteq G$ and for a vertex $x \in V(H)$, let $\delta_H(x)$ be the degree of x in the induced subgraph H of G, if $H = G$ then $\delta_G(x) = \delta(x)$. We denote $E_H(x) = \{\{x, u\} \in E(G) : u \in H\}$. Similarly, $N_H(x)$ denotes the set of nodes from H adjacent to x. For any subgraph $H \subseteq G$, $\delta_G(H) = \sum_{x \in H} \delta_G(x)$. If H is an independent set of G then $\delta_G(H)$ is the number of edges of G incident to any node of H.

A path from a vertex v to a vertex w in a graph is a sequence of edges: $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$ such that $v = v_0$, $v_n = w$, v_k is adjacent to v_{k+1} and the length of the path is n. A simple path is a path such that $v_0, v_1, \ldots, v_{n-1}, v_n$ are all distinct. A cycle is just a nonempty path such that the first and last vertices are identical, and a simple cycle is a cycle in which no vertex is repeated, except the first and last vertices.

A k-cycle is a cycle of length k, that is, a k-cycle has k edges. A cycle of odd length is called an odd cycle, while a cycle of even length is called an even cycle. A graph G is acyclic if it has not cycles.

A connected component of G is a maximal induced subgraph of G, that is, a connected subgraph which is not a proper subgraph of any other connected subgraph of G. Note that, in a connected component, for every pair of its vertices x, y, there is a path from x to y. If an acyclic graph is also connected, then it is called a free tree. Let G be a connected graph, a node $v \in V(G)$ is called a no articulation point if $G \setminus v$ is a connected graph. A subset $S \subset V(G)$ is called a no articulation set if $G \setminus S$ is a connected graph.

A colouring of a graph $G = (V, E)$ is an assignment of colours to its vertices. A colouring is *proper* if adjacent vertices always have different colours. A k-colouring of G is a mapping from V into the set $\{1, 2, \ldots, k\}$ of k "colours". The chromatic number of G denoted by $\chi(G)$ is the minimum value k such

that G has a proper k-colouring. If $\chi(G) = k$, G is then said to be k-chromatic. To determine the value $\chi(G)$ is polynomial computable when $\chi(G) \leq 2$, but when $\chi(G) \geq 3$, the problem becomes NP-complete, even for graphs G with degree $\Delta(G) \geq 3$.

Given a graph $G = (V, E)$, $S \subseteq V$ is an independent set in G if for whatever two vertices v_1, v_2 in S, $\{v_1, v_2\} \notin E$. Let $I(G)$ be the set of all independent sets of G. An independent set $S \in I(G)$ is *maximal*, abbreviated as MIS, if it is not a subset of any larger independent set and, it is *maximum* if it has the largest size among all independent sets in $I(G)$. The *independence number* $\alpha(G)$ is the cardinality of the maximum independent set of G.

Let $G = (V, E)$ be a graph, G is a *bipartite graph* if V can be partitioned into two subsets U_1 and U_2, called *partite sets*, such that every edge of G joins a vertex of U_1 and a vertex of U_2. If G is a k-chromatic graph, then it is possible to partition V into k independent sets $V_1, V_2, ..., V_k$, called *colour classes*, but it is not possible to partition V into $k-1$ independent sets.

3 An Approximate Algorithm for the Chromatic Number

Given an input connected graph $G = (V, E)$, let $n = |V|, m = |E|$ be the number of nodes and edges, respectively. A depth-first search (dfs) on G is applied starting the search with the node $v \in V$ of minimum degree, and selecting among different potential nodes to visit the node with minimum degree first and with minimum value in its label as a second criterion.

At the beginning of our procedure, a set I_B which consists of nodes not part of none odd cycle from G can be computed in polynomial time on the size $(n + m)$ of G. We show that I_B is a bipartite subgraph of G, and then I_B can be coloured at the end of the colouring process by the two last colours used for the last bipartite subgraph from G (subprocedure 2-colouring).

If $\delta(G) = (2m/n) \leq 2$ then G has not intersected cycles and it can be coloured in linear time on the number of nodes. Otherwise, if $\delta(G)$ is close to n, e.g. $\delta(G) \geq n - 4$, the complement graph of G, denoted as $\overline{G}$, shows the different colour classes of G. Then, the hard cases for colouring G is when $2 < \delta(G) < n - 3$.

Let $G_0 = G'$ be the initial graph, and let $G_{i+1} = (G_i - K_i)$ be the remaining subgraph after the i-iteration of our procedure. Let us denote as δ_i to $\delta(G_i)$ - the average degree of G_i, $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$. In each iteration, the procedure looks for a new MIS K_{i+1} in the remaining subgraph G_{i+1}.

We show that our procedure builds a MIS K_i of G_i holding that if G_i is a connected graph, then there exists a maximal independent set K_i of G_i such that $\sum_{x \in K_i} \delta(x) \geq |V(G_i)| - 1$. The algorithm $Build\_MIS(G)$ builds a MIS K_i of its input G, and it consists on the following steps:

1. Contraction process: while $|E(G_i)| > |V(G_i)|$ choose a no articulation node $x \in G_i$, push x to a stack and remove x from G_i, $G_{i+1} = G_i - \{x\}$. To

store the original edges of x on the subgraph G_i. To speed up the process choose no articulation nodes with maximum degree in G_i. At the end you have a graph G_j such that $|E(G_j)| \leq |V(G_j)|$.
2. Building a MIS of G_j: an initial MIS K_0 of G_j is built such that $\delta_{G_j}(K_0) \geq |V(G_j)| - 1$.
3. Building process: pop x from the stack, and recover its relating edges upon the subgraph G_{j-1}. If $N_{G_{j-1}}(x) \cap K_0 = \emptyset$ then $K_0 = K_0 \cup \{x\}$. Repeat step 3 until the stack is empty.
4. Return K_0. At this point $\delta_G(K_0) \geq |V(G)| - 1$.

We describe the general strategy of our proposal for colouring G, called *Seek_Chromatic_Number(G)*, as:
Firstly, in each main iteration of the procedure, G_i is tested to be polynomial-time 2-colourable and in this case, the procedure finishes and a polynomial-time 2-colouring procedure is executed.
Secondly, a MIS K_i such that $\delta_{G_i}(K_i) \geq |V(G_i)| - 1$ is formed.
Thirdly, we colour the nodes in K_i with the active colour, the nodes in K_i and the edges upon them are deleted from the current graph and the process is repeated.

Algorithm Seek_Chromatic_Number(G)
Input: G a non directed graph
Output: An approximate value for $\chi(G)$
Procedure:

$k = 3$; /* Starting with class colour $k = 3$ */
$G = dfs(G)$; $I_B = \{u \in V(G) : u$ is not part of any odd cycle of $G\}$;
$G = G - I_B$; /* Consider only nodes forming odd or intersected cycles */
if(G is Polynomial_3Colourable)returns("$\chi(G)$ is 3");
while(is_bipartite(G)==false) /* While there are odd cycles in G */
{$K = Build\_MIS(G)$;
$G = G - K$; $k++$; /* Updating for the next MIS */
}
$G = G \cup I_B$; /* returns the first bipartite component */
Call 2-colouring(G); /*At the end, the remaining graph is 2-colourable*/
Returns("$\chi(G)$ is $k + 2$");

4 Complexity Analysis

Given a connected initial graph G, let $G_0 = (G - I_B)$ be the input graph without its intial bipartite component (I_B), $G_0 = (V, E)$ with $n = |V|$ and $m = |E|$. Let us assume that $m = t \cdot n, t > 1$, and that G_0 has intersected odd cycles, then $m > n$.

Let T_i be the MIS formed in the iteration i of the loop of the procedure *Seek_Chromatic_Number*. Let $G_{i+1} = (V_{i+1}, E_{i+1})$ be the remaining subgraph generated from G_i after finishing the iteration i, that is, $G_{i+1} = G_i - T_i$. Let $n_{i+1} = |V_{i+1}|, m_{i+1} = |E_{i+1}|$ and let $\delta_i = \frac{2m_i}{n_i}$ be the average degree of each subgraph G_i. In each iteration, the number of nodes and edges are updated as: $n_{i+1} = n_i - |T_i|$ and $m_{i+1} = m_i - |E_{G_i}(T_i)|$, since in each iteration the nodes in T_i (a MIS of the current graph) are deleted as well as its incident edges: $E_{G_i}(T_i)$.

In each iteration the procedure *Build_MIS* builds a MIS T_i of the current graph G_i such that $\sum_{x\in T_i} \delta_{G_i}(x) \geq (n_i - 1)$ under the assumptions that G_i is connected and $m_i \geq n_i$.

In the first iteration it holds: $\sum_{x\in T_1} \delta_G(x) \geq (n-1)$. In the second iteration $\sum_{x\in T_2} \delta_{G-T_1}(x) \geq n_1 - 1$ which is equivalent to $\sum_{x\in T_2} \delta_G(x) - |T_1| \geq (n - |T_1|) - 1$ since each node in T_2 was originally adjacent to some node in T_1, T_1 is the first MIS of G and $n_1 = n - |T_1|$. Thus $\sum_{x\in T_2} \delta_G(x) \geq n - 1$.

The same analysis holds for the third iterations $\sum_{x\in T_3} \delta_{G-(T_1\cup T_2)}(x) \geq n_2 - 1$ which is equivalent to $\sum_{x\in T_3} \delta_G(x) - (|T_1| + |T_2|) \geq (n - |T_1| - |T_2|) - 1$, since each node in T_3 was originally adjacent to some node in T_1 and some node in T_2, $n_2 = n - |T_1| - |T_2|$. Thus, $\sum_{x\in T_3} \delta_G(x) \geq n - 1$.

The main cycle in the procedure *Seek_Chromatic_Number* stop until the graph G_k is polynomially 3-coloring (recognized in the polynomial procedure presented in [3]) and which includes the case of bipartite graphs (2-coloring graphs). Thus, in the iteration k, it holds $\sum_{x\in T_k} \delta_G(x) \geq n - 1$.

Adding the left and right parts of the inequality respectively

$$\sum_{x\in \cup_{i=1}^{k} T_i} \delta_G(x) = \sum_{x\in V} \delta_G(x) = 2m \geq k \cdot (n-1)$$

since $T_i, i = 1, \ldots, k$ is a partition of V and the sum of the degree of the nodes of a connected graph is the double of the number of edges.

The last inequality establishes an order of $k \leq (2m)/(n-1)$ iterations for the while in the procedure *Seek_Chromatic_Number*. Then, $[(2m)/(n-1)] + 1$ colours are enough (because of the two colours used in the last iteration) to colour the initial graph G, that is $\delta(1 + [1/(n-1)]) + 1$, or $\left\lceil \delta \cdot \frac{n}{n-1} \right\rceil + 1$ colours, $\delta = 2m/n$ being the average degree of the initial graph G_0.

Thus, If each G_{i+1} generated by our heuristic is connected, then $\lceil \delta(G_0) \rceil + 2$ colours are enough for colouring the input graph G, where $\delta(G_0)$ is the average degree of the input graph G without its first bipartite component.

This last result can be further improved if in addition to the condition that G_i is connected in each iteration, it also holds that $m_i \geq n_i + 1$ and then a MIS T_i can be built such that: $\sum_{x\in T_i} \delta_{G_i}(x) \geq n_i$, hence $\sum_{x\in \cup_{i=1}^{k} T_i} \delta_G(x) = \sum_{x\in V} \delta_G(x) = 2m \geq k \cdot n$. Thus, the number of iterations of the *Seek_Chromatic_Number* procedure would be $k \leq (2m)/n = \delta$, and $\lceil \delta(G_0) \rceil + 1$ colours would be required to colour the input graph G.

Notice that the main purpose to consider $G_0 = (G - I_B)$ for starting the iterative procedure *Seek_Chromatic_Number* is to reduce the possibilities that G_i will be a disconnected subgraph, $i = 1, \ldots, k$. But, if during an iteration of our procedure G_i is disconnected, then $\chi(G_i) = max\{\chi(H_1), \ldots, \chi(H_t)\}$, where $H_i, \ldots, H_t$ are the different connected components from G_i and the number of colours for colouring G could not be upper bounded by $\lceil \delta(G_0) \rceil + 2$.

One of the most expensive time task included in $Build\_MIS$ is to recognize articulation points (or cut vertices) on the current subgraph, this task is done in time $O(m + n)$, and assuming $m > n$ (which are the cases when $Build\_MIS$ is executed) the total time for recognizing articulation points is $O(m) = O(2m)$.

The number of iterations of the step 1 of $Build\_MIS$ (which concides with the number of iterations in the step 3) is at most $\lceil n/3 \rceil$ because at most $\lceil n/3 \rceil$ nodes can be removed from the original graph in order to form an acyclic graph. And to determine the articulation points in the step 1 is of order $O(m)$. And the second step (to build the initial MIS) requires at most time $O(n)$. Then, $Build\_MIS$ has a time complexity in the worst case of $O(n^2) = \lceil n/3 \rceil \cdot n$.

The most expensive step, with respect to the time complexity, of the procedure $Seek\_Chromatic\_Number$, is the "while" whose body has a time complexity of $O(n^2)$ because this consists of performing $Build\_MIS$. The number of iterations in $Seek\_Chromatic\_Number$ is proportional to $\delta(G) \cdot \frac{n}{n-1}$, then in the worst case the total time of our procedure will be $n^2 \cdot \frac{2m}{n} \cdot \frac{n}{n-1} = 2\frac{n^2 \cdot m}{n-1} \approx 2 \cdot m \cdot n$. Thus, an upper bound for the time complexity of our procedure is $O(m \cdot n)$, which is a polynomial value on the size of the input graph G.

5 Conclusions

We have presented a novel polynomial-time algorithm for determining the chromatic number of a graph $\chi(G)$. Given an input connected graph G, our heuristic discards a first bipartite component of G, denoted by I_B, formed by the nodes which are no part of any odd cycle in G since those nodes can be coloured at the end of the process with the first two basic colours. Let $G_0 = G - I_B$ be the remaining subgraph.

Our proposal is based on selecting, in an iterative manner, a MIS K_i from the current subgraph G_i such that $\delta_{G_i}(K_i) \geq |V(G_i)| - 1$. That lower bound on the number of edges in the current graph G_i, with an endpoint in any node of K_i, allow us to design an iterative procedure such that if each remained subgraph $G_{i+1} = (G_i - K_i)$ is connected, then we obtain an upper bound to colour a graph; given that $\lceil \delta(G_0) \rceil + 2$ colours are enough to colour the input graph G, where $\delta(G_0)$ is the average degree of the initial subgraph without its first bipartite component.

On the other hand, if any G_i is disconnected then $\chi(G_i) = max\{\chi(H_1), \ldots, \chi(H_t)\}$ where $H_i, \ldots, H_t$ are the different connected components from G_i.

References

[1] Beigel R., and Eppstein, D. 3-colouring in time $O(1.3289^n)$. *Journal of Algorithms.*, 54:2, 168–204, 2005.

[2] Byskov J.M. Exact Algorithms for graph colouring and exact satisifiability. Phd thesis, University of Aarbus, Denmark, 2005.

[3] De Ita Guillermo, and Castillo Javier A. Recognizing 3-colorings cycle-patterns on graphs. *Pattern Recognition Letters.*, Vol.34,Issue 4, 433–438, 2013.

[4] Golumbic M.C. *Algorithmic Graph Theory and Perfect Graphs.* 2nd edn. North Holland, 2004.

[5] Kocay William, and Kreher Donald L. *Graphs, Algorithms, and Optimization.* Chapman & Hall/CRC, 2005.

[6] Wilson R. A. *Graphs, Colourings and the Four-colour Theorem.* Oxford University Press, 2002.

An Enumerative Algorithm for #2SAT *

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Abstract. Counting models for two conjunctive forms (2-CF), problem known as #2SAT, is a classic #P problem. We determine different discrete structures on the constrained graph of the 2-CF fomula F allowing the efficient computation of #2SAT(F).

We show that if the constrained graph of a 2-CF F is acyclic or it has only cycles, which are independent one to other, then #2SAT(F) is computed efficiently. On the other hand, we design a procedure with an orientation down-top adequate to decompose the computation of #2SAT(F) in an incremental way. We start our procedure with $F_0 = A_F$, which is the maximum subformula of F without intersecting cycles. In each iteration of the procedure, a new clause $C_i \in (F - A_F)$ is considered in order to form $F_i = (F_{i-1} \wedge C_i)$ and then to compute #2SAT(F_i) based on the computation of #2SAT(F_{i-1}).

Key words: #SAT Problem, Counting models, Enumerative Algorithm.

1 Introduction

#SAT (the problem of counting models for a Boolean formula) is of special concern to Artificial Intelligence (AI), and it has a direct relationship to Automated Theorem Proving, as well as to approximate reasoning [3,4,7]. #SAT can be reduced to several different problems in approximate reasoning. For example, in the cases of: estimating the degree of belief in propositional theories, the generation of explanations to propositional queries, repairing inconsistent databases, in Bayesian inference, in a truth maintenance systems [1,3,4,7,8]. The previous problems come from several AI applications such as planning, expert systems, approximate reasoning, etc.

To count combinatorial objects on graphs is an interesting and important area of research in Mathematics, Physics, and Computer Sciences. Counting problems, being mathematically relevant by themselves, are closely related to

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practical problems. Although many relevant counting problems are hard time-complexity problems. For example, the maximum polynomial class recognized for #2SAT is the class $(\leq 2, 2\mu)$-CF (2-CF where each variable appears twice at most) [7,8].

Contrary to the traditional procedures for computing #2SAT with an orientation top-down, as the Davis & Putnam counting procedure and some of its adaptations [1,2,6], we have designed a novel procedure with an orientation down-top which is more adequate to apply dinamic programming for solving the #2SAT problem.

We extend here some of the procedures presented in [5,4] for the #2-SAT problem and show how to apply them to the computation of the number of models in a propositional theory. Furthermore, we show different structural patterns on the constrained graph of the formula which allow the efficient computation of the number of models for some classes of 2-CF's. And we consider the general case, showing that this problem can be adequated parameterized according to the number of intersected cycles appearing in the constrained graph of the input formula.

2 Preliminaries

Let $X = \{x_1, \ldots, x_n\}$ be a set of n Boolean variables. A literal is either a variable x_i or a negated variable $\overline{x}_i$. As usual, for each $x_i \in X$, $x_i^0 = x_i$ and $x_i^1 = x_i$. A clause is a disjunction of different literals (sometimes, we also consider a clause as a set of literals). For $k \in N$, a k-clause is a clause consisting of exactly k literals and, a $(\leq k)$-clause is a clause with at most k literals. A variable $x \in X$ appears in a clause c if either x or $\overline{x}$ is an element of c.

A Conjunctive Form (CF) F is a conjunction of clauses (we also consider a CF as a set of clauses). We say that F is a positive monotone CF if all of its variables appear in unnegated form. A k-CF is a CF containing only k-clauses and, $(\leq k)$- CF denotes a CF containing clauses with at most k literals. A $k\mu$-CF is a formula in which no variable occurs more than k times. A $(k, j\mu)$-CF ($(\leq k, j\mu)$-CF) is a k-CF ($(\leq k)$-CF) such that each variable appears no more than j times.

We use $\nu(Y)$ to express the set of variables involved in the object Y, where Y could be a literal, a clause or a Boolean formula. For instance, for the clause $c = \{\overline{x}_1, x_2\}, \nu(c) = \{x_1, x_2\}$. And $Lit(F)$ is the set of literals which appear in a CF F, i.e. if $X = \nu(F)$, then $Lit(F) = X \cup \overline{X} = \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$. We also denote $\{1, 2, \ldots, n\}$ by $[[n]]$.

An assignment s for F is a Boolean function $s : \nu(F) \to \{0, 1\}$. An assignment can be also considered as a set of non complementary pairs of literals. If $l \in s$, being s an assignment, then s turns l true and $\bar{l}$ false. Considering a

clause c and assignment s as a set of literals, c is satisfied by s if and only if $c \cap s \neq \emptyset$, and if for all $l \in c$, $\bar{l} \in s$ then s falsifies c.

Let F be a Boolean formula in Conjunctive Form (CF), F is satisfied by an assignment s if each clause in F is satisfied by s. F is contradicted by s if any clause in F is contradicted by s. A model of F is an assignment for $\nu(F)$ that satisfies F. Given F a CF, the SAT problem consists of determining if F has a model. The #SAT problem consists of counting the number of models of F defined over $\nu(F)$. #2-SAT denotes #SAT for formulas in 2-CF.

2.1 The constrained graph of a 2-CF

Among the graphical representations of a conjunctive form (see e.g. [9]), we took the signed primal graph of a two conjuctive form (2-CF) and use it to design a recursive enumerative procedure to solve the #2SAT problem. Let F be a 2-CF with set of variables $\upsilon(F)$. The constrained graph (signed primal graph) of F is denoted by $G_F = (V(F), E(F))$, with $V(F) = \upsilon(F)$ and $E(F) = \{\{\upsilon(x), \upsilon(y)\} : \{x, y\} \in F\}$, that is, the vertices of G_F are the variables of F, and for each clause $\{x, y\}$ in F there is an edge $\{\upsilon(x), \upsilon(y)\} \in E(F)$. We say that a 2-CF F is a path, cycle, or a tree, if its constrained graph G_F represents a path, cycle, or a tree, respectively.

Each edge $c = \{\upsilon(x), \upsilon(y)\} \in E$ is associated with an ordered pair (s_1, s_2) of signs which are used as labels of the edge that connect the variables appearing in the clause. For example, the clause $\{x^0, y^1\}$ determines the labelled edge: "$x \overset{-+}{—} y$" which is equivalent to the edge "$y \overset{+-}{—} x$". The signs $s_1, s_2 \in \{+, -\}$ are related to the variables x and y, respectively.

Let $S = \{+, -\}$ be a set of signs. A graph with labelled edges on a set S is a pair (G, ψ), where $G = (V, E)$ is a graph, and ψ is a function with domain E and range S. $\psi(e)$ is called the label of the edge $e \in E$. Let $G = (V, E, \psi)$ be a constrained graph with labelled edges on SxS. Let x and y be nodes in V . If $e = \{x, y\}$ is an edge and $\psi(e) = (s, s')$, then $s(s')$ is called the adjacent sign to $x(y)$.

Notice that a constrained graph of a 2-CF can be a multigraph, because two same fixed variables can appear in more than one clause of the formula, and this forms parallel edges. Furthermore, an unitary clause is represented by a loop in the constrained graph (an edge to join a vertex to itself).

3 Linear Counting Procedure for #2SAT

Let $G_F = (V(F), E(F), \{+, -\})$ be a signed connected graph of an input formula F in 2-CF, with $n = |V|$ and $m = |E|$.

A depth-first search (abbreviated as dfs) is applied over G_F. The dfs starts with the node $v_r \in V$ of minimum degree; when a new node is visited, we select first the nodes with minimum degree, and as a second criterion, we select the

one with minimum value in its label. The result of applying a dfs on G is a depth-first graph G', which we will denote as $G' = dfs(G_F)$ and a spanning tree T_G with v_r as the root node.

The dfs allows us to detect if G has or not cycles and the parity of such cycles in time $O(m+n)$. The edges in T_G are called *tree edges*. An edge $e \in E(G) \backslash E(T_G)$ is called *back edge*, and we call to its repective clause $C_e \in F$ a *back clause*. Let $e \in E(G) \backslash E(T_G)$ be a back edge, the union of the path in T_G between the endpoints of e with the edge e itself forms a simple cycle, such cycle is called a basic cycle of G with respect to T_G.

Let $\mathcal{C} = \{C_1, C_2, ..., C_k\}$ be the set of back edges found during the depth-first search. If two distinct base cycles C_i and C_j from $\mathcal{C}$ have common edges then we say that both cycles are *intersected*, and then, $C_i \triangle C_j$ form a new cycle, where $\triangle$ denotes the symmetric difference operation between the set of edges of both cycles. If two cycles are non-intersected we say that they are *independent*.

Let A_F be the graph formed by the union of a spanning tree from G_F and all back edges from $\mathcal{C}$ which form only independent cycles in A_F. And let $CC = E(G_F) - E(A_F) = \{C_1, C_2, \ldots, C_t\} \subset \mathcal{C}$ be the set of back edges forming intersecting cycles with any cycle in A_F. If A_F has not intersecting cycles (if $CC = \emptyset$), then we have designed a procedure (see [4]) for computing #2SAT(F) in linear time on the size of A_F.

A knot, denoted by κ, is a set of intersecting cycles such that each two cycles of κ are intersected. The cardinality $|\kappa|$ of a knot is the number of cycles forming such knot. Notice that any knot has at least one edge e that is common to all cycle in the knot.

The knots of the graph are identified by traversing through the marks of the edges, and also the cardinality of each knot can be computed at the same time. We call *knotting number* of a graph to the maximum cardinality of the knots in the graph. We denote the *knotting number* of a graph G as $\varpi(G)$.

We present in the following section, a recursive procedure for computing #2SAT(F) based on the topology of the constrained graph G_F, and its set of knots (if there exists).

4 Incremental Computation of #2SAT

Contrary to the traditional procedures for computing #2SAT with an orientation top-down, as the Davis & Putnam counting procedure and some of its adaptations [1,2,6], we have designed a novel procedure with an orientation down-top which is more adequate to apply dinamic programming for solving the #2SAT problem.

Our procedure starts with $F_0 = A_F$ and the computation of #2SAT(F_0) which is performed in linear time on the size of A_F [4]. In each iteration, a new clause $C_i \in CC, i = 1, \ldots, t$ is considered in order to form

$F_i = F_{i-1} \wedge C_i$, and then #2SAT(F_i) is computed. Notice that #2SAT(F) = #2SAT($A_F \bigwedge_{C_i \in CC} C_i$).

Definition 1. *Let H be a CF and x a literal of H. The reduction of H by x, also called forcing x and denoted by $H[x]$, is the formula generated from H by the following two rules:*

a) removing from H the clauses containing x (subsumption rule),
b) removing $\overline{x}$ from the remaining clauses (unit resolution rule).

A reduction is also sometimes called a *unit reduction*. The reduction by a set of literals can be inductively established as follows: let $s = \{l_1, l_2, \ldots, l_k\}$ be a partial assignment of $\upsilon(H)$. The reduction of H by s is defined by successively applying definition 1 for l_i, $i = 1, \ldots, k$, that is, the reduction of H by l_1 gives the formula $H[l_1]$, following a reduction of $H[l_1]$ by l_2, giving as a result the formula $H[l_1, l_2]$ and so on. The process continues until $H[s] = H[l_1, ..., l_k]$ is reached. In case that $s = \emptyset$ then $H[s] = H$.

Example 1. Let $H = \{\{x_1, \overline{x}_2\}, \{x_1, x_2\}, \{x_1, x_3\}, \{\overline{x}_1, x_3\}, \{\overline{x}_2, x_4\}, \{\overline{x}_2, \overline{x}_4\}, \{x_2, x_5\}, \{x_3, \overline{x}_5\}\}$. Then, $H[\overline{x}_2] = \{\{x_1\}, \{x_1, x_3\}, \{\overline{x}_1, x_3\}, \{x_5\}, \{x_3, \overline{x}_5\}\}$, and for $s = \{x_2, \overline{x}_3\}$ we have that $H[s] = \{\{x_1\}, \{x_1\}, \{\overline{x}_1\}, \{x_4\}, \{\overline{x}_4\}, \{\overline{x}_5\}\}$.

Let H be a CF and s a partial assignment of H. If a pair of contradictory unitary clauses is obtained while $H[s]$ is being computed then $\#SAT(H[s]) = 0$, because under no circumstances a pair of complementary unit clauses can be set to true at the same time. Thus, $H[s]$ does not have models.

During the computation of $H[s]$ new unitary clauses can be generated. Thus, the partial assignment s is extended by adding the unitary clauses found, that is, $s = s \cup \{u\}$ where $\{u\}$ is a unitary clause. So, $H[s]$ can be again reduced using the new unitary clauses. We call to this iterative process $Eval(H, s)$.

The application of definition 1 on a 2-CF formula F, could remove variables which have to be considered in the models of F. Let us present an example.

Example 2. Let $F = \{\{x\}, \{x, y\}\}$, in other words $F = x \wedge (x \vee y)$, it follows that $F[x] = \emptyset$. It can be noticed that $\{y\} \notin F[x]$, however y can take any logical value in the models of F.

The variables which are removed from F during the application of $Eval(F, s)$, form a set, which will be denoted by $Elim\_Vars(s)$. In fact, it can be checked that $|Elim\_Vars(s)| = |\upsilon(F)| - |\upsilon(F')| - |s'|$, where s' is the assignment of s extended by the unit clauses processed during the iterative process $Eval(F, s)$, and F' is the subformula from F resulting of the aplication of $Eval(F, s)$. We will denote to those results of $Eval(F, s)$, the subformula F' and the extended assignment s', as: $(F', s') = Eval(F, s)$. However, when we want to

do reference to only one element of the pair, we use $F' = Eval(F, s)$, or $s' = Eval(F, s)$.

Let F be a 2-CF and s a partial assignment on $\upsilon(F)$, after to obtain $(F', s') = Eval(F, s)$, we have that

$$\#SAT(F) = \#SAT(F') \times 2^{|Elim\_Vars(S(F))|}. \quad (1)$$

Every model of A_F had already determined truth values for all variable $\upsilon(F)$ of the total formula F. Given a back clause $C = \{x_j, x_k\}$ in $F - A_F$, for any model s of A_F such that $\overline{x}_j \in s$ and $\overline{x}_k \in s$, then s is not a model of $A_F \wedge C$, because s falsifies C. Thus, if we know SAT(A_F) then

$$\#SAT(A_F \wedge C) = \#SAT(A_F) - |\{s \in SAT(A_F) : s \text{ falsifies } C\}|. \quad (2)$$

Let $Y = \{s \in SAT(A_F) : s$ falsifies $C\}$. A way to compute $|Y|$ is to form the partial assignment $s_C = \overline{C} = \{\overline{x}_j, \overline{x}_k\}$ and to evaluate that partial assignment on A_F, that is, $F' = A_F[s_C]$. Then, $|Y| = \#SAT(F')$. Furthermore, if A_F has not intersecting cycles, then $G_{F'}$ has not intersecting cycles too.

We can iterate the procedure (2) for computing new back clauses in F, that is, #2SAT($A_F \bigwedge_{C_i \in CC} C_i$) is computed in an iterative way, forming in each iteration a new formula $F' = Eval(F_{i-1}, s_i)$, which is the resulting formula of the evaluation of the partial assignment s_i on the formula $F_{i-1} = (A_F \wedge \bigwedge_{j=1}^{i-1} C_i)$. In our case, $s_i = \overline{C}_i$ denotes the partial assignment on $\upsilon(F)$ falsifying C_i, that is, if $C_i = (l_1, l_2)$ then $s_i = \{\bar{l}_1, \bar{l}_2\}$.

We presente now, the pseudo-code for the incremental computation of #2SAT($A_F \bigwedge_{C_i \in CC} C_i$).

Procedure $Count\_Models(F)$
Let $F_0 = A_F$; /* F_0 is the spanning tree union independent cycles */
Let $A_0 = \#2SAT(F_0)$;
$\forall C_i \in CC$ do {
Let $F' = Eval(F_{i-1}, \overline{C_i})$;
Let $A_i = A_{i-1} - \#2SAT(F')$;
Let $F_i = (F_{i-1} \wedge C_i)$;
} Returns(A_i) /* $\#2SAT(F) = A_i$ */

Thus, our procedure computes #2SAT(F) based on the folllowing equation:

$$\#2SAT(F) = \#2SAT(A_F) - \sum_{i=0}^{t-1} \#2SAT((A_F \bigwedge_{j=1}^{i} C_j)[\overline{C_{i+1}}]) \quad (3)$$

In fact, if the constrained graph of each resulting subformula $F_{i+1} = (A_F \bigwedge_{j=1}^{i} C_j)[\overline{C}_{i+1}], i = 0, \ldots, t-1$ has not intersecting cycles, then each #2SAT(F_i) is computed in linear time on the size of F_i, which in the worst

case is of order $O(m)$, and then the sum $\sum_{i=0}^{t-1} \#2SAT(F_{i+1})$ will be computed in a complexity time of $O(m \cdot t) = O(m(m-n))$ and all the computation of $\#2\text{SAT}(F)$ will have a polynomial time complexity of order $O(m \cdot (m-n))$. In this case, we can consider the computation of $\#2\text{SAT}(F)$ by $t+1$ linear-time logical subformulas F_i (where $F_0 = A_F$ and the computation of each $\#2\text{SAT}(F_i), i = 1, \ldots, t$ is done in linear-time complexity). Notice that in this case, the initial graph G_F has knots but each one of the constrained graph of each $F_i, i = 1, \ldots, t$ has not knots.

If the constrained graph of any $F_i, i = 1, \ldots, t$ has knots (intersecting cycles) then the same equation (3) can be applied recursively in order to compute $\#2\text{SAT}(F_i)$. Then, we could design a recursive procedure based on the procedure $Count\_Models$ for computing $\#2\text{SAT}(F)$ independent from the number of knots and its cardinality that the constrained graph G_F has, as well as any of its subformulas $F_{i+1} = (A_F \bigwedge_{j=1}^{i} C_j)[\overline{C}_{i+1}], i = 0, \ldots, t-1$.

Of course, an ordering on the set of clauses $CC = F - A_F$ is crucial for accelerating the computation of $\#2\text{SAT}(F)$. In this aspect, if during the computation of $F_{i+1} = (A_F \bigwedge_{j=1}^{i} C_j)[\overline{C}_{i+1}], i = 0, \ldots, t-1$, the graph of any F_{i+1} has intersecting cycles then it is preferable to move the clause C_{i+1} at the end of the set CC, avoiding so to drag a factor with an expensive time-complexity in the remaining iterations of the iterative procedure.

5 Conclusions

#SAT problem for the class of Boolean formulas in 2-CF is a classical #P-complete problem. However, there are several instances of 2-CF's for which #2SAT can be solved efficiently.

We have shown different polynomial-time procedures for counting models of Boolean formulas for subclasses of 2-CF's. For example, for formulas whose contrained graph is acyclic, its corresponding number of models is computed in linear time. Furthermore, if the set of cycles in a constrained graph G_F can be arranged as independent and embedded cycles, then we can count efficiently the number of models of F.

When G_F has intersected cycles then an incremental deterministic procedure is presented in order to compute $\#2\text{SAT}(F)$. This procedure has an orientation down-top adequate to decompose the computation of $\#2\text{SAT}(F)$ in an incremental way. We start our procedure with $F_0 = A_F$, which is the maximum subformula of F without intersecting cycles. In each iteration of the procedure, a new clause $C_i \in (F - A_F)$ is considered in order to form $F_i = (F_{i-1} \wedge C_i)$ and then to compute $\#2\text{SAT}(F_i)$ based on the computation of $\#2\text{SAT}(F_{i-1})$.

References

[1] Angelsmark O., and Jonsson P. Improved Algorithms for Counting Solutions in Constraint Satisfaction Problems. In ICCP: Int. Conf. on Constraint Programming, Springer, 81–95, 2003.

[2] Dahllöf, V., Jonsonn, P., and Wahlström, M. Counting models for 2SAT and 3SAT formulae. *J. Theoretical Computer Sciences*, 332(1-3):265–291, 2005.

[3] Darwiche A. On the Tractability of Counting Theory Models and its Application to Belief Revision and Truth Maintenance. *J. of Applied Non-classical Logics*, 11(1-2):11–34, 2001.

[4] De Ita G., Bello P., and Contreras M. New Polynomial Classes for #2SAT Established via Graph-Topological Structure. *Engineering Letters*, Vol. 15, Issue 2:250–258, 2007.

[5] De Ita G., Bello P., and Contreras M. Recognizing Structural Patterns for the Efficient Computation of #2SAT. *LNCS, Springer-Verlag*, 7914:1611–3349, 2013.

[6] Fürer, M., and Prasad, S. K. *Algorithms for Counting 2-SAT Solutions and Coloring with Applications.* Technical Report No. 33, Electronic Colloqium on Comp. Complexity, 2005.

[7] Roth D. On the hardness of approximate reasoning. *J. Artificial Intelligence*, 82:273–302, 1996.

[8] Russ B. *Randomized Algorithms: Approximation, Generation, and Counting.* Distingished dissertations, Springer, 2001.

[9] Szeider Stefan. On Fixed-Parameter Tractable Parameterizations of SAT. *LNCS, Springer-Verlag*, 2919:188–202, 2004.

A problem of Shapozenko on Johnson graphs*

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Abstract. The Johnson graph $J(n,m)$ has the m–subsets of $[n]$ as vertices and two subsets are adjacent in the graph if they share $m-1$ elements. Shapozenko asked about the isoperimetric function of Johnson graphs, that is, the cardinality of the smallest boundary of sets with k vertices in $J(n,m)$ for each $1 \leq k \leq \binom{n}{m}$. We give an upper bound for the isoperimetric function of Johnson graphs. We show that, for given t and all sufficiently large n, the given bound is tight for $k = \binom{t}{m}$. We also show that the bound is tight for the small values of $k \leq m+1$ and for all values of k when $m = 2$.

Key words: Johnson graph, Isoperimetric, Colexicographic, Shift compression.

1 Introduction

Let $G = (V, E)$ be a graph. Given a set $X \subset V$ of vertices, we denote by

$$\partial X = \{y \in V \setminus X : d(X, y) = 1\},$$

the *boundary* of X, and by

$$B(X) = \{y \in V : d(X, y) \leq 1\} = X \cup \partial X,$$

the *ball* of X. We write ∂_G and B_G when the reference to G has to be made explicit.

The *isoperimetric function* of G is defined as

$$\mu(k) = \min\{|\partial X| : X \subset V,\ |X| = k\},$$

that is, $\mu(k)$ is the size of the smallest boundary among sets of vertices with cardinality precisely k.

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The isoperimetric function is known only for some few classes of graphs. One of the seminal results is the exact determination of the isoperimetric function for the n–cube obtained by Harper [11] in 1966. Analogous results were obtained for cartesian products of chains by Bollobás and Leader [7] and Bezrukov [1], cartesian products of even cycles by Karachanjan [16] and Riordan [18] (see also [3]) and some other cartesian products by Bezrukov and Serra [4]. The survey on discrete isoperimetric problems by Bezrukov [2] gives a panoramic view of the isoperimetric problem in discrete spaces.

In his monography on discrete isoperimetric problems Leader [17] mentions the isoperimetric problem for Johnson graphs as one of the intriguing open problems in the area. Later on, in his extense monography on isoperimetric problems, Harper [12] atributes the problem to Shapozenko, and recalls that the problem is still open.

The Johnson graph $J(n, m)$ is defined as the graph with vertex set the family of all m–subsets of $[n]$ where two m–subsets are adjacent in the graph whenever their symmetric difference has cardinality 2. It follows from the definition that, for $m = 1$, the Johnson graph $J(n, 1)$ is the complete graph K_n. For $m = 2$ the Johnson graph $J(n, 2)$ is the line graph of the complete graph on n vertices, also known as the triangular graph $T(n)$. Thus, for instance, $J(5, 2)$ is the complement of the Petersen graph, displayed in Figure 1. More generally, $J(n, 2)$ is the complement of the Kneser graph $K(n, 2)$, the graph which has the 2–subsets of $[n]$ as vertices and two pairs are adjacent whenever they are disjoint.

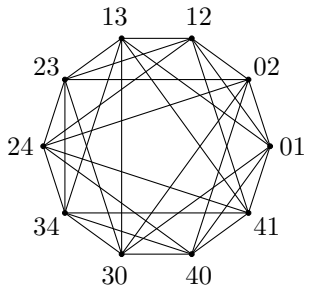

Fig. 1: The Johnson graph $J(5, 2)$.

Johnson graphs arise from the association schemes named after Johnson who introduced them, see e.g. [10]. The Johnson graphs form one of the classes of distance transitive graphs (we recall that a graph is distance transitive if, for every two pairs x, y and x', y' of vertices at the same mutual distance $d(x, y) = d(x', y')$, there is an automorphism f of the graph such that $f(x) = x'$ and $f(y) = y'$.)

Recall that the colex order in the set of m–subsets of n is defined as $\mathbf{x} \leq \mathbf{y}$ if and only if $\min(\mathbf{x} \oplus \mathbf{y}) \in \mathbf{x}$, where $\oplus$ denote the symmetric difference of the two m–sets $\mathbf{x}, \mathbf{y}$. The computation of the boundary of initial segments for the colex order provides the following upper bound for the isoperimetric function of Johnson graphs:

Proposition 1. *Let $\mu : N \to \mathbb{N}$ denote the isoperimetric function of the Johsnon graph $J(n,m)$, where $N = \binom{n}{m}$. Let*

$$k = \binom{k_0}{m} + \cdots + \binom{k_r}{m-r},\ k_0 > \ldots > k_r > 0,$$

be the m–binomial representation of k. Then

$$\mu(k) \leq \binom{k_0}{m-1}(n-k_0) + \sum_{i=1}^{r} \left(\binom{k_i}{m-i-1}(n-k_0-1) - \binom{k_i}{m-i} \right).$$

Our main purpose is to show that the above upper bound for the isoperimetric function of Johsnon graphs is tight in several cases.

We call a set S of vertices of $J(n,m)$ *optimal* if $|\partial(S)| = \mu(|S|)$.

Given a family S of m–sets of an n–set, its lower shadow $\Delta(S)$ is the family of $(m-1)$–sets which are contained in some m–set in S. The upper shadow $\nabla(S)$ of S is the family of $(m+1)$-sets which contain some m–set in S. The ball of S in the Johnson graph $J(n,m)$ can be written as

$$B(S) = \nabla\Delta(S) = \Delta\nabla(S).$$

These equalities establish a connection between the isoperimetric problem in the Johnson graph with the Shadow Minimization Problem (SMP), which consists in finding, for a given k, the smallest cardinality of $\Delta(S)$ among all families S of m–sets with cardinality k. The latter problem is solved by the well–known Kruskal–Katona theorem, which establishes that the initial segments of the colex order provide a family of extremal sets for the SMP. However, this family of initial segments does not provide in general a solution to the isoperimetric problem in $J(n,m)$. Nevertheless this is the case for the cardinalities of the form $\binom{t}{m}$ for some fixed t and every sufficiently large n:

Theorem 1. *Let $t \geq m$ be positive integers. There is $n(t,m)$ such that for all $n \geq n(t,m)$ the set $\binom{[t]}{m}$ is an optimal set of the Johnson graph $J(n,m)$.*

One can check that the ball of radius one is an optimal set of $J(n,m)$ for $n \leq f(m)$ and some function of m. It has a smaller boundary than the corresponding initial segment in colex order. In particular, in $J(6,3)$ the ball has cardinality $\binom{5}{3}$ and its boundary is one less than the initial segment with the same cardinality. This shows that $n(5,3) > 6$.

For small values of k the colex order still provides the solution to the isoperimetric problem in $J(n,m)$:

Theorem 2. *Let m be a positive integer. For every $k \leq m+1$ the k first elements in the colex order provide an optimal set of $J(n,m)$ for every $n \geq m$.*

Finally, for $m = 2$, the colex order always provides a family of optimal sets in $J(n,2)$ for all $n \geq 2$:

Theorem 3. *Let $I(k)$ denote the initial segment of length k in the colex order of the family of 2–subsets of an n–set. For each set X of k vertices of the Johnson graph $J(n,2)$ we have*

$$|\partial(I(k))| \leq |\partial(X)|.$$

2 Shifting techniques

Shifting techniques are one of the key tools in the study of set systems. They were initially introduced in the original proof of the Erdos–Ko–Rado theorem [8] and have been particularly used in the solution by Frankl and Füredi [9] of the isoperimetric problem for hypercubes.

In what follows we identify subsets of $[n]$ with their characteristic vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \{0,1\}^n$ where $x_i = 1$ if i is in the corresponding set and $x_i = 0$ otherwise. We denote the *support* of $\mathbf{x}$ by

$$\overline{\mathbf{x}} = \{i : x_i = 1\}.$$

and the (ℓ_1) *norm* of $\mathbf{x}$ by

$$|\mathbf{x}| = \sum_i x_i.$$

The sum $\mathbf{x} + \mathbf{y} = (x_1 + y_1 \pmod 2, \ldots, x_n + y_n \pmod 2)$, of characteristic vectors is meant to be performed in the field $\mathcal{F}_2^n$ and it corresponds to the symmetric difference of the corresponding sets. We also denote by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the unit vectors with 1 in the i–th coordinate and zero everywhere else.

With the above notation, the set of vertices of the Johnson graph are all vectors with norm m and the neighbors of $\mathbf{x}$ in $J(n,m)$ are the vectors

$$\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j,$$

for each pair i, j such that $x_i + x_j = 1$.

We shall use the shifting transformations defined below.

Definition 1. *Let $\mathbf{u}, \mathbf{v}$ be two vectors with the same weight and disjoint support. For a set $S \subset \{0,1\}^n$ define*

$$S_{uv} = \{\mathbf{x} \in S : \overline{\mathbf{v}} \subset \overline{\mathbf{x}} \text{ and } \overline{\mathbf{x}} \cap \overline{\mathbf{u}} = \emptyset\},$$

and

$$T_{uv}(\mathbf{x}, S) = \begin{cases} \mathbf{x}+\mathbf{u}+\mathbf{v}\ \mathbf{x} \in S_{uv} \text{ and } \mathbf{x}+\mathbf{u}+\mathbf{v} \notin S \\ \mathbf{y} \qquad\qquad \text{otherwise} \end{cases}$$

The uv–shift is

$$T_{uv}(S) = \{T_{uv}(\mathbf{y}, S) : \mathbf{y} \in S\}.$$

It follows from the definition that the shifting T_{uv} of a set preserves its cardinality and the norm of its elements. Moreover, if the norm of $\mathbf{u}$ and $\mathbf{v}$ is r then it sends every vertex to a vertex at distance at most r. It is easily checked that the shifting is also monotone with respect to inclusion.

Lemma 1. *If $X \subset Y$ then $T_{uv}(X) \subset T_{uv}(Y)$.*

The weight of a vector $\mathbf{x}$ is

$$w(\mathbf{x}) = \sum_{i=1}^{n} i x_i,$$

and the weight of a set S is

$$w(S) = \sum_{\mathbf{x} \in S} w(\mathbf{x}).$$

We note that, if $w(\mathbf{u}) < w(\mathbf{v})$ then $w(T_{uv}(S)) \leq w(S)$. Moreover, equality holds if and only if $T_{uv}(S) = S$. Thus, successive application of transformations T_{uv} using pairs of vectors $\mathbf{u}, \mathbf{v}$ with $w(\mathbf{u}) < w(\mathbf{v})$ eventually produces a set which is stable by any of such transformations. This fact leads to the following definition.

Definition 2. *Given a set $W \subset \binom{[n]}{r}$ of vectors, we say that a set S is W–compressed if $T_{uv}(S) = S$ for each pair $u, v \in W$ with $\max \overline{\mathbf{u}} < \min \overline{\mathbf{v}}$.*

A set S is said to be r–compressed if it is compressed for any set $W \subset \binom{[n]}{i}$ of vectors with $i \leq r$.

Let $\mathbf{u} = \{i_1 < \cdots < i_r\}$ and $\mathbf{v} = \{j_1 < \cdots < j_r\}$. Let $\sigma \in Sym(n)$ be the permutation which exchanges $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ by preserving the local order, and leaves invariant the remaining elements: $\sigma(i_1) = j_1, \ldots, \sigma(i_r) = j_r$ and $\sigma(j_1) = i_1, \ldots, \sigma(j_r) = i_r$. We denote by $\sigma_{uv} : \{0,1\}^n \to \{0,1\}^n$ the map

$$\sigma_{uv}(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Thus, for example, if $\mathbf{x} \in S_{uv}$ for some set S then $\sigma_{uv}(\mathbf{x}) = \mathbf{x}+\mathbf{u}+\mathbf{v}$.

The main property of the uv–shifting of a set S is that, when $r = |u| = |v|$ and S is $(r-1)$–compressed, then it does not increase the cardinality of the ball $B(S)$.

Lemma 2. *Let* $\mathbf{u}, \mathbf{v} \in \{0,1\}^n$ *with norm* 1 *and disjoint support and write* $T = T_{uv}$. *For each set* S *of vertices in the Johnson graph* $J(n,m)$ *we have*

$$B(T(S)) \subseteq T(B(S)). \tag{1}$$

From this lemma, it follows that

$$|B(T(S))| \leq |T(B(S))| = |B(S)|. \tag{2}$$

3 The case $m = 2$

Proposition 2. *Given a set* S *of vertices of the graph* $J(n,2)$ *with cardinality* $\binom{t-1}{2} < |S| \leq \binom{t}{2}$, *it is optimal if and only if* $S \subset \binom{[t]}{2}$.

As a consequence of this Proposition, we can see that the sets of the kind $\binom{[t]}{2}$ are the unique optimal set of this cardinality (and also are unique, up to isomorphisms, the optimal sets of cardinality $\binom{t}{2} - 1$).

4 The case $m + 1$

Consider the partition

$$V(J(n,m)) = (V(J(n,m)) \cap \{x_1 = 0\}) \cup (V(J(n,m)) \cap \{x_1 = 1\}).$$

The subgraph induced by $V_0 = V(J(n,m)) \cap \{x_1 = 0\}$ is isomorphic to $J(n-1,m)$ and the subgraph induced by $V_1 = V(J(n,m)) \cap \{x_1 = 1\}$ is isomorphic to $J(n-1,m-1)$. There is an edge in $J(n,m)$ joining $x \in V_1$ with $y \in V_0$ if and only if $\sigma x \subset \sigma y$.

For every subset $S \subset V = V(J(n,m))$ let $S_0 = S \cap V_0$ and $S_1 = S \cap V_1$.

Lemma 3. *Let* S *be a set of vertices in* $J(n,m)$. *Let* $S_0 = S \cap \{x_1 = 0\}$ *and* $S_1 = S \cap \{x_1 = 1\}$. *If* S *is* 1*–compressed then*

$$B(S) = B(S_1) \;\; \textit{and} \; |B(S)| = |B'(S_1')| + |\nabla(S_1')|,$$

where $S_1' = \{\mathbf{x} \in \{0,1\}^{n-1} : (1,\mathbf{x}) \in S_1\}$ *and* B' *denotes the ball in* $J(n-1,m-1)$.

The above Lemma shows that if a set $S' \in V(J(n-1,m-1))$ minimizes both the ball and the upper shadow then a 1–compressed set in $J(n,m)$ such that its $\{x_1 = 1\}$–section is S' might be optimal in $J(n,m)$. Initial segments in the lex order do minimize the upper shadow but these sets usually do not minimize the size of the ball.

A similar argument can be used with sections of $\{x_n = 0\}$ and $\{x_n = 1\}$.

Lemma 4. *Let S be a set of vertices in $J(n,m)$. Let $S_0 = S \cap \{x_n = 0\}$ and $S_1 = S \cap \{x_n = 1\}$. If S is compressed then*

$$B(S) = B(S_0) \quad \text{and } |B(S)| = |B'(S_0')| + |\Delta(S_0')|,$$

where $S_0' = \{\mathbf{x} \in \{0,1\}^{n-1} : (\mathbf{x}, 0) \in S_0\}$ and B' denotes the ball in $J(n-1,m)$.

Proposition 3. *For $k \leq m+1$ the initial segment of the colex order in $J(n,m)$ is an optimal set. Moreover, for $n > m+2$, the solution is unique up to automorphisms of the graph.*

References

[1] S.L. Bezrukov, An isoperimetric problem for Manhattan lattices, in: Proc. Int. Conf. Finite and Infinite Sets, Budapest 1990.

[2] S. L. Bezrukov. Isoperimetric problems in discrete spaces. Extremal problems for finite sets (Visegrád, 1991), 59–91, Bolyai Soc. Math. Stud., 3, János Bolyai Math. Soc., Budapest, 1994.

[3] S.L. Bezrukov, U. Leck. A simple proof of the Karakhanyan-Riordan theorem on the even discrete torus. *SIAM J. Discrete Math.* 23 (2009), no. 3, 1416–1421.

[4] S. L. Bezrukov; O. Serra. A local-global principle for vertex-isoperimetric problems. Discrete Math. 257 (2002), no. 2-3, 285–309.

[5] Z. Füredi; J. R. Griggs. Families of finite sets with minimum shadows. Combinatorica (1986), no. 6, 355–363.

[6] M. Mörs. A Generalization of a Theorem of Kruskal. Graphs and Combinatirocs (1985), no. 1, 167–183.

[7] B. Bollobás, I. Leader, An isoperimetric inequality on the discrete torus, *SIAM J. Appl. Math.*, **3** (1990), 32–37.

[8] P. ErdHos, C. Ko, R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser.* (2) 12 (1961) 313–320.

[9] P. Frankl, Z. Füredi. A short proof for a theorem of Harper about Hamming-spheres. Discrete Math. 34 (1981), no. 3, 311–313.

[10] C. D. Godsil *Algebraic combinatorics* Chapman and Hall Mathematics Series I Chapman & Hall. New York, 1993.

[11] L.H. Harper. Optimal numberings and isoperimetric problems on graphs, *J. Comb. Theory*, **1** (1966), 385–393.

[12] L. H. Harper, L. H. Global methods for combinatorial isoperimetric problems. Cambridge Studies in Advanced Mathematics, 90. Cambridge University Press, Cambridge, 2004.

[13] G. O. H. Katona. A theorem of finite sets. Theory of graphs (Proc. Colloq., Tihany, 1966), pp. 187–207. Academic Press, New York, 1968.

[14] V.M. Karachanjan, A discrete isoperimetric problem on multidimensional torus, (in Russian), *Doklady AN Arm. SSR*, vol. **LXXIV** (1982), No. 2, 61–65.

[15] J. B. Kruskal. The number of simplices in a complex. Mathematical optimization techniques. Univ. of California Press, Berkeley, Calif. (1963) 251–278.

[16] V.M. Karachanjan, A discrete isoperimetric problem on multidimensional torus, (in Russian), *Doklady AN Arm. SSR*, vol. **LXXIV** (1982), No. 2, 61–65.

[17] I. Leader. Discrete isoperimetric inequalities. Probabilistic combinatorics and its applications (San Francisco, CA, 1991), 57–80, Proc. Sympos. Appl. Math., 44, Amer. Math. Soc., Providence, RI, 1991.

[18] O. Riordan, An ordering on the discrete even torus, *SIAM J. Discr. Math.*, **11** (1998), No. 1, 110–127.

k-metric resolvability in graphs

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Abstract. A set S of vertices of a graph G is said to be a k-metric generator for G if for any $u, v \in V(G)$, $u \neq v$, there exists $S_{uv} \subseteq S$ such that $|S_{uv}| \geq k$ and for every $w \in S_{uv}$, $d_G(u, w) \neq d_G(v, w)$. A metric generator of minimum cardinality is called a k-metric basis and its cardinality the k-metric dimension of G. We give a necessary and sufficient condition for the existence of a k-metric basis of a graph and we obtain several results on the k-metric dimension.

Key words: k-metric dimension, metric dimension, resolving set, locating set.

1 Introduction

Given a simple and connected graph $G = (V, E)$, a vertex $v \in V$ is said to distinguish two different vertices $x, y \in V$, if $d_G(v, x) \neq d_G(v, y)$. A set $S \subseteq V$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. This name for such a set raises from the concept of *generator* of metric spaces, that is, a set S of points in the space with the property that every point of the space is uniquely determined by its "distances" from the elements of S. For our specific case, in a simple and connected graph $G = (V, E)$, we consider the metric $d_G : V \times V \to \mathbb{N}$, where $d_G(x, y)$ is the length of a shortest path between the vertices x and y. With this metric, (V, d_G) is clearly a metric space.

According to the concept above, it is said that a metric generator of minimum cardinality is a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$. The primary notions of metric dimension of a graph were motivated by the problem of uniquely determining the location of an intruder in a network and were introduced by Slater in [8,10], where the metric generators were called *locating sets*. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [3], where metric generators were called *resolving sets*. Further on, several variations of

such a generators were presented and studied, remarkable cases include for instance, resolving dominating sets [1], independent resolving sets [3], local metric sets [7], and strong resolving sets [5,6,8].

The concept of metric generator has a weakness related with the possible uniqueness of the vertex identifying a pair of vertices of the graph. That is, consider for instance some robots which are navigating, moving from node to node of a network. On a graph, however, there is neither the concept of direction nor that of visibility. We assume that robots have communication with a set of landmarks S (a subset of nodes) which provide them the distance to the landmarks in order to facilitate the navigation. In this sense, one aim is that each robot is uniquely determined by the landmarks. Suppose that in a specific moment there are two robots x, y whose positions are only distinguished by one landmark $s \in S$. If the communication between x and s is "unexpectedly blocked", then the robot x will get "lost‘" in the sense that it can assume that it has the position of y. So, for more realistic settings, it would be desirable to avoid such a uniqueness situations while the metric basis identifies a pair of vertices of the graph. Roughly speaking, we could consider a set of landmarks where each pair of nodes is distinguished by a minimum number of landmarks greater than or equal to two.

From now on, according to our description above, we present an extension of the concept of metric generators in the following way. As above, we consider a simple and connected graph $G = (V, E)$ and a positive integer k, we say that a set $S \subseteq V$ is a *k-metric generator* for G if and only if any pair of vertices of G is distinguished by at least k elements of S. A k-metric generator of minimum cardinality in G will be called a *k-metric basis* and its cardinality the *k-metric dimension* of G, which will be denoted by $\dim_k(G)$. Note that every k-metric generator S satisfies that $|S| \geq k$ and, if $k > 1$, then S is also a $(k-1)$-metric generator. Moreover, 1-metric generators are the standard metric generators.

2 k-metric dimensional graphs.

A connected graph G is said to be a *k-metric dimensional graph* if k is the largest integer such that there exists a k-metric basis. Notice that if G is a k-metric dimensional graph, then for every positive integer $k' \leq k$, G has at least a k'-metric basis. Since for every pair of vertices x, y of a graph G we have that they are distinguished at least by themselves, it follows that the whole vertex set $V(G)$ is a 2-metric generator for G and, as a consequence it follows that every graph G is k-metric dimensional for some $k \geq 2$. On the other hand, for any connected graph G of order $n > 2$ there exists at least one vertex $v \in V(G)$ of degree $\delta(v) \geq 2$ and, since v does not distinguish any pair $x, y \in N_G(v)$, there is no n-metric dimensional graph of order $n > 2$.

Remark 1. Let G be a k-metric dimensional graph of order n. If $n \geq 3$ then, $2 \leq k \leq n-1$. Moreover, G is n-metric dimensional if and only if $G \cong K_2$.

Given two vertices $x, y \in V(G)$, we say that the set of *distinctive vertices* of x, y is $\mathcal{D}(x,y) = \{z \in V(G) : d_G(x,z) \neq d_G(y,z)\}$ and the set of *non-trivial distinctive vertices* of x, y is $\mathcal{D}^*(x,y) = \mathcal{D}(x,y) - \{x,y\}$.

Theorem 1. *A connected graph G is k-metric dimensional if and only if $k = \min_{x,y \in V(G)} |\mathcal{D}(x,y)|$.*

The open neighborhood of a vertex x will be denoted by $N(x)$. Two vertices x, y are called *false twins* if $N(x) = N(y)$, and x, y are called *true twins* if $N[x] = N[y]$. Two vertices x, y are *twins* if they are false twin vertices or true twin vertices. Notice that two vertices x, y are twins if and only if $\mathcal{D}^*(x,y) = \emptyset$.

Corollary 1. *A connected graph G of order $n \geq 2$ is 2-metric dimensional if and only if G has twin vertices.*

As a direct consequence of Corollary 1 we have that K_2 and P_3 are 2-metric dimensional trees. Also, a specific characterization for 2-dimensional trees is obtained from Theorem 1 (and from Corollary 1). A *leaf* in a tree is a vertex of degree one, while a *support vertex* is a vertex adjacent to a leaf.

Corollary 2. *A tree T of order $n \geq 4$ is 2-metric dimensional if and only if T contains a support vertex which is adjacent to at least two leaves.*

A vertex of degree at least three in a graph G will be called a *major vertex* of G. Any end-vertex (a vertex of degree one) u of G is said to be a *terminal vertex* of a major vertex v of G if $d_G(u,v) < d_G(u,w)$ for every other major vertex w of G. The terminal degree $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\mathcal{M}(G)$ be the set of exterior major vertices of G having terminal degree greater than one. Given $w_i \in \mathcal{M}(G)$ and a terminal vertex u_{ij} of w_i, we denote by $P(u_{ij}, w_i)$ the shortest path that starts at u_{ij} and ends at w_i. Now, given $w_i \in \mathcal{M}(G)$ and two terminal vertices u_{ij}, u_{ir} of w_i we denote by $P(u_{ij}, w_i, u_{ir}) = u_{ij} - w_i - u_{ir}$ the shortest path from u_{ij} to u_{ir} containing w_i and by $\varsigma(u_{ij}, u_{ir})$ the length of $P(u_{ij}, w_i, u_{ir})$. Finally, given $w_i \in \mathcal{M}(G)$ and the set of terminal vertices $U_i = \{u_{i1}, u_{i2}, \ldots, u_{ik}\}$ of w_i, for $j \neq r$ we define $\varsigma(w_i) = \min_{u_{ij}, u_{ir} \in U_i} \{\varsigma(u_{ij}, u_{ir})\}$. From the above local parameters we define the following global parameters. $\varsigma(G) = \min_{w_i \in \mathcal{M}(G)} \{\varsigma(w_i)\}$ and $\mu(G) = \sum_{v \in \mathcal{M}(G)} ter(v)$.

Theorem 2. *Let G be a connected graph such that $\mathcal{M}(G) \neq \emptyset$. If G is k-metric dimensional, then $k \leq \varsigma(G)$. Moreover, for any k-metric dimensional tree T, different from a path, $k = \varsigma(T)$.*

Theorem 3. *Let G be a graph, different from the complete graph, of order n and clique number $\omega(G)$. If G is k-metric dimensional, then $k \leq n-\omega(G)+1$.*

Examples where the above bound is achieved are those connected graphs G of order n and $\omega(G) = n-1$, so $n-\omega(G)+1 = 2$. Notice that in this case there exists at least two twin vertices. Hence, by Corollary 1 these graphs are 2-metric dimensional.

Proposition 1. *Let C_n be a cycle graph of order n. If n is odd, then C_n is $(n-1)$-metric dimensional and if n is even, then C_n is $(n-2)$-metric dimensional.*

Theorem 4. *A graph G of order $n \geq 3$ is $(n-1)$-metric dimensional if and only if G is a path or G is an odd cycle.*

3 The k-metric dimension of graphs.

The problem of computing the value k' for which a given graph is k'-metric dimensional is polynomial [11]. Now, we study the problem of deciding whether the k-metric dimension, $k \leq k'$, of G is less than r, for some $r \geq k+1$, *i.e.*, the following decision problem.

k-METRIC DIMENSION PROBLEM
INSTANCE: A connected k'-metric dimensional graph G of order n and integers k, r such that $k \leq k'$ and $r \geq k+1$
PROBLEM: Deciding whether the k-metric dimension of G is less than r

Theorem 5. [11] *k-METRIC DIMENSION PROBLEM is NP-complete.*

Considering the previous theorem, we present some results that allow to compute the k-metric dimension of several families of graphs. We also give some tight bounds on the k-metric dimension of a graph.

Theorem 6 (Monotony of the k-metric dimension). *Let G be a k-metric dimensional graph and let k_1, k_2 be two integers. If $1 \leq k_1 < k_2 \leq k$, then $\dim_{k_1}(G) < \dim_{k_2}(G)$.*

Corollary 3. *Let G be a k-metric dimensional graph of order n.*

(i) *For every $r \in \{1, ..., k\}$, $\dim_r(G) \geq \dim(G) + (r-1)$.*
(ii) *For every $r \in \{1, ..., k-1\}$, $\dim_r(G) < n$.*

Theorem 7. *Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = 2$ if and only if $G \cong P_n$.*

Corollary 3 (i) and the fact that $dim(G) = 1$ if and only if $G \cong P_n$ immediately lead to the following result.

Theorem 8. *Let G be a k-metric dimensional graph different from a path. Then for any $r \in \{2, ..., k\}$, $\dim_r(G) \geq r + 1$.*

Let $\mathcal{D}_k(G)$ be the set obtained as the union of the sets of distinctive vertices $\mathcal{D}(x, y)$ whenever $|\mathcal{D}(x, y)| = k$.

Remark 2. If G is a k-metric dimensional graph, then $\dim_k(G) \geq |\mathcal{D}_k(G)|$.

The bound given in Remark 2 is tight. For instance, Proposition 2 shows that there exists a family of trees attaining this bound for every k. Other examples can be derived from the following result.

Theorem 9. *Let G be a k-metric dimensional graph of order n. Then $\dim_k(G) = n$ if and only if $V(G) = \mathcal{D}_k(G)$.*

We recall that the *join graph* $G + H$ of the graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G + H) = V_1 \cup V_2$ and edge set $E(G + H) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. We give now some examples of graphs satisfying the assumptions of Theorem 9. Let $W_{1,n} = C_n + K_1$ be the wheel graph and $F_{1,n} = P_n + K_1$ be the fan graph. The vertex of K_1 is called the central vertex of the wheel or the fan, respectively. Since $V(F_{1,4}) = \mathcal{D}_3(F_{1,4})$ and $V(W_{1,5}) = \mathcal{D}_4(W_{1,5})$, by Theorem 9 we have that $\dim_3(F_{1,4}) = 5$ and $\dim_4(W_{1,5}) = 6$ respectively.

Corollary 4. *Let G be a connected graph of order $n \geq 2$. Then $\dim_2(G) = n$ if and only if every vertex is a twin.*

Remark 3. Let G and H be two nontrivial graphs of order n_1 and n_2, respectively. If all the vertices of G and H are twin vertices, then $G + H$ is 2-metric dimensional and $\dim_2(G + H) = n_1 + n_2$.

We define the *twin equivalence relation* $\mathcal{R}$ on $V(G)$ as follows: $x\mathcal{R}y$ if and only if $N_G[x] = N_G[y]$ or $N_G(x) = N_G(y)$. We have three possibilities for each twin equivalence class U: (a) U is singleton, or (b) $N_G(x) = N_G(y)$, for any $x, y \in U$, or (c) $N_G[x] = N_G[y]$, for any $x, y \in U$. We will refer to the type (c) classes as the *true twin equivalence classes i.e., U* is a true twin equivalence class if and only if U is not singleton and $N_G[x] = N_G[y]$, for any $x, y \in U$.

Remark 4. Let G be a connected graph and let $U_1, U_2, ..., U_t$ be the non-singleton twin equivalence classes of G. Then $dim_2(G) \geq \sum_{i=1}^{t} |U_i|$.

Notice that the above result leads to Corollary 4, so this bound is tight.

We recall that the *strong product graph* $G \boxtimes H$ of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G \boxtimes H) = V_1 \times V_2$, where two distinct vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent in $G \boxtimes H$ if and only if one of the following holds: ($x_1 = y_1$ and $x_2 \sim y_2$) or ($x_1 \sim y_1$ and $x_2 = y_2$) or ($x_1 \sim y_1$ and $x_2 \sim y_2$).

Theorem 10. *Let G and H be two nontrivial connected graphs of order n and n', respectively. Let $U_1, U_2, ..., U_t$ be the true twin equivalence classes of G. Then $dim_2(G \boxtimes H) \geq n' \sum_{i=1}^{t} |U_i|$. Moreover, if every vertex of G is true twin, then $dim_2(G \boxtimes H) = nn'$.*

Now, we present a lower bound for the k-metric dimension of a graph. To do so, we will use some notation introduced on Page 307 and we first define the following function for any exterior major vertex $w_i \in V(G)$ having terminal degree greater than one, *i.e.*, $w_i \in \mathcal{M}(G)$.

$$I_r(w_i) = \begin{cases} (ter(w_i) - 1)(r - l(w_i)) + l(w_i), & \text{if } l(w_i) \leq \lfloor \frac{r}{2} \rfloor, \\ (ter(w_i) - 1)\lceil \frac{r}{2} \rceil + \lfloor \frac{r}{2} \rfloor, & \text{otherwise.} \end{cases}$$

Theorem 11. *If G is a k-metric dimensional graph such that $|\mathcal{M}(G)| \geq 1$, then for every $r \in \{1, ..., k\}$, $\dim_r(G) \geq \sum_{w_i \in \mathcal{M}(G)} I_r(w_i)$.*

Theorem 12 shows that the lower bound of Theorem 11 is tight. Also, notice that for $k = 1$ Theorem 11 leads to the bound on the metric dimension of a graph, established by Chartrand et al. in [2]. In such a case, $I_1(w_i) = ter(w_i) - 1$ for all $w_i \in \mathcal{M}(G)$ and thus, $\dim(G) \geq \sum_{w_i \in \mathcal{M}(G)} (ter(w_i) - 1) = \mu(G) - |\mathcal{M}(G)|$.

Corollary 5. *If G is a connected graph, then* $\dim_2(G) \geq \mu(G)$.

Corollary 6. *If G is k-metric dimensional for some $k \geq 3$, then* $\dim_3(G) \geq 2\mu(G) - |\mathcal{M}(G)|$.

In Theorem 7, we noticed that the 2-metric dimension of a path $P_n (n \geqslant 2)$ is two. Here we give a formula for the k-metric dimension of any tree.

Theorem 12. *If T is a k-metric dimensional tree which is not a path, then for any $r \in \{1, ..., k\}$, $\dim_r(T) = \sum_{w_i \in \mathcal{M}(T)} I_r(w_i)$. Moreover, for any $k \geq 3$ and $n \geq k + 1$, it holds $\dim_k(P_n) = k + 1$.*

The formula of Theorem 12 leads to $\dim(T) = \mu(T) - |\mathcal{M}(T)|$ in the case $r = 1$, which is the result obtained in [2]. Other interesting particular cases are the following ones for $r = 2$ and $r = 3$, respectively.

Corollary 7. *Let T be a tree different from a path. Then,* $\dim_2(T) = \mu(T)$ *and if T is k-metric dimensional with $k \geq 3$, then* $\dim_3(T) = 2\mu(T) - |\mathcal{M}(T)|$.

Proposition 2. *Let T be a tree different from a path and let $k \geq 2$ be an integer. If $ter(w_i) = 2$ and $\varsigma(w_i) = k$ for every $w_i \in \mathcal{M}(T)$, then $\dim_k(T) = |\mathcal{D}_k(T)|$.*

References

[1] R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, Mathematica Bohemica 128 (1) (2003) 25–36.

[2] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 (1-3) (2000) 99–113.

[3] G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, Computers & Mathematics with Applications 39 (12) (2000) 19–28.

[4] F. Harary, R. A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195.

[5] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of corona product graphs and join graphs, Discrete Applied Mathematics 161 (7–8) (2013) 1022–1027.

[6] O. R. Oellermann, J. Peters-Fransen, The strong metric dimension of graphs and digraphs, Discrete Applied Mathematics 155 (3) (2007) 356–364.

[7] F. Okamoto, B. Phinezy, P. Zhang, The local metric dimension of a graph, Mathematica Bohemica 135 (3) (2010) 239–255.

[8] A. Sebö, E. Tannier, On metric generators of graphs, Mathematics of Operations Research 29 (2) (2004) 383–393.

[9] P. J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549–559.

[10] P. J. Slater, Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences 22 (4) (1988) 445–455.

[11] I. G. Yero, A. Estrada-Moreno, J. A. Rodríguez-Velázquez, The k-metric dimension of a graph: Complexity and algorithms, arXiv:1401.0342 [math.CO].

The number of empty four-gons in random point sets

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Abstract. Let $\mathcal{S}$ be a set of n points distributed uniformly and independently in the unit square. Then the expected number of empty four-gons with vertices from $\mathcal{S}$ is $\Theta(n^2 \log n)$. A four-gon is empty if it contains no points of $\mathcal{S}$ in its interior.

Key words: random point set, four-gon, empty polygon, geometric probability

1 Introduction

Throughout this paper let $\mathcal{S}$ be a set of n points distributed uniformly and independently in the unit square. Since with probability 1 the n points will be in general position (no three points are collinear), we may and will assume this throughout the paper. All asymptotics in this paper are w.r.t. the number of points n, that is, when $n \to \infty$. As our results are asymptotic, we may ignore also rounding issues throughout the paper, that is, if for some constant $c > 0$, cn is not an integer, depending on the context, we may and will consider $\lfloor cn \rfloor$ or $\lceil cn \rceil$ without changing the results. A four-gon whose vertices are from $\mathcal{S}$ is empty if it contains no other point from $\mathcal{S}$ in its interior. A four-gon can be convex or non-convex, see Figure 1.

Denote by N_4 the random variable that counts the number of empty non-convex four-gons with vertices from $\mathcal{S}$. Our main result is the following:

Theorem 1. $\mathbb{E}[N_4] = \Theta(n^2 \log n)$.

Denote by C_4 the random variable that counts the number of empty convex four-gons with vertices from $\mathcal{S}$. Complementing Theorem 1, we obtain the

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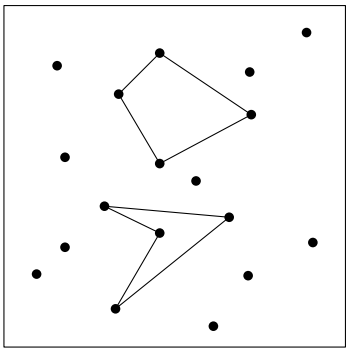

Fig. 1: An empty convex four-gon and an empty non-convex four-gon in $\mathcal{S}$.

following result, which might be known, but we only found a proof for the lower bound [6]. There, also another random construction with $O(n^2)$ empty convex four-gons is given.

Theorem 2. $\mathbb{E}[C_4] = \Theta(n^2)$.

We omit the proof of Theorem 2 in this abstract due to lack of space. We also remark that Theorems 1 and 2 also hold for other convex, bounded sets, not only the square. In related works, for point sets of n points distributed uniformly and independently in a convex, bounded set, Valtr [13] proved that the expected number of empty triangles is at most $2n^2 - 2n$, and Bárány and Füredi [6] proved that, in R^d, the number of empty simplices is at most $K\binom{n}{d}$, for some constant K. Balogh et al. [5] showed that the expected number of vertices of the largest empty convex polygon in $\mathcal{S}$ (and in any convex, bounded set in the plane) is $\Theta(\frac{\log n}{\log \log n})$. A lot of research has been done to determine the minimum number $f_k(n)$ of empty convex k-gons among all sets of n points in general position in the plane (not only random point sets). For the case of empty triangles, Katchalski and Meir [11] showed that $f_3(n)$ is of order $\Theta(n^2)$. Later, this bound has been refined [2,6,7,8,9,13]; the currently best bounds are $n^2 - \frac{32n}{7} + \frac{22}{7} \leq f_3(n) \leq 1.6196n^2 + o(n^2)$. Concering empty convex four-gons, Bárány and Füredi [6] established that $f_4(n)$ is of order $\Theta(n^2)$, and the currently best bounds on $f_4(n)$ are $\frac{n^2}{2} - \frac{9}{4}n - o(n) \leq f_4(n) \leq 1.9397n^2 + o(n^2)$, see [2,7]. Research mainly focussed on empty convex polygons. Only recently the number of convex and non-covex polygons in point sets has been studied [1,3,4]. In [1] it is shown that every set of n points in general position in the plane determines at least $\frac{5n^2}{2} - \Theta(n)$ empty four-gons and a point set with only $O(n^{5/2} \log n)$ empty four-gons is given. Our result improves this bound to $O(n^2 \log n)$.

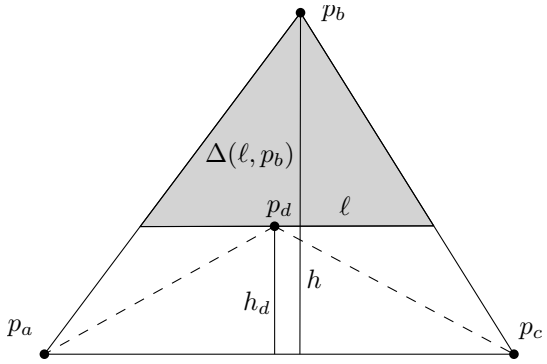

Fig. 2: The triangles $\Delta(p_a, p_b, p_c)$, $\Delta(p_a, p_d, p_c)$ and $\Delta(\ell, p_b)$.

2 Proof of Theorem 1

The proof of Theorem 1 is implied by the following lemmas, for which we need some definitions. Fix three points p_a, p_b, p_c and denote by $\Delta(p_a, p_b, p_c)$ the triangle spanned by them. Let $\mathcal{P}$ be a set of $k \geq 1$ points distributed uniformly and independently in $\Delta(p_a, p_b, p_c)$. Denote by $\mathcal{E}^{\mathcal{P}}_{p_a p_b, p_b p_c}$ the event that $\mathcal{P} \cup \{p_a, p_b, p_c\}$ contains an empty non-convex four-gon with $p_a p_b$ and $p_b p_c$ among its edges.

Lemma 1. $\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a p_b, p_b p_c}) = \frac{2}{k+1}$.

Proof. First observe that the points $\{p_a, p_b, p_c\}$ together with a fourth point $p_d \in \mathcal{P}$ form an empty non-convex four-gon with $p_a p_b$ and $p_b p_c$ among its edges if and only if the triangle $\Delta(p_a, p_d, p_c)$ contains $\mathcal{P} \backslash \{p_d\}$ in its interior. We now determine the distribution of the height h_d, which is the distance from p_d to the segment $p_a p_c$. Denote by h the height of $\Delta(p_a, p_b, p_c)$ with respect to the edge $p_a p_c$. Let ℓ be the segment parallel to the edge $p_a p_c$, at distance h_d from this edge, and with endpoints on the edges $p_a p_b$ and $p_b p_c$ respectively. Assume w.l.o.g. that ℓ is a horizontal line with p_a and p_c below it. Define then $\Delta(\ell, p_b)$ to be the triangle coming from the intersection of $\Delta(p_a, p_b, p_c)$ and all points lying on or above ℓ, as shown in Figure 2. Since only relative heights between h and h_d matter, we may assume w.l.o.g. that $h = 1$. By the intercept theorem we have $\frac{|p_a p_c|}{h} = \frac{|\ell|}{h - h_d}$, where $|p_a p_c|$ and $|\ell|$ are the lengths of the segments $p_a p_c$ and ℓ. It follows that

$$\frac{\text{area}(\Delta(\ell, p_b))}{\text{area}(\Delta(p_a, p_b, p_c))} = \frac{\frac{|\ell|(h - h_d)}{2}}{\frac{|p_a p_c| h}{2}} = (1 - h_d)^2.$$

Hence, the distribution function F_{h_d} for the height h_d satisfies $F_{h_d}(x) = \mathbb{P}(h_d \leq x) = 1 - (1 - x)^2$ and the density of the height h_d is $f_{h_d}(x) = 2 - 2x$ for $x \in [0, 1]$.

Fix any $p \in \mathcal{P} \setminus \{p_d\}$. Since the points are distributed uniformly at random inside $\Delta(p_a, p_b, p_c)$, we have

$$\mathbb{P}(p \in \Delta(p_a, p_d, p_c)) = \frac{\text{area}(\Delta(p_a, p_d, p_c))}{\text{area}(\Delta(p_a, p_b, p_c))} = h_d.$$

Therefore, integrating over all possible heights $0 \leq h_d \leq 1$,

$$\mathbb{P}(\mathcal{P}\setminus\{p_d\} \in \Delta(p_a, p_d, p_c)) = \int_0^1 x^{k-1}(2-2x)\, dx = \frac{2}{k(k+1)}.$$

As there are k choices for the point p_d, by a union bound, we have

$$\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a p_b, p_b p_c}) \leq \frac{2}{k+1}.$$

On the other hand, there is at most one point $p_d \in \mathcal{P}$ such that $\mathcal{P}\setminus\{p_d\} \in \Delta(p_a, p_d, p_c)$: indeed, if this were true for another point $p_e \neq p_d$, then $p_d \notin \Delta(p_a, p_e, p_c)$, contradicting the assumption. Hence,

$$\mathbb{P}((\mathcal{P}\setminus\{p_d\} \in (\Delta(p_a, p_d, p_c)) \wedge (\mathcal{P}\setminus\{p_e\} \in \Delta(p_a, p_e, p_c))) = 0,$$

and thus

$$\begin{aligned}\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a p_b, p_b p_c}) &= \bigcup_{p_d \in \mathcal{P}} (\mathbb{P}(\mathcal{P}\setminus\{p_d\} \in (\Delta(p_a, p_d, p_c))) \\ &= \sum_{p_d \in \mathcal{P}} (\mathbb{P}(\mathcal{P}\setminus\{p_d\} \in (\Delta(p_a, p_d, p_c))) = \frac{2}{k+1}.\end{aligned}$$

∎

For the next lemma, we need one more definition. Let T_k denote the random variable that counts the number of triangles with vertices in $\mathcal{S}$ containing exactly $k \geq 0$ points from $\mathcal{S}$ in its interior.

Lemma 2. *For any $k = k(n) \geq 0$, $\mathbb{E}[T_k] \leq 2n^2 - 2n$.*

Proof. The density function $f_{area(\Delta)}(v)$ for the area v of a triangle Δ formed by three points chosen uniformly and independently in the unit square is given in [10], and also in [12]. From the results there one can see that for any area $v \geq 0$, $f_{area(\Delta)}(v) \leq 12$. Fix now three points $p_a, p_b, p_c \in \mathcal{S}$, let as before $\Delta(p_a, p_b, p_c)$ be the triangle spanned by them, and let $int(\Delta(p_a, p_b, p_c))$ denote the interior of this triangle. Denote also for $x, y > 0$ by $\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ the beta function of x and y. Integrating over all possible areas v of the triangle $\Delta(p_a, p_b, p_c)$, we obtain

$$\begin{aligned}\mathbb{P}(|int(\Delta(p_a,p_b,p_c))\cap\mathcal{S}|=k) &= \int_0^{1/2}\binom{n-3}{k}v^k(1-v)^{n-3-k}f_{area(\Delta)}(v)\,dv\\ &\le 12\int_0^{1/2}\binom{n-3}{k}v^k(1-v)^{n-3-k}\,dv\\ &\le 12\int_0^{1}\binom{n-3}{k}v^k(1-v)^{n-3-k}\,dv\\ &= 12\binom{n-3}{k}\beta(k+1,n-k-2)\\ &= 12\binom{n-3}{k}\frac{k!(n-k-3)!}{(n-2)!}=\frac{12}{n-2}.\end{aligned}$$

Hence, by linearity of expectation, for any $k=k(n)\ge 0$,

$$\mathbb{E}\left[T_k\right]\le\binom{n}{3}\frac{12}{n-2}=2n^2-2n.$$

∎

Remark. The special case $k=0$ of Lemma 2 was also proved by Valtr in [13].

We are now ready to prove the upper bound of Theorem 1, which is encapsulated in the following lemma.

Lemma 3. $\mathbb{E}\left[N_4\right]=O(n^2\log n)$.

Proof. Note that each triangle $\Delta(p_a,p_b,p_c)$ with vertices $p_a,p_b,p_c\in\mathcal{S}$ determines at most three empty non-convex four-gons with p_a,p_b,p_c among its vertices: indeed, any pair of edges from $\{p_ap_b,p_bp_c,p_ap_c\}$ can be chosen and might possibly give rise to an empty non-convex four-gon. Let $\mathcal{P}\subseteq\mathcal{S}$ denote the set of points in the interior of $\Delta(p_a,p_b,p_c)$, and let $\mathcal{E}^{\mathcal{P}}_{p_a,p_b,p_c}$ be the event that any of the three pairs of edges gives rise to an empty non-convex four-gon. By a union bound,

$$\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a,p_b,p_c})\le\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_ap_b,p_bp_c})+\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_ap_c,p_bp_c})+\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_ap_b,p_ap_c}),$$

and thus, by Lemma 1, since the points of $\mathcal{P}$ are uniformly distributed in $\Delta(p_a,p_b,p_c)$ (if a point is distributed uniformly at random in the unit square, then conditional under knowing that it is inside a subarea of that square, it is still uniform in this subarea),

$$\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a,p_b,p_c}\mid|\mathcal{P}|=k)\le\frac{6}{k+1}.$$

By Lemma 2, for each $k\ge 0$ in expectation there are at most $2n^2$ triangles with k interior points and since once again, conditioned on having k interior points, their distribution is uniform inside the triangle, we obtain

$$\mathbb{E}[N_4] \leq \sum_{k=0}^{n} \mathbb{E}[T_k]\, \mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a,p_b,p_c} \mid |\mathcal{P}| = k)$$
$$\leq 2n^2 \sum_{k=0}^{n} \frac{6}{k+1} = O(n^2 \log n).$$

∎

We now proceed to prove the corresponding lower bound of Theorem 1. We first prove the following lower bound on T_k.

Lemma 4. *For every $\epsilon > 0$ there exists some $\alpha = \alpha(\epsilon) > 0$ such that $\mathbb{E}[T_k] \geq (2-\epsilon)n^2$ for any $k = 0, 1, \ldots, \alpha n$.*

Proof. By [10,12], the density function $f_{area(\Delta)}(v)$ for the area of the triangle $\Delta = \Delta(p_a, p_b, p_c)$, formed by three points p_a, p_b, p_c from $\mathcal{S}$, satisfies $f_{area(\Delta)}(0) = 12$ and is then strictly monotonically decreasing. In particular, for every small $\epsilon > 0$ there exists $v_0 > 0$ such that $f_{area(\Delta)}(v_0) = 12 - \epsilon$. We define $\alpha = 0.6v_0$.

$$\mathbb{P}(|int(\Delta) \cap \mathcal{S}| = k) = \int_0^{1/2} \binom{n-3}{k} v^k (1-v)^{n-3-k} f_{area(\Delta)}(v)\, dv$$
$$\geq (12-\epsilon) \int_0^{v_0} \binom{n-3}{k} v^k (1-v)^{n-3-k}\, dv.$$

As in the proof of Lemma 2 we have

$$\int_0^1 \binom{n-3}{k} v^k (1-v)^{n-3-k}\, dv = \frac{1}{n-2}.$$

We will show that $\mathbb{P}(|int(\Delta) \cap \mathcal{S}| = k) \geq \frac{12-\epsilon}{n-2} - o(\frac{1}{n})$, for $k \leq \alpha n$. To this end, it is sufficient to show that

$$\int_{v_0}^1 \binom{n-3}{k} v^k (1-v)^{n-3-k}\, dv = o(\frac{1}{n}). \tag{1}$$

Computing the derivative of the function $g(v) := v^k(1-v)^{n-3-k}$ in $[v_0, 1]$, we see that $g'(v) \leq 0$, implying that in $[v_0, 1]$, $g(v)$ is maximized at $v = v_0$. Thus,

$$\int_{v_0}^1 \binom{n-3}{k} v^k (1-v)^{n-3-k}\, dv \leq \binom{n-3}{k} {v_0}^k (1-v_0)^{n-3-k}(1-v_0).$$

It is easily verified that

$$\binom{n-3}{k} {v_0}^k (1-v_0)^{n-3-k} < \binom{n-3}{k+1} {v_0}^{k+1} (1-v_0)^{n-3-k-1}$$

holds for $k < v_0(n-2) - 1$. If we can show that

$$\binom{n-3}{k} {v_0}^k (1-v_0)^{n-3-k}(1-v_0) = o(\frac{1}{n})$$

holds for $k = \alpha n$, then it holds for all smaller values of k as well, and then also (1) holds for all $k = 0, 1, \ldots, \alpha n$. Using $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$, we get for some constant $C > 0$

$$\begin{aligned}\binom{n-3}{\alpha n} {v_0}^{\alpha n}(1-v_0)^{n-3-\alpha n}(1-v_0) &\leq C\left(\frac{e}{0.6v_0}\right)^{0.6v_0 n} v_0^{0.6v_0 n}(1-v_0)^{(1-0.6v_0)n} \\ &= C\left((e/0.6)^{0.6v_0}(1-v_0)^{1-0.6v_0}\right)^n \\ &= o(\frac{1}{n}),\end{aligned}$$

where the last line follows from the fact that

$$f(v_0) := (\frac{e}{0.6})^{0.6v_0}(1-v_0)^{1-0.6v_0}$$

is monotone decreasing for $v_0 \in [0,1]$, $v_0 > 0$, and that $f(0) = 1$. Thus,

$$\begin{aligned}\mathbb{P}(|int(\Delta) \cap \mathcal{S}| = k) &\geq (12-\epsilon)\int_0^{v_0} \binom{n-3}{k} v^k (1-v)^{n-3-k}\, dv \\ &\geq \frac{12-\epsilon}{n-2} - o(\frac{1}{n}).\end{aligned}$$

As before, by linearity of expectation,

$$\mathbb{E}[T_k] \geq \binom{n}{3}\left(\frac{12-\epsilon}{n-2} - o(\frac{1}{n})\right) \geq (2-\epsilon)n^2$$

for any $k = 0, 1, \ldots, \alpha n$, thus concluding the proof. ∎

The lower bound of Theorem 1 now also follows easily.

Lemma 5. $\mathbb{E}[N_4] = \Omega(n^2 \log n)$.

Proof. As before, define for three points $p_a, p_b, p_c \in \mathcal{S}$ by $\Delta(p_a, p_b, p_c)$ the triangle with vertices p_a, p_b, p_c. Let $\mathcal{P} \subseteq \mathcal{S}$ denote the set of points in the interior of $\Delta(p_a, p_b, p_c)$. Using the notation of Lemma 3, it is clear that

$$\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a,p_b,p_c}) \geq \mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_ap_b,p_bp_c}).$$

Hence, by Lemma 1,

$$\mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a,p_b,p_c}) \geq \frac{2}{k+1}.$$

By Lemma 4, for $k = 0, 1, \ldots \alpha n$, in expectation there are at least $(2-\epsilon)n^2$ triangles with k interior points and since once again, conditioned on having k interior points, their distribution is uniform inside the triangle, we obtain

$$\mathbb{E}[N_4] \geq \sum_{k=0}^{\alpha n} \mathbb{E}[T_k]\, \mathbb{P}(\mathcal{E}^{\mathcal{P}}_{p_a,p_b,p_c} \mid |\mathcal{P}| = k)$$
$$\geq (2-\epsilon)n^2 \sum_{k=0}^{\alpha n} \frac{2}{k+1} = \Omega(n^2 \log n).$$

∎

References

[1] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M.A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. *Comput. Geom.*, 47(6): 644–650, 2014.

[2] O. Aichholzer, R. Fabila Monroy, T. Hackl, C. Huemer, A. Pilz, and B. Vogtenhuber. Lower bounds for the number of small convex k-holes. *Comput. Geom.*, 47(5): 605–613, 2014.

[3] O. Aichholzer, R. Fabila-Monroy, H- González-Aguilar, T. Hackl, M.A. Heredia, C. Huemer, J. Urrutia, P. Valtr, and B. Vogtenhuber. On k-gons and k-holes in point sets. *23rd Annual Canadian Conference on Computational Geometry CCCG 2011*, 21–26. Toronto, Canada, 2011.

[4] O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-gons and 5-holes. *XIV Encuentros de Geometría Computacional ECG2011*, *Lecture Notes in Comput. Sci.*, vol. 7579:1–13. Springer, 2012.

[5] J. Balogh, H. González-Aguilar, and G. Salazar. Large convex holes in random point sets. *Comput. Geom.* 46(6): 725–733, 2013.

[6] I. Bárány and Z. Füredi. Empty simplices in Euclidean space. *Canad. Math. Bull.* 30(4): 436–445, 1987.

[7] I. Bárány and P. Valtr. Planar point sets with a small number of empty convex polygons. *Stud. Sci. Math. Hung.* 41: 243–266, 2004.

[8] A. Dumitrescu. Planar sets with few empty convex polygons. *Stud. Sci. Math. Hung.*, 36: 93–109, 2000.

[9] A. García. A note on the number of empty triangles. *XIV Encuentros de Geometría Computacional ECG2011*, *Lecture Notes in Comput. Sci.*, vol. 7579:249–257. Springer, 2012.

[10] N. Henze. Random triangles in convex regions. *J. Appl. Prob.* 20: 111–125, 1983.

[11] M. Katchalski and A. Meir. On empty triangles determined by points in the plane. *Acta Math. Hung.*, 51: 323–328, 1988.

[12] J. Philip. The area of a random convex polygon. Technical Report, 2004. `http://www.math.kth.se/~johanph/area12.pdf`.

[13] P. Valtr. On the minimum number of empty polygons in planar point sets. *Stud. Sci. Math. Hungar.*, 30: 155–163, 1995.

Searching for Large Multi-Loop Networks

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Abstract. We describe and implement a computer-based method to find large multi-loop graphs with given degree and diameter. For some values of degree and diameter, our algorithm produces the largest known circulant graphs. We summarize our findings in a table.

Key words: Network design, Degree/Diameter Problem, circulant graphs, multi-loop networks, Delannoy numbers

1 Introduction

The Degree/Diameter Problem (or DDP, for short) asks for constructing the largest possible graph, in terms of the number of vertices, subject to an upper bound on the maximum degree Δ and an upper bound on the diameter D.

We are interested in some particular (undirected) versions of the problem, namely when the graphs are restricted to a certain class, such as the class of bipartite graphs [9,10], Cayley graphs [6,12], or circulant graphs [22,17]. In this paper we are concerned with a special case of circulant graphs, i.e. Cayley graphs of finite cyclic groups.

Most of the research, both in the general version, as well as in these restricted versions of DDP, falls into one of two main categories:

1. Proving the non-existence of graphs with order close or equal to the upper bound, or
2. Giving constructions of large graphs, whose order approach the upper bounds as much as possible.

Research in the second category has been substantially boosted by the compliation of record graph tables, containing the largest known graphs for

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several combinations of degree and diameter. They provide benchmarks to test construction methods and computer search algorithms. Thus, after the first compilation started by Comellas in 1995 [3], other record graph tables have been collected (see [4]).

The study of circulant graphs began in 1970 with Elspas and Turner [8]. Coincidentally, it was also Elspas who had formulated the Degree/Diameter Problem back in 1964 [7], but apparently he missed the connection between both topics.

Even though the abelian property of the underlying group prevents abelian Cayley graphs in general (and circulant graphs in particular) to grow as large as their non-abelian counterparts, these graphs have been widely used as topologies for computer networks and parallel computers, due to their other nice properties. However, up to now there are no comprehensive tables for these graphs. Our goals in this paper are twofold:

1. Design and implement a computer search algorithm to find large circulant graphs of a special type, with small degree and diameter.
2. Start the compilation of a table containing the largest known circulant graphs for some small values of degree and diameter, including the results obtained with the aforementioned algorithm.

Additionally, we explore the sharpness of the existing upper bounds for some particular combinations of degree and diameter.

2 Definitions and basic facts

An undirected circulant graph $C(n; S)$ is a Cayley graph on the cyclic group $\mathbb{Z}_n$, with a symmetric connection set S (i.e. $S = S^{-1}$, or $S = -S$, in additive notation). The degree of $C(n; S)$ is $\Delta = |S|$, and its order is obviously n. A circulant graph can also be defined as a graph of n vertices whose adjacency matrix is circulant [7].

Regarding the degree, we distinguish two cases:

1. Even degree: $\Delta = 2t$. In that case, $S = \{\pm s_1, \ldots, \pm s_t\}$, where $1 \leq s_1 < \ldots < s_t < \frac{n}{2}$. $C(n; S)$ is connected if, and only if, $\gcd(n, s_1, \ldots, s_t) = 1$.
2. Odd degree: $\Delta = 2t + 1$. In that case, $S = \{\pm s_1, \ldots, \pm s_t, \frac{n}{2}\}$, where $1 \leq s_1 < \ldots < s_t < \frac{n}{2}$. It follows that odd degree is only possible when n is even. Connectedness is similar to the previous case.

Now let $N^{circ}_{\Delta,D}$ be the number of vertices of the largest circulant graph with degree Δ and diameter D. It was proved in [2] that, if $\Delta = 2t$, then

$$N^{circ}_{\Delta,D} \leq F(t, D) = \sum_{i=0}^{t} 2^i \binom{t}{i} \binom{D}{i} \tag{1}$$

This upper bound was later rediscovered by Muga [17]. The quantity $F(t, D)$ also turns out to be an upper bound for $N^{AC}_{\Delta,D}$, the order of the largest Cayley graph over an abelian group, with degree Δ and diameter D [6].

The numbers $F(t, D)$ of Eq. 1 are known as *Delannoy numbers* (sequence A008288 of [18]), and they arise in a variety of combinatorial and geometric problems [20]. For example, they correspond to the volume of the ball of radius $D/2$ in the L^1 metric in t dimensions [6,13,21]. Other exact and asymptotic formulas for these numbers are given in [13,19], such as:

$$F(t, D) = F(t-1, D) + F(t, D-1) + F(t-1, D-1) \tag{2}$$

$$F(t, D) = \sum_{i=0}^{t} \binom{t}{i}\binom{D+i}{t} = \sum_{i=0}^{t} \binom{D+i}{i}\binom{D}{t-i} \tag{3}$$

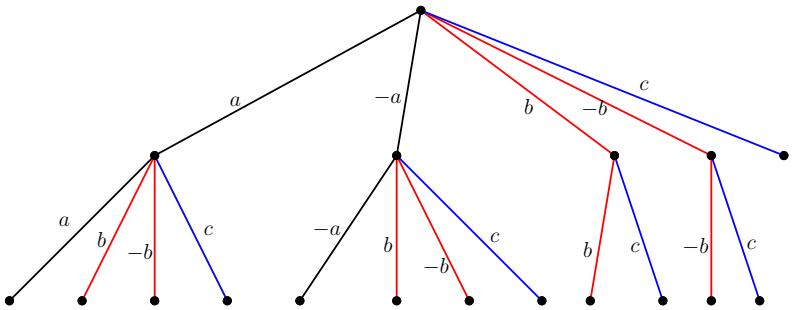

Fig. 1: Tree representation of a maximal abelian Cayley graph of degree 5 and diameter 2.

In the case of odd Δ (i.e. $\Delta = 2t + 1$) we have the generator $\frac{n}{2}$, which is its own inverse. Figure 1 provides a graphical example of that case, for $\Delta = 5$ and $D = 3$. We have denoted the generators as a, b, c, where $c = -c$. We can see in this example that the edges labeled with c duplicate every vertex, except those in the lowest level (level D). Therefore, an upper bound for the maximum number of vertices in this case is:

$$F'(t, D) = F(t, D) + F(t, D-1) \tag{4}$$

From Equations 2 and 4 we can deduce that the numbers $F'(t, D)$ also satisfy the same recurrence 2. The first few values of $F(t, D)$ and $F'(t, D)$ are collected in Table 1.

Circulant graphs of the form $C(n; \{\pm 1, \pm s_2, \ldots, \pm s_t\})$ are called *multi-loop graphs*. In particular, for $t = 2$ and $t = 3$ they are called *double-loop graphs*

and *triple-loop graphs*, respectively. According to [16], the maximum order of a triple-loop network $C(n; \{\pm 1, \pm s_2, \pm s_3\})$ is:

$$T_{6,D} = \begin{cases} \frac{32}{27}D^3 + \frac{16}{9}D^2 + 2D + 1 & \text{if } D \equiv 0 \bmod 3 \\ 32\lfloor \frac{D}{3} \rfloor^3 + 48\lfloor \frac{D}{3} \rfloor^2 + 30\lfloor \frac{D}{3} \rfloor + 7 & \text{if } D \equiv 1 \bmod 3 \\ 32\lfloor \frac{D}{3} \rfloor^3 + 80\lfloor \frac{D}{3} \rfloor^2 + 70\lfloor \frac{D}{3} \rfloor + 21 & \text{if } D \equiv 2 \bmod 3 \end{cases} \quad (5)$$

3 A search procedure

Let n, Δ and D be given. We now describe a systematic search procedure that attempts to find a multi-loop graph with the given parameters. Our search space consists of all connection sets S. The basic idea is to consider a tree-like organization of the search space, and then perform a Depth-First Search with backtracking on the tree, while pruning significant portions of it.

For even (resp. odd) Δ, a node at depth t in the search tree corresponds to a multi-loop graph of order n and degree $2t$ (resp. $2t + 1$). The root of the search tree is the graph $C(n; \{\pm 1\})$ (resp. $C(n; \{\pm 1, n/2\})$). Let $C(n; S)$ be a node of degree smaller than Δ, and let m be the largest element in S such that $1 \leq m < n/2$. Then $C(n; S)$ has offspring $C(n; S \cup \{\pm g\})$ for every $m + k \leq g \leq (n - k)/2$ (resp. $m + k \leq g \leq n/2 - k$), where k is a control parameter.

In the general Degree/Diameter Problem, it is well-known that large graphs tend to have large girth with respect to their diameter. This observation does not apply to circulant graphs, as their girth is always at most 4 when $\Delta \geq 3$. Thus, we have generalized the girth criterion as follows:

For a circulant graph $C(n; S)$ and a length d, define $\mathcal{P}_w^d$ $(0 \leq w < N)$ as the set of vertex-disjoint paths of length d from a fixed vertex v (say 0) to the vertex w.

Whether the offspring of a given node $C(n; S)$ are explored or not depends on a function on the cardinals of the sets $\mathcal{P}_w^d$. Basically, we fix constants c_i^d (for $0 \leq i \leq d < D$) which stand for the maximum number of paths allowed in $\mathcal{P}_w^d$ whenever w is at distance $i \leq d$ from the vertex 0. That is, for every vertex w in the graph, if w is at distance $i < D$ from the vertex 0, we require that $|\mathcal{P}_w^d| \leq c_i^d$ for every d with $i \leq d < D$. By varying the constants c_i^d it is possible to adjust the amount of pruning. Exhaustive search, when feasible, is performed by setting large enough values for the constants.

Algorithm 1 formalizes the above ideas in a recursive fashion. The parameters n, Δ and D are global. The input to the algorithm is the connection set S. Initially, the algorithm is called with $S = \{\pm 1\}$ (resp. $S = \{\pm 1, n/2\}$). If a multi-loop graph $C(n; S)$ with de desired parameters is found, the algorithm will print it out. Otherwise, the algorithm terminates without producing any output.

For a given generator $g \in S$, the algorithm computes a g-path, which is a path in $C(n;S)$ containing g. From a g-path P, the algorithm then computes the parameters w, d, and $\mathcal{P}_w^d$, as explained above.

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Input: A set of generators S.
Output: A circulant graph C(n;S) with degree Δ and diameter D.

m:= Largest element in S;
for g:= m+k to (n-k)/2 do begin
     S':= S ∪ {±g};
     success := true;
     for each g-path P in C(n;S'), with P ∈ P_w^d do begin
          i:= Distance from vertex 0 to w in C(n;S');
          if d > i and |P_w^d| > c_i^d then begin
               success := false;
               break;
          end
          if d = i and |P_w^j| > c_i^j for some j (i ≤ j < D) then begin
               success := false;
               break;
          end
     end
     if success then
          Call SEARCH(S');
     else
          if diameter of C(n;S') is D then
               Print C(n;S');
end
```

Algorithm 1: SEARCH

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4 The largest known circulant graphs

Prior to our computer-based search, the main sources of large circulant graphs were:

1. For $t = 2$ an optimal circulant graph $C(n; \pm s_1, \pm s_2)$, is achieved for $s_1 = \lfloor \frac{1}{2}(\sqrt{2n-1}-1) \rfloor$ and $s_2 = s_1 + 1$ [1,2,14].
2. Monakhov and Monakhova used an evolutionary algorithm to find dense families of undirected circulant graphs. In particular, with the aid of this algorithm they found some families of large triple-loop graphs [15].

3. For larger degrees we have the construction $C(n; \pm 1, \pm s, \ldots, \pm s^{t-1})$, where $n = s^t$, and s is an odd integer, which yields good circulant graphs of diameter $\frac{t}{2}(s-1) = \frac{t}{2}n^{1/t} - \frac{t}{2}$ [22].
4. Applying the methods described in [6] for abelian Cayley graphs, Charles Delorme and Robert Lewis have recently obtained several circulant graphs of large order [5,11].

Table 1 summarizes our knowledge about the largest known circulant graphs, including the ones obtained by Algorithm 1 above. Each entry contains the order of the largest circulant graph known to-date (upper left), a reference (upper right), the best-known upper bound (lower left), and the percentage of the upper bound achieved (lower right).

The new graphs obtained by Algorithm 1 are given in Table 2. The table also indicates which of these graphs are optimal multi-loop graphs. Optimality has been verified for some small values of Δ and D, by performing a systematic search through all multi-loop graphs with the given parameters Δ and D.

	Diameter D									
	2	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
3	8	12	16	20	24	28	32	36	40
	8 100%	12 100%	16 100%	20 100%	24 100%	28 100%	32 100%	36 100%	40 100%
4	13	25	41	61	85	113	145	181	221
	13 100%	25 100%	41 100%	61 100%	85 100%	113 100%	145 100%	181 100%	221 100%
5	16	36 [12]	64 [12]	100 [12]	144 [12]	196 [12]	256 [12]	324 [12]	400 [12]
	18 89%	38 95%	66 97%	102 98%	146 99%	198 99%	258 99%	326 99%	402 99%
6	21 [6,5]	55 [6,5]	117 [6,5]	203 [6,5]	333 [6,12]	515 [6,12]	737 [6,12]	1027 [6,12]	1393 [6,16]
	25 84%	63 87%	129 91%	231 88%	377 88%	575 90%	833 88%	1159 89%	1561 89%
7	26 [6,5]	76 [6,12]	160 [6,12]	308 [6,12]	536 [6]	828 [6]	1232 [6]	1764 [6]	2392 [6]
	32 81%	88 86%	192 83%	360 86%	608 88%	952 87%	1408 87%	1992 88%	2720 88%
8	35 [5]	104 T.2	248 T.2	528 T.2	984 [11]	1712 [11]	2768 [11]	4280 [11]	6320 [11]
	41 85%	129 77%	321 75%	681 77%	1289 76%	2241 76%	3649 76%	5641 76%	8361 76%
Δ **9**	42 [5]	130 [12]	320 T.2	700 [11]	1416 [11]	2548 [11]	4304 [11]	6804 [11]	10320 [11]
	50 84%	170 76%	450 71%	1002 70%	1970 72%	3530 72%	5890 73%	9290 73%	14002 74%
10	51 [5]	177 [12]	457 T.2	1071 T.2	—	—	—	—	—
	61 84%	231 77%	681 67%	1683 64%	3653 —	7183 —	13073 —	22363 —	36365 —
11	56 [5]	210 T.2	576 T.2	—	—	—	—	—	—
	72 78%	292 72%	912 63%	2364 —	5336 —	10836 —	20256 —	35436 —	58728 —
12	67 [5]	275 T.2	—	—	—	—	—	—	—
	85 79%	377 73%	1289 —	3653 —	8989 —	19825 —	40081 —	75517 —	134245 —
13	80 T.2	312 T.2	—	—	—	—	—	—	—
	98 82%	462 68%	1666 —	4942 —	12642 —	28814 —	59906 —	115598 —	209762 —

Table 1: Orders of the largest known circulant graphs.

Δ	D	Multi-loop graph	Δ	D	Multi-loop graph
8	3	$104\{\pm1, \pm16, \pm20, \pm27\}$ $\star\dagger$	11	3	$210\{\pm1, \pm49, \pm59, \pm84, \pm89, 105\}$ $\star$
8	4	$248\{\pm1, \pm61, \pm72, \pm76\}$ $\star\dagger$	11	4	$576\{\pm1, \pm9, \pm75, \pm155, \pm179, 288\}$
8	5	$528\{\pm1, \pm89, \pm156, \pm162\}$ $\star\dagger$	12	3	$275\{\pm1, \pm16, \pm19, \pm29, \pm86, \pm110\}$
9	4	$320\{\pm1, \pm15, \pm25, \pm83, 160\}$ $\star\dagger$	13	2	$80\{\pm1, \pm3, \pm9, \pm20, \pm25, \pm33, 40\}$
10	4	$457\{\pm1, \pm20, \pm130, \pm147, \pm191\}$	13	3	$312\{\pm1, \pm14, \pm74, \pm77, \pm130, \pm138, 156\}$
10	5	$1071\{\pm1, \pm101, \pm168, \pm378, \pm509\}$			

Table 2: New multi-loop networks obtained by Algorithm 1. The graphs marked with $\star$ are optimal. The graphs marked with † have been obtained independently by Lewis [11].

5 Open problems

The apparent simplicity of circulant graphs is deceiving, and they are a reservoir of interesting open problems. From our point of view, the main open questions here have to do with the sharpness of the upper bound given in Eq. 1. Is there a smaller general upper bound for $N^{circ}_{\Delta,D}$? If not, can we find smaller upper bounds for some special cases?

As for the lower bounds, there are few constructions for $\Delta > 4$. In that respect, the methods of [6,5,11] seem very promising. Additionally, other computer-based methods in the style of [15] could be devised, to obtain families of large circulant graphs with low diameter.

Table 1 still has several vacant entries at the upper values of Δ and D, where the execution time of Algorithm 1 becomes prohibitive with the computing power at our disposal. In principle, some of these vacancies could be filled by increasing the computing power. Alternatively, the vacancies could be filled with the aid of the construction described by Wong and Coppersmith [22], or later variants. However, we rather chose not to do that, since the values resulting from these latter constructions turn out to be smaller than the ones already found in lower entries of the table. We prefer to leave those entries blank until they can be occupied by graphs that conform to the steadily growing pattern of the table.

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References

[1] J.C. Bermond, G. Illiades, and C. Peyrat. An Optimization Problem in Distributed Loop Computer Networks. *Annals of the New York Academy of Sciences* 555: 45–55, 1989.

[2] F.T. Boesch, and J.F. Wang. Reliable circulant networks with minimal transmission delay. *IEEE Transactions on Circuits and Systems* 32: 1286–1291, 1985.

[3] F. Comellas. The (Degree,Diameter) Problem for Graphs. `http://maite71.upc.es/grup_de_grafs/grafs/taula_delta_d.html`.

[4] E. Loz, H. Pérez-Rosés, and G. Pineda-Villavicencio. The Degree Diameter Problem. `http://combinatoricswiki.org/wiki/The_Degree/Diameter_Problem`.

[5] C. Delorme. Some remarks about circulant Cayley graphs. *Personal communication*, 2013.

[6] R. Dougherty and V. Faber. The Degree Diameter Problem for Several Varieties of Cayley Graphs I: The Abelian Case. *SIAM Journal of Discrete Mathematics* 17(3): 478–519, 2004.

[7] B. Elspas. Topological constraints on interconnection-limited logic. *Procs. of the 5th Annual IEEE Symposium on Switching Circuit Theory and Logical Design, Princeton, New Jersey, USA*, 133–137, 1964.

[8] B. Elspas and J. Turner. Graphs with circulant adjacency matrices. *Journal of Combinatorial Theory* 9: 297–307, 1970.

[9] R. Feria-Puron and G. Pineda-Villavicencio. On bipartite graphs of defect at most 4. *Discrete Applied Mathematics* 160: 140–154, 2012.

[10] R. Feria-Puron, M. Miller, and G. Pineda-Villavicencio. On large bipartite graphs of diameter 3. *Discrete Mathematics* 313: 381–390, 2013.

[11] R. Lewis. The degree-diameter problem for circulant graphs of degree 8 and 9. *arXiv:1404.3948v1*, 2014.

[12] H. Machbeth, J. Siagiová, J. Sirán, T. Vetrík. Large Cayley graphs and vertex-transitive non-Cayley graphs of given degree and diameter. *Journal of Graph Theory* 64: 87–98, 2009.

[13] M. Miller, H. Pérez-Rosés and J. Ryan. The maximum degree and diameter-bounded subgraph in the mesh. *Discrete Applied Mathematics* 160: 1782–1790, 2012.

[14] E.A. Monakhova. On analytical representation of optimal two-dimensional Diophantine structures of homogeneous computer systems (in Russian). *Computing systems* 90: 81–91, 1981.

[15] O. Monakhov and E.A. Monakhova. Using evolutionary algorithm for generation of dense families of circulant networks. *Proceedings of the IEEE Congress on Evolutionary Computation, Honolulu, Hawai*: 1854–1859, 2002.

[16] E.A. Monakhova. A survey on undirected circulant graphs. *Discrete Mathematics, Algorithms and Applications* 4 (1), 2012.

[17] F.P. Muga II. Undirected circulant graphs. *Procs. of the IEEE Int. Symp. on Parallel Architectures, Algorithms and Networks (ISPAN)*: 113–118, 1994.

[18] *OEIS: The On-Line Encyclopedia of Integer Sequences.* `http://oeis.org/classic/index.html`.

[19] R.G. Stanton and D.D. Cowan. Note on a square functional equation. *SIAM Review* 12: 277–279, 1970.

[20] R. Sulanke. Objects counted by the central Delannoy numbers. *J. Integer Sequences* 6, Art. 03.1.5, 2003.

[21] M.V. Vassilev-Missana and K.T. Atanassov. On Delanoy [*sic*] numbers. *Annuaire Univ. Sofia Fac. Math. Inform.* 81: 153–162, 1987.

[22] C.K. Wong and D. Coppersmith. A combinatorial problem related to multimodule memory organizations. *Journal of the ACM* 21: 392–402, 1974.

Caracterizaciones combinatorias y algebraicas de grafos distancia-regulares

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Resumen. Los grafos distancia-regulares aparecen a menudo en el estudio de estructuras matemáticas con un alto grado de simetría y/o regularidad. Un ejemplo bien conocido de tales grafos son los esqueletos de los sólidos platónicos. Desde que fueron propuestos por Norman Biggs, los grafos distancia-regulares han sido caracterizados por numerosos resultados, tanto de carácter combinatorio como algebraico. Como ejemplo del primer caso, sabemos que un grafo es distancia-regular si, y sólo si, el número de caminos de una longitud dada entre dos vértices sólo depende de la distancia entre dichos vértices. En esta charla se van a presentar y comparar las diferentes caracterizaciones conocidas, tanto las más clásicas como las que han sido recientemente descubiertas por el conferenciante y algunos de sus colaboradores. Entre las últimas, cabe destacar el que ya es conocido en la literatura com el 'teorema del exceso espectral'. Este resultado puede considerarse como una caracterización casi-espectral, y afirma que un grafo es distancia-regular si, y sólo si, su exceso espectral (una cantidad calculable a partir de su matriz de adyacencia) es igual a su exceso medio (el número medio de vértices a distancia máxima de cada vértice).

1 Preliminaries

Recordemos primero algunas notaciones y resultados básicos. Para más información sobre teoría espectral de grafos y distancia-regularidad consultar, por ejemplo, [3,5,6,8,10,20,22,23,27]. En este artículo, Γ denota un grafo con conjunto $V = V(G)$, de $n = |V|$ vertices, matriz de adyacencia $\boldsymbol{A}$ y espectro $\operatorname{sp}\Gamma = \operatorname{sp}\boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, donde los distintos autovalores de Γ estan en orden decreciente, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, y los supraíndices denotan sus multiplicidades $m_i = m(\lambda_i)$, $i = 0, 1, \ldots, d$. Entonces, como es bien conocido, λ_0, con multiplicidad $m_0 = 1$, coincide con el radio espectral de $\boldsymbol{A}$, y tiene un autovector (columna) positivo (el *vector de Perron*) $\boldsymbol{\alpha}$, que normalizamos de manera que $\|\boldsymbol{\alpha}\|^2 = n$. Para cada $i = 0, \ldots, d$, la matriz idempotente $\boldsymbol{E}_i$ representa la proyección ortogonal sobre el espacio propio asociado al autovalor λ_i. Denotamos por $\operatorname{dist}(u, v)$ la distancia entre los vértices $u, v \in V$. Entonces, con $\Gamma_i(u) = \{v | \operatorname{dist}(u, v) = i\}$, la *eccentricidad* de un vértice u es $\operatorname{ecc}(u) = \operatorname{máx}\{\operatorname{dist}(u, v) | v \in V\}$, y el *diámetro* de Γ es

$D = \max\{\mathrm{ecc}(u) | u \in V\}$. Para cada $0 \le i \le D$, la *matriz distancia-i* $\boldsymbol{A}_i$ tiene elementos $(\boldsymbol{A}_i)_{uv} = 1$ si $\mathrm{dist}(u,v) = i$, y $(\boldsymbol{A}_i)_{uv} = 0$, en otro caso.

1.1 Polinomios predistancia y números de preintersección

Dado un grafo Γ con espectro indicado anteriormente, los *polinomios predistancia* $p_0, \ldots, p_d$, introducidos por Fiol y Garriga en [25], constituyen una sequencia de polinomios ortogonales con respecto al producto escalar

$$\langle f, g \rangle_\Gamma = \frac{1}{n} \operatorname{tr}(f(\boldsymbol{A})g(\boldsymbol{A})) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\lambda_i) g(\lambda_i), \tag{1}$$

normalizados de tal manera que $\|p_i\|_\Gamma^2 = p_i(\lambda_0)$ (es conocido que $p_i(\lambda_0) > 0$ para todo $i = 0, \ldots, d$). Algunas propiedades de estos polinomios son las siguientes (ver Cámara, Fàbrega, Fiol, y Garriga [9]):

(a) $p_0 = 1$, $p_1 = x$, y las constantes de la relación de recurrencia de tres términos

$$x p_i = \beta_{i-1} p_{i-1} + \alpha_i p_i + \gamma_{i+1} p_{i+1}, \tag{2}$$

donde $\beta_{-1} = \gamma_{d+1} = 0$, satisfacen $\alpha_i + \beta_i + \gamma_i = \lambda_0$ para cada $i = 0, \ldots, d$.

(b) El valor de p_d en λ_0 es:

$$p_d(\lambda_0) = n \left(\sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1}, \tag{3}$$

donde $\pi_i = \prod_{j \ne i} |\lambda_i - \lambda_j|$, para $i = 0, \ldots, d$.

(c) $H = p_0 + p_1 + \cdots + p_d$ es el *polinomio de Hoffman* que caracteriza la regularidad de Γ a través de la condición $H(\boldsymbol{A}) = \boldsymbol{J}$, donde $\boldsymbol{J}$ denota la matrix todo 1's (ver Hoffman [30]).

A partir de los polinomios predistancia, definimos los *números de preintersección* ξ_{ij}^h [13] como los coeficientes de Fourier de $p_i p_j$ en términos de la base $\{p_h\}_{0 \le h \le d}$, es decir,

$$\xi_{ij}^h = \frac{\langle p_i p_j, p_h \rangle_\Gamma}{\|p_h\|_\Gamma^2} = \frac{1}{n p_h(\lambda_0)} \sum_{r=0}^{d} m(\lambda_r) p_i(\lambda_r) p_j(\lambda_r) p_h(\lambda_r). \tag{4}$$

Notar que, en particular, los coeficientes de la relación de recurrencia (2) son $\alpha_i = \xi_{1,i}^i$, $\beta_i = \xi_{1,i+1}^i$, y $\gamma_i = \xi_{1,i-1}^i$. Como era de esperar, cuando Γ es distancia-regular, los polinomios predistancia y los números de preintersección resultan ser, respectivamente, los *polinomios distancia*, que evaluados en $\boldsymbol{A}$ dan las matrices distancia $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$, y que satisfacen

$p_i(\lambda_0) = k_i = \Gamma_i(u)$ para todo $u \in V$ e $i = 0, \ldots, D$; y los *números de intersección* $p_{ij}^k = |\Gamma_i(u) \cap \Gamma_j(v)|$, donde $\mathrm{dist}(u,v) = k$. Para un grafo cualquiera, decimos que el número p_{ij}^k está *dien definido* cuando los números $p_{ij}^k(u,v) = |\Gamma_i(u) \cap \Gamma_j(v)|$ son idénticos para todos los pares de vértices u, v a distancia k, y, en particular, escribimos $a_i = p_{1,i}^i$, $b_i = p_{1,i+1}^i$, y $c_i = p_{1,i-1}^i$.

2 Caracterizaciones combinatorias

Los dos resultados siguientes dan caracterizaciones puramente combinatorias de la distancia-regularidad.

Teorema 1. *Un grafo Γ con diámetro D es distancia-regular si y sólo si se cumple cualquiera de las condiciones siguientes:*

(*a*) *Dados dos vértices u, v a distancia $k = \mathrm{dist}(u,v)$, el número de vértices*

$$p_{ij}^k(u,v) = |\Gamma_i(u) \cap \Gamma_j(v)|$$

sólo depende de i, j, k. Entonces $p_{ij}^k(u,v) = p_{ij}^k$ són los llamados números de intersección. (*Es decir, los p_{ij}^k's estan bien definidos*).

(*b*) *Dados dos vértices u, v a distancia $k = \mathrm{dist}(u,v)$, $0 \le k \le D$, los números*

$$c_k(u,v) = |\Gamma_{k-1}(u) \cap \Gamma(v)|,\ a_k(u,v) = |\Gamma_k(u) \cap \Gamma(v)|,\ b_k(u,v) = |\Gamma_{k+1}(u) \cap \Gamma(v)|$$

sólo dependen de k, en cuyo caso los denotamos por c_k, a_k y b_k, respectivamente. (*Es decir, a_k, b_k y c_k estan bien definidos*).

Teorema 2 ([33,23,13]). *Un grafo Γ con diámetro D es distancia-regular si y sólo si se cumple cualquiera de las condiciones siguientes:*

(*a*) *Para cada número no negativo ℓ, el número $a_{uv}^\ell = (\boldsymbol{A}^\ell)_{uv}$ de recorridos de longitud ℓ entre dos vértices u, v sólo depende de $k = \mathrm{dist}(u,v)$.*

(*b*) *Γ es regular y, para cada $\ell \in \{k, k+1\}$, $\ell \le D$, el número a_{uv}^ℓ de recorridos de longitud ℓ entre dos vértices u, v sólo depende de $k = \mathrm{dist}(u,v)$.*

(*c*) *Γ es regular, bipartito, y $a_{uv}^{(\ell)} = a_i^{(\ell)}$ para $\ell = i \le D-2$ y algunas constantes $a_i^{(\ell)}$.*

3 Caracterizaciones algebraicas

Usando las matrices distancia, los Teoremas 1 y 2 pueden enunciarse de la forma siguiente:

Teorema 3. *Un grafo Γ con diámetro D es distancia-regular si, y sólo si, se cumple cualquiera de las condiciones siguientes:*

(*a*) *El producto de matrices distancia satisface*

$$\boldsymbol{A}_i\boldsymbol{A}_j = \sum_{k=0}^{D} p_{ij}^k \boldsymbol{A}_k \qquad (0 \le i,j \le D)$$

para algunas constantes p_{ij}^k *(los números de intersección).*
(*b*) *Las matrices distancia satisfacen la siguiente relación de recurrencia:*

$$\boldsymbol{A}\boldsymbol{A}_k = b_{k-1}\boldsymbol{A}_{k-1} + a_k\boldsymbol{A}_k + c_{k+1}\boldsymbol{A}_{k+1} \qquad (0 \le k \le D)$$

para algunas constantes a_k, b_k *y* c_k *(con* $b_{-1} = c_{D+1} = 0$*).*
(*c*) *Las potencias de la matriz de adyacencia* $\boldsymbol{A}$ *sumplen:*

$$\boldsymbol{A}^\ell \circ \boldsymbol{A}_i = a_i^{(\ell)}\boldsymbol{A}_i \qquad (0 \le i,\ell \le d)$$

para algunas constantes $a_i^{(\ell)}$.

Por otra parte, resulta que

$$\mathcal{A} = \mathbb{R}_d[\boldsymbol{A}] = \mathrm{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\}$$

es un álgebra con el producto ordinario de matrices y bases ortogonales $\{\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d\}$ y $\{p_0(\boldsymbol{A}), p_1(\boldsymbol{A}), \ldots, p_d(\boldsymbol{A})\}$, denominada *álgebra de adyacencia*. Por otra parte,

$$\mathcal{D} = \mathrm{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_D\}$$

constituye un álgebra con el producto de Hadamard "$\circ$"de matrices, definido por $(\boldsymbol{X} \circ \boldsymbol{Y})_{uv} = (\boldsymbol{X})_{uv}(\boldsymbol{Y})_{uv}$. Llamamos a $\mathcal{D}$ el $\circ$-*algebra de distancia*. Notar que, cuando Γ es regular, $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{J} \in \mathcal{A} \cap \mathcal{D}$ ya que $\boldsymbol{J} = H(\boldsymbol{A}) = \sum_{i=0}^{D} \boldsymbol{A}_i$. Así, $\dim(\mathcal{A} \cap \mathcal{D}) \ge 3$ siempre que Γ no sea un grafo completo (en este caso excepcional, $\boldsymbol{J} = \boldsymbol{I} + \boldsymbol{A}$).

Una caracterización clásica de la distancia-regularidad que usa dichas álgebras es la siguiente:

Teorema 4 ([5,6]). *Un grafo* Γ *regular con* $d+1$ *autovalores distintos, diámetro* $D = d$*, álgebra de adyacéncia* $\mathcal{A}$ *y* $\circ$*-álgebra de distancia* $\mathcal{D}$ *es distancia-regular si, y sólo si, se cumple cualquiera de las condiciones siguientes:*

(*a*) $\mathcal{A} = \mathcal{D}$.
(*b*) $\dim(\mathcal{A} \cap \mathcal{D}) = d + 1$.

En el siguiente resultado, que constitute una formulación equivalente al teorema anterior, p_{ji} y q_{ij} son constantes:

Teorema 5 ([5,6]). *Un grafo* Γ *regular con* $d+1$ *autovalores distintos, diámetro* $D = d$*, e idempotentes* $\boldsymbol{E}_0, \ldots, \boldsymbol{E}_d$ *es distancia-regular si y sólo si se cumple cualquiera de las condiciones siguientes:*

$(a)\, \boldsymbol{A}_i \boldsymbol{E}_j = p_{ji} \boldsymbol{E}_j, \quad i,j = 0,1,\ldots,d(=D).$
$(b)\, \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij} \boldsymbol{A}_i, \quad i,j = 0,1,\ldots,d.$

Cuando se cumples las condiciones (a), Los polinomios predistancia resultan ser los *polinomios distancia* que satisfacen $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$, $i = 0,1,2,\ldots,d$ (ver, por ejemplo, Bannai y Ito [3]). Además, para grafos en general que no cumplen necesariamente $D = d$, las condiciones (a) son una caracterización de los llamados *grafos distancia-polinomiales*, introducidos por Weichsel [34] (ver también Beezer [4] y Dalfó, van Dam, Fiol, Garriga y Gorissen [12]). Esto es equivalente a $\mathcal{D} \subset \mathcal{A}$ (pero no necesariamente $\mathcal{D} = \mathcal{A}$); es decir, cada matriz distancia $\boldsymbol{A}_i$ es un polinomio (no necesariamente p_i) en $\boldsymbol{A}$. En contraste con esta situación, la condición (b) es equivalente a $\mathcal{A} \subset \mathcal{D}$ y, por tanto, a $\mathcal{A} = \mathcal{D}$ (lo cual implica $d = D$) ya que $\dim \mathcal{A} \geq D + 1 = \dim \mathcal{D}$.

Teorema 6 ([27,22,12]). *Un grafo Γ regular con $d+1$ autovalores distintos, diáametro $D = d$, e idempotentes $\boldsymbol{E}_1, \boldsymbol{E}_d$ es distancia-regular si y sólo si se cumple cualquiera de las condiciones siguientes:*

$(a)\, \boldsymbol{A}_i = p_i(\boldsymbol{A})$ *para* $i = 0,1,\ldots,D(=d)$
$(b)\, \boldsymbol{A}_d \in \mathcal{A}$.
$(c)\, \boldsymbol{A}_d = p_d(\boldsymbol{A})$.
$(d)\, \boldsymbol{A}_i = p_i(\boldsymbol{A})$ *para* $i = d-2, d-1$.
$(e)\, \Gamma$ *es bipartito y* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *para* $i = d-3, d-2$.
$(f)\, \Gamma$ *es bipartito y* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *para* $i = d-4, d-2$.
$(g)\, \boldsymbol{E}_j \in \mathcal{D}$ *para* $j = 1, d$.
$(h)\, \Gamma$ *es bipartito y* $\boldsymbol{E}_1 \in \mathcal{D}$.
$(i)\, \boldsymbol{A}^\ell \circ \boldsymbol{A}_i \in \mathcal{D}$ *para* $\ell = i, i+1$ *y* $i \leq d-1$.

Notar que, como muestran los resultados anteriores, las condiciones para tener distancia-regularidad pueden ser relajadas si asumimos que el grafo es bipartito, obteniéndose así caracterizaciones más 'económicas'.

4 Caracterizaciones espectrales

Un tema central en la teoría espectral de grafos es el estudio de hasta que punto el espectro de un grafo lo determina unívocamente o, al menos, algunas de sus características, ver [29,16,17]. En particular se ha prestado una atención especial a la formulación de caracterizaciones espectrales o cuasi-espectrales de la distancia-regularidad. En este contexto, existen tres clasees de condiciones que nos permiten concluir que un grafo conexo Γ es distancia-regular. A saber:

(A) El espectro de Γ satisface unas determinadas condiciones.
(B) Γ es coespectral con un grafo distancia-regular que satisface unas determinadas condiciones combinatorias.
(C) Γ satisface unas determinadas condiciones espectrales y combinatorias.

En realidad, las clases (B) y (C) estan estrechamente relacionadas, en el sentido de que la condición sobre Γ de ser coespectral con un grafo distancia-regular implica que el espectro de Γ satisface unas condiciones bien definidas.

Algunos resultados dentro de cada categoría son los siguientes:

(a1) Un grafo conexo Γ es distancia-regular si es regular (una propiedad que puede ser determinada a partir del espectro) y tiene $d = 3$ autovalores distintos. (Folclore, ver por ejemplo Godsil [28]). Ejemplo: El grafo de Petersen.

(a2) El 'teorema del cuello impar': Un grafo conexo Γ es distancia-regular si tiene $d+1$ autovalores distintos y cuello impar (es decir, el ciclo más corto de longitud impar) $2d+1$, (éste también puede ser deducido a partir del espectro). (Ver Van Dam y Haemers [18], Lee y Weng [32], y Van Dam y Fiol [14]). Ejemplo: Los 'cubos plegados', tal como el grafo de Clebsch.

(b1) Si el grafo Γ es coespectral con el grafo del dodecaedro o del icosahedro, entonces Γ es tal grafo. (Ver Brouwer y Haemers [7]).

(b2) Un grafo Γ es distance-regular si es cospectral con un grafo distancia-regular graph con diámetro d y números de intersección $c_1 = \cdots = c_{d-1} = 1$ (ver Van Dam y Haemers [15]). Ejemplo: El grafo de Coxeter.

(c1) Un grafo conexo y regular Γ con $d+1$ autovalores distintos es r-antipodal y distancia-regular si, y sólo si, su distancia-d grafo esta constituido por copias disjuntas del grafo completo K_r, con r determinado por el spectro. (See Fiol [21]). Examplo: El grafo de Klein.

(c2) El 'teorema del exceso espectral': Un grafo conexo y regular Γ es distancia-regular si, y sólo si, su 'exceso espectral' (un número que puede ser calculado a partir del espectro) es igual a su 'exceso medio' (la media de los números de vértices a distancia máxima de cada vértice). (Ver Fiol y Garriga [25], o Van Dam [11], y Fiol, Gago, y Garriga [24] para demostraciones cortas del resultado).

Los resultados conocidos de clase (B) vienen recogidos en el siguiente resultado:

Teorema 7 ([8]). *Si G es un grafo distancia-regular con diámetro $D = d$, cuello g, y satisface una de las propiedades siguientes, entonces, cada grafo Γ cospectral con G es tambien distance-regular, y tiene los mismos números de intersección que G.*

(*i*) $g \geq 2d-1$,
(*ii*) $g \geq 2d-2$ *y* G *es bipartito,*
(*iii*) $g \geq 2d-2$ *y* $c_{d-1}c_d < -(c_{d-1}+1)(\lambda_1 + \cdots + \lambda_d)$,
(*iv*) G *es un grafo 'odd' generalizado [31], esto es,* $a_1 = \cdots = a_{d-1} = 0$, $a_d \neq 0$,
(*v*) $c_1 = \cdots = c_{d-1} = 1$,
(*vi*) G *es el grafo del dodecaedro, o del icosaedro,*
(*vii*) G *es el grafo 'coset' del código de Golay ternario extendido,*

(*viii*) G *es el grafo de Ivanov-Ivanov-Faradjev.*

Los casos (i), (iv), y (vi) (Ejemplo (b1)) son debidos a Brouwer y Haemers [7]. Los casos (ii), (iii), (v) (Ejemplo (b2)), y (vii) fueron demostrados por Van Dam y Haemers [15], mientras que el caso $(viii)$ fue probado por Van Dam, Haemers, Koolen, y Spence [19]. Tal como se menciona en el texto de Brouwer y Haemers [8], (ii) es un caso especial de (iii), los ciclos C_n y los grafos fuertemente regulares $(d = 2)$ son consecuencia de (i), y los grafos bipartitos distancia-regulares $(d = 3)$ son un caso especial de (ii). Un ejemplo de (iii) es el grafo 'coset' del código de Golay binario doblemente truncado; ejemplos de (iv) son los grafos 'odd' y los cubos plegados, y un ejemplo de (v) es el grafo línea del grafo de Petersen.

Los dos primeros resultados del Theorem 7 pueden ser generalizados como resultados de clase (A). (La condición de ser bipartito es deducible a partir del espectro por la condición $\lambda_d = -\lambda_0$):

Teorema 8 ([1]). *Un grafo regular Γ, con $d+1$ autovalores distintos y cuello g, es distance-regular si se cumple cualquiera de las condiciones siguientes:*

(a) $g \geq 2d - 1$,
(b) $g \geq 2d - 2$ *y* Γ *es bipartito.*

Asimismo, el siguiente resultado corresponde a la versión de clase (A) del Theorem 7(v) y a su contrapartida para grafos bipartitos. (Recuérdese que los números de preintersección estan determinados por el espectro del grafo.)

Teorema 9 ([1]). *Sea Γ un grafo regular con $d + 1$ autovalores distintos y números de preintersección $\gamma_1, \ldots, \gamma_d$. Entonces, Γ es distancia-regular si se cumple cualquiera de las condiciones siguientes:*

(a) $\gamma_1 = \cdots = \gamma_{d-1} = 1$.
(b) Γ *es bipartito y* $\gamma_1 = \cdots = \gamma_{d-2} = 1$.

Para terminar, enunciamos formalmente el 'teorema del exceso espectral' como ejemplo de resultado de clase (C):

Teorema 10 ([25,11,24]). *Sea Γ un grafo conexo y regular con $d + 1$ autovalores distintos, exceso espectral $p_d(\lambda_0)$ (dado por (3)) y exceso medio $\overline{k}_d = \frac{1}{n}\sum_{u\in V} |\Gamma_d(u)|$. Entonces Γ es distancia-regular si, y sólo si,*

$$\overline{k}_d = p_d(\lambda_0).$$

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Referencias

[1] A. Abiad, C. Dalfó, and M.A. Fiol, Algebraic characterizations of regularity properties in bipartite graphs, *European J. Combin.* **34** (2013) 1223–1231 (Special Issue in memory of Yahya Ould Hamidoune).

[2] A. Abiad, C. Dalfó, E.R. van Dam, and M.A. Fiol, On some spectral and quasi-spectral characterizations of distance-regular graphs, *Electro. J. Algebra*, submitted (2013).

[3] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings, London, 1974, 1984.

[4] R.A. Beezer, Distance polynomial graphs, in *Proceedings of the Sixth Caribbean Conference on Combinatorics and Computing, Trinidad, 1991*, 51–73.

[5] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1974, second edition, 1993.

[6] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin-New York, 1989.

[7] A.E. Brouwer and W.H. Haemers, The Gewirtz graph: An exercise in the theory of graph spectra, *European J. Combin.* **14** (1993) 397–407.

[8] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, 2012; available online at `http://homepages.cwi.nl/~aeb/math/ipm/`.

[9] M. Cámara, J. Fàbrega, M.A. Fiol, and E. Garriga, Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes, *Electron. J. Combin.* **16(1)** (2009), #R83.

[10] D.M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs. Theory and Application*, VEB Deutscher Verlag der Wissenschaften, Berlin, second edition, 1982.

[11] E.R. van Dam, The spectral excess theorem for distance-regular graphs: a global (over)view, *Electron. J. Combin.* **15(1)** (2008), #R129.

[12] C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga, and B.L. Gorissen, On almost distance-regular graphs, *J. Combin. Theory Ser. A* **118** (2011), no. 3, 1094–1113.

[13] C. Dalfó, M.A. Fiol, and E. Garriga, Characterizing (ℓ, m)-walk-regular graphs, *Linear Algebra Appl.* **433** (2010), no. 9, 1821–1826.

[14] E.R. van Dam and M.A. Fiol, A short proof of the odd-girth theorem, *Electron. J. Combin.* **19(3)** (2012) #P12.

[15] E.R. van Dam and W.H. Haemers, Spectral characterizations of some distance-regular graphs, *J. Algebraic Combin.* **15** (2002) 189–202.

[16] E.R. van Dam and W.H. Haemers, Which graphs are determined by their spectrum?, *Linear Algebra Appl.* **373** (2003) 241–272.

[17] E.R. van Dam and W.H. Haemers, Developments on spectral characterizations of graphs, *Discrete Math.* **309** (2009) 576–586.

[18] E.R. van Dam and W.H. Haemers, An odd characterization of the generalized odd graphs, *J. Combin. Theory Ser. B* **101** (2011) 486–489.

[19] E.R. van Dam, W.H. Haemers, J.H. Koolen, and E. Spence, Characterizing distance-regularity of graphs by the spectrum, *J. Combin. Theory Ser. A* **113** (2006) 1805–1820.

[20] E.R. van Dam, J.H. Koolen, and H. Tanaka, Distance-regular graphs, manuscript (2012), available online at `http://lyrawww.uvt.nl/~evandam/files/drg.pdf`.

[21] M.A. Fiol, An eigenvalue characterization of antipodal distance-regular graphs, *Electron. J. Combin.* **4** (1997), no. 1, #R30.

[22] M.A. Fiol, On pseudo-distance-regularity. *Linear Algebra Appl.* **323** (2001), 145–165.

[23] M.A. Fiol, Algebraic characterizations of distance-regular graphs, *Discrete Math.* **246** (2002), 111–129.

[24] M.A. Fiol, S. Gago, and E. Garriga, A simple proof of the spectral excess theorem for distance-regular graphs, *Linear Algebra Appl.* **432** (2010), no. 9, 2418–2422.

[25] M.A. Fiol and E. Garriga, From local adjacency polynomials to locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **71** (1997), 162–183.

[26] M.A. Fiol and E. Garriga, On the algebraic theory of pseudo-distance-regularity around a set, *Linear Algebra Appl.* **298** (1999), 115–141.

[27] M.A. Fiol, E. Garriga, and J.L.A. Yebra, Locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **68** (1996), 179–205.

[28] C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, NewYork, 1993.

[29] W.H. Haemers, Distance-regularity and the spectrum of graphs, *Linear Algebra Appl.* **236** (1996) 236–278.

[30] A.J. Hoffman, On the polynomial of a graph, *Amer. Math. Monthly* **70** (1963) 30–36.

[31] T. Huang and C. Liu, Spectral characterization of some generalized odd graphs, *Graphs Combin.* **15** (1999) 195–209.

[32] G.-S. Lee and C.-W. Weng, The spectral excess theorem for general graphs, *J. Combin. Theory, Ser. A* **119** (2012) 1427–1431.

[33] P. Rowlinson, Linear algebra, in *Graph Connections* (L.W. Beineke and R.J. Wilson, eds.), Oxford Lecture Ser. Math. Appl., Vol. 5, 86–99, Oxford Univ. Press, New York, 1997.

[34] P.M. Weichsel, On distance-regularity in graphs, *J. Combin. Theory Ser. B* **32** (1982) 156–161.

Breaking symmetries of graphs with resolving sets ⋆

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Abstract. We undertake a study on the maximum value of the difference between the metric dimension and the determining number of a graph as a function of its order. Our results include lower and upper bounds on that maximum, and exact computations when restricting to some specific families of graphs. Although our technique is mainly based on locating-dominating sets, it also requires very diverse tools and relationships with well-known objects in graph theory; among them: a classical result in graph domination by Ore, a Ramsey-type result by ErdHos and Szekeres, a polynomial time algorithm to compute distinguishing sets and dominating sets of twin-free graphs, k-dominating sets, and matchings.

Key words: determining set, determining number, resolving set, metric dimension, locating-dominating set.

1 Introduction and preliminaries

In this paper we focuss on two graph parameters that have attracted much attention in recent years: the determining number and the metric dimension. Concretely, we deal with the following question proposed by Boutin [2]: *Can the difference between the determining number and the metric dimension of a graph be arbitrarily large?* We begin with some definitions and notations.

Let G be a finite, connected, undirected, and simple graph[2]. The *stabilizer* of a set $S \subseteq V(G)$ is $\mathrm{Stab}(S) = \{\phi \in \mathrm{Aut}(G) : \phi(u) = u, \forall u \in S\}$, and S is a

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[2] The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively; the order of G is $n = |V(G)|$. As usual, $\overline{G}$ denotes the complement of G. An *automorphism* of G is a bijective mapping of $V(G)$ onto itself such that $f(u)f(v) \in E(G)$ if and only if $uv \in E(G)$. The *automorphism group* of G is written as $\mathrm{Aut}(G)$, and its identity element is id_G. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest u-v path. Finally, we write $N(u)$ for the open neighbourhood of a vertex $u \in V(G)$.

determining set of G if $\mathrm{Stab}(S)$ is trivial. The minimum cardinality of a determining set is the *determining number* of G, denoted by $\mathrm{Det}(G)$. On the other hand, a vertex $u \in V(G)$ *resolves* a pair $\{x, y\} \subseteq V(G)$ if $d(u, x) \neq d(u, y)$. When every pair of vertices of G is resolved by some vertex in S, it is said that S is a *resolving set* of G. The minimum cardinality of a resolving set is the *metric dimension* of G, written as $\dim(G)$.

Determining sets of graphs are particular cases of *bases* of permutation groups, defined by Sims [9] in 1971 as subsets of elements whose stabilizer is trivial. Much later, Boutin [2] and Erwin and Harary [5] used the terms determining set and *fixing set*, respectively, to refer to a base of the automorphism group of a graph. Also in the 1970s, Harary and Melter [7], and independently Slater [8], introduced the notion of resolving set. For more references on these topics, we refer the reader to the survey of Bailey and Cameron [1].

The above-mentioned question posed by Boutin comes from the fact that every resolving set of a graph G is also a determining set, and so $\mathrm{Det}(G) \leq \dim(G)$ (see [2,5]). To study this question, we define the function $(\dim - \mathrm{Det})(n)$ as the maximum value of $\dim(G) - \mathrm{Det}(G)$ over all graphs G of order n, and develop a technique based on locating-dominating sets, whose definition (recalled from [11]) is provided below. The following result, written in terms of our function, is the best approach to date on the problem.

Proposition 1. [3] *For every* $n \geq 8$,

$$\lfloor \frac{2}{5} n \rfloor - 2 \leq (\dim - \mathrm{Det})(n) \leq n - 2.$$

A pair $\{x, y\} \subseteq V(G)$ is *distinguished* by a vertex $u \in V(G)$ if either $u \in \{x, y\}$ or $N(x) \cap \{u\} \neq N(y) \cap \{u\}$, and a set $D \subseteq V(G)$ is a *distinguishing set* of G if every pair of $V(G)$ is distinguished by some vertex in D. If D is also a *dominating set*, i.e., $N(x) \cap D \neq \emptyset$ for every $x \in V(G) \setminus D$, then D is a *locating-dominating set*. The minimum cardinality of a locating-dominating set is the *locating-domination number* of G, denoted by $\lambda(G)$.

Note that distinguishing sets and locating-dominating sets are in essence the same concept: one can easily check that every distinguishing set becomes a dominating set by adding at most one vertex. Thus,

Remark 1. Let D be a distinguishing set of a graph G. Then, $\lambda(G) \leq |D| + 1$.

Every locating-dominating set is clearly a resolving set, and so $\mathrm{Det}(G) \leq \dim(G) \leq \lambda(G)$ for any graph G. Let $(\lambda - \mathrm{Det})(n)$ and $\lambda(n)$ be the maximum values of, respectively, $\lambda(G) - \mathrm{Det}(G)$ and $\lambda(G)$ over all graphs G of order n. Obviously,

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda(n).$$

The function $\lambda(n)$ equals $n-1$ (attained by the complete graph K_n) but the non-trivial restriction of this function to the class $\mathcal{C}^*$ of twin-free graphs (i.e.,

graphs that do not contain twin vertices) denoted by $\lambda_{|\mathcal{C}^*}(n)$ is fundamental in this work.

The paper is organized as follows. In Section 2, we provide lower bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ by constructing appropriate families of graphs. Section 3 establishes first that $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ are bounded above by $\lambda_{|\mathcal{C}^*}(n)$ and then provides two upper bounds on this last function. The first bound is obtained by combining a variant of a classical theorem in domination theory due to Ore [8] and a Ramsey-type result of ErdHos and Szekeres [4]. The second upper bound on $\lambda_{|\mathcal{C}^*}(n)$ comes from a greedy algorithm to compute distinguishing sets and determining sets of bounded size, which in addition gives an upper bound on the determining number of a twin-free graph. In Section 4, we study the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ restricted to the class $\mathcal{C}_4$ of graphs not containing the cycle C_4 as a subgraph, and its subclass $\mathcal{T}$ of trees. For this purpose, we use two well-known invariants of graphs: the k-domination number and the matching number. We conclude in Section 5 with some remarks and open problems.

2 Lower bounds on $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$

Cáceres et al. [3] used the wheel graph $W_{1,n}$ to obtain the lower bound on $(\dim - \mathrm{Det})(n)$ given in Proposition 1. In order to get a better approach, we construct two adequate families of graphs that, in addition, give a lower bound on the function $(\lambda - \mathrm{Det})(n)$.

Let T_r, $r \geq 6$, be a tree that consists of a path $(u_1, ..., u_r)$ and a pendant vertex u_0 adjacent to u_3, and let G_r be the corona product $T_r \circ K_1$, i.e., the graph with vertex set $V(G_r) = \{u_0, u_1, ..., u_r, v_0, v_1, ..., v_r\}$ and edge set $E(G_r) = E(T_r) \cup \{u_i v_i : i \in \{0, 1, ..., r\}\}$. By adding another pendant vertex v_0' to u_0 in G_r we obtain the graph H_r (see Figure 1).

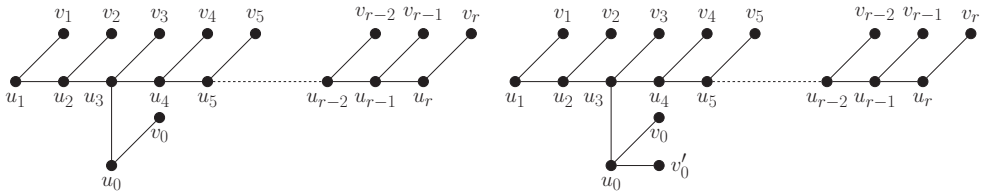

Fig. 1: The graphs (a) G_r and (b) H_r.

The following lemma is the key tool to obtain lower bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$.

Lemma 1. *For every $r \geq 6$, the following statements hold:*

(i) $\mathrm{Det}(G_r) = 0$ *and* $\mathrm{Det}(H_r) = 1$.
(ii) $\dim(\overline{G}_r) = r$ *and* $\dim(\overline{H}_r) = r + 1$.
(iii) $\lambda(G_r) = r + 1$ *and* $\lambda(H_r) = r + 2$.

With this lemma in hand, one can prove that G_r, H_r, and their complements (for appropriate r) are the above-mentioned adequate families of graphs which provide the following lower bounds.

Theorem 1. *For every $n \geq 14$,*

$$(\dim - \mathrm{Det})(n) \geq \lfloor \frac{n}{2} \rfloor - 1 \qquad \textit{and} \qquad (\lambda - \mathrm{Det})(n) \geq \lfloor \frac{n}{2} \rfloor.$$

We shall exhibit large classes of graphs in which the restrictions of $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ do not exceed $\frac{n}{2}$. Thus, we believe that the preceding bounds are in fact the exact values of our functions.

Conjecture 1. There exists a positive integer n_0 such that, for every $n \geq n_0$,

$$(\dim - \mathrm{Det})(n) = \lfloor \frac{n}{2} \rfloor - 1 \qquad \textit{and} \qquad (\lambda - \mathrm{Det})(n) = \lfloor \frac{n}{2} \rfloor.$$

3 Upper bounds on $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$

In order to obtain explicit upper bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$, our first step is to prove that they are bounded above by $\lambda_{|\mathcal{C}^*}(n)$. To do this, we associate to every graph G a twin-free graph $\widetilde{G}$, from which one can extract locating-dominating sets of G. We then prove that $\lambda(G) - \mathrm{Det}(G) \leq \lambda(\widetilde{G})$, whose maximum taken over all graphs of order n leads us to the desired inequality. This process comprises a complex machinery of technical results that we omit for the sake of brevity.

Theorem 2. *For every $n \geq 4$,*

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda_{|\mathcal{C}^*}(n).$$

Theorems 4 and 2 give $\lfloor \frac{n}{2} \rfloor \leq \lambda_{|\mathcal{C}^*}(n)$. Further, in [6] we find numerous conditions for a twin-free graph G to satisfy $\lambda(G) \leq \lfloor \frac{n}{2} \rfloor$ (here we will indicate only some of them). Thus, we believe that the following conjecture, which implies most of Conjecture 1, is true.

Conjecture 2. There exists a positive integer n_1 such that, for every $n \geq n_1$,

$$\lambda_{|\mathcal{C}^*}(n) = \lfloor \frac{n}{2} \rfloor.$$

In the following two subsections, we are concerned with obtaining upper bounds on $\lambda_{|\mathcal{C}^*}(n)$ since, by Theorem 2, these are also bounds on the functions $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$.

3.1 From minimal dominating sets to locating-dominating sets

A set $D \subseteq V(G)$ is a *minimal dominating set* if no proper subset of D is a dominating set of G. The following theorem, due to Ore [8], is one of the first results in the field of graph domination. Theorem 4 below is a variant of this result for twin-free graphs.

Theorem 3. [8] *Let G be a graph without isolated vertices and let $D \subseteq V(G)$ be a minimal dominating set of G. Then, $V(G) \setminus D$ is a dominating set of G.*

Theorem 4. *Let G be a twin-free graph and let $D \subseteq V(G)$ be a minimal dominating set of G. Then, $V(G) \setminus D$ is a locating-dominating set of G.*

The preceding theorem implies that minimal dominating sets of twin-free graphs G provide bounds on $\lambda(G)$. By showing that minimal dominating sets can be constructed from independent sets and cliques of maximum size, we reach the following corollary that supports Conjecture 2. Recall that the *independence number* $\alpha(G)$ and the *clique number* $\omega(G)$ are, respectively, the maximum cardinality of an independent set and a clique of G.

Corollary 1. *Let G be a twin-free graph. Then, $\lambda(G) \leq n - \max\{\alpha(G), \omega(G) - 1\}$. In particular, $\lambda(G) \leq \frac{n}{2}$ when either $\alpha(G) \geq \frac{n}{2}$ or $\omega(G) \geq \frac{n}{2} + 1$.*

ErdHos and Szekeres [4] proved that every graph of order n contains either a clique or an independent set of cardinality at least $\lceil \frac{\log_2 n}{2} \rceil$. Applying this result to Corollary 1, we obtain an upper bound on $\lambda_{|\mathcal{C}^*}(n)$ that, on account of Theorem 2, improves significantly the upper bound of Proposition 1, due to Cáceres et al. [3].

Corollary 2. *For every $n \geq 4$,*

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda_{|\mathcal{C}^*}(n) \leq n - \lceil \frac{\log_2 n}{2} \rceil + 1.$$

3.2 A greedy algorithm for twin-free graphs

Our second upper bound on $\lambda_{|\mathcal{C}^*}(n)$, that is one of the main contributions of this work, comes from a polynomial time algorithm that produces distinguishing sets of twin-free graphs. In addition, this algorithm computes determining sets of bounded size, and thus an upper bound on the determining number of a twin-free graph. To present this algorithm, we first provide some notation.

For any set $D \subseteq V(G)$, let us define a relation on $V(G)$ given by $u \sim_D v$ if and only if either $u = v$ or $\{u, v\}$ is distinguished by no vertex of D. It is easy to check that this is an equivalence relation, and so we denote by $[u]_D$ the set of vertices $v \in V(G)$ such that $u \sim_D v$. Thus, the sets D, $D^1 = \{u \in V(G) \setminus D : |[u]_D| = 1\}$ and $D^{>1} = V(G) \setminus (D \cup D^1)$ form a partition of $V(G)$, where any of these sets may be empty.

The following greedy algorithm gives a partition of $V(G)$ into three sets so that, combining them properly, one obtains distinguishing sets and determining sets of G of bounded size, as stated in Lemma 2 below.

Algorithm 1

Input: A twin-free graph G and a vertex $u_0 \in V(G)$.
Output: An appropriate partition of $V(G)$ into three subsets A, B, C.

$A \leftarrow \{u_0\}$
$B \leftarrow A^1$
$C \leftarrow A^{>1}$
while $\exists\ u, x, y \in C$ such that $[x]_A = [y]_A$ and $[x]_{A\cup\{u\}} \neq [y]_{A\cup\{u\}}$ **do**
    $A \leftarrow A \cup \{u\}$
    $B \leftarrow A^1$
    $C \leftarrow A^{>1}$
end
return A, B, C

Lemma 2. *Let A, B, C be the sets obtained by application of Algorithm 1 to a twin-free graph G and a vertex $u_0 \in V(G)$. Then, the following statements hold:*

(i) $A \cup B$, $A \cup C$ and $B \cup C$ are distinguishing sets of G.
(ii) A and $B \cup C$ are determining sets of G.

The pigeonhole principle ensures that one set among $A \cup B, A \cup C, B \cup C$ has cardinality at most $\lfloor \frac{2}{3}n \rfloor$. Then, by Lemma 2 and Remark 1, we obtain the following theorem that improves the upper bound of Corollary 2, and consequently the upper bound of Proposition 1 by Cáceres et al. [3].

Theorem 5. *Let G be a twin-free graph of order $n \geq 4$. Then,*

$$(\dim - \mathrm{Det})(n) \leq (\lambda - \mathrm{Det})(n) \leq \lambda_{|\mathcal{C}^*}(n) \leq \lfloor \frac{2}{3}n \rfloor + 1.$$

We want to point out that, this result and Theorem 4 give, as far as we know, the best bounds on the function $(\dim - \mathrm{Det})(n)$.

By applying again the pigeonhole principle, one gets that either A or $B \cup C$ has cardinality at most $\lfloor \frac{n}{2} \rfloor$ and so, by Lemma 2, we obtain the following bound on the determining number of a twin-free graph.

Theorem 6. *Let G be a twin-free graph of order $n \geq 4$. Then,*

$$\mathrm{Det}(G) \leq \lfloor \frac{n}{2} \rfloor.$$

4 Restriction to specific families of graphs

Let $(\dim - \mathrm{Det})_{|\mathcal{C}}(n)$ and $(\lambda - \mathrm{Det})_{|\mathcal{C}}(n)$ be the restrictions of our functions to a class of graphs $\mathcal{C}$. Here, we study these restrictions to the classes $\mathcal{C}_4$ and $\mathcal{T}$. To do this, we relate the locating-domination number to two well-known graph parameters: the k-domination number and the matching number.

Given a positive integer k, a set $D \subseteq V(G)$ is said to be a *k-dominating set* if $|N(x) \cap D| \geq k$ for every $x \in V(G) \setminus D$. The minimum cardinality of a k-dominating set is the *k-domination number* of G, denoted by $\gamma_k(G)$. On the other hand, the *matching number* of G, written as $\alpha'(G)$, is the cardinality of a maximum matching in G.

Let $\mathcal{K}_{2,k}$ denote the class of graphs not containing $K_{2,k}$ as a (not necessarily induced) subgraph (observe that $\mathcal{K}_{2,2} = \mathcal{C}_4$). The following proposition gives two bounds on the locating-domination number of a graph of $\mathcal{K}_{2,k}$ in terms of, respectively, its k-domination number and its matching number.

Proposition 2. *For every $G \in \mathcal{K}_{2,k}$ of order $n \geq 4$, the following statements hold:*

(i) $\lambda(G) \leq \gamma_k(G)$.
(ii) $\lambda(G) \leq \alpha'(G)$ *whenever* $k = 2$ *and* $G \in \mathcal{C}^*$.

As an application of this proposition, we obtain the exact value of the function $(\lambda - \mathrm{Det})_{|\mathcal{C}_4}(n)$ and give bounds on $(\dim - \mathrm{Det})_{|\mathcal{C}_4}(n)$.

Theorem 7. *For every $n \geq 49$, it holds that*

$$\lfloor \frac{2}{7}n \rfloor \leq (\dim - \mathrm{Det})_{|\mathcal{C}_4}(n) \leq \lfloor \frac{n}{2} \rfloor \quad \textit{and} \quad (\lambda - \mathrm{Det})_{|\mathcal{C}_4}(n) = \lfloor \frac{n}{2} \rfloor.$$

Regarding the family of trees, Cáceres et al. [3] proved that $(\dim - \mathrm{Det})_{|\mathcal{T}}(n) = \Omega(\sqrt{n})$. The following theorem provides the exact value of this function (thereby closing the study initiated by those authors) and also the value of $(\lambda - \mathrm{Det})_{|\mathcal{T}}(n)$.

Theorem 8. *For every $n \geq 49$, it holds that*

$$(\dim - \mathrm{Det})_{|\mathcal{T}}(n) = \lfloor \frac{2}{7}n \rfloor \quad \textit{and} \quad (\lambda - \mathrm{Det})_{|\mathcal{T}}(n) = \lfloor \frac{n}{2} \rfloor.$$

Observe that Theorems 7 and 8 also support Conjecture 2.

5 Concluding remarks

We develop a technique to study the function $(\dim - \mathrm{Det})(n)$ that involves two other functions related to locating-dominating sets: $(\lambda - \mathrm{Det})(n)$ and $\lambda_{|\mathcal{C}^*}(n)$. This technique uses tools that go from results by Ore, and ErdHos

and Szekeres to matchings, k-domination, and the design of a polynomial time algorithm to obtain distinguishing sets and determining sets of twin-free graphs. We want to stress that our technique requires many auxiliary results that are of independent interest; here, they are omitted for the sake of brevity but we refer the interested reader to [6] for more details.

It would be interesting to settle Conjectures 1 and 2, which deal with the exact values of our functions. Further, it remains open the computation of the function $(\dim - \mathrm{Det})_{|\mathcal{C}_4}(n)$. It would be also interesting to find specific families of graphs where the restrictions of $(\dim - \mathrm{Det})(n)$ and $(\lambda - \mathrm{Det})(n)$ may be computed. Finally, the maximum value of the difference between the metric dimension and the locating-domination number is still unknown and a study on this function may be proposed.

References

[1] R. F. Bailey and P. J. Cameron. Base size, metric dimension and other invariants of groups and graphs. *Bull. Lond. Math. Soc.*, 43(2):209–242, 2011.

[2] D. L. Boutin. Identifying graph automorphisms using determining sets. *Electron. J. Combin.*, 13(1):Research Paper 78, 12, 2006.

[3] J. Cáceres, D. Garijo, M. L. Puertas, and C. Seara. On the determining number and the metric dimension of graphs. *Electron. J. Combin.*, 17(1):Research Paper 63, 20, 2010.

[4] P. ErdHos and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[5] D. Erwin and F. Harary. Destroying automorphisms by fixing nodes. *Discrete Math.*, 306(24):3244–3252, 2006.

[6] D. Garijo, A. González, and A. Márquez. Resolving sets for breaking symmetries of graphs. Preprint. 2014. arXiv:1401.3686v1.

[7] F. Harary and R. A. Melter. On the metric dimension of a graph. *Ars Combinatoria*, 2:191–195, 1976.

[8] O. Ore. *Theory of graphs.* American Mathematical Society Colloquium Publications, Vol. 38. American Mathematical Society, Providence, R.I., 1962.

[9] C. C. Sims. Determining the conjugacy classes of a permutation group. In *Computers in algebra and number theory*, pages 191–195. SIAM–AMS Proc., Vol. IV. Amer. Math. Soc., Providence, R.I., 1971.

[10] P. J. Slater. Leaves of trees. *Congressus Numerantium*, 14:549–559, 1975.

[11] P. J. Slater. Dominating and reference sets in a graph. *J. Math. Phys. Sci.*, 22(4):445–455, 1988.

Monitoring maximal outerplanar graphs

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Abstract. In this paper we define a new concept of monitoring the elements of triangulation graphs by faces. Furthermore, we analyze this, and other monitoring concepts (by vertices and by edges), from a combinatorial point of view, on *maximal outerplanar graphs*.

Key words: Domination, covering, guarding, triangulation graphs.

1 Introduction

In Graph Theory, the notion of monitoring vertices, edges of graphs by other vertices or edges has been widely studied (see, e.g., [4]). For instance, monitoring vertices by other vertices or edges by other edges leads to well known parameters concerned with *vertex domination* or *edge domination*, respectively. When vertices are to monitor edges we have the well known notions of *vertex covering*. Finally, when edges are to monitor vertices we obtain parameters associated with *edge covering*. A *dominating set* is a set $D \subseteq V$ such that every vertex not in D is adjacent to a vertex in D. The *domination number*, $\gamma(G)$, is the number of vertices in a smallest dominating set for G. A set $D \subseteq E$ is an *edge dominating set* if each edge in E is either in D or is adjacent to an edge in D. The *edge domination number*, $\gamma'(G)$, is the smallest cardinality among all minimal edge dominating sets for G. A *vertex cover* is a set $C \subseteq V$ if each edge of the graph is incident to at least one vertex of the C. The *covering number*, $\beta(G)$, is the size of a minimum vertex cover. Finally, an edge $e = uv$ is said to cover the vertices u and v. An *edge cover* is set $C \subseteq E$ which

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cover every vertex in V. The *edge covering number*, $\beta'(G)$, is the minimum cardinality of an edge cover for G. In Computational Geometry, for triangulations or quadrangulations, a different monitoring notion was established – the notion of monitoring bounded faces (*faces*, for short). When the faces are monitored by vertices or edges, we obtain the parameters associated with *vertex guarding* or *edge guarding*, respectively. Being $G = (V, E)$ a triangulation, a *guarding set* is a set $L \subseteq V$ such that every face has a vertex in S. The *guarding number*, $g(G)$, is the number of vertices in a smallest guarding set for G. Concerning edge guarding, an edge $e = uv$ is said to guard a face of F of G if u or v are vertices of F. An *edge guarding set* $L \subseteq E$ is a set which guards every face in G. The *edge guarding number*, $g^e(G)$, is the minimum cardinality of an edge guarding set for G. All the previous described monitoring notions were extended to include its distance versions on plane graphs. For example, domination was extended to *distance domination* and guarding to *distance guarding* [2].

Regarding combinatorial bounds, there are many results about domination and covering for graphs and for triangulations graphs (that is, the graph of a triangulation of a set of points in the plane). In this paper we analyze these monitoring concepts (domination, covering and guarding) from a combinatorial point of view, for a special class of triangulation graphs, the *maximal outerplanar graphs*. A *maximal outerplanar graph* embedded in the plane corresponds to a triangulation of a polygon. Concerning plane graphs, it is natural to extend the notions of monitoring by faces. So, in this paper we also define three new concepts: *face-vertex guarding*, *face guarding* and *face-edge guarding* in triangulation graphs (*triangulations*, for short), that is, when faces are monitored by vertices, faces and edges, respectively. Furthermore, we establish tight bounds for the "normal" and distance versions of monitoring by faces on maximal outerplanar graphs. In the next section we describe some definitions and terminology that will be used throughout this paper.

2 Definitions

As stated above, given a triangulation $T = (V, E)$ its elements (vertices, edges and faces) can be monitored by other vertices, edges or faces (i.e., triangles). First, we start by presenting the terminology that we use when the elements of T are monitored by vertices, at its distance version (see [2], for details). Let $T = (V, E)$ be a triangulation. A *kd-dominating set* for T is a subset $D \subset V$ such that each vertex $u \in V - D$, $dist_T(u, v) \leq k$ for some $v \in D$. The *kd*-domination number $\gamma_{kd}(T)$ is the number of vertices in a smallest *kd*-dominating set for T. Given $n \in \mathbb{N}$, we define $\gamma_{kd}(n) = max\{\gamma_{kd}(T) : T \text{ is triangulation } T = (V, E) \text{ with } |V| = n\}$. We say that a triangle T_i of T is *kd-guarded* from by a vertex $v \in V$, if there is a vertex $x \in T_i$ such that $dist_T(x, v) \leq k - 1$. A *kd-guarding set* for T is a subset $L \subseteq V$ such

that every triangle of T is kd-guarded by an element of L. The *kd-guarding number* $g_{kd}(T)$ is the number of vertices in a smallest kd-guarding set for T. Given $n \in \mathbb{N}$, we define $g_{kd}(n) = max\{g_{kd}(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. A *kd-vertex cover* of T, is a subset $C \subseteq V$ such that for each edge $e \in E$ there is a path of length at most k, which contains e and a vertex of C. The *kd-covering number* $\beta_{kd}(T)$ is the number of vertices in a smallest kd-vertex cover of T. Given $n \in \mathbb{N}$, we define $\beta_{kd}(n) = max\{\beta_{kd}(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$.

In the following, we introduce the terminology that we use when the monitoring of the elements of T, at its distance version, is done by edges. A *kd-edge cover* of T, is a subset $C \subseteq E$ such that each vertex, $v \in V$ $dist_T(v, e) \leq k - 1$, for some $e \in C_{e,kd}$, where $dist_T(v, e)$ is the minimum distance between the endpoints of e and v (see Fig. 1(a)), for a sketch). If the monitored elements are triangles we have the notions of *kd-edge guarding* and $g^e_{kd}(T)$ (see Fig. 1(b)). And if they are edges, kd-edge covering and $\beta'_{kd}(T)$ (see Fig. 1(c)). Given $n \in \mathbb{N}$, the values $\gamma'_{kd}(n), g^e_{kd}(n)$ and $\beta'_{kd}(n)$, are defined similarly to the previous case (monitoring by vertices, in their distance versions).

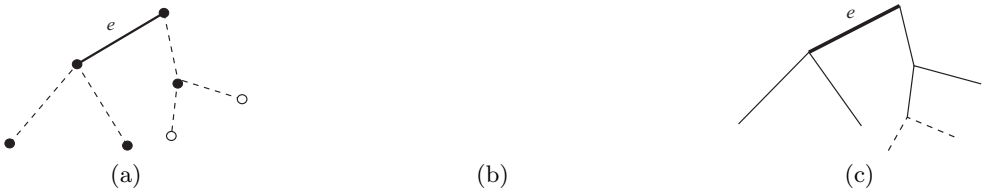

Fig. 1: The edge e: (a) $2d$-edge dominate the black vertices; (b) $2d$-edge guard the shadow triangles; (c) $2d$-edge cover the filled edges vertices.

Next, we will define new monitoring concepts: monitoring the elements of a triangulation by its faces (the "normal" and the distance versions). Let $T = (V, E)$ be a triangulation. A *face-vertex cover* is a subset C of faces of T such that each vertex $v \in V$ is a vertex of some $T_i \in C$. The *face-vertex covering number* $f^v(T)$ is the number of vertices in a smallest face-vertex cover for T. Given $n \in \mathbb{N}$, we define $f^v(n) = max\{f^v(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. We say that a triangle T_j of T is *guarded* by a triangle T_i of T if they share some vertex. A *face guarding set* is a subset L of triangles of T such that every triangle of T is guarded by an element of L. The *face guarding number*, $g^f(T)$, is the number of triangles in a smallest face guarding

set for T. Given $n \in \mathbb{N}$, we define $g^f(n) = max\{g^f(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$. Finally, a subset C of triangles of T is *face-edge cover* if each edge $e \in E$ has an endpoint on some $T_i \in C$. The *face-edge covering number* $f^e(T)$ is the minimum cardinality of a face-edge cover for T. Given $n \in \mathbb{N}$, we define $f^e(n) = max\{f^e(T) : T$ is triangulation $T = (V, E)$ with $|V| = n\}$.

The above defined concepts were extended to its distance versions, "similarly to the monitoring by edges" (see Fig.2). Given $n \in \mathbb{N}$, the values $f^v_{kd}(n), g^f_{kd}(n)$ and $f^e_{kd}(n)$ are defined similarly to $f^v(n), g^f(n)$ and $f^e(n)$, respectively.

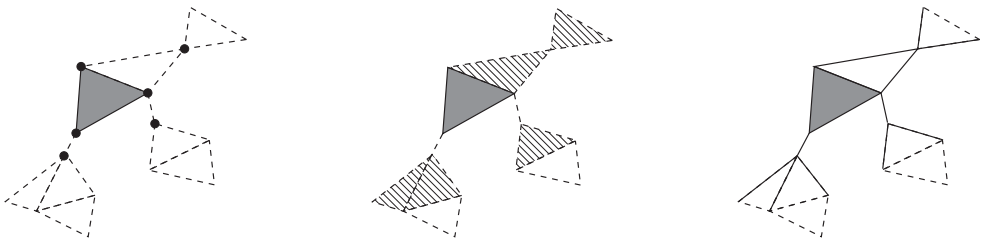

Fig. 2: The gray face:(a) $2d$-face-face guard the shadow triangles; (b) $2d$-face-vertex cover the filled edges; (c) $2d$-face-edge cover the black vertices.

Our main goal is to obtain combinatorial bounds related to the monitoring numbers on triangulation. That is, establish tight bounds on $\gamma(n), g(n), \beta(n)$, $\ldots$, $g^f_{kd}(n)$, $f^e_{kd}(n)$, on triangulations. As stated in the introduction some of these bounds are already known, so, as is evident, we will study the unknown ones. Particularly, $\beta(n), \beta^{'}(n), \beta^{'}_{kd}(n), g^e_{kd}(n), \gamma^{'}_{kd}(n)$ and, obviously, all the values concerning the new defined concepts related to monitoring by faces. We start by studying a special class of triangulations, namely the maximal outerplanar graphs; and concerning the the distance versions we begin with distance 2, that is, $d = 2$.

3 Monitoring maximal outerplanar graphs

In this section we establish tight bounds for the minimum number of vertices, edges and faces that monitor the different elements (vertices, edges and faces) of a special class of triangulation graphs – the maximal outerplanar graphs – which correspond, as stated above, to triangulations of polygons. We call the edges on the exterior face *exterior edges*, otherwise they are *interior edges*. In the following tables are summarized our, and related, results concerning the monitoring the different elements of maximal outerplanar graphs. In Table 1

we present the results regarding "normal" monitoring versions, that is, monitoring at distance 1, and in Table 2 we show the results related to monitoring versions at distance 2.

		Monitored Elements		
		Vertices	Faces	Edges
Monitored by	Vertices	**Dominating** $\gamma(n) = \lfloor \frac{n+n_2}{4} \rfloor$[1] [1], [8]	**Guarding** $g(n) = \lfloor \frac{n}{3} \rfloor$ [3]	**Covering** $\beta(n) = \lfloor \frac{2n}{3} \rfloor$ (here)
	Edges	**Edge-Covering** $\beta'(n) = \lceil \frac{n}{2} \rceil$ (here)	**Edge-Guarding** $g^e(n) = \lfloor \frac{n}{2} \rfloor$ [6]	**Edge-Dominating** $\gamma'(n) = \lfloor \frac{n+1}{3} \rfloor$ [5]
	Faces	**Face-vertex Covering** $f^v(n) = \lfloor \frac{n}{2} \rfloor$ (here)	**Face-face Guarding** $g^f(n) = \lfloor \frac{n}{4} \rfloor$ (here)	**Face-edge Covering** $f^e(n) = \lfloor \frac{n}{3} \rfloor$ (here)

Table 1: A summary of new and related results for "normal" monitoring.

		Monitored elements		
		Vertices	Faces	Edges
Monitored by	Vertices	**Dominating** $\gamma_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ [2]	**Guarding** $g_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ [2]	**Covering** $\beta_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ [2]
	Edges	**Edge-Covering** $\beta'_{2d}(n) = \lfloor \frac{n}{4} \rfloor$ (here)	**Edge-Guarding** $g^e_{2d}(n) = \lfloor \frac{n}{6} \rfloor$ (here)	**Edge-Dominating** $\gamma'_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ (here)
	Faces	**Face-vertex Covering** $f^v_{2d}(n) = \lfloor \frac{n}{4} \rfloor$ (here)	**Face-face Guarding** $g^f_{2d}(n) = \lfloor \frac{n}{6} \rfloor$ (here)	**Face-edge Covering** $f^e_{2d}(n) = \lfloor \frac{n}{5} \rfloor$ (here)

Table 2: A summary of new results for monitoring at distance 2.

In the following, due to lack of space, we will present only two proofs of the results shown in the tables.

First, we will prove that $g^f(n) = \lfloor \frac{n}{4} \rfloor$, for $n \geq 4$. In order to do this, and following the ideas of O'Rourke [7], we first need to introduce some lemmas.

Lemma 1. *Suppose that $f(m)$ triangles are always sufficient to guard any outerplanar maximal graph with m vertices. Let T be a m-vertex outerplanar maximal graph and e an exterior edge. Then with $f(m-1)$ triangles and an additional "collapsed triangle" at the edge e are sufficient to guard T.*

Lemma 2. *Let T be an outerplanar maximal graph with $n \geq 2k$ vertices. There is an interior edge e of T that partitions G into two pieces, one of which contains $m = k, k+1, \ldots, 2k-1$ or $2k-2$ exterior edges of T.*

Theorem 1. *Every n-vertex maximal outerplanar graph, with $n \geq 4$, can be face-guarded by $\lfloor \frac{n}{4} \rfloor$ triangles. And there are maximal outerplanar graphs that*

[1] n_2 is the number of vertices of degree 2.

need at least $\lfloor \frac{n}{4} \rfloor$ *triangles to be face-guarded. That is,* $g^f(n) = \lfloor \frac{n}{4} \rfloor$ *for all* $n \geq 4$.

Proof.

For $4 \leq n \leq 7$, the truth of the theorem can be easily established. Assume that $n \geq 8$ and that the theorem holds for all $n' < n$. Lemma 2 guarantees the existence of an interior edge e that divides T into two maximal outerplanar graphs T_1 and T_2, such that T_1 has m exterior edges of T with $5 \leq m \leq 8$. The vertices of T are labeled with $0, 1, \ldots, n-1$ such that e is $(0, m)$. Each value of m, which is minimal is considered separately. Here, we present the cases $m = 5$ and $m = 8$.

Case $m = 5$. T_1 has $m + 1 = 6$ exterior edges, thus it can be face-guarded with one triangle. T_2 has $n - 4$ exterior edges including e, and by induction hypothesis, it can be face-guarded with $\lfloor \frac{n-4}{4} \rfloor = \lfloor \frac{n}{4} \rfloor - 1$ guards. Thus T_1 and T_2 together can be face-guarded by $\lfloor \frac{n}{4} \rfloor$ guards.

Case $m = 8$. The presence of any of the internal edges (0,7), (0,6), (0,5), (7,1), (7,2) and (7,3) would violate the minimality of m. Thus, the triangle T' in T_1 that is bounded by e is (0,4,8). Consider the maximal outerplanar graph $T^* = T_2 + (0, 4, 5, 6, 7, 8)$ (see Fig. 3(a)). T^* has $n - 3$ exterior edges, applying lemma 1 it can be face-guarded with $f(n-3)$ triangles, that is $\lfloor \frac{n}{4} \rfloor - 1$ triangles, and an additional "collapsed triangle" at the edge (0,4). This "collapsed triangle" also face-guards the pentagon (0,1,2,3,4) regardless of the way how it is triangulated. Thus, T is face-guarded by $\lfloor \frac{n}{4} \rfloor$ triangles.

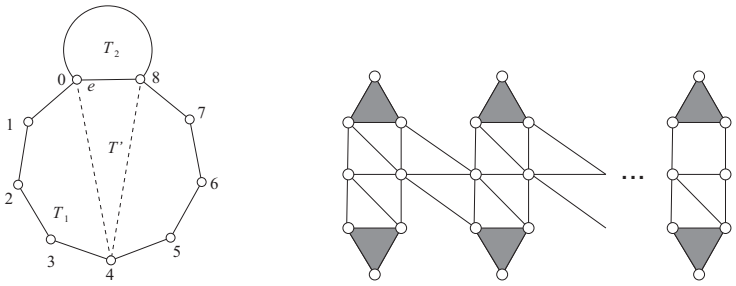

Fig. 3: (a) The triangle T' is (0,4,8); (b) a maximal outerplanar graph T for which $g^f(n) \geq \lfloor \frac{n}{4} \rfloor$.

To prove that this upper bound is tight we need to construct a n-vertex maximal outerplanar graph T such that $g^f(n) \geq \lfloor \frac{n}{4} \rfloor$. Fig. 3(b) shows a maximal outerplanar graph T for which $g^f(n) \geq \lfloor \frac{n}{4} \rfloor$, since two shadowed triangles can only be face-guarded by different triangles.

□ ∎

Next, we will show that $\beta'_{2d}(n) = \lfloor \frac{n}{4} \rfloor$, for any n-vertex maximal outerplanar graph, with $n \geq 4$. In order to do this, we first need to introduce the next lemma.

Lemma 3. *Suppose that $f(m)$ edges are always sufficient to $2d$-edge cover any outerplanar maximal graph with m vertices. Let T be a m-vertex outerplanar maximal graph and $e = (u, v)$ an exterior edge. Then with $f(m-1)$ edges and an additional "collapsed edge" at the vertex u or v are sufficient to $2d$-edge cover T.*

Theorem 2. *Every n-vertex maximal outerplanar graph, with $n \geq 4$, can be $2d$-edge covered by $\lfloor \frac{n}{4} \rfloor$ edges. And this bound is tight in the worst case.*

Proof.

The proof is done by induction on n. For $4 \leq n \leq 9$, the truth of the theorem can be easily established. Assume that $n \geq 10$, and that the theorem holds for $n' < n$. Let T be a maximal outerplanar graph with n vertices. The vertices of T are labeled with $0, 1, 2, \ldots, n-1$. Lemma 2 guarantees the existence of an interior edge e (which can be labeled $(0, m)$) that divides T into maximal outerplanar graphs T_1 and T_2, such that T_1 has m exterior edges of T with $5, 6, 7$ or 8. Each value of m, which is minimal, is considered separately. Here, we present the cases $m = 6$ and $m = 7$.

Case $m = 6$. T_2 has $n - 5$ exterior edges including e, and by induction hypothesis, $\lfloor \frac{n-5}{4} \rfloor \leq \lfloor \frac{n}{4} \rfloor - 1$ edges are sufficient to $2d$-edge-covered it. T_1 has 7 exterior edges, so it can be $2d$-edge-covered with 1 edge. In this way, T can be $2d$-edge covered with $\lfloor \frac{n}{4} \rfloor$ edges.

Case $m = 7$. By the minimality of m the triangle T' supported by the internal edge (0,7) is (0,3,7) or (0,4,7). Since these are equivalent cases, we suppose that T' is (0,3,7) as shown in Fig. 4(a). Consider $T^* = T_2 + (0, 1, 2, 3, 7)$. T^* has $n - 3$ exterior edges, applying lemma 3 it can be $2d$-edge covered with $\lfloor \frac{n}{4} \rfloor - 1$ edges, and an additional "collapsed edge" at the vertex 3 or 7. This "collapsed edge" also $2d$-edge cover the pentagon (3,4,5,6,7). Thus, T can be $2d$-edge covered by $\lfloor \frac{n}{4} \rfloor$ edges.

Now, we will prove that this upper bound is tight. No two black vertices of the maximal outerplanar graph illustrated in Fig. 4(b) can be $2d$-edge covered by the same edge, and therefore $\beta'_{2d}(n) \geq \lfloor \frac{n}{4} \rfloor$.

□ ∎

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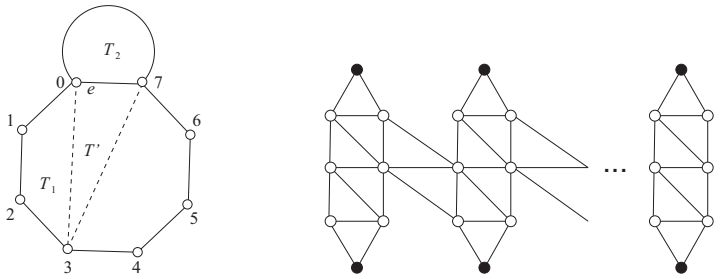

Fig. 4: (a) The triangle $T^{'}$ is (0,3,7); (b) a maximal outerplanar graph T for which $\beta^{'}_{2d}(n) \geq \lfloor \frac{n}{4} \rfloor$.

References

[1] C.N. Campos and Y. Wakabayashi, On dominating sets of maximal outerplanar graphs, *Discrete Appl.Math*, 161(3) (2013), 330–335.

[2] Canales, S., Hernández, G., Martins M., Matos, I., Distance domination, guarding and vertex cover for maximal outerplanar graphs, *in Proc. of the XV Spanish Meeting on Computational Geometry (EGC'13)* (2013), 51–54.

[3] S. Fisk. A short proof of Chvatal's watchman theorem, *J. Combin. Theory Ser. B*, 24 (1978), 374.

[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York (1998).

[5] M.I. Karavelas, "Guarding curvilinear art galleries with edge or mobile guards via 2-dominance of triangulation graphs". Computational Geometry: Theory and Applications, 44 (2011) 20–51

[6] T. Shermer, A tight bound on the combinatorial edge guarding problem, *Snapshots of Comp. and Discrete Geom*, 3 (1994), 191–223.

[7] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, Inc., New York, USA (1987).

[8] S. Tokunaga, Dominating sets of maximal outerplanar graphs, *Discrete Appl.Math*, 161(18) (2013), 3097–3099.

Bounds on the Hyperbolicity Constant

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Abstract. If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X. The space X is δ-hyperbolic in the Gromov sense if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. If X is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e. $\delta(X) = \inf\{\delta \geq 0 : \quad X \text{ is } \delta\text{-hyperbolic}\}$. To compute the hyperbolicity constant is a very hard problem. Then it is natural to try to bound the hyperbolycity constant in terms of some parameters of the graph. Denote by $\mathcal{G}(n, m)$ the set of graphs G with n vertices and m edges, and such that every edge has length 1. In this work we estimate $A(n, m) := \min\{\delta(G) \mid G \in \mathcal{G}(n, m)\}$ and $B(n, m) := \max\{\delta(G) \mid G \in \mathcal{G}(n, m)\}$. We compute the precise value of $A(n, m)$ in many cases and we obtain upper and lower bounds for $B(n, m)$. In addition, we obtain an upper bound of the size of any graph in terms of its diameter and its order.

Key words: Gromov hyperbolicity, hyperbolicity constant, finite graphs, geodesic.

1 Introduction

Gromov hyperbolicity was introduced by the Russian mathematician Mikhail Leonidovich Gromov in the setting of geometric group theory [18], [17], [16], [14], but has played an increasing role in analysis on general metric spaces [6], [7], [3], with applications to the Martin boundary, invariant metrics in several complex variables [2] and extendability of Lipschitz mappings [24].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. Another important application of these spaces is the secure transmission of information by internet. The hyperbolicity is also useful in the study of DNA data (see [9]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance, [4,9,10,11,12,15,21,22,23,25,26,31].

Now, let us introduce the concept of Gromov hyperbolicity and the main results concerning this theory. For detailed expositions about Gromov hyperbolicity, see e.g. [1], [16], [14], [8, II.H] or [33].

Let (X,d) be a metric space and let $\gamma : [a,b] \longrightarrow X$ be a continuous function. We define the *length* of $\mathtt{g}$ as

$$L(\delta) := \sup\Big\{ \sum_{i=1}^{n} d(\mathtt{g}(t_{i-1}), \mathtt{g}(t_i)) : \ a = t_0 < t_1 < \cdots < t_n = b \Big\}.$$

If X is a metric space we say that the curve $\mathtt{g} : [a,b] \longrightarrow X$ is a *geodesic* if we have $L(\mathtt{g}|_{[t,s]}) = d(\mathtt{g}(t), \mathtt{g}(s)) = |t-s|$ for every $s,t \in [a,b]$ (then γ is equipped with an arc-length parametrization). The metric space X is said *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a graph, then the edge joining the vertices u and v will be denoted by $[u,v]$.

In order to consider a graph G as a geodesic metric space, identify (by an isometry) any edge $[u,v] \in E(G)$ with the interval $[0,1]$ in the real line; then the edge $[u,v]$ (considered as a graph with just one edge) is isometric to the interval $[0,1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any graph G has a natural distance defined on its points, induced by taking shortest paths in G, and we can see G as a metric graph. Throughout this paper, $G = (V,E)$ denotes a simple connected graph such that every edge has length 1. These properties guarantee that any graph is a geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality, since [5, Theorems 8 and 10] reduce the problem of compute the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$ is a polygon with sides $J_j \subseteq X$, we say that J is *δ-thin* if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. In other words, a polygon is δ-thin if each of its sides is contained in the δ-neighborhood of the union of the other sides. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geq 0 |\, J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is *δ-hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) | \, T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then

$\delta(X) = \inf\{\delta \geq 0 | \, X \text{ is } \delta\text{-hyperbolic}\}$.

The following are interesting examples of hyperbolic spaces. Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. The real line $\mathbb{R}$ is 0-hyperbolic due to any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic, since the midpoint of a side on a large equilateral triangle is far from all points in the other two sides. A normed vector space E is hyperbolic if and only if $\dim\ E = 1$. The main examples of hyperbolic graphs are trees.

For a general graph deciding whether or not a space is hyperbolic seems an unabordable problem. Therefore, it is interesting to study the hyperbolicity of particular classes of graphs. The papers [4,9,11,26,28,29,30,32,34] study the hyperbolicity of, respectively, complement of graphs, chordal graphs, strong product graphs, corona and join of graphs, line graphs, Cartesian product graphs, cubic graphs, tessellation graphs, short graphs, median graphs and k-chordal graphs. In this work we study the hyperbolicity constant of the graphs with n vertices and m edges.

Let $\mathcal{G}(n, m)$ be the set of graphs G with n vertices and m edges, and such that every edge has length 1. Let us define

$$A(n, m) := \min\{\delta(G) \mid G \in \mathcal{G}(n, m)\},$$

$$B(n, m) := \max\{\delta(G) \mid G \in \mathcal{G}(n, m)\}.$$

Our aim in this work is to estimate $A(n, m)$ and $B(n, m)$. In particular, we compute the precise value of $A(n, m)$ in many cases and we obtain good bounds for $B(n, m)$. In addition, we obtain an upper bound of the size of any graph in terms of its diameter and its order.

We state now the main results and the lemmas used in their proofs. See [19] for the detailed proofs.

2 Bounds for the hyperbolicity constant of graphs with n vertices and m edges

If $m = n-1$, then every $G \in \mathcal{G}(n, m)$ is a tree and $\delta(G) = 0$. If $n \leq m \leq n+3$, then there exists $G \in \mathcal{G}(n, m)$ with $\delta(G) = \delta(C_n) = n/4$ (see [31, Theorem 11]). Since $\delta \leq n/4$ (see [31, Theorem 11]), we have:

$$B(n, m) = \begin{cases} 0 & \text{if} \quad n = m - 1, \\ \frac{n}{4} & \text{if } \ n \leq m \leq n + 3. \end{cases}$$

Then we can consider just the case $m \geq n + 4$.

2.1 Upper bound for the hyperbolicity constant

In order to prove our main result, we need some technical lemmas.

Lemma 1. *Consider the following optimization problem:*

$$\Delta_r := \min_{x\in W} f_r, \quad \textit{with} \quad f_r(k_1,k_2,\ldots,k_r) := \Sigma_{t=2}^{r} k_t \Sigma_{s=0}^{t-2} k_s, \quad 2 \le r \le \frac{n}{2},$$

$$\textit{and } W := \{k_0 = 1, \quad k_j \ge 2, \quad \textit{if} \quad 1 \le j \le r-1, \quad k_r \ge 1,$$
$$1 + k_1 + k_2 + \cdots + k_r = n\}.$$

Then $\Delta_2 = 1$, $\Delta_3 = n-1$ *and* $\Delta_r = 2n(r-3) - 2r^2 + 6r + 5$ *for* $r \ge 4$.

We say that a vertex v of a graph G is a *cut-vertex* if $G \setminus \{v\}$ is not connected. A graph is *two-connected* if it is connected and it does not contain cut-vertices.

Given a graph G, we say that a family of subgraphs $\{G_s\}$ of G is a *T-decomposition* of G if $\cup_s G_s = G$ and $G_s \cap G_r$ is either a *cut-vertex* or the empty set for each $s \neq r$.

Given any edge in G, let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal *two-connected* subgraphs $\{G_s\}_s$ the *canonical T-decomposition* of G, and we define

$$\operatorname{diameff} V(G) := \sup_s \operatorname{diam} V(G_s), \quad \operatorname{diameff} G := \sup_s \operatorname{diam} G_s.$$

The following result allows to obtain global information about the hyperbolicity of a graph from local information (see [5, Theorem 3] and [31, Theorem 8]).

Lemma 2. *Let G be any graph. Then* $\delta(G) \le \frac{1}{2} \operatorname{diameff}(G)$.

We define $M(n,r) := \binom{n}{2} - \Delta_r$, for $2 \le r \le n/2$. We have the following Lemmas.

Lemma 3. *If* $G \in \mathcal{G}(n,m)$ *and* $\operatorname{diameff} V(G) = \operatorname{diam} V(G) = r$, *then* $m \le M(n,r)$.

Lemma 4. *The inequality*

$$\binom{n-n_0+1}{2} \le M(n,r) - M(n_0,r)$$

holds for $2 \le r \le n_0/2$ *and* $n > n_0$.

Lemma 5. *If* $G \in \mathcal{G}(n,m)$ *and* $\operatorname{diameff} V(G) = r$, *then* $m \le M(n,r)$.

We obtain the following Corollary as a consequence of Lemma 5.

Corollary 1. *If* $G \in \mathcal{G}(n,m)$, $2 \leq r \leq r/2$ *and* $m > M(n,r)$, *then* $\operatorname{diameff} V(G) \neq r$.

In fact, this result can be improved.

Theorem 1. *If* $G \in \mathcal{G}(n,m)$, $2 \leq r \leq n/2$ *and* $m > M(n,r)$, *then* $\operatorname{diameff} V(G) < r$.

Since $\operatorname{diameff} V(G) < r$ implies $\operatorname{diameff} G \leq r$, Lemma 2 and Theorem 1 imply the following Theorem.

Theorem 2. *If* $G \in \mathcal{G}(n,m)$, $2 \leq r \leq n/2$ *and* $M(n,r) < m \leq M(n,r-1)$, *then* $\delta(G) \leq r/2$.

Remark 1. Theorem 2 implies $B(n,m) \leq r/2$ for $M(n,r) < m \leq M(n,r-1)$.

2.2 An inequality involving the diameter of a graph

We consider a similar optimization problem to the one in Lemma 1.

Lemma 6. *Consider the following optimization problem:*

$$\Lambda'_r = \min_{x \in W_0} f_r, \quad \textit{with} \quad f_r(k_1, k_2, \ldots, k_r) := \Sigma_{t=2}^{r} k_t \Sigma_{s=0}^{t-2} k_s, \quad 2 \leq r \leq n-1,$$

$$\textit{and} \quad W_0 := \{k_0 = 1, \quad k_j \geq 1 \quad \textit{if} \quad 1 \leq j \leq r,$$
$$1 + k_1 + k_2 + \cdots + k_r = n\}.$$

Then $\Lambda'_2 = 1$ *and* $\Lambda'_r = nr - \frac{1}{2}r^2 + \frac{1}{2}r + 2$ *for* $r \geq 3$.

To find inequalities relating the diameter, the order and another parameter of any graph is an important problem in graph theory (see, e.g., [13] and [27]). If the third parameter is a bound of the degree, we have the well-known Moore bounds (see, e.g., [13] and [27]). We present here an upper bound of the size of the graph in terms of its diameter and its order, that is a consequence of Lemma 6.

Theorem 3. *Let* G *be any graph with* n *vertices and* m *edges.*

- *If* $\operatorname{diam} V(G) = 2$, *then*
$$m \leq \binom{n}{2} - 1.$$
- *If* $3 \leq \operatorname{diam} V(G) \leq n-1$, *then*
$$m \leq \binom{n}{2} - 2 + \frac{1}{2}\big(\operatorname{diam} V(G)\big)^2 - \Big(n + \frac{1}{2}\Big)\operatorname{diam} V(G).$$

2.3 A lower bound for $B(n, m)$

Theorem 4. *If* $5 \leq n_0 \leq n$ *and*

$$n + \binom{n_0 - 2}{2} < m \leq n + \binom{n_0 - 1}{2},$$

then

$$B(n, m) \geq \frac{n - n_0 + 3}{4}.$$

2.4 Estimation of $A(n, m)$

Denote by Γ_3 the set of graphs such that every cycle has length 3 and every edge belongs to some cycle.

Proposition 1. *Consider a graph* $G \in \mathcal{G}(n, m) \cap \Gamma_3$. *If* k *denotes the number of cycles of* G, *then* $n = 2k + 1$ *and* $m = 3k$.

Proposition 2. *Let* $G \in \mathcal{G}(n, m)$ *be a graph such that every cycle has length* 3. *Then* $2m \leq 3n - 3$.

The previous results have the following consequence.

Theorem 5. *If* $m = n - 1$, *then* $A(n, m) = 0$.
If $m \geq n$ *and* $2m \leq 3n - 3$, *then* $A(n, m) = 3/4$.
If $m \geq n$ *and* $2m > 3n - 3$, *then* $1 \leq A(n, m) \leq 3/2$.

3 Conclusions

The main result in this work provides an upper bound for $B(n, m)$ (see Theorem 2). Also, we find a lower bound for $B(n, m)$ (see Theorem 4). In addition, we obtain an upper bound of the size of any graph in terms of its diameter and its order (see Theorem 3). Another important result of this work is Theorem 5, which estimates the value of $A(n, m)$.

References

[1] Alonso, J., Brady, T., Cooper, D., Delzant, T., Ferlini, V., Lustig, M., Mihalik, M., Shapiro, M. and Short, H. *Notes on word hyperbolic groups*, in: E. Ghys, A. Haefliger, A. Verjovsky (Eds.), Group Theory from a Geometrical Viewpoint, World Scientific, Singapore, 1992.

[2] Balogh, Z. M. and Bonk, M. *Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains*, Comment. Math. Helv. **75** (2000), 504-533.

[3] Balogh, Z. M. and Buckley, S. M. *Geometric characterizations of Gromov hyperbolicity*, Invent. Math. **153** (2003), 261-301.

[4] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Tourís, E. *Hyperbolicity and complement of graphs*, Appl. Math. Letters **24** (2011), 1882-1887.

[5] Bermudo, S., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M. *Gromov hyperbolic graphs*, Discr. Math. 313, (2013), 1575-1585.

[6] Bonk, M., Heinonen, J. and Koskela, P. *Uniformizing Gromov hyperbolic spaces*, Astérisque **270** (2001).

[7] Bonk, M. and Schramm, O. *Embeddings of Gromov hyperbolic spaces*, Geom. Funct. Anal. **10** (2000), 266-306.

[8] Bridson, M. R. and Haefliger, A. "Metric spaces of non-positive curvature", Springer-Verlag, Berlin, 1999.

[9] Brinkmann, G., Koolen J. and Moulton, V. *On the hyperbolicity of chordal graphs*, Ann. Comb. **5** (2001), 61-69.

[10] Carballosa, W., Pestana, D., Rodríguez, J. M. and Sigarreta, J. M. *Distortion of the hyperbolicity constant of a graph*, Electr. J. Comb. **19** (2012), P67.

[11] Carballosa, W., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M. *On the Hyperbolicity Constant of Line Graphs*, Electr. J. Comb. **18** (2011), P210.

[12] Chepoi, V., Dragan, F. F., Estellon, B., Habib, M. and Vaxes Y. *Notes on diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs*, Electr. Notes Discrete Math. **31** (2008), 231-234.

[13] Chung Fan, R.K., *Diameters of graphs: old problems and new results*, Congress Numerantium, 60:295-317, 1987.

[14] Coornaert, M., Delzant, T. and Papadopoulos, A. *Géometrie et théorie des groupes*, Lecture Notes in Mathematics 1441, Springer-Verlag, Berlin, 1990.

[15] Frigerio, R. and Sisto, A. *Characterizing hyperbolic spaces and real trees*, Geom. Dedicata **142** (2009), 139-149.

[16] Ghys, E. and de la Harpe, P. *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Progress in Mathematics 83, Birkhäuser Boston Inc., MA, 1990.

[17] Gromov, M. *Asymptotic invariants of infinite groups*, Geometric Group Theory, London Math. Soc. Lecture Notes Series 182, 1993.

[18] Gromov, M. " Hyperbolic groups, in Essays in group theory". Edited by S. M. Gersten, in Math. Sci. Res. Inst. Publ. **8**. Springer, 1987, 75-265.

[19] Hernández, V., Pestana, D. and Rodríguez, J. M. *Bounds of diameter and Gromov hyperbolicity constant of graphs*, Submitted.

[20] Jonckheere, E. A. *Controle du trafic sur les reseaux a geometrie hyperbolique– Une approche mathematique a la securite de l'acheminement de l'information*, J. Europ. de Syst. Autom. **37** (2003), 145-159.

[21] Jonckheere, E. A. and Lohsoonthorn, P. *Geometry of network security*, American Control Conference **ACC** (2004), 111-151.

[22] Jonckheere, E. A., Lohsoonthorn, P. and Bonahon, F. *Scaled Gromov hyperbolic graphs*, J. Graph Theory **2** (2007), 157-180.

[23] Koolen, J. H. and Moulton, V. *Hyperbolic Bridged Graphs*, Europ. J. Comb. **23** (2002), 683-699.

[24] Lang, U. *Extendability of large-scale Lipschitz maps*, Trans. Amer. Math. Soc. **351** (1999), 3975-3988.

[25] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M. *Hyperbolicity and parameters of graphs*, Ars Comb. **100** (2011), 43-63.

[26] Michel, J., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M. *Gromov hyperbolicity in Cartesian product graphs*, Proc. Indian Acad. Sci. Math. Sci. **120** (2010), 1-17.

[27] Miller, M., Sirán, J. *Moore graphs and beyond: A survey of the degree/diameter problem*, Electr. J. Comb. **20(2)** (2013), DS14v2.

[28] Pestana, D., Rodríguez, J. M., Sigarreta, J. M. and Villeta, M. *Gromov hyperbolic cubic graphs*, Central Europ. J. Math. **10(3)** (2012), 1141-1151.

[29] Portilla, A., Rodríguez, J. M., Sigarreta, J. M. and Vilaire, J.-M. *Gromov hyperbolic tessellation graphs*, to appear in Utilitas Math. Preprint in http://gama.uc3m.es/index.php/jomaro.html

[30] Rodríguez, J. M. *Characterization of Gromov hyperbolic short graphs*, Ars Comb. **100** (2011), 43-63.

[31] Rodríguez, J. M., Sigarreta, J. M., Vilaire, J.-M. and Villeta, M. *On the hyperbolicity constant in graphs*, Discrete Math. **311** (2011), 211-219.

[32] Sigarreta, J. M. *Hyperbolicity in median graphs*, Proc. Math. Sci. **123** (2013), 455-467.

[33] Väisälä, J. *Gromov hyperbolic spaces*, Expo. Math. **23** (2005), 187-231.

[34] Wu, Y. and Zhang, C. *Chordality and hyperbolicity of a graph*, Electr. J. Comb. **18** (2011), P43.

LD-graphs and global location-domination in bipartite graphs *

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Abstract. A dominating set S of a graph G is a *locating-dominating-set*, *LD-set* for short, if every vertex v not in S is uniquely determined by the set of neighbors of v belonging to S. Locating-dominating sets of minimum cardinality are called *LD-codes* and the cardinality of an LD-code is the *location-domination number*, $\lambda(G)$. An LD-set S of a graph G is *global* if it is an LD-set for both G and its complement, $\overline{G}$. One of the main contributions of this work is the definition of the *LD-graph*, an edge-labeled graph associated to an LD-set, that will be very helpful to deduce some properties of location-domination in graphs. Concretely, we use LD-graphs to study the relation between the location-domination number in a bipartite graph and its complement.

Key words: domination, location, complement graph, bipartite graph.

1 Introduction

In this work, $G = (V, E)$ stands for a simple, finite graph. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is $N_G[v] = \{u \in V : uv \in E\} \cup \{v\}$. We write $N(v)$ or $N[v]$ if the graph G is clear from the context. The *complement* of a graph G, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A set $D \subseteq V$ is a *dominating set* if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set is *global* if it is a dominating set for both G and its complement graph, $\overline{G}$. If D is a subset of V and $v \in V \setminus D$, we say that v *dominates* D if $D \subseteq N(v)$.

A dominating set $S \subseteq V$ is a *locating-dominating set*, *LD-set* for short, if for every two different vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The

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location-domination number of G, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is an *LD-code* [11]. LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1,2,5,6,7] A complete and regularly updated list of papers on locating dominating codes can be found in [9].

In the following section we introduce the LD-graph associated to an LD-set. After that, we study the relation between LD-sets and the location-domination number in a graph and its complement. Finally, we consider this parameter for connected bipartite graphs. We omit proofs due to space limitations.

2 The LD-graph associated to an LD-set

We introduce in this section the so-called *LD-graph*, an edge-labeled graph associated to an LD-set. This graph will allow us to deduce some properties of LD-sets and the location-domination number of graphs.

Let S be an LD-set of a graph G of order n. Consider $z \notin V(G)$ and define $N_G(z) = \emptyset$. Let $\triangle$ denote the symmetric difference set operation. The *LD-graph* associated to S, denoted by G^S, is the edge-labeled graph defined as follows:

i) $V(G^S) = (V \setminus S) \cup \{z\}$;
ii) $E(G^S) = \{xy \,|\, x, y \in V(G^S)\,, |\big(N_G(x) \cap S\big) \triangle \big(N_G(y) \cap S\big)| = 1\}$;
iii) The label of edge $xy \in E(G^S)$ is $\ell(xy) = \big(N_G(x) \cap S\big) \triangle \big(N_G(y) \cap S\big) \in S$.

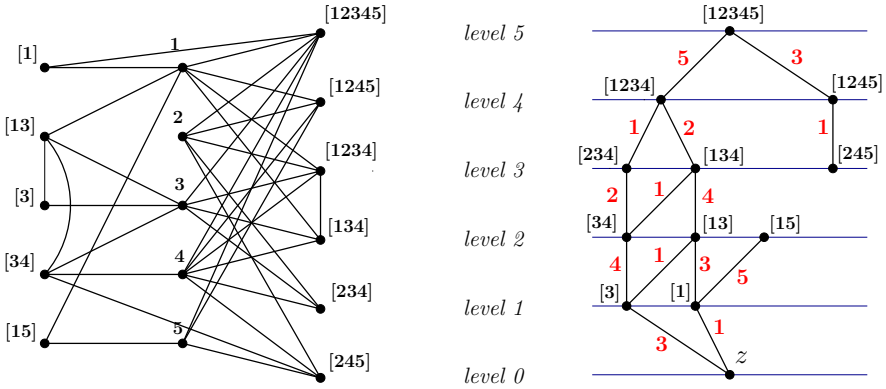

Fig. 1: Left: a graph G. Right: the LD-graph G^S associated to the LD-set $S = \{1, 2, 3, 4, 5\}$.

Notice that two vertices of $V \setminus S$ are adjacent in G^S if their neighborhood in S differ in exactly one vertex, the label of the edge, and z is adjacent to

vertices of $V \setminus S$ with exactly a neighbor in S. Therefore, we can represent the graph G^S with the vertices lying on $|S|+1$ levels, from bottom (level 0) to top (level $|S|$), in such a way that vertices with exactly k neighbors in S are at level k. There is at most one vertex at level $|S|$ and, if it is so, this vertex is adjacent to all vertices of S. The vertices at level 1 are those with exactly one neighbor in S and z is the unique vertex at level 0. An edge of G^S has its endpoints at consecutive levels. Moreover, if $e = xy \in E(G^S)$, with $\ell(e) = u \in S$, and x is at exactly one level higher than y, then $N(x) \cap S = (N(y) \cap S) \cup \{u\}$, i.e., x and y have the same neighborhood in $S \setminus \{u\}$. Therefore, the existence of an edge in G^S with label $u \in S$ means that $S \setminus \{u\}$ is not an LD-set. Hence, if S is an LD-code, then for every $u \in S$ there exists at least an edge in G^S with label u. See an example of an LD-graph in Figure 1.

The following proposition states some properties of the LD-graph.

Proposition 1. *Let S be an LD-set with exactly r vertices of a connected graph $G = (V, E)$ of order n. Let G^S be the LD-graph associated to S. Then:*

i) $|V(G^S)| = n - r + 1$.
ii) G^S *is bipartite.*
iii) Incident edges of G^S *have different labels.*
iv) Every cycle of G^S *contains an even number of edges labeled* v*, for all* $v \in S$.
v) Let ρ *be a walk with no repeated edges in* G^S*. If, for every* $v \in S$*,* ρ *contains an even number of edges labeled* v*, then* ρ *is a closed walk.*
vi) If $\rho = x_i x_{i+1} \dots x_{i+h}$ *is a path satisfying that vertex* x_j *lies at level* j*, for any* $j \in \{i, i+1, \dots, i+h\}$*, then*
  (a) the labels of the edges of ρ *are different;*
  (b) for all $j \in \{i+1, i+2, \dots, i+h\}$*,* $N(x_j) \cap S$ *contains the vertex* $\ell(x_k x_{k+1})$*, for any* $k \in \{i, i+1, \dots, j-1\}$.

3 Global location domination

This section is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph G.

Notice that $N_{\overline{G}}(x) \cap S = S \setminus N_G(x)$ for any set $S \subseteq V$ and any vertex $x \in V \setminus S$. A straightforward consequence of this fact are the following results.

Proposition 2 ([8]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$.*

Proposition 3 ([7]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ dominating S in G.*

Proposition 4 ([7]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$ then there is at most one vertex $u \in V \setminus S$ dominating S, and in the case it exists, $S \cup \{u\}$ is an LD-set of $\overline{G}$.*

Theorem 1 ([7]). *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

According to the preceding result, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$ for every graph G, all cases being feasible for some connected graph G. For example, it is straightforward to check that the complete graph K_n of order $n \geq 2$ satisfies $\lambda(\overline{K_n}) = \lambda(K_n) + 1$; the star $K_{1,n-1}$ of order $n \geq 2$ satisfies $\lambda(\overline{K_{1,n-1}}) = \lambda(K_{1,n-1})$, and the bi-star $K_2(r, s)$, $r, s \geq 2$, obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,s}$, satisfies $\lambda(K_2(r, s)) = \lambda(\overline{K_2(r, s)}) + 1$.

We intend to obtain either necessary or sufficient conditions for a graph G to satisfy $\lambda(\overline{G}) > \lambda(G)$, i.e., $\lambda(\overline{G}) = \lambda(G) + 1$. After noticing that this fact is closely related to the existence or not of sets that are simultaneously locating-dominating sets in both G and its complement $\overline{G}$, the following definition was introduced in [8].

A set S of vertices of a graph G is a *global LD-set* if S is an LD-set for both G and its complement $\overline{G}$ and it is a *global LD-code* if it is an LD-code of G and an LD-set of $\overline{G}$. Next results follow immediately from the definition of global LD-set and global LD-code.

Proposition 5 ([8]). *If G is a graph with a global LD-code, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 6 ([8]). *Let S be an LD-set of a graph G. Then, S is a non-global LD-set if and only if there exists a (unique) vertex $u \in V \setminus S$ such that $S \subseteq N(u)$.*

In Table 1, the location-domination number of some families of graphs is displayed, along with the location-domination number of their complement graphs. Concretely, we consider the path P_n of order $n \geq 7$; the cycle C_n of order $n \geq 7$; the wheel W_n of order $n \geq 8$, obtained by joining a new vertex to all vertices of a cycle of order $n-1$; the complete graph K_n of order $n \geq 2$; the complete bipartite graph $K_{r,n-r}$ of order $n \geq 4$, with $2 \leq r \leq n - r$ and stable sets of order r and $n - r$, respectively; the star $K_{1,n-1}$ of order $n \geq 4$, obtained by joining a new vertex to $n - 1$ isolated vertices; and finally, the bi-star $K_2(r, n - r)$ of order $n \geq 6$ with $2 \leq r \leq n - r$, obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,n-r}$ respectively.

4 The bipartite case

In this section we study the relation between $\lambda(G)$ and $\lambda(\overline{G})$ in bipartite connected graphs. Bipartite connected graphs of order at most 3 are the path

G	P_n	C_n	W_n	K_n	$K_{1,n-1}$	$K_{r,n-r}$	$K_2(r,n-r)$
n	$n \geq 7$	$n \geq 7$	$n \geq 8$	$n \geq 2$	$n \geq 4$	$2 \leq r \leq n-r$	$2 \leq r \leq n-r$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$n-1$	$n-1$	$n-2$	$n-2$
$\lambda(\overline{G})$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-3$

Table 1: The values of $\lambda(G)$ and $\lambda(\overline{G})$ for some families of graphs.

graphs P_1, P_2 and P_3, and for these graphs, $\lambda(P_1) = \lambda(\overline{P_1}) = 1$; $1 = \lambda(P_2) < \lambda(\overline{P_2}) = 2$; $\lambda(P_3) = \lambda(\overline{P_3}) = 2$. In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n = r + s \geq 4$, such that $V = U \cup W$, being U,W its stable sets and $1 \leq |U| = r \leq s = |W|$.

Proposition 7. *Let S be an LD-code of G. Then, $\lambda(\overline{G}) \leq \lambda(G)$ if any of the following conditions holds*

i) $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$;
ii) $r < s$ and $S = W$.
iii) $2^r \leq s$.

Corollary 1. *If $\lambda(\overline{G}) = \lambda(G) + 1$, then $r \leq s \leq 2^r - 1$. Moreover, if $r < s$ then U is the unique LD-code of G, and if $r = s$ we may assume that U is a non-global LD-code of G.*

Proposition 8. *If $1 \leq r \leq 2$, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Notice that bipartite connected graphs G of order at least 4 such that $\lambda(G) \leq 2$ are P_4, P_5 and C_4. These graphs satisfy $\lambda(\overline{G}) \leq \lambda(G)$.

Next, we approach the case $\lambda(\overline{G}) = \lambda(G) + 1$, when $\lambda(G) \geq 3$, using LD-graphs. We may assume that $r \geq 3$ by Proposition 8. First we give some properties of LD-graphs for bipartite graphs satisfying the preceding equality.

Lemma 1. *If $\lambda(\overline{G}) = \lambda(G) + 1$ and U is an LD-code of G, then G^U contains at least two edges with label u, for all $u \in U$.*

In the study of LD-sets of a connected bipartite graph, a family of graphs is particularly useful, the *cactus graphs*. A *block* of a graph is a maximal connected subgraph with no cut vertices. A connected graph G is a *cactus* if all its blocks are cycles or edges. Cactus are also characterized as those connected graphs with no edge shared by two different cycles. The following lemma gives some properties relating parameters of bipartite graphs having cactus as connected components.

Lemma 2. *Let H be a bipartite graph of order at least 4 such that all its connected components are cactus having $cc(H)$ connected components and $cy(H)$ cycles. Let $ex(H) = |E(H)| - 4\,cy(H)$. Then H satisfies:*

i) $|V(H)| = |E(H)| - cy(H) + cc(H)$.
ii) $ex(H) \geq 0$ *and* $|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}ex(H) + cc(H)$.
iii) $|V(H)| \geq \frac{3}{4}|E(H)| + 1$.
iv) $|V(H)| = \frac{3}{4}|E(H)| + 1$ *if and only if* H *is connected and all blocks are cycles of order 4.*

Lemma 3. *Let* $\lambda(\overline{G}) = \lambda(G) + 1$ *and assume that* U *is an LD-code of* G. *Consider a subgraph* H *of* G^U *induced by a set of edges containing exactly two edges with label* u, *for each* $u \in U$. *Then, all the connected components of* H *are cactus.*

Proposition 9. *If* $r \geq 3$ *and* $\lambda(\overline{G}) = \lambda(G) + 1$, *then* $\frac{3r}{2} \leq s \leq 2^r - 1$.

Lemma 4. *If* $\lambda(\overline{G}) = \lambda(G) + 1$ *and* U *is an LD-code of* G, *let* z *be the vertex of* G^U *introduced in the definition of this graph and let* H *be a subgraph of* G^U *with exactly two edges with label* u, *for each* $u \in U$. *Then:*

i) If H *has at least two connected components, then* $s \geq \frac{3r}{2} + 1$.
ii) If z *is an isolated vertex in* G^U, *then* $s \geq \frac{3r}{2} + 1$.
iii) z *is an non-isolated vertex in* G^U *if and only if there is at least a vertex in* $V \setminus U$ *of degree 1 in* G.
iv) If G *has no vertex of degree 1 in* W, *then* $s \geq \frac{3r}{2} + 1$.

Proposition 10. *There are no bipartite graphs* G *satisfying* $\lambda(\overline{G}) = \lambda(G) + 1$ *if* $\frac{3r}{2} \leq s < \frac{3r}{2} + 1$.

Proposition 11. *For every pair* (r, s), $r, s \in \mathbb{N}$, *such that* $3 \leq r$ *and* $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, *there exists a bipartite graph* $G(r, s)$ *such that* $\lambda(\overline{G}) = \lambda(G) + 1$.

Graphs satisfying the conditions of Proposition 11 can be constructed from the LD-graph described in Figure 2 associated to the LD-code $U = \{1, 2, \ldots, r\}$ when $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$.

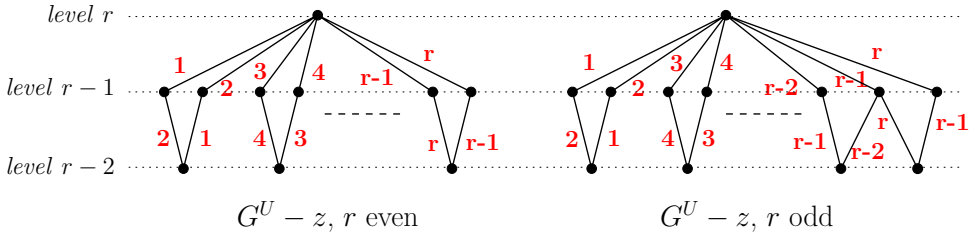

Fig. 2: The labeled graph $G^U - z$, for $G = G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ and $U = \{1, \ldots, r\}$.

For $s > \lceil \frac{3r}{2} + 1 \rceil$, we can add up to $2^r - 1 - r$ vertices to the set W of the graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ taking into account that the neighborhoods in U of the vertices of W must be different and non-empty.

Proposition 12. *Let G be a bipartite connected graph with $V(G) = U \cup W$, $|U| = r$, $|W| = s$, $3 \leq r \leq s$.*

i) If $r \leq s < \frac{3r}{2} + 1 \Rightarrow \lambda(G) - \lambda(\overline{G}) \in \{0, 1\}$ and there are examples of both cases.

ii) If $\frac{3r}{2} + 1 \leq s \leq 2^r - 1 \Rightarrow \lambda(G) - \lambda(\overline{G}) \in \{-1, 0, 1\}$ there are graphs satisfying all cases.

iii) If $2^r - 1 \leq s \Rightarrow \lambda(G) - \lambda(\overline{G}) \in \{0, 1\}$ and there are examples of both cases.

Examples of graphs satisfying the different cases of Proposition 12 are the following. For every r, s, $3 \leq r \leq s$, complete bipartite graphs $K_{r,s}$, satisfy

$$\lambda(K_{r,s}) = \lambda(\overline{K_{r,s}}) = r + s - 2$$

and bistars $K_2(r-1, s-1)$ satisfy

$$r + s - 2 = \lambda(K_2(r, s)) > \lambda(\overline{K_2(r, s)}) = r + s - 3.$$

By Proposition 9, we know that the equality $\lambda(G) - \lambda(\overline{G}) = -1$ is possible only in the case that $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$ and by Proposition 11 we know examples for any s satisfying $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.

References

[1] N. Bertrand, I. Charon, O. Hudry, A. Lobstein. Identifying and locating-dominating codes on chains and cycles. *Eur. J. Combin.*, 25:969–987, 2004.

[2] M. Blidia, M. Chellali, F. Maffray, J. Moncel, A. Semri. Locating-domination and identifying codes in trees. *Australas. J. Combin.*, 39:219–232, 2007.

[3] R. C. Brigham, J. R. Carrington. Global domination, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.). *Domination in Graphs, Advanced Topics.* Marcel Dekker, New York, 1998, pp. 30–320.

[4] R. C. Brigham, R. D. Dutton. Factor domination in graphs. *Discrete Math.*,86(1-3):127–136, 1990.

[5] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas. Locating-dominating codes: Bounds and extremal cardinalities. *Appl. Math. Comput.*, 220:38–45, 2013.

[6] C. Chen, R. C. Lu, Z. Miao. Identifying codes and locating-dominating sets on paths and cycles. *Discrete Appl. Math.*, 159(15):1540–1547, 2011.

[7] C. Hernando, M. Mora, I. M. Pelayo. Nordhaus-Gaddum bounds for locating domination. *Eur. J. Combin.*, 36:1–6, 2014.

[8] C. Hernando, M. Mora, I. M. Pelayo, On global location-domination in graphs, submitted.

[9] A. Lobstein. Watching systems, identifying, locating-dominating and discriminating codes in graphs.
`http://www.infres.enst.fr/~lobstein/debutBIBidetlocdom.pdf`

[10] E. Sampathkumar. The global domination number of a graph. *J. Math. Phys. Sci.* 23:377–385, 1989.

[11] P. J. Slater. Dominating and reference sets in a graph. *J. Math. Phys. Sci.* 22:445–455, 1988.

On the strong metric dimension of product graphs

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Abstract. Let G be a connected graph. A vertex $w \in V(G)$ strongly resolves two vertices $u, v \in V(G)$ if there exists some shortest $u - w$ path containing v or some shortest $v - w$ path containing u. A set S of vertices is a strong metric generator for G if every pair of vertices of G is strongly resolved by some vertex of S. The smallest cardinality of a strong metric generator for G is called the strong metric dimension of G. In this paper we study the problem of finding sharp bounds for the strong metric dimension of Cartesian product graphs, rooted product graphs and strong product graphs, and express these in terms of invariants of the factor graphs.

Key words: strong metric dimension, Cartesian product graphs, rooted product graphs, strong product graphs.

1 Introduction

The problem of uniquely recognizing the position of an intruder in a network was the principal motivation of introducing the concept of metric generators in graphs by Slater in [8]. An analogous concept was also introduced independently by Harary and Melter in [4]. A vertex v of a connected graph G is said to distinguish two vertices x and y of G if $d_G(v, x) \neq d_G(v, y)$, *i.e.*, the distance between v and x is different from the distance between v and y. A set $S \subseteq V(G)$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. A minimum generator is called a *metric basis*, and its cardinality the *metric dimension* of G.

Another invariant, more restricted than the metric dimension, was presented by SebHo and Tannier in [7], and studied further in several articles. That is, a vertex $w \in V(G)$ *strongly resolves* two vertices $u, v \in V(G)$ if $d_G(w, u) = d_G(w, v) + d_G(v, u)$ or $d_G(w, v) = d_G(w, u) + d_G(u, v)$, *i.e.*, there exists some shortest $w - u$ path containing v or some shortest $w - v$ path containing u. A set S of vertices in a connected graph G is a *strong metric generator* for G if every two vertices of G are strongly resolved by some vertex of S. The smallest cardinality of a strong metric generator of G is called the

strong metric dimension and is denoted by $dim_s(G)$. A *strong metric basis* of G is a strong metric generator for G of cardinality $dim_s(G)$.

On the other hand, studies about graph operations are being frequently presented and published in the last years. The book [3] is a very rich compendium of the existent theory of product graphs. It is therefore natural that several graphs invariants are studied for such classes of graph products, in the sense of computing or bounding its exact value, or deducing its behavior with respect to its factors. The strong metric dimension of product graphs has not, of course, escaped to these kind of investigations.

We recall that the *Cartesian product of two graphs* G and H is the graph $G\Box H$, such that $V(G\Box H) = V(G) \times V(H)$ and two vertices $(a,b),(c,d)$ are adjacent in $G\Box H$ if and only if, either ($a = c$ and $bd \in E(H)$) or ($b = d$ and $ac \in E(G)$).

The *direct product of two graphs* G and H is the graph $G \times H$, such that $V(G \times H) = V(G) \times V(H)$ and two vertices (a,b) and (c,d) are adjacent in $G \times H$ if and only if $ac \in E(G)$ and $bd \in E(H)$.

A *rooted graph* is a graph in which one vertex is labeled in a special way so as to distinguish it from other vertices. The special vertex is called the *root* of the graph. Let G be a labeled graph on n vertices. Let $\mathcal{H}$ be a sequence of n rooted graphs H_1, H_2,...,H_n. The *rooted product graph* $G(\mathcal{H})$ is the graph obtained by identifying the root of H_i with the i^{th} vertex of G [2]. In this paper we consider the particular case of rooted product graph where $\mathcal{H}$ consists of n isomorphic rooted graphs. More formally, assuming that $V(G) = \{u_1,...,u_n\}$ and that the root vertex of H is v, we define the rooted product graph $G \circ_v H$, where $V(G \circ_v H) = V(G) \times V(H)$ and

$$E(G \circ_v H) = \bigcup_{i=1}^{n} \{(u_i,b)(u_i,y) :\ by \in E(H)\} \cup \{(u_i,v)(u_j,v) :\ u_iu_j \in E(G)\}.$$

The *strong product of two graphs* G and H is the graph $G \boxtimes H$, such that $V(G \boxtimes H) = V(G) \times V(H)$ and two vertices (a,b) and (c,d) are adjacent in $G \boxtimes H$ if and only if either ($a = c$ and $bd \in E_2$), or ($ac \in E_1$ and $b = d$), or ($ac \in E_1$ and $bd \in E_2$).

Also, the *Cartesian sum of two graphs* G and H, denoted by $G \oplus H$, has as the vertex set $V(G \oplus H) = V(G) \times V(H)$ and two vertices $(a,b),(c,d)$ are adjacent in $G \oplus H$ if and only if $ac \in E(G)$ or $bd \in E(H)$.

It was shown in [6] that the problem of computing $dim_s(G)$ is NP-hard. This suggests obtaining closed formulaes for the strong metric dimension of specific families of graphs or bounding the value of this invariant as tight as possible. In this article we are interested in the study of strong metric generators of Cartesian product graphs, rooted product graphs and strong product graphs.

2 Basic tools

Oellermann and Peters-Fransen [6] showed that the problem of finding the strong metric dimension of a graph G can be transformed into a more well-known problem, the vertex cover problem. We recall that a set S of vertices of G is a *vertex cover* of G if every edge of G is incident with at least one vertex of S. The *vertex cover number* of G, denoted by $\beta(G)$, is the smallest cardinality of a vertex cover of G.

A vertex u of G is *maximally distant* from v if for every vertex w in the open neighborhood of u, $d_G(v,w) \leq d_G(u,v)$. If u is maximally distant from v and v is maximally distant from u, then we say that u and v are *mutually maximally distant*. The *boundary* of G is defined as $\partial(G) = \{u \in V(G) : \text{there exists } v \in V(G) \text{ such that } u, v \text{ are mutually maximally distant}\}$. For some basic graph classes, such as complete graphs K_n, complete bipartite graphs $K_{r,s}$, cycles C_n and hypercube graphs Q_k, the boundary is simply the whole vertex set. Notice that the boundary of a tree consists exactly of the set of its leaves. A vertex of a graph is a *simplicial vertex* if the subgraph induced by its neighbors is a complete graph. Given a graph G, we denote by $\sigma(G)$ the set of simplicial vertices of G. Notice that $\sigma(G) \subseteq \partial(G)$.

The *strong resolving graph* of G is a graph G_{SR} which has the same vertex set as G and two vertices u, v are adjacent in G_{SR} if and only if u and v are mutually maximally distant in G.

Oellermann and Peters-Fransen [6] showed that finding the strong metric dimension of a connected graph G is equivalent to finding the vertex cover number of G_{SR}. The following result is an important tool for study the strong metric dimension.

Theorem 1. [6] *For any connected graph G, $dim_s(G) = \beta(G_{SR})$.*

Recall that the largest cardinality of a set of vertices of G, no two of which are adjacent, is called the *independence number* of G and is denoted by $\alpha(G)$. We refer to an $\alpha(G)$-set in a graph G as an independent set of cardinality $\alpha(G)$. The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph.

Theorem 2. (Gallai's theorem) *For any graph G of order n, $\alpha(G)+\beta(G) = n$.*

Thus, for any graph G of order n, by using Theorems 1 and 2, we immediately obtain that

$$dim_s(G) = n - \alpha(G_{SR}) \tag{1}$$

There are some families of graphs for which the strong resolving graphs can be obtained relatively easily. We state some of these here, since we need to refer to these in other sections of the paper.

Remark 1.

(a) $(K_n)_{SR} \cong K_n$.
(b) For any tree T of order n with $l(T)$ leaves, $T_{SR} \cong K_{l(T)} \cup \bigcup_{i=1}^{n-l(T)} K_1$.
(c) For any 2-antipodal[3] graph G of order n, G_{SR} is isomorphic to the graph $\bigcup_{i=1}^{\frac{n}{2}} K_2$. In particular, $(C_{2k})_{SR} \cong \bigcup_{i=1}^{k} K_2$.
(d) $(C_{2k+1})_{SR} \cong C_{2k+1}$.
(e) For any complete k-partite graph such that at least all but one $p_i \geq 2$, $i \in \{1, 2, ..., k\}$, $(K_{p_1,p_2,...,p_k})_{SR}$ is isomorphic to the graph $\bigcup_{i=1}^{k} K_{p_i}$.

3 Cartesian product of graphs

We begin this section with an easily verified bounds relating the strong metric dimension of a graph with the number of its simplicial vertices and the number of its mutually maximally distant vertices.

Lemma 1. *For every graph G, $|\sigma(G)| - 1 \leq dim_s(G) \leq |\partial(G)| - 1$.*

Note that if $\sigma(G) = \partial(G)$, then by Lemma 1, $dim_s(G) = |\partial(G)| - 1$.

Theorem 3. *Let G and H be two connected graphs. If $\partial(G) = \sigma(G)$ and $\partial(H) = \sigma(H)$, then*

$$dim_s(G\Box H) = \min\{|\partial(G)|(|\partial(H)| - 1), |\partial(H)|(|\partial(G)| - 1)\}.$$

With the theorem above presented, it is possible to see that by using Remark 1 in such a result we deduce the following values of $dim_s(G\Box H)$ for other specific families of graphs G and H.

Corollary 1.

(a) $dim_s(K_n\Box K_r) = \min\{n(r-1), r(n-1)\}$.
(b) *For any trees T_1 and T_2 with $l(T_1)$ and $l(T_2)$ leaves, respectively,* $dim_s(T_1\Box T_2) = \min\{l(T_1)(l(T_2) - 1), l(T_2)(l(T_1) - 1)\}$.
(c) *For any tree T with $l(T)$ leaves,* $dim_s(K_n\Box T) = \min\{l(T)(n-1), n(l(T) - 1)\}$.

We continue with establishing an interesting connection between the strong resolving graph of the Cartesian product of two graphs and the direct product of the strong resolving graphs of its factors.

Theorem 4. *Let G and H be two connected graphs. Then*

$$(G\Box H)_{SR} \cong G_{SR} \times H_{SR}.$$

[3] The diameter of G is defined as $D(G) = \max_{u,v \in V(G)}\{d(u, v)\}$. We recall that G is 2-antipodal if for each vertex $x \in V(G)$ there exists exactly one vertex $y \in V(G)$ such that $d_G(x, y) = D(G)$.

As a consequence of Theorems 1 and 4 we have the next result.

Corollary 2. *Let G and H be two connected graphs. Then $dim_s(G\Box H) = \beta(G_{SR} \times H_{SR})$.*

The following lemmas present two useful relationships between the independence numbers of the Cartesian product of two graphs and that of its factors.

Lemma 2. [5] *For any graphs G and H of orders n_1 and n_2, respectively,*

$$\alpha(G \times H) \geq \max\{n_2 \cdot \alpha(G),\ n_1 \cdot \alpha(H)\}.$$

Lemma 3. [9] *For any graphs G and H of orders n_1 and n_2, respectively,*

$$\alpha(G \times H) \leq n_2 \cdot \alpha(G) + n_1 \cdot \alpha(H) - \alpha(G) \cdot \alpha(H).$$

The next result gives lower and upper bounds for the strong metric dimension of the Cartesian product of graphs in terms of the strong metric dimension of its factors and the size of their boundaries.

Theorem 5. *For any connected graphs G and H, $dim_s(G\Box H) \geq dim_s(G) \cdot dim_s(H)$ and $dim_s(G\Box H) \leq \min\{|\partial(H)|dim_s(G),\ |\partial(G)|dim_s(H)\}$.*

Proof. The lower bound is obtained from Lemma 3, Theorem 2 and Corollary 2. By using Lemma 2, Theorem 2 and Corollary 2 we have the upper bound. ∎

We do observe that there is an infinite family of Cartesian products for which the strong dimension is "close" to the lower bound. In particular, $dim_s(K_n\Box K_2) = n = dim_s(K_n) \cdot dim_s(K_2) + 1$. Moreover, several examples of pairs of graphs where the upper bound is attained are given in Corollary 1 and in the following proposition.

Proposition 1.

(a) $dim_s(K_n\Box C_{2r+1}) = \min\{n(r+1), (2r+1)(n-1)\}$.
(b) *For any tree T with $l(T)$ leaves,*
$dim_s(T\Box C_{2r+1}) = \min\{l(T)(r+1), (2r+1)(l(T)-1)\}$.
(c) [6] $dim_s(C_{2n+1}\Box C_{2r+1}) = \min\{(2n+1)(r+1), (2r+1)(n+1)\}$.

4 Rooted product of graphs

We start by presenting closed formulae for the strong metric dimension of rooted product graphs $G \circ_v H$, where $\partial(H) = \sigma(H)$.

Theorem 6. *Let G be a connected graph of order $n \geq 2$ and let H be a connected graph such that $\partial(H) = \sigma(H)$.*

(i) *If $v \in \partial(H)$, then $dim_s(G \circ_v H) = n(|\partial(H)| - 1) - 1$.*
(ii) *If $v \notin \partial(H)$, then $dim_s(G \circ_v H) = n|\partial(H)| - 1$.*

We emphasize the following particular cases of Theorem 6.

Corollary 3. *Let G be a connected graph of order $n \geq 2$.*

(a) *For any complete graph of order n', $dim_s(G \circ K_{n'}) = n(n' - 1) - 1$.*
(b) *For any tree T with $l(T)$ leaves,*

$$dim_s(G \circ_v T) = \begin{cases} n(l(T) - 1) - 1, \text{ if } v \text{ is a leaf of } T, \\ \\ n \cdot l(T) - 1, \qquad \text{if } v \text{ is an inner vertex of } T. \end{cases}$$

The next result gives the lower and upper bounds for the strong metric dimension of rooted product graphs $G \circ_v H$ in terms of the order of graph G, the strong metric dimension of graph H and the size of its boundary.

Theorem 7. *Let G be a connected graph of order $n \geq 2$ and let H be a connected graph.*

(i) *If $v \in V(H)$ belongs to a strong metric basis of H, then*

$$n \cdot dim_s(H) - 1 \leq dim_s(G \circ_v H) \leq (|\partial(H)| - 1)(n - 1) + dim_s(H) - 1.$$

(ii) *If $v \in V(H)$ does not belong to any strong metric basis of H, then*

$$n \cdot dim_s(H) \leq dim_s(G \circ_v H) \leq \begin{cases} |\partial(H)|(n - 1) + dim_s(H), & \text{if } v \notin \partial(H), \\ \\ (|\partial(H)| - 1)(n - 1) + dim_s(H), & \text{if } v \in \partial(H). \end{cases}$$

As Corollary 3 shows, the bounds of Theorem 7 (i) are tight and the upper bound $dim_s(G \circ_v H) \leq |\partial(H)|(n - 1) + dim_s(H)$ of Theorem 7 (ii) is tight.

5 Strong product of graphs

We start this section with a description of the structure of the strong resolving graph of $G \boxtimes H$.

Lemma 4. *Let G and H be two connected nontrivial graphs. Let u, x be two vertices of G and let v, y be two vertices of H. Then (u, v) and (x, y) are mutually maximally distant vertices in $G \boxtimes H$ if and only if one of the following conditions holds:*

(a) *u, x are mutually maximally distant in G and v, y are mutually maximally distant in H;*
(b) *u, x are mutually maximally distant in G and $v = y$;*
(c) *v, y are mutually maximally distant in H and $u = x$;*
(d) *u, x are mutually maximally distant in G and $d_G(u, x) > d_H(v, y)$;*
(e) *v, y are mutually maximally distant in H and $d_G(u, x) < d_H(v, y)$.*

Notice that the lemma above leads to the following relationship, where $G \sqsubseteq H$ means that the graph G is a subgraph of the graph H.

Theorem 8. *For any connected graphs G and H,*

$$G_{SR} \boxtimes H_{SR} \sqsubseteq (G \boxtimes H)_{SR} \sqsubseteq G_{SR} \oplus H_{SR}.$$

Corollary 4. *For any connected graphs G and H,*

$$\alpha(G_{SR} \boxtimes H_{SR}) \geq \alpha((G \boxtimes H)_{SR}) \geq \alpha(G_{SR} \oplus H_{SR}).$$

The next three known results will be useful for our purposes.

Theorem 9. [5] *For any graphs G and H,*

$$\alpha(G) \cdot \alpha(H) \leq \alpha(G \boxtimes H) \leq \alpha(G \square H).$$

Theorem 10. (Vizing's theorem) *For any graphs G and H,*

$$\alpha(G \square H) \leq \min\{\alpha(G)|V(H)|,\ \alpha(H)|V(G)|\}.$$

Lemma 5. [1] *For any graphs G and H, $\alpha(G \oplus H) = \alpha(G) \cdot \alpha(H)$.*

The next result gives lower and upper bounds for the strong metric dimension of strong product of graphs in terms of the strong metric dimension of its factors and theirs orders.

Theorem 11. *Let G and H be two connected nontrivial graphs of order n_1, n_2, respectively. Then $dim_s(G \boxtimes H) \geq \max\{n_2 \cdot dim_s(G),\ n_1 \cdot dim_s(H)\}$ and $dim_s(G \boxtimes H) \leq n_2 \cdot dim_s(G) + n_1 \cdot dim_s(H) - dim_s(G) \cdot dim_s(H)$.*

Proof. By using Corollary 4, equality (1), Theorem 9 and Theorem 10 we obtain the lower bound. On the other hand, from Corollary 4, equality (1) and Lemma 5 we have the upper bound. ∎

To see the tightness of the upper bound of Theorem 11 we define a *$\mathcal{C}$-graph* as a graph G whose vertex set can be partitioned into $\beta(G)$ cliques. Notice that there are several graphs which are $\mathcal{C}$-graphs. For instance, we emphasize the following cases: complete graphs and cycles of even order.

Theorem 12. *Let G and H be two connected nontrivial graphs of order n_1, n_2, respectively. If G_{SR} is a $\mathcal{C}$-graph, then*

$$dim_s(G \boxtimes H) = n_2 \cdot dim_s(G) + n_1 \cdot dim_s(H) - dim_s(G) \cdot dim_s(H).$$

We emphasize the following particular cases of Theorem 12

Corollary 5. *Let G and H be two connected nontrivial graphs of order n_1 and n_2, respectively.*

(a) $dim_s(K_{n_1} \boxtimes H) = n_2(n_1 - 1) + n_1 \cdot dim_s(H) - (n_1 - 1)dim_s(H)$.
(b) *If G is a complete k-partite graph, then*

$$dim_s(G \boxtimes H) = n_2(n_1 - k) + n_1 \cdot dim_s(H) - (n_1 - k)dim_s(H).$$

(c) *If G is a tree with $l(G)$ leaves, then*

$$dim_s(G \boxtimes H) = n_2(l(G) - 1) + n_1 \cdot dim_s(H) - (l(G) - 1)dim_s(H).$$

(d) *If G is a 2-antipodal graph, then*

$$dim_s(G \boxtimes H) = \frac{n_2 \cdot n_1}{2} + n_1 \cdot dim_s(H) - \frac{n_1}{2} \cdot dim_s(H).$$

References

[1] N. Cizek and S. Klavzar. On the chromatic number of the lexicographic product and the Cartesian sum of graphs. *Discrete Math.*, 134(1-3):17–24, 1994.

[2] C. D. Godsil and B. D. McKay. A new graph product and its spectrum. *Bull. Austral. Math. Soc.*, 18(1):21–28, 1978.

[3] R. Hammack, W. Imrich and S. Klavzar. *Handbook of Product Graphs, Second Edition.* CRC Press, Boca Raton, FL, 2011.

[4] F. Harary and R. A. Melter. On the metric dimension of a graph. *Ars Combin.*, 2:191–195, 1976.

[5] P. K. Jha and G. Slutzki. Independence numbers of product graphs. *Appl. Math. Lett.*, 7(4):91–94, 1994.

[6] O. R. Oellermann and J. Peters-Fransen. The strong metric dimension of graphs and digraphs. *Discrete Appl. Math.*, 155:356–364, 2007.

[7] A. SebHo and E. Tannier. On metric generators of graphs. *Math. Oper. Res.*, 29(2):383–393, 2004.

[8] P. J. Slater. Leaves of trees. *Proceeding of the 6^{th} Southeastern Conference on Combinatorics, Graph Theory, and Computing. Congr. Numer.*, 14:549–559, 1975.

[9] S. Spacapan. The k-independence number of direct products of graphs and Hedetniemi's conjecture. *European J. Combin.*, 32:1377–1383, 2011.

Decomposing almost complete graphs by random trees

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Abstract. An old conjecture of Ringel states that every tree with m edges decomposes the complete graph K_{2m+1}. A more general version of the Ringel's conjecture says that every tree with m edges decomposes K_{rm+1} for each $r \geq 2$ provided that r and $m+1$ are not both odd. The best lower bound for the order of a complete graph decomposed by a given tree with m edge is $O(m^3)$. We show that asymptotically almost surely a random tree with m edges and $p = 2m+1$ is a prime decomposes the complete graph minus one edge $K_{3p} - e$. We also show that, for every prime of the form $2km+1$ a random tree with m edges asymptotically almost surely decomposes the graph $K_{2km+1}(3)$ obtained from the complete graph by replacing each vertex by the complement of a triangle.

Key words: Ringel Conjecture, random trees.

1 Introduction

Given two graphs H and G we say that H decomposes G if G is the edge–disjoint union of isomorphic copies of H. The following is a well–known conjecture of Ringel.

Conjecture 1 (Ringel [15]). Every tree with m edges decomposes the complete graph K_{2m+1}.

The conjecture has been verified by a number of particular classes of trees, see the dynamic survey of Gallian [8]. By using the polynomial method, the conjecture was verified by Kézdy [10] for the more general class of so–called *stunted* trees. As mentioned by the author, this class is still small among the set of all trees.

Robinson and Schwenk [16] proved that the average number of leaves in an (unlabelled) random tree with m edges is asymptotically cm with $c \approx 0.438$. Drmota and Gittenberger [5] showed that the distribution of the number of leaves in a random tree with m edges is asymptotically normal with variance

c_2m for some positive constant c_2. Thus, asymptotically almost surely a random tree with m edges has more than $2m/5$ leaves. Drmota and the author [7] used further structural results on random trees to show that asymptotically almost surely a random tree with m edges decomposes the complete bipartite graph $K_{2m,2m}$, thus providing an aproximate result to another decomposition conjecture by Graham and Haggkvist which asserts that in fact $K_{m,m}$ can be decomposed by a given tree with m edges.

Let $g(m)$ be the smallest integer n such that any tree with m edges decomposes the complete graph K_n. It was shown by Yuster [18] that $g(m) = O(m^{10})$ and the upper bound was reduced by Kezdy and Snevily [11] to $g(m) = O(m^3)$. Since $K_{2m,2m}$ decomposes the complete graph K_{8m^2+1} (see Snevily [17]), the above mentioned result on the decomposition of $K_{2m,2m}$ shows that $g(m) = O(m^2)$ asymptotically almost surely.

In this note we use a similar approach to prove that one can decompose smaller almost complete graphs by random trees, getting much closer to the original conjecture by Ringel. Our main result is the following one.

Theorem 1. *For every m such that $p = 2m - 1$ is a prime, asymptotically almost surely a random tree with m edges decomposes the complete graph minus one edge $K_{6m-1} - e$.*

By using the same techniques we prove the following general result on decomposition of almost complete graphs. For a graph G and a positive integer m we denote by $G(m)$ the graph obtained form G by replacing each vertex with a coclique of order m and joining two vertices if they belong to cocliques corresponding to adjacent vertices in G.

Theorem 2. *For every prime of the form $2km+1$ a random tree with m edges asymptotically almost surely decomposes the graph $K_{2km+1}(3)$.*

The above mentioned conjecture of Ringel has a general form according to which every tree with m edges decomposes each complete graph K_{rm+1}, $r \geq 2$, provided that r and $m + 1$ are not both odd. Theorem 2 provides the means of decomposing an infinite family of almost complete graphs.

Theorem 3. *Let q be a prime power different from 2. For every m such that $p = 2km - (q - 1)$ is a prime, $k \geq 1$, asymptotically almost surely a random tree with m edges decomposes the $K_{qp} - K_{q-1}$.*

2 Rainbow embeddings

The general approach to show that a tree T decomposes a complete graph or a complete bipartite graph consists in showing that T cyclically decompose the corresponding graphs. We next recall the basic principle behind this approach in slightly different terminology.

A rainbow embedding of a graph H into an oriented arc–colored graph X is an injective homomorphism f of some orientation $\vec{H}$ of H in X such that no two arcs of $f(\vec{H})$ have the same color.

Let $X = Cay(G, S)$ be a Cayley digraph of an abelian group G with respect to an antisymmetric subset $S \subset G$ (that is, $S \cap -S = \emptyset$). We consider X as an arc–colored oriented graph, by giving to each arc $(x, x + s)$, $x \in G, s \in S$, the color s.

Lemma 1. *Suppose that the graph H admits a rainbow embedding f in $X = Cay(G, S)$, where S is an antisymmetric subset of G. Then T decomposes the underlying graph of X.*

Proof. For each $a \in G$ the translation $x \to x+a$, $x \in G$, is an automorphism of X which preserves the colors and has no fixed points. Therefore, each translation sends $f(\vec{H})$ to an isomorphic copy which is edge disjoint from it. Thus the sets of translations for all $a \in G$ give rise to $n := |G|$ edge–disjoint copies of $\vec{H}$ in X. By ignoring orientations and colors, we thus have n edge disjoint copies of H in the underlying graph of X. □ ∎

The following result was proved by Alon [2] by using the Combinatorial Nullsetellensatz.

Theorem 4. *Let p be a prime. For every sequence $a_1, \ldots, a_k$ and every set $\{b_1, \ldots, b_k\}$ there is a permutation $\sigma \in Sym(k)$ such that the sums $a_1 + b_{\sigma(1)}, \ldots, a_k + b_{\sigma(k)}$ are pairwise distinct.*

One consequence of the above result is that every forest of stars with m edges admits a rainbow embedding in $Cay(\mathbb{Z}_p, S)$ for every antisymmetric set S with m elements. Moreover, the centers of the stars in the forest can be placed at prescribed vertices.

Corollary 1. *Let p be a prime. Let F be a forest of k stars centered at $x_1, \ldots, x_k$ and $m \leq (p-1)/2$ edges. Let S be an antisymmetric subset with m elements. Every injection $f : \{x_1, \ldots, x_k\} \to \mathbb{Z}_p$ can be extended to a rainbow embedding of F in $Cay(\mathbb{Z}_p, S)$.*

Proof. Consider the sequence $A = (f(x_1)^{m_1}, \ldots, f(x_k)^{m_k})$ where m_i is the number of edges incident to x_i, $1 \leq i \leq k$. By Alon's theorem There is a numbering $\{s_1, \ldots, s_m\}$ of the elements of S such that the sums $a_i + s_i$ are pairwise distinct. By defining $f(y_i) = a_i + s_i$ we obtain the desired rainbow embedding. □ ∎

3 The case $2m+1$ a prime

Let p be a prime and let T be a tree with $m=(p-1)/2$ edges. Let $S \subset \mathbb{Z}_p$ be an antisymmetric subset of $\mathbb{Z}_p$ with $|S|=(p-1)/2$. We will use the above approach with the Cayley graph $X = Cay(\mathbb{Z}_p \times \mathbb{Z}_3, S \times \mathbb{Z}_3)$. We note that the underlying graph of X is isomorphic to $K_{2m+1}(3)$. The strategy of the proof is to show first that the base tree T_0 of a random tree with m edges admits a rainbow embedding f into X in such a way that $f(E(T_0)) \subset S \times \{0\}$.

Lemma 2. *Let T be a tree with m edges. Let $p \geq 3m$ be a prime. There is a rainbow embedding of T into $Y = Cay(\mathbb{Z}_p, S)$ for some antisymmetric set $S \subset \mathbb{Z}_p$ with $|S| = (p-1)/2$.*

Proof. The rainbow embedding can be defined greedily. Let xy be a leave of T and suppose that f is a rainbow embedding of the subtree $T' = T - x$ of T. Let $U = \mathbb{Z}_p \setminus (f(E(T')) \cup (-f(E(T'))))$ and $W = \mathbb{Z}_p \setminus f(V(T'))$. Since $|U| \geq p - 2m + 2 \geq m + 1$ and $|W| \geq p - m + 1$, there is $u \in U$ such that $f(y) + u \in W$. By extending f to x as $f(x) = f(y) + u$ we obtain a rainbow embedding of T. □ ∎

Some ideas in the proof of the next Lemma can be found in [9].

Lemma 3. *Let $p = 2m+1$ be a prime. Let T be a tree with m edges and at least $2m/5$ end vertices. Then T decomposes $K_{2m+1}(3)$.*

Proof. Let T_0 be the subtree of T obtained by removing its endvertices. Since T_0 has at most $3m/5 < 3(p-1)/10 < (p-1)/3$ edges there is a rainbow embedding of T_0 in $Y_\alpha = Cay(\mathbb{Z}_p \times \mathbb{Z}_3, S_0 \times \{\alpha\})$ for some antisymmetric set $S_0 \subset \mathbb{Z}_p$ with $|S_0| = |E(T_0)|$ and each $\alpha \in \mathbb{Z}_3$. Let us call $(T_0)_\alpha$ such rainbowly embedded graph.

Let S be an antisymmetric set with $(p-1)/2$ elements containing S_0 and let $S_1 = S \setminus S_0$. Since $F = T - T_0$ is a forest of stars, it admits a rainbow embedding in $Z_\beta = Cay(\mathbb{Z}_p \times \mathbb{Z}_3, S_1 \times \{\beta\})$ for each $\beta \in \mathbb{Z}_3$ for which the centers of the stars are placed according to the previous embeddings of the base tree T_0 in Y_α for $\alpha = \beta + 1$. Let us call F_β be such rainbowly embedded forest.

Thus we get a rainbow graph

$$((T_0)_0 \oplus F_1) \oplus ((T_0)_1 \oplus F_2) \oplus ((T_0)_2 \oplus F_0)$$

which has $3p$ edges and hence it decomposes $K_{2m+1}(3)$, the underlying graph of $Cay(\mathbb{Z}_p \times \mathbb{Z}_3, \mathbb{Z}_p^* \times \mathbb{Z}_3)$.

It is clear that $(T_0)_0 \oplus F_1$ is a rainbow embedding of the tree T in $Cay(\mathbb{Z}_p \times \mathbb{Z}_3, (S_0 \times \{0\}) \cup (S_1 \times \{1\}))$. If each of $(T_0)_1 \oplus F_2$ and $(T_0)_2 \oplus F_0$ is also isomorphic with T our job is finished.

It is however possible that $(T_0)_1 \oplus F_2$ is no longer isomorphic to the original tree T since it may contain some cycles. More precisely, it may happen that some edge of F_2 joins a vertex $(x,i) \in (T_0)_1$ with a vertex $(x+s, i+2) \in (T_0)_1$. In that case, the edge (s,β) in F_2 will also join vertices in the translated copies $(T_0)_1 + (0,i)$, $i \in \mathbb{Z}_3$, of $(T_0)_1$. Denote by u, v the vertices of the original tree T_0 which are mapped to $(x,i), (x+s, i+2)$ by the embedding.

We consider the subgraph induced by the edges of $Cay(\mathbb{Z}_p \times \mathbb{Z}_3, \mathbb{Z}_p^* \times \mathbb{Z}_3)$ which are colored $(s,\beta), \beta \in \mathbb{Z}_3$. They form an oriented $K_{3,3}$ isomorphic to $Z_s = Cay(\mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \{1\})$. Each vertex $(i,0)$ in Z_s contains three translated copies of T_0 in which the vertices u, v of T_0 are mapped to (x,i) and $(x+s, i+2)$, which once combined with the rainbow embedding of the forest F, create a cycle. We will redistribute the edges of these copies locally to undo the created cycle. By doing so in each embedded edge of F we end up with a decomposition of $Cay(\mathbb{Z}_p \times \mathbb{Z}_3, \mathbb{Z}_p^* \times \mathbb{Z}_3)$, and hence of $K_{2m+1}(3)$ with isomorphic copies of T. This local redistribution can be made by using two mutually orthogonal latin squares of order 3, one labelled with the copies of the nine trees meeting the vertices of Z_s and an orthogonal mate according to which the vertices are distributed in the corresponding copies of forests. By orthogonality, no edge in a forest will join two vertices in its base tree. □ ∎

Lemma 3 leads to the proof of Theorem 1 as follows.

Proof of Theorem 1 Let T be a random tree with m edges such that $p = 2m-1$ is a prime. Asymptotically almost surely T has at least $2m/5$ edges. Let T' be the graph obtained from T by removing one leave uv, where v is a pendant vertex of T. By Lemma 3, T' decomposes $K_{2m-1}(3)$. Every vertex in this graph contains precisely three copies of T' in which this vertex occupies the position of u in T'. Add two vertices to $K_{2m-1}(3)$ joined from every vertex in $K_{2m-1}(3)$ and a directed triangle in each coclique with size three. This way we can use one of the three outgoing edges to each such copy and complete the embedding of the original tree T in the resulting graph, which happens to be K_{6m-1} minus the edge joining the two added vertices. □

The proofs of Theorem 2 and Theorem 3 are extensions of the above arguments and are not included on this extended abstract.

References

[1] N. Alon, Combinatorial Nullstellensatz, *Combinatorics, Probability and Computing* **8** (1999), 7–29.

[2] N. Alon, Additive Latin Transversals, *Israel J. Math.* **117** (2000), 125–130.

[3] M. Càmara, A. Lladó, J. Moragas. On a conjecture of Graham and Haggkvist with the polynomial method, *European J. Combin.* 30 (7) (2009) 1585–1592.

[4] M. Drmota. Random Trees. Springer-Verlag, 2009.

[5] M. Drmota, B. Gittenberger. The distribution of nodes of given degree in random trees, *Journal of Graph Theory* 31 (1999) 227–253.

[6] M. Drmota, B. Gittenberger. The shape of unlabeled rooted random trees. *European J. Combin.* 31 (2010) 2028–2063

[7] M. Drmota, A. Lladó. Almost Every Tree With m Edges Decomposes $K_{2m,2m}$. *Combinatorics, Probability and Computing* **23** 1 (2014) 50–65.

[8] J. A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics* 5 (2007) # DS6.

[9] R. L. Häggkvist, Decompositions of Complete Bipartite Graphs, *Surveys in Combinatorics*, Johannes Siemons Ed., Cambridge University Press (1989) 115–146.

[10] A.E. Kézdy. ρ–valuations for some stunted trees. *Discrete Math.* 306 (21) (2006) 2786–2789

[11] A. E. Kézdy, H. S. Snevily. Distinct Sums Modulo n and Tree Embeddings. *Combinatorics, Probability and Computing*, 1, Issue 1, (2002)

[12] A. Lladó, S.C. López. Edge-Decompositions of $K_{n,n}$ Into Isomorphic Copies of a Given Tree, *J. Graph Theory* 48 (2005), no. 1, 1–18.

[13] A. Lladó, S.C. López, J. Moragas. Every tree is a large subtree of a tree that decomposes K_n or $K_{n,n}$, *Discrete Math.* 310 (2010) 838–842

[14] R. Otter, The number of trees. *Ann. of Math.* (2) 49, (1948). 583–599.

[15] G. Ringel, Problem 25, Theory of Graphs and its Applications, Nakl. CSAV, Praha, (1964) pp. 162.

[16] R.W. Robinson, A.J. Schwenk. The distribution of degrees in a large random tree. *Discrete Math.* 12(4) (1975) 359-372.

[17] H. Snevily. New families of graphs that have α–labelings. *Discrete Math.* 170 (1997) 185–194.

[18] R. Yuster. Packing and decomposition of graphs with trees. *J. Combin. Theory Ser. B* 78 (2000) 123–140.

Grafos mezclados de Moore de tipo Cayley*

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Resumen. Los grafos construidos a partir de grupos (grafos de Cayley) han sido ampliamente estudiados y gozan de propiedades que los hacen muy útiles en el contexto de los grafos y digrafos de Moore. En el año 1978, Bosák generalizó los grafos y digrafos de Moore a grafos mezclados de Moore, es decir, grafos donde coexisten aristas y arcos simultaneamente y para los que existe un único recorrido entre dos pares de vértices cualesquiera a distancia más pequeña o igual que el diámetro. En este trabajo extendemos el concepto de grafo de Cayley para los grafos mezclados, mostramos como algunos grafos mezclados de Moore son de tipo Cayley y descartamos la existencia de este tipo de grafos en otros casos.

Palabras clave: Grafo, Digrafo, Moore, Cayley.

1 Introducción, terminología y notación.

Los grafos mezclados (mixed) resultan ser un buen modelo para ciertas redes donde conexiones unidirecciones y bidireccionales coexisten en la topología de la propia red. Para los propósitos de este trabajo, los grafos (mezclados) pueden contener tanto arcos como aristas. Las definiciones relativas a *recorridos*, *caminos* y *distancia* son conocidas en este contexto y pueden leerse, por ejemplo, en [2]. El *diámetro* de un grafo G, denotado por $\mathrm{diam}(G)$, es la máxima de las distancias entre pares de vértices de G. Por otro lado, la *cintura* de un grafo es la longitud del ciclo más corto del grafo. Sea $(\mathcal{G}, \cdot)$ un grupo con elemento unitario $1_{\mathcal{G}}$. El conjunto de todos los elementos que conmutan con $a \in \mathcal{G}$, $\mathcal{N}_a = \{x \in \mathcal{G} \mid xa = ax\}$, se denomina *centralizador* de a. El *orden* de un elemento a lo denotaremos por $\mathrm{ord}_{\mathcal{G}}(a)$. Para otros conceptos relacionados con la teoría de grupos seguiremos la notación de Lang [5].

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2 Grafos mezclados de Moore

Los *grafos parcialmente dirigidos de Moore* también denominados *grafos mezclados de Moore* fueron definidos e investigados por Bosák en [1,2]. Así, un grafo mezclado de Moore se define como aquel grafo simple, finito y totalmente regular (cada vértice comparte r aristas y z arcos de entrada y de salida, donde r y z son independientes de la elección del vértice en cuestión) de forma que existe exactamente un recorrido desde un vértice u a un vértice v de longitud como mucho el diámetro del grafo. Existe una cota superior para el orden de un grafo de diámetro k, grado máximo (no dirigido) r y grado máximo de salida (dirigido) z,

$$1+(r+z)+[z(r+z)+r(r+z-1)]+\cdots+[z(r+z)^{k-1}+r(r+z-1)^{k-1}]. \quad (1)$$

Este valor, denotado por $M_{r,z,k}$, se denomina *cota de Moore para grafos mezclados* (ver [8]). La cota se obtiene fácilmente si 'colgamos' un grafo con las características anteriores de un vértice cualquiera v y contamos el número máximo de vértices que aparecen a distancia $\leq k$ de v (ver Fig. 1) Los gra-

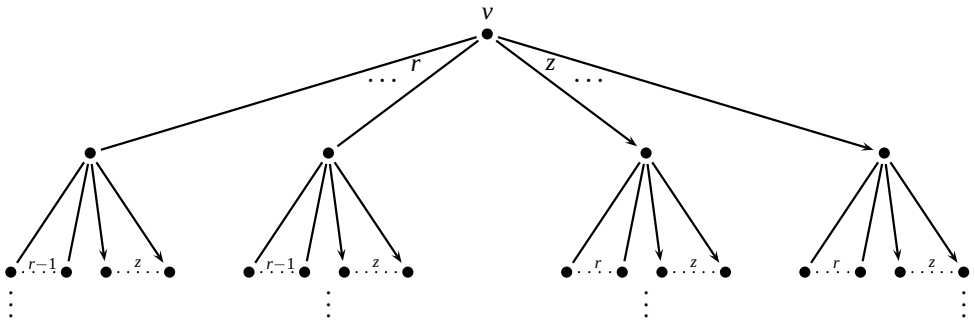

Figura 1: Árbol generador de un grafo mezclado de Moore 'colgado' desde v.

fos mezclados con $M_{r,z,k}$ vértices, diámetro k, grado máximo (no dirigido) r y grado máximo de salida (dirigido) z se denominan *Grafos mezclados de Moore*. Estos grafos mezclados extremales son totalmente regulares de grado $d = r + z$ y tienen la propiedad que dados dos vértices cualesquiera, existe un único camino que los une de longitud $\leq k$. Los casos extremos $r = 0$ y $z = 0$ corresponden a los *Digrafos de Moore* y *Grafos de Moore*, respectivamente. Estas estructuras han sido ampliamente estudiadas (ver [7] para un resumen completo sobre ellas). En general, podemos afirmar que los grafos y digrafos de Moore existen sólo para escasos valores del diámetro y/o el grado. Los grafos mezclados de Moore que contienen al menos un arco ($z \geq 1$) y una arista ($r \geq 1$) se denominan *grafos mezclados de Moore propios*. Nguyen, Miller y Gimbert demuestran en [9] que no hay grafos mezclados de Moore propios

de diámetro $k \geq 3$. Así pues, su existencia se focaliza en el caso de diámetro $k = 2$, donde Bosák da una condición necesaria sobre los parámetros r y z, para la existencia de estos grafos:

Teorema 1 ([2]). *Sea G un grafo mezclado de Moore (propio) de diametro dos. Entonces, G es totalmente regular con grado dirigido $z \geq 1$ y grado no dirigido $r \geq 1$. Además, existe un entero impar c tal que*

$$r = \frac{1}{4}(c^2 + 3) \quad y \quad c|(4z - 3)(4z + 5). \tag{2}$$

Para $r = 1$ y $z \geq 1$ la familia de *digrafos de Kautz* $\mathrm{Ka}(d, 2)$ resultan ser grafos mezclados de Moore (ver [9]). Otro ejemplo es el conocido como *grafo de Bosák* de parámetros $r = 3$ y $z = 1$ que contiene 18 vértices (ver Fig. 2). En el

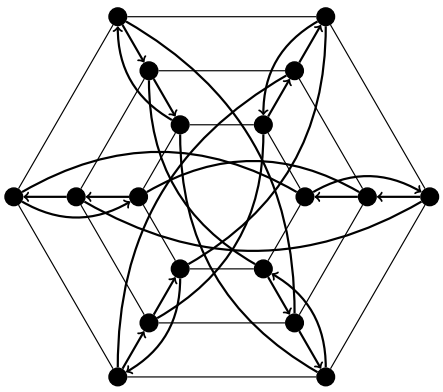

Figura 2: El grafo de Bosák es el único grafo mezclado de Moore de 18 vértices, corresondiente al caso $r = 3$ y $z = 1$.

$M_{r,z,2}$	r	z	d	Existencia	Unicidad
6	1	1	2	Ka(2, 2)	Sí
12	1	2	3	Ka(3, 2)	Sí
18	3	1	4	Bosák	Sí
20	1	3	4	Ka(4, 2)	Sí
30	1	4	5	Ka(5, 2)	Sí
40	3	3	6	?	?
42	1	5	6	Ka(6, 2)	Sí
54	3	4	7	?	?
56	1	6	7	Ka(7, 2)	Sí
72	1	7	8	Ka(8, 2)	Sí
84	7	2	9	?	?
88	3	6	9	?	?
90	1	8	9	Ka(9, 2)	Sí
108	3	7	10	Jørgensen	No
110	1	9	10	Ka(10, 2)	Sí

Figura 3: Valores que satisfacen la ecuación (2) para ordenes $M_{r,z,2} \leq 110$.

año 2012, Jørgensen [4] encuentra dos grafos mezclados de Moore de 108 vértices, construyendolos a partir de dos grupos distintos de ese mismo cardinal. Recientemente, López y Pujolàs [6] caracterizan todos los grafos mezclados de Moore con $z = 1$, demostrando la no existencia de un grafo mezclado de Moore de 486 vértices. En general, existen una infinidad de valores r y z que satisfacen las condiciones de la ecuación (2) y para los cuales se desconoce la existencia de un grafo mezclado de Moore (ver Fig. 3).

3 Grafos mezclados de Moore de tipo Cayley

Los grafos y digrafos de Cayley admiten una particularización natural en el mundo de los grafos mezclados. Sea $(\mathcal{G}, \cdot)$ un grupo y sean $\mathcal{A}$ y $\mathcal{B}$ dos subconjuntos disjuntos sin elemento unidad de $\mathcal{G}$, de forma que $\mathcal{A}$ es cerrado por inversos, es decir $\mathcal{A} = \mathcal{A}^{-1}$. El *grafo mezclado de Cayley*, denotado por $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$, tiene por vértices los elementos de $\mathcal{G}$, las aristas se generan entre los vértices $g \in \mathcal{G}$ y gx, para todo $x \in \mathcal{A}$, y existe un arco desde un vértice $g \in \mathcal{G}$ hacia gx, para todo $x \in \mathcal{B}$. Por definición, el grafo mezclado $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$ tiene $|\mathcal{G}|$ vértices y es totalmente regular de grado no dirigido $|\mathcal{A}|$ y grado dirigido $|\mathcal{B}|$. Además, $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$ tendrá diámetro ≤ 2 si para todo $u, v \in \mathcal{G}$, existen $g_1, g_2 \in \mathcal{A} \cup \mathcal{B} \cup \{1_\mathcal{G}\}$ tales que $v = ug_1g_2$. Los grafos de Cayley tienen la propiedad de ser vertice-transitivos, lo que permite simplificar la condición anterior de la manera siguiente:

Proposición 1. *Sea* $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$*, donde* $|\mathcal{G}| = M_{r,z,2}$*,* $|\mathcal{A}| = r$ *y* $|\mathcal{B}| = z$*. Entonces* $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$ *es un grafo mezclado de Moore (de diámetro* 2*) si y sólo si existen* $g_1, g_2 \in \mathcal{A} \cup \mathcal{B} \cup \{1_\mathcal{G}\}$ *tales que* $v = g_1g_2$ *para todo* $v \in \mathcal{G}$*.*

Nuestro objetivo es describir algunas condiciones necesarias para que una terna $(\mathcal{G}; \mathcal{A}, \mathcal{B})$ pueda generar un grafo mezclado de Moore. En el caso $z = 0$, los grafos de Moore tienen una cintura relativamente grande (por ej. 5, para un diámetro igual a 2). Existen cotas superiores para la cintura que se puede lograr en un grafo de Cayley sobre un grupo nilpotente o resoluble $\mathcal{G}$ [3]. Estas cotas también serían válidas en el caso de los grafos mezclados, y en principio nos permitirían descartar algunos grupos $\mathcal{G}$, suponiendo que conociéramos la cintura que debería tener el grafo mezclado de Moore que estamos buscando. Sin embargo, los grafos mezclados de Moore siempre tienen una cintura igual a 3, muy por debajo de las cotas proporcionadas por [3], con lo cual este criterio carecería de valor práctico en nuestro caso. No obstante, el resultado que damos a continuación nos permite acotar la búsqueda de ternas $(\mathcal{G}; \mathcal{A}, \mathcal{B})$ que puedan generar grafos mezclados de Moore.

Proposición 2. *Sea* $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$ *un grafo mezclado de Moore de tipo Cayley, donde* $|\mathcal{G}| = M_{r,z,2}$*,* $|\mathcal{A}| = r$ *y* $|\mathcal{B}| = z$*. Pongamos* $\mathcal{A} = \{g_1, \ldots, g_r\}$ *y* $\mathcal{B} = \{h_1, \ldots, h_z\}$*. Entonces,*

(a) $\mathcal{B} \cap \mathcal{B}^{-1} = \emptyset$*, es decir,* $\mathcal{B}$ *no puede contener un elemento y su inverso.*

(b) *Los conjuntos* $\mathcal{A}$ *y* $\mathcal{B}$ *son* libres de centralizadores*, es decir,*
- *sea* $g \in \mathcal{A}$*, entonces* $x \notin \mathcal{N}_g$*, para todo* $x \in \mathcal{A} \cup \mathcal{B} \setminus \{g, g^{-1}\}$*,*
- *sea* $h \in \mathcal{B}$*, entonces* $x \notin \mathcal{N}_h$*, para todo* $x \in \mathcal{A} \cup \mathcal{B} \setminus \{h\}$*.*

En particular $\mathcal{G}$ *es un grupo no abeliano.*

(c) $\mathcal{A}$ *contiene un número impar de involuciones de* $\mathcal{G}$*.*

(d) *Para cada* $h_i \in \mathcal{B}$*, existe un único* $h_j \in \mathcal{B}$ *tal que* $h_ih_j = h_{\sigma(i)}^{-1}$*, donde* $\sigma \in S_z$ *es una permutación fija e independiente de la elección de* i*. Además,* $\mathrm{ord}_\mathcal{G}(g_i) > 3$ *para todo* g_i *que no es una involución de* $\mathcal{A}$*.*

Proof. (a) Supongamos que existiera $h \in \mathcal{B}$ tal que $h^{-1} \in \mathcal{B}$. Desde un punto de vista geométrico, el arbol generador que se obtendría al 'colgar' el grafo $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$ tendría un máximo de $r(r-1+z)+(z-2)(r+z)+2(r+z-1)$ vértices a distancia 2 del vértice $1_{\mathcal{G}}$, ya que cada uno de los vértices h y h^{-1} aportarÃan $r+z-1$ vértices a esta cantidad. Sin embargo, todo grafo mezclado de Moore tiene exactamente $r(r-1+z)+z(r+z)$ vértices a distancia 2 de cualquier vértice (ver Fig. 1), por lo que $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$ no podría ser un grafo mezclado de Moore.
(b) Sea $g \in \mathcal{A}$ y supongamos que existe $x \in \mathcal{A} \cup \mathcal{B} \setminus \{g, g^{-1}\}$ tal que $x \in \mathcal{N}_g$. Entonces g y x son distintos y conmutan, es decir, $gx = xg$, además $gx \neq 1_{\mathcal{G}}$. Veamos como se traduce este efecto en el grafo mezclado $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$:

- Si $x \in \mathcal{A}$, entonces tenemos dos caminos distintos
  $1_{\mathcal{G}} \leftrightarrow g \leftrightarrow gx$
  $1_{\mathcal{G}} \leftrightarrow x \leftrightarrow xg$
  compuestos por dos aristas desde el vértice $1_{\mathcal{G}}$ hasta el vértice gx.
- Si $x \in \mathcal{B}$, entonces tenemos también dos caminos distintos
  $1_{\mathcal{G}} \leftrightarrow g \rightarrow gx$
  $1_{\mathcal{G}} \rightarrow x \leftrightarrow xg$
  compuestos por una arista y un arco desde el vértice $1_{\mathcal{G}}$ hasta gx.

En cualquier caso tendríamos dos caminos distintos desde el vértice $1_{\mathcal{G}}$ hasta gx, contradiciendo la unicidad de caminos que por definición tiene un grafo mezclado de Moore. De forma análoga se hace el caso $h \in \mathcal{B}$, teniendo en cuenta que $h^{-1} \notin \mathcal{B}$ por el apartado anterior.
(c) Primero demostramos que $|\mathcal{A}| = r$ es un número impar. En efecto, de la ecuación (2) tenemos que $r = \frac{1}{4}(c^2+3)$ para un $c \in \mathbb{Z}$ impar. Así, poniendo $c = 2n+1$, tenemos que $c^2 = 4n^2+4n+1$, por lo que $r = n^2+n+1 = n(n+1)+1$ de donde se deduce que r es siempre un número impar. Ahora como $\mathcal{A} = \mathcal{A}^{-1}$, por cada elemento $g \in \mathcal{A}$ tenemos su inverso $g^{-1} \in \mathcal{A}$, por lo que estos elementos aparecen en parejas en $\mathcal{A}$ y aportan una cantidad par al cardinal de $\mathcal{A}$, siempre que $g \neq g^{-1}$. El resto de elementos de $\mathcal{A}$ sólo pueden ser involuciones ($g = g^{-1}$), por lo que tendremos un número impar de éstas.
(d) Por un lado, observamos que los z arcos de entrada que tiene el vértice $1_{\mathcal{G}}$ provienen de los elementos del conjunto $\mathcal{B}^{-1} = \{h_1^{-1}, \ldots, h_z^{-1}\}$. Por otro lado, desde un punto de vista geométrico, estos z arcos sólo pueden provenir de vértices a distancia 2 de $1_{\mathcal{G}}$, formando caminos constituidos unicamente por arcos, ya que si una arista apareciera en alguno de los z caminos llegariamos a contradicción con la unicidad de caminos que define a los grafos mezclados de Moore. Los caminos constituidos por arcos sólo pueden generarse con elementos de $\mathcal{B}$, por lo que por cada $h_i \in \mathcal{B}$ debe existir $h_j \in \mathcal{B}$ tal que $h_i h_j \in \mathcal{B}^{-1}$. Como hay z caminos distintos, tenemos que $h_i h_j = h_{\sigma(i)}^{-1}$, donde $\sigma \in S_z$ es una permutación fija e independiente de la elección de i (ver Fig. 4). A consecuencia de este hecho $\mathrm{ord}_{\mathcal{G}}(g_i) \neq 3$, ya que en caso contrario tendríamos

que $g_i^3 = 1_{\mathcal{G}}$ para un determinado $g_i \in \mathcal{A}$, lo que significaría que uno de los z caminos estaría constituido por aristas, lo que es imposible. □ ∎

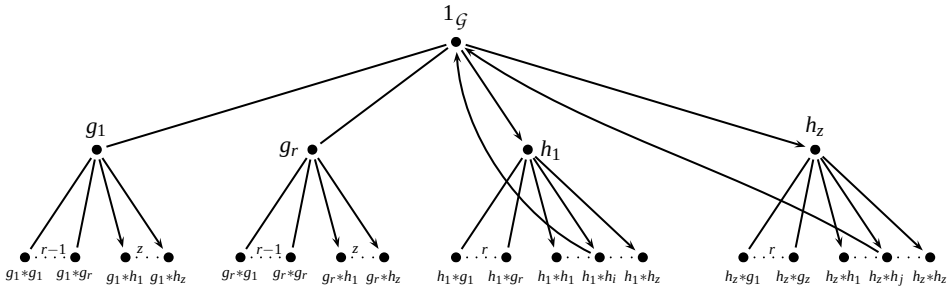

Figura 4: Grafo mezclado de Moore de tipo Cayley $\mathrm{Cay}(\mathcal{G}; \mathcal{A}, \mathcal{B})$ 'colgado' desde el elemento unidad del grupo $\mathcal{G}$, donde $\mathcal{A} = \{g_1, \ldots, g_r\}$ y $\mathcal{B} = \{h_1, \ldots, h_z\}$.

El grafo de Bosák como grafo mezclado de Cayley

Podemos aplicar la proposición 2 para intentar encontrar ternas $(\mathcal{G}; \mathcal{A}, \mathcal{B})$ que posibiliten la construcción de grafos mezclados de Moore de tipo Cayley de 18 vértices, correspondiente al caso $r = 3$ y $z = 1$. Respecto al grupo $\mathcal{G}$, existen un total de 5 grupos de 18 elementos, pero unicamente 3 de ellos son no abelianos:

- El grupo dihédrico de 18 elementos: D_{18}.
- El producto directo del grupo de permutaciones de tres elementos con $\mathbb{Z}_3$, es decir: $S_3 \times \mathbb{Z}_3$.
- Un producto semidirecto de $\mathbb{Z}_3 \times \mathbb{Z}_3$ con $\mathbb{Z}_2$, es decir: $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Sobre cada uno de estos grupos candidatos, hay que buscar conjuntos $\mathcal{A} = \{g_1, g_2, g_3\}$ y $\mathcal{B} = \{h_1\}$ de forma que se cumplan las condiciones necesarias descritas en la proposición 2. Por el apartado c) de dicha proposición se tiene que o bien g_1, g_2 y g_3 son tres involuciones o bien g_1 es una involución y $g_3 = g_2^{-1}$. Además, por el apartado d), tenemos que $h_1 h_1 = h_1^{-1}$, es decir, $\mathrm{ord}_{\mathcal{G}}(h_1) = 3$. Además, $\mathrm{ord}_{\mathcal{G}}(g_2) > 3$ (cuando g_2 no fuera una involución). Así pues nos reducimos a los siguientes casos:

1. $\mathcal{A}$=$\{g_1, g_2, g_3\}$ y $\mathcal{B}$=$\{h_1\}$, con $\mathrm{ord}_{\mathcal{G}}(g_i)$=2 y $\mathrm{ord}_{\mathcal{G}}(h_1)$=3, para todo $1 \leq i \leq 3$.
2. $\mathcal{A}$=$\{g_1, g_2, g_2^{-1}\}$ y $\mathcal{B}$=$\{h_1\}$, con $\mathrm{ord}_{\mathcal{G}}(g_1)$=2, $\mathrm{ord}_{\mathcal{G}}(g_2)$>3 y $\mathrm{ord}_{\mathcal{G}}(h_1)$=3.

A los pares $(\mathcal{A}, \mathcal{B})$ que satisfacen la primera condición les diremos pares 'tipo 1', y a los que satisfacen la segunda condición pares 'tipo 2'. Hemos implemen-

tado un algorítmo combinando los programas SAGE[2] y GAP[3] que construye todos los pares de tipo 1 y tipo 2 para los tres grupos no abelianos de 18 vértices y que además verifican la condición (b) de la proposición 2. Para cada uno de estos pares se ha construido el correspondiente grafo mezclado de Cayley y se ha comprobado si dicho grafo tenía diámetro 2. Hemos obteniendo los resultados representados en la tabla 1. En el caso del grupo dihédrico D_{18} hay

Grupo $\mathcal{G}$	Tipo 1 [de Moore]		Tipo 2 [de Moore]	
D_{18}	168	[0]	0	[0]
$S_3 \times \mathbb{Z}_3$	6	[0]	48	[16]
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	672	[144]	0	[0]

Tabla 1: Parejas $(\mathcal{A}, \mathcal{B})$ de tipo 1 y 2 para los tres grupos no abelianos de 18 vértices. En algunos casos estas parejas dan lugar grafos mezclados de Moore.

un total de 168 de parejas de tipo 1 y ninguna del tipo 2. Estas $168 = 84 \cdot 2$ parejas estan formadas de la siguiente forma: $\mathcal{A}$ es cualquiera de los $\binom{9}{3} = 84$ conjuntos de tamaño 3 formados por las 9 involuciones de D_{18} y $\mathcal{B}$ sólo puede ser 2 conjuntos (los formados por los dos únicos elementos de orden 3 que tiene D_{18}). En cualquier caso, ninguna de las 168 parejas $(\mathcal{A}, \mathcal{B})$ da como resultado un grafo de diámetro 2. En cambio, los otros dos grupos no abelianos sí generan grafos mezclados de Moore. Los $144 + 16$ grafos de mezcados de Moore obtenidos son todos isomorfos entre sí, dando lugar al grafo de Bosák (ver Fig. 2).

Proposición 3. *El grafo de Bosák es un grafo mezclado de Cayley que puede obtenerse a partir de* $S_3 \times \mathbb{Z}_3$ *o bien* $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$.

Imposibilidad de otros grafos mezclados de Moore de tipo Cayley

En la linea de lo realizado para el caso de 18 vértices, hemos buscado exhaustivamente grafos mezclados de Moore de tipo Cayley para otros casos donde se desconoce la existencia de grafos mezclados de Moore. Así, para el caso de 40 y 54 vértices, no hemos hallado ninguna terna $(\mathcal{G}; \mathcal{A}, \mathcal{B})$ que diera lugar a un grafo mezclado de Moore. Hay un total de 11 grupos no abelianos con 40 elementos, pero únicamente tres de ellos producen pares $(\mathcal{A}, \mathcal{B})$ que satisfacen condiciones de la proposición 2: $\mathbb{Z}_5 \times Q_8$, $\mathbb{Z}_2 \times (\mathbb{Z}_5 \ltimes \mathbb{Z}_4)$ y $\mathbb{Z}_5 \times D_8$, donde Q_8 es el grupo de cuaterniones de 8 elementos. Cabe destacar que en $\mathbb{Z}_2 \times (\mathbb{Z}_5 \ltimes \mathbb{Z}_4)$ se han encontrado del orden de 10^5 pares $(\mathcal{A}, \mathcal{B})$ verificando condiciones de la proposición 2. Ninguno de estos pares ha dado lugar a un grafo mezclado de Moore. Algo parecido ocurre con los grupos de 54 elementos.

[2] http://www.sagemath.org
[3] http://www.gap-system.org

Proposición 4. *No existen grafos mezclados de Moore de tipo Cayley de* 40 *y* 54 *vértices.*

La proposición anterior no descarta la posibilidad de la existencia de grafos mezclados de Moore de 40 y 54 vértices, pero, en estos casos, dichos grafos extremales deberían ser de tipo no Cayley. Dado que el grafo de Bosák, así como los grafos de Jørgensen y algunos grafos de la familia $\mathrm{Ka}(d, 2)$, son grafos de Cayley, es plausible pensar en la no existencia de grafos mezclados de Moore de tipo no Cayley.

Conjetura 1. No existen grafos mezclados de Moore de 40 y 54 vértices.

Dedicatoria

Dedicamos este trabajo a nuestro añorado compañero y gran amigo *Joan Gimbert*, a quién la relación entre el álgebra y la teoría de grafos atrajo tanto como a nosotros su personalidad.

Referencias

[1] J. Bosák. Graphs with unique walks, trails or paths of given lengths. *in: Theory and Applications of Graphs. Proceedings of the Conference, Kalamazoo, 1976, Springer Berlin.* (1978), 75–85.

[2] J. Bosák, Partially directed Moore graphs, *Math. Slovaca* 29 (1979), 181–196.

[3] R. Jajcay y G. Exoo. Properties of groups for the cage and degree/diameter problem. *Elec. Notes Disc. Math.* 34 (2009), 341–345.

[4] Leif K. Jørgensen, Variations and generalizations of Moore graphs. *International Workshop on Network Topologies*, Bandung (2012).

[5] S. Lang, Algebra. *Addison-Wesley Publishing* (1993)

[6] N. López y J. Pujolàs, On the existence of mixed Moore graphs of directed degree one. *Submitted*, (2014).

[7] M. Miller y J. Širáň. Moore graphs and beyond: A survey. *Electronic J. Combin.*, **DS14** (2005).

[8] M.H. Nguyen, M. Miller. Moore bounds for mixed networks. *Discrete Mathematics*, 308 (2008), 5499–5503.

[9] M.H. Nguyen, M. Miller, J. Gimbert. On mixed Moore graphs. *Discrete Mathematics*, **307** (2007)

Sobre condiciones suficientes para el RNIEP y sus márgenes de realizabilidad

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Resumen. En este trabajo se enuncian nuevas relaciones de inclusión entre varias condiciones suficientes para el RNIEP y se estudian la negatividad y el margen de realizabilidad de un espectro respecto de dichas condiciones.

Palabras clave: matrices no negativas, RNIEP, condiciones suficientes.

1 Consideraciones previas

En el contexto de la Teoría Espectral de Grafos, estamos interesados en problemas espectrales inversos sobre digrafos. En particular, tratamos de encontrar condiciones necesarias y suficientes para que una familia de números reales sea el espectro de la matriz de adyacencia de un digrafo pesado. Este problema se conoce como problema espectral real inverso no negativo (RNIEP) en el contexto de matrices no negativas y sólo está resuelto para espectros reales de tamaño menor o igual que cuatro.

En este trabajo, si $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ es una familia de n números reales que es el espectro de una matriz no negativa (digrafo pesado) de tamaño n escribiremos $\sigma \in Spec_n$ o simplemente $\sigma \in Spec$.

A continuación describiremos cronológicamente algunas condiciones suficientes, que llamaremos por el nombre de sus autores, para que σ sea el espectro de una matriz no negativa:

- **Suleimanova**, 1949: Si $\sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$:

$$\left.\lambda_0 \geq |\lambda| \ , \ \lambda \in \sigma \ , \quad \text{y} \quad \lambda_0 + \sum_{\lambda_i < 0} \lambda_i \geq 0 \right\} \Rightarrow \sigma \in Spec \, .$$

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- **Suleimanova-Perfect**, 1953: Si $\sigma = \{\lambda_0, \lambda_{01}, \ldots, \lambda_{0t_0}\} \cup \{\lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\}$:

$$\left.\lambda_0 \geq |\lambda|,\ \lambda \in \sigma\ ,\quad \text{y} \quad \lambda_j + \sum_{\lambda_{ji}<0} \lambda_{ji} \geq 0\ ,\ 0 \leq j \leq r \right\} \Rightarrow \sigma \in Spec\,.$$

- **Perfect 1**, 1953: Si $\sigma = \{\lambda_0\} \cup \{\lambda_1, \lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_r, \lambda_{r1}, \ldots, \lambda_{rt_r}\} \cup \{\lambda_n\}$, con $\lambda_j \geq 0$ y $\lambda_{ji} \leq 0$ para $j = 1, \ldots, r$ e $i = 1, \ldots, t_j$ y $\lambda_n \leq 0$:

$$\left.\begin{array}{r} \lambda_0 \geq |\lambda|,\ \lambda \in \sigma\ ,\ \sum\limits_{\lambda\in\sigma} \lambda \geq 0 \\ \lambda_j + \lambda_n \leq 0 \quad \text{y} \quad \lambda_j + \sum\limits_{1\leq i\leq t_j} \lambda_{ji} \leq 0\ ,\ 1 \leq j \leq r \end{array}\right\} \Rightarrow \sigma \in Spec\,.$$

- **Perfect 2**, 1955: Si $\sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_r\} \cup \{\lambda_{01}, \ldots, \lambda_{0t_0}\} \cup \{\lambda_{11}, \ldots, \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{r1}, \ldots, \lambda_{rt_r}\}$, con $\{\lambda_0, \lambda_1, \ldots, \lambda_r\}$ el espectro de una matriz no negativa con diagonal $d_0, d_1, \ldots, d_r$ y $\lambda_{ji} \leq 0$ para $j = 0, \ldots, r$ e $i = 1, \ldots, t_j$:

$$\left.\begin{array}{r} \lambda_0 \geq |\lambda|,\ \lambda \in \sigma\ ,\ \sum\limits_{\lambda\in\sigma} \lambda \geq 0 \\ d_j + \sum\limits_{1\leq i\leq t_j} \lambda_{ji} \geq 0\ ,\ 0 \leq j \leq r \end{array}\right\} \Rightarrow \sigma \in Spec\,.$$

  Cuando $\lambda_j \geq 0$, para $j = 0, \ldots, r$, la llamaremos condición **Perfect 2**$^+$.

- **Ciarlet**, 1968: Si $\sigma = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$:

$$\left.|\lambda_j| \leq \frac{\lambda_0}{n}\ ,\ j = 1, \ldots, n \right\} \Rightarrow \sigma \in Spec\,.$$

- **Kellogg**, 1971: Si $\sigma = \{\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n\}$, con $\lambda_0 \geq |\lambda|$ para $\lambda \in \sigma$, $M = \text{máx}\{j \in \{0, 1, \ldots, n\}\ :\ \lambda_j \geq 0\}$ y $K = \{i \in \{1, \ldots, \lfloor n/2 \rfloor\}\ :\ \lambda_i \geq 0,\ \lambda_i + \lambda_{n+1-i} < 0\}$:

$$\left.\begin{array}{r} \lambda_0 + \sum\limits_{i\in K,\, i<s} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-s} \geq 0 \quad \forall s \in K \\ \lambda_0 + \sum\limits_{i\in K} (\lambda_i + \lambda_{n+1-i}) + \sum\limits_{j=M+1}^{n-M} \lambda_j \geq 0 \end{array}\right\} \Rightarrow \sigma \in Spec\,.$$

- **Salzmann**, 1972: Si $\sigma = \{\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n\}$:

$$\left.\sum_{\lambda\in\sigma} \lambda \geq 0 \quad \text{y} \quad \frac{\lambda_i + \lambda_{n-i}}{2} \leq \frac{1}{n+1} \sum_{0\leq j\leq n} \lambda_j\ ,\ 1 \leq i \leq \lfloor n/2 \rfloor \right\} \Rightarrow \sigma \in Spec\,.$$

- **Fiedler**, 1974: Si $\sigma = \{\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n\}$:

$$\left.\lambda_0 + \lambda_n + \sum_{\lambda\in\sigma} \lambda \geq \frac{1}{2} \sum_{1\leq i\leq n-1} |\lambda_i + \lambda_{n-i}| \right\} \Rightarrow \sigma \in Spec\,.$$

- **Borobia**, 1995: Sea $\sigma = \{\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n\}$, $M = \text{máx}\{j \in \{0, \ldots, n\} : \lambda_j \geq 0\}$. Si existe una partición $J_1 \cup \cdots \cup J_t$ de $\{\lambda_{M+1}, \ldots, \lambda_n\}$ con $\{\lambda_0 \geq \cdots \geq \lambda_M \geq \sum_{\lambda \in J_1} \lambda \geq \cdots \geq \sum_{\lambda \in J_t} \lambda\}$ verificando la condición de Kellogg, entonces $\sigma \in Spec$.

- **Soto 2**, 2003: Sea $\sigma = \{\lambda_{11} \geq \cdots \geq \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{r1} \geq \cdots \geq \lambda_{rt_r}\}$, con $\lambda_{11} \geq |\lambda|$ para $\lambda \in \sigma$ y $\lambda_{i1} \geq 0$ para $i = 1, \ldots, r$. Sean $S_{ij} = \lambda_{ij} + \lambda_{i,t_{i-j+1}}$ para $j = 2, \ldots, \lfloor t_i/2 \rfloor$, $S_{(t_i+1)/2} = \text{mín}\{\lambda_{(t_i+1)/2}, 0\}$ si t_i es impar para $i = 1, \ldots, r$, $T_i = \lambda_{i1} + \lambda_{it_i} + \sum_{S_{ij}<0} S_{ij}$ para $i = 1, \ldots, r$ y $\lambda_M = \text{máx}\{-\lambda_{1t_1} - \sum_{S_{1j}<0} S_{1j}; \text{máx}_{2 \leq i \leq r}\{\lambda_{i1}\}\}$:

$$\left.\lambda_{11} \geq \lambda_M - \sum_{T_i<0,\ 2 \leq i \leq r} T_i \right\} \Rightarrow \sigma \in Spec\,.$$

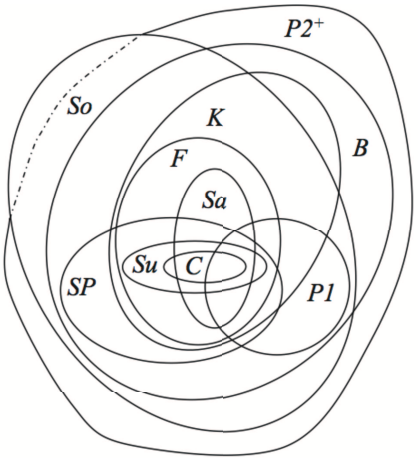
- **Soto-Rojo**, 2006: Si $\sigma = \{\lambda_{11} \geq \cdots \geq \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{r1} \geq \cdots \geq \lambda_{rt_r}\}$, con $\lambda_{11} \geq |\lambda|$ para $\lambda \in \sigma$, $\lambda_{i1} \geq 0$ para $i = 1, \ldots, r$ y $\{\lambda_{11}, \lambda_{21}, \ldots, \lambda_{r1}\}$ el espectro de una matriz no negativa con diagonal $d_1, \ldots, d_r$:

$$\left. d_i \geq \lambda_{i2} \quad \text{y} \quad \{d_i, \lambda_{i2}, \ldots, \lambda_{it_i}\} \in Spec\,,\ 1 \leq i \leq r \right\} \Rightarrow \sigma \in Spec\,.$$

Si una familia de números reales satisface una condición suficiente X diremos que es X-**realizable**. Por ejemplo, si σ verifica la condición de Fiedler diremos que σ es Fiedler-realizable.

Si un espectro σ está dado por una partición, al elemento i-ésimo de la partición lo denotaremos por σ_i.

El siguiente diagrama muestra las relaciones entre las anteriores condiciones suficientes y aparece en [3]. La línea punteada indica el desconocimiento que entonces tenían los autores sobre si Soto implica o no Perfect 2^+.



Suleimanova = Su
Suleimanova-Perfect = SP
Perfect 1 = $P1$
Perfect 2^+ = $P2^+$
Ciarlet = C
Kellogg = K
Salzmann = Sa
Fiedler = F
Borobia = B
Soto 2 = So ($S2$ en la sección 3)

2 Negatividad y margen de realizabilidad

Sea X una condición suficiente, siguiendo [1], definimos la X-**negatividad** de una familia $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \boldsymbol{R}$ con $\lambda_1 \geq \lambda_j$ para $j = 2, \ldots, n$, como

$$\mathcal{N}_X(\sigma) = \text{mín}\{\epsilon \geq 0 : \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\} \text{ es } X\text{-realizable}\}$$

y como $+\infty$ si $\{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ no es X-realizable para ningún $\epsilon \geq 0$. Si σ es X-realizable, definimos el **X-margen de realizablidad** de σ como

$$\mathcal{M}_X(\sigma) = \text{máx}\left\{\epsilon \geq 0 : \begin{array}{c}\{\lambda_1 - \epsilon, \lambda_2, \ldots, \lambda_n\} \text{ es } X\text{-realizable} \\ \text{y } \lambda_1 - \epsilon \geq |\lambda_j| \text{ para } j = 2, \ldots, n\end{array}\right\}.$$

Observemos que la X-negatividad de un espectro σ mide, en cierto sentido, cuánto le falta a σ para ser X-realizable. Una interpretación análoga puede darse al margen de realizabilidad. Se verifican las siguientes propiedades:

- σ es X-realizable si y solo si $\mathcal{N}_X(\sigma) = 0$.
- Si X e Y son condiciones suficientes con X incluida en Y, entonces $\mathcal{N}_Y(\sigma) \leq \mathcal{N}_X(\sigma)$ y $\mathcal{M}_Y(\sigma) \geq \mathcal{M}_X(\sigma)$.
- $\mathcal{N}_X(\sigma) \geq \text{máx}\{0, -\sum_{j=1}^{n} \lambda_j, |\lambda_2| - \lambda_1, \ldots, |\lambda_n| - \lambda_1\}$.
- $\mathcal{M}_X(\sigma) \leq \text{mín}\{\sum_{j=1}^{n} \lambda_j\ , \lambda_1 - |\lambda_2|\ , \ldots, \lambda_1 - |\lambda_n|\}$.
- Si σ es X-realizable con traza cero, entonces $\mathcal{M}_X(\sigma) = 0$.
- Si σ es X-realizable con radio espectral múltiple, entonces $\mathcal{M}_X(\sigma) = 0$.
- Si σ es X-realizable con índice de ciclicidad 2, entonces $\mathcal{M}_X(\sigma) = 0$.

Para un espectro real σ, con las notaciones de las condiciones suficientes y del diagrama dados en la sección anterior, tenemos:

$$\mathcal{N}_{\text{Su}}(\sigma) = \text{máx}\{0, -\lambda_0 - \sum_{\lambda_i < 0} \lambda_i\},\ \mathcal{M}_{\text{Su}}(\sigma) = \text{mín}\{\lambda_0 + \sum_{\lambda_i < 0} \lambda_i\ , \lambda_0 - |\lambda_1|, \ldots, \lambda_0 - |\lambda_n|\},$$

$$\mathcal{N}_{\text{SP}}(\sigma) = \text{mín}\{\mathcal{N}_{\text{Su}}(\sigma_0) : \sigma = \bigcup_{i=0}^{r} \sigma_i\},\ \mathcal{M}_{\text{SP}}(\sigma) = \text{mín}\{\sum_{\lambda \in \sigma} \lambda, \text{mín}\{\lambda_0 - |\lambda| : \lambda \in \sigma - \{\lambda_0\}\}, a\},$$

donde $a = \text{máx}\{\mathcal{M}_{\text{Su}}(\sigma_0) : \sigma = \bigcup_{i=0}^{r} \sigma_i \text{ es } SP\text{-realizable}\}$,

$$\mathcal{N}_{\text{P1}}(\sigma) = \begin{cases} +\infty \ \text{ si } \forall \text{ partición de } \sigma,\ \exists j \in \{1, \ldots, r\} \text{ con } \lambda_j + \lambda_n > 0 \text{ ó } \lambda_j + \sum_{i=1}^{t_j} \lambda_{ji} > 0 \\ \text{máx}\left\{0, -\sum_{\lambda \in \sigma} \lambda, \text{máx}\{|\lambda| - \lambda_0 : \lambda \in \sigma - \{\lambda_0\}\}\right\} \ \text{ en caso contrario,} \end{cases}$$

$$\mathcal{M}_{\text{P1}}(\sigma) = \text{mín}\left\{\sum_{\lambda \in \sigma} \lambda\ , \text{mín}\{\lambda_0 - |\lambda|\ : \lambda \in \sigma - \{\lambda_0\}\}\right\},$$

$$\mathcal{M}_{\text{P2}}(\sigma),\ \mathcal{M}_{\text{P2}^+}(\sigma) \geq \text{mín}\left\{\sum_{i=0}^{r} \text{mín}\{d_i + \sum_{j=1}^{t_i} \lambda_{ij}, m_i\}, \text{mín}\{\lambda_0 - |\lambda|\ : \lambda \in \sigma - \{\lambda_0\}\}\right\},$$

donde m_i, $0 \leq i \leq r$, es el menor elemento de la columna $i+1$ de la matriz no negativa con espectro $\{\lambda_0, \ldots, \lambda_r\}$ considerada, y surge del uso del teorema de Brauer.

$$\mathcal{N}_{\mathrm{C}}(\sigma) = n\,\mathrm{máx}\{|\lambda_1|, \ldots, |\lambda_n|\} - \lambda_0,\ \mathcal{M}_{\mathrm{C}}(\sigma) = \lambda_0 - n\,\mathrm{máx}\{|\lambda_1|, \ldots, |\lambda_n|\},$$

$$\mathcal{N}_{\mathrm{K}}(\sigma) = -\,\mathrm{mín}\left\{0, \mathrm{mín}\{\lambda_0 + \sum_{i\in K,\, i<s} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-s} : k \in K\}, \right.$$
$$\left. \lambda_0 + \sum_{i\in K}(\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \right\},$$

$$\mathcal{M}_{\mathrm{K}}(\sigma) = \mathrm{mín}\left\{\lambda_0 - \lambda_1, \lambda_0 + \lambda_n, \mathrm{mín}\{\lambda_0 + \sum_{i\in K,\, i<s} (\lambda_i + \lambda_{n+1-i}) + \lambda_{n+1-s} : k \in K\}, \right.$$
$$\left. \lambda_0 + \sum_{i\in K}(\lambda_i + \lambda_{n+1-i}) + \sum_{j=M+1}^{n-M} \lambda_j \right\},$$

$$\mathcal{N}_{\mathrm{Sa}}(\sigma) = -\,\mathrm{mín}\left\{0, \sum_{j=0}^{n} \lambda_j, \mathrm{mín}\left\{\sum_{j=0}^{n} \lambda_j - \frac{n+1}{2}(\lambda_i + \lambda_{n-i}) : 1 \leq i \leq \lfloor n/2 \rfloor\right\}\right\},$$

$$\mathcal{M}_{\mathrm{Sa}}(\sigma) = \mathrm{mín}\left\{\lambda_0 - |\lambda_n|, \lambda_0 - |\lambda_1|, \sum_{j=0}^{n} \lambda_j, \mathrm{mín}\{\sum_{j=0}^{n} \lambda_j - \frac{n+1}{2}(\lambda_i + \lambda_{n-i}) : 1 \leq i \leq \lfloor n/2 \rfloor\}\right\},$$

$$\mathcal{N}_{\mathrm{F}}(\sigma) = \mathrm{máx}\left\{0, -\sum_{\lambda\in\sigma} \lambda, -\frac{1}{2}\left[\lambda_0 + \lambda_n + \sum_{\lambda\in\sigma} \lambda - \frac{1}{2}\sum_{i=1}^{n-1} |\lambda_i + \lambda_{n-i}|\right]\right\},$$

$$\mathcal{M}_{\mathrm{F}}(\sigma) = \mathrm{mín}\left\{\sum_{\lambda\in\sigma} \lambda, \lambda_0 - |\lambda_1|, \ldots, \lambda_0 - |\lambda_n|, \frac{1}{2}\left[\lambda_0 + \lambda_n + \sum_{\lambda\in\sigma} \lambda - \frac{1}{2}\sum_{i=1}^{n-1} |\lambda_i + \lambda_{n-i}|\right]\right\}.$$

Observemos que la negatividad de todas las condiciones suficientes consideradas es finita, excepto para Perfect 1. Para Borobia y Soto 2 sólo disponemos de procedimientos de fuerza bruta, para la construcción de todas las particiones posibles de un espectro bajo las restricciones correspondientes. La obtención de la negatividad y el margen de realizabilidad para Perfect 2 y Soto-Rojo implica, por un lado, fuerza bruta para la construcción de las particiones y, por otro lado, la determinación de diagonales y de los elementos mínimos de las columnas de las matrices que los realizan. En relación con estos hechos, podemos considerar los siguientes problemas abiertos:

Problema 1: Hallar el conjunto de diagonales de todas las realizaciones no negativas de un espectro real.
Problema 2: Hallar el conjunto de espectros realizables con diagonal dada.

Estos dos problemas pueden ser tan complejos como el propio RNIEP, de hecho, el primero de ellos está resuelto para $n \leq 3$ y para $n > 3$ sólo se conocen condiciones suficientes. Si la diagonal es nula el problema 2 es el RNIEP con traza cero, sólo resuelto para $n \leq 5$. Parecen igualmente complejos los problemas relacionados con la determinación de los elementos mínimos de las columnas que realizan los espectros, y su relación con los elementos diagonales, y la determinación del elemento diagonal máximo. Véase el siguiente ejemplo.

Ejemplo 1. El espectro $\sigma = \{18, 6, -6, -2, -2, -3, -3\}$ es P2-realizable: $\sigma_0 = \{18, 6, -6\}$ es el espectro de

$$A = \begin{pmatrix} 12 & 3 & 3 \\ 0 & 6 & 12 \\ 12 & 6 & 0 \end{pmatrix} \quad \text{con} \quad \begin{matrix} m_0 = 0 \\ m_1 = 3 \\ m_2 = 0 \end{matrix} \quad \text{ó} \quad B = \begin{pmatrix} 12 & 5 & 1 \\ 6 & 6 & 6 \\ 6 & 12 & 0 \end{pmatrix} \quad \text{con} \quad \begin{matrix} m_0 = 6 & d_0 = 12 \\ m_1 = 5 \text{ y } & d_1 = 6 \\ m_2 = 0 & d_2 = 0 \end{matrix} .$$

Para $\{-2, -2, -3, -3\} = \sigma_1 \cup \sigma_2 \cup \sigma_3$ tenemos varias particiones posibles

Partición $\sigma_1 \cup \sigma_2 \cup \sigma_3$	cota $\mathcal{M}_{\mathrm{P2}}(\sigma)$ con A	cota $\mathcal{M}_{\mathrm{P2}}(\sigma)$ con B
$\{-2, -2\} \cup \{-3, -3\} \cup \emptyset$	0	6
$\{-3, -3\} \cup \{-2, -2\} \cup \emptyset$	2	8
$\{-2, -2, -3\} \cup \{-3\} \cup \emptyset$	3	8
$\{-2, -2, -3, -3\} \cup \emptyset \cup \emptyset$	3	7

El espectro σ también es P2$^+$-realizable: $\sigma_0 = \{18, 6\}$ es el espectro de

$$C_a = \begin{pmatrix} 18 & 0 \\ a & 6 \end{pmatrix}, \quad a \geq 0, \quad \text{y} \quad D = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}.$$

Según las particiones de $\{-2, -2, -3, -3, -6\} = \sigma_1 \cup \sigma_2$ obtenemos diferentes cotas para $\mathcal{M}_{\mathrm{P2}}(\sigma)$. Podemos concluir que $\mathcal{M}_{\mathrm{P2}}(\sigma) = \mathcal{M}_{\mathrm{P2}^+}(\sigma) = 8$. Obsevemos que el conocimiento de la diagonal no garantiza obtener una cota óptima.

3 Nuevas condiciones suficientes y nuevas relaciones

En [4] los autores prueban que Soto 2 implica estrictamente P2$^+$ (ver línea discontinua en el diagrama de la sección 1) y consideran las siguientes nuevas condiciones suficientes para que σ sea el espectro de una matriz no negativa:

- **Juego**, 2008: Si σ con n elementos puede alcanzarse empezando con los n espectros $\{0\},\ldots,\{0\}$ aplicando sucesivamente, en cualquier orden y cualquier número de veces, las reglas 1, 2 ó 3, entonces $\sigma \in Spec$, donde
  Regla 1: Si $\sigma = \{\lambda_1,\ldots,\lambda_n\} \in Spec$ con $\lambda_1 \geq |\lambda|$ para $\lambda \in \sigma$, entonces $\{\lambda_1+\epsilon, \lambda_2-\epsilon, \lambda_3,\ldots,\lambda_n\} \in Spec\,, \quad \forall \epsilon > 0$.
  Regla 2: Si $\sigma = \{\lambda_1,\ldots,\lambda_n\} \in Spec$ con $\lambda_1 \geq |\lambda|$ para $\lambda \in \sigma$, entonces $\{\lambda_1+\epsilon, \lambda_2,\ldots,\lambda_n\} \in Spec\,, \quad \forall \epsilon > 0$.
  Regla 3: Si $\sigma_1, \sigma_2 \in Spec$, entonces $\sigma_1 \cup \sigma_2 \in Spec$.
  Observemos que la presencia o no de elementos nulos en σ no modifica la juego-realizabilidad.
  Esta condición suficiente en [2] es llamada C-realizabilidad y hemos cambiado el nombre para evitar confusiones con las notaciones y nombres de otras condiciones suficientes.
- **Soto 3 = S3**, 2013: Si $\sigma = \{\lambda_{11} \geq \cdots \geq \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{r1} \geq \cdots \geq \lambda_{rt_r}\}$, con $\lambda_{11} \geq |\lambda|$ para $\lambda \in \sigma$, $\lambda_{i1} \geq 0$, para $i = 1,\ldots,r$, $\{\lambda_{11},\ldots,\lambda_{1t_1}\}$ Soto 2 realizable y $\gamma = \text{máx}\{\lambda_{11} - \mathcal{M}_{S2}(\sigma_1), \text{máx}_{2\leq i\leq r}\{\lambda_{i1}\}\}$:

$$\left.\lambda_{11} \geq \gamma + \sum_{i=2}^{r} \mathcal{N}_{S2}(\sigma_i)\right\} \Rightarrow \sigma \in Spec\,.$$

- **Soto p = Sp**, 2013: Si $\sigma = \{\lambda_{11} \geq \cdots \geq \lambda_{1t_1}\} \cup \cdots \cup \{\lambda_{r1} \geq \cdots \geq \lambda_{rt_r}\}$, con $\lambda_{11} \geq |\lambda|$ para $\lambda \in \sigma$, $\lambda_{i1} \geq 0$ para $i = 1,\ldots,r$, y $\{\lambda_{11},\ldots,\lambda_{1t_1}\}$ Soto $p-1$ realizable con $p \geq 3$ y $\gamma = \text{máx}\{\lambda_{11} - \mathcal{M}_{Sp-1}(\sigma_1),\ \text{máx}_{2\leq i\leq r}\{\lambda_{i1}\}\}$:

$$\left.\lambda_{11} \geq \gamma + \sum_{i=2}^{r} \mathcal{N}_{Sp-1}(\sigma_i)\right\} \Rightarrow \sigma \in Spec\,.$$

  En la práctiva, no es necesario conocer $\mathcal{M}_{Sp-1}(\sigma_1)$, basta conocer una cota inferior suya para poder aplicar la condición Soto p, ver [4,5].

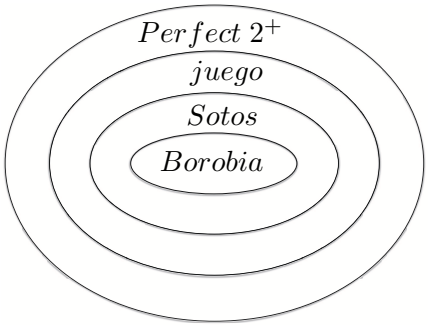
Los siguientes resultados aparecen en [4] y los enunciaremos sin demostración.

Teorema 1. *Juego implica Perfect 2^+ y la inclusión es estricta.*

Teorema 2. *1. La inclusión de Perfect 2^+ en Soto-Rojo es estricta.*
2. Soto p está estrictamente contenido en Soto $p+1$, para $p \geq 3$.
3. Kellogg y Borobia son independientes de Soto p, para $p \geq 3$.
4. Soto p implica juego, para $p \geq 3$, y la inclusión es estricta.

Teorema 3. *Si σ es Borobia realizable, entonces σ es Soto p realizable para algún p (p depende de σ).*

A continuación resumimos los resultados anteriores en un diagrama y damos ejemplos que lo justifican.



$$Sotos = \bigcup_{p\geq 2} Soto\ p$$

$\overline{X}$ = la condición X no se cumple

$Sotos \cap \overline{Borobia}$: $\{3, 3, 1, 1, -2, -2, -2, -2\}$

$juego \cap \overline{Sotos}$: ?

$Perfect\ 2^+ \cap \overline{juego}$: $\{6, 1, 1, -4, -4\}$

En [4] se describe una sucesión de espectros que permiten discriminar la validez de las condiciones Soto p para distintos valores de p. No la incluimos aquí porque su extensión excede la capacidad de este documento.

La condición suficiente juego es difícilmente algoritmizable y no disponemos de expresiones explícitas para $\mathcal{N}_{\text{juego}}(\sigma)$ y $\mathcal{M}_{\text{juego}}(\sigma)$. Una condición necesaria para juego es que, en el espectro ordenado, los elementos positivos mayoricen débilmente a los negativos. Este resultado permite dar la siguiente cota para un espectro σ juego-realizable que, sin pérdida de generalidad, podemos suponer de la forma $\sigma = \{\lambda_1 \geq \cdots \geq \lambda_p \geq 0 \geq -\mu_p \geq \cdots \geq -\mu_1\}$:

$$\mathcal{M}_{\text{juego}}(\sigma) \geq \text{mín}\left\{\lambda_1 - \lambda_2, \lambda_1 - \mu_1, \sum_{\lambda\in\sigma} \lambda, \text{mín}\{\sum_{i=1}^{k}(\lambda_i - \mu_i)\ :\ k = 1, \ldots, p\}\right\}.$$

Para la obtención de la negatividad y el margen de realizabildad para Soto p, de nuevo, sólo disponemos de la fuerza bruta para la construcción de las particiones.

Ejemplo 2. El espectro $\sigma = \{6, 3, 3, -5, -5\}$ sólo es realizable por Perfect 2^+ (por ello también por Perfect 2 y por Soto-Rojo) y no por ninguna de las condiciones suficientes incluidas en Perfect 2^+. Para este espectro tenemos

$$\mathcal{N}_{\text{Su}}(\sigma) = 4\,, \quad \mathcal{N}_{\text{SP}}(\sigma) = 4\,, \quad \mathcal{N}_{\text{P1}}(\sigma) = +\infty\,, \quad \mathcal{N}_{\text{C}}(\sigma) = 14\,,$$
$$\mathcal{N}_{\text{Sa}}(\sigma) = 13\,, \quad \mathcal{N}_{\text{F}}(\sigma) = 1\,, \quad \mathcal{N}_{\text{juego}}(\sigma) = 1\,, \quad \mathcal{M}_{\text{P2}^+}(\sigma) = 0\,.$$

Como Kellogg, Borobia y Soto p están comprendidos entre Fiedler y juego obtenemos que todas sus negatividades son 1.

Ejemplo 3. Para el espectro $\sigma = \{12, 6, 1, 1, 1, 1, -2, -3, -3, -4, -4\}$ se tiene

$$\mathcal{N}_{\text{C}}(\sigma) = 48\,, \quad \mathcal{N}_{\text{Su}}(\sigma) = 4\,, \quad \mathcal{N}_{\text{P1}}(\sigma) = +\infty\,, \quad \mathcal{N}_{\text{Sa}}(\sigma) = 5\,,$$
$$\mathcal{M}_{\text{SP}}(\sigma) = 2\,, \quad \mathcal{M}_{\text{F}}(\sigma) = 3\,, \quad \mathcal{M}_{\text{K}}(\sigma) = 3\,, \quad \mathcal{M}_{\text{B}}(\sigma) = 4\,,$$
$$\mathcal{M}_{\text{Sp}}(\sigma) = 5\,, \quad p \geq 2\,, \quad \mathcal{M}_{\text{juego}}(\sigma) = 5\,, \quad \mathcal{M}_{\text{P2}^+}(\sigma) = 6\,.$$

Referencias

[1] A. Borobia, J. Moro, R. L. Soto, Negativity compensation in the nonnegative inverse eigenvalue problem, *Linear Algebra Appl.* 393 (2004) 73-89.

[2] A.Borobia, J. Moro, R. L. Soto, A unified view on compensation criteria in the real nonnegative inverse eigenvalue problem, *Linear Algebra Appl.* 428 (2008) 2574-2584.

[3] C. Marijuán, M. Pisonero, R. L. Soto, A map of sufficient conditions for the real nonnegative inverse eigenvalue problem, *Linear Algebra Appl.* 426 (2007) 690-705.

[4] C. Marijuán, M. Pisonero, R. L. Soto, Updating the map of sufficient conditions for the RNIEP, *preprint* 2014.

[5] R. L. Soto, A family of realizability criteria for the real and symmetric nonnegative inverse eigenvalue problem, *Numer. Linear Algebra Appl.* 20 (2013) 336-348.

Linearly dependent vectorial decomposition of clutters

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Abstract. This paper deals with the question of completing a monotone increasing family of subsets Γ of a finite set Ω to obtain the linearly dependent subsets of a family of vectors of a vector space. Specifically, we demonstrate that such vectorial completions of the family of subsets Γ exist and, in addition, we show that the minimal vectorial completions of the family Γ provide a decomposition of the clutter Λ of the inclusion-minimal elements of Γ. The computation of such vectorial decomposition of clutters is also discussed in some cases.

Key words: Clutter, Antichain, Hypergraph, Matroid, Decomposition.

1 Introduction

A *monotone increasing* family of subsets Γ of a finite set Ω is a collection of subsets of Ω such that any superset of a set in the family Γ must be in Γ. All the inclusion-minimal elements of Γ determine a *clutter* Λ, that is, a collection of subsets of Ω none of which is a proper subset of another. Clutters are also known as *antichains*, *Sperner systems* or *simple hypergraphs*.

A wide variety of examples of monotone increasing families exist, among which we find the collection of the linearly dependent subsets of vectors in a vector space. We say that a clutter Λ is *LD-vectorial* if its elements are the inclusion-minimal linearly dependent subsets of an indexed family of vectors of a vector space. In other words, the LD-vectorial clutters are exactly those corresponding to the set of *circuits of representable matroids*.

In some cases it is convenient to use clutters that are either LD-vectorial or are closed to be LD-vectorial. Examples of this situation can be found in the context of secret-sharing schemes [3,5], or in the framework of algebraic combinatorics and commutative algebra [1,6]. For instance, in the context of secret-sharing schemes, the LD-vectorial clutters become a crucial issue

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for providing general bounds on the optimal information rate of the scheme, while in the framework of algebraic combinatorics and commutative algebra, they are useful for controlling certain arithmetic properties of either monomial ideals or the face rings of simplicial complexes.

In general, a clutter is far from being LD-vectorial. Therefore it is of interest to determine how it can be transformed into an LD-vectorial clutter. This paper deals with this issue; that is, with the question of finding *LD-vectorial completions of a clutter*. Specifically, the goal of this paper is to prove that these completions exist and that the *minimal* ones provide a *decomposition* of the clutter.

The outline of the paper is as follows. In Section 2 we recall some definitions and basic facts about clutters and present the problem of the LD-vectorial completion of a clutter. Our main results are gathered in Section 3; namely, we present two theorems concerning LD-vectorial decomposition of clutters (Theorem 1 and Theorem 2). Finally, Section 4 is devoted to analyzing the computation of such decompositions (Proposition 1). Due to limitations of space, the proofs are omitted.

2 LD-vectorial clutters and LD-vectorial completions

In this section we present the definitions and basic facts concerning families of subsets, clutters and LD-vectorial clutters that are used in the paper.

Let Ω be a finite set. A family of subsets Γ of Ω is *monotone increasing* if any superset of a set in Γ must be in Γ; that is, if $A \in \Gamma$ and $A \subseteq A' \subseteq \Omega$, then $A' \in \Gamma$. A *clutter* of Ω is a collection of subsets Λ of Ω, none of which is a proper subset of another; that is, if $A, A' \in \Lambda$ and $A \subseteq A'$ then $A = A'$.

Observe that if Γ is a monotone increasing family of subsets of Ω, then the collection $\min(\Gamma)$ of its inclusion-minimal elements is a clutter; while if Λ is a clutter on Ω, then the family $\Lambda^+ = \{A \subseteq \Omega : A_0 \subseteq A \text{ for some } A_0 \in \Lambda\}$ is a monotone increasing family of subsets. Clearly $\Gamma = (\min(\Gamma))^+$ and $\Lambda = \min\left(\Lambda^+\right)$. So a monotone increasing family of subsets Γ is determined uniquely by the clutter $\min(\Gamma)$, while a clutter Λ is determined uniquely by the monotone increasing family Λ^+.

Let Λ_1, Λ_2 be two clutters on Ω. It is clear that if $\Lambda_1 \subseteq \Lambda_2$ then $\Lambda_1^+ \subseteq \Lambda_2^+$. However, the converse is not true; that is, there exist clutters with $\Lambda_1 \not\subseteq \Lambda_2$ and $\Lambda_1^+ \subseteq \Lambda_2^+$. For instance, on the finite set $\Omega = \{1, 2, 3\}$, let us consider the clutters $\Lambda_1 = \{\{1, 2\}, \{2, 3\}\}$ and $\Lambda_2 = \{\{1\}, \{2, 3\}\}$. Then $\Lambda_1 \not\subseteq \Lambda_2$, while $\Lambda_1^+ = \{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\} \subseteq \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} = \Lambda_2^+$.

This fact leads us to consider a binary relation $\leq$ defined on the set of clutters on Ω. Namely, if Λ_1 and Λ_2 are two clutters on Ω, then we say that $\Lambda_1 \leq \Lambda_2$ if and only if $\Lambda_1^+ \subseteq \Lambda_2^+$. The following lemma will be used several times throughout the paper.

Lemma 1. *Let Ω be a finite set. The following statements hold:*

1. *If Λ_1, Λ_2 are two clutters on Ω then, $\Lambda_1 \leq \Lambda_2$ if and only if for all $A_1 \in \Lambda_1$ there exists $A_2 \in \Lambda_2$ such that $A_2 \subseteq A_1$.*
2. *The binary relation $\leq$ is a partial order on the set of clutters of Ω.*

There are many interesting families of clutters that can be considered. However, because of their applications, we are interested in those clutters that are *LD-vectorial*.

Let $\Omega = \{x_1, \ldots, x_n\}$ be a finite set of n elements. A monotone increasing family Γ of subsets of Ω is said to be an *LD-vectorial family* if there exists an indexed family of vectors $v_1, \ldots, v_n$ of a $\mathbb{K}$-vector space (that can be $v_i = v_j$) such that $\{x_{i_1}, \ldots, x_{i_r}\} \in \Gamma$ if and only if $\{v_{i_1}, \ldots, v_{i_r}\}$ is a linearly dependent set of vectors. A clutter Λ on Ω is said to be an *LD-vectorial clutter* if the monotone increasing family Λ^+ is an LD-vectorial family.

In other words, a monotone increasing family of subsets Γ is LD-vectorial if Γ is the family of the *dependent sets of a representable matroid* $\mathcal{M}$ with ground set Ω; whereas a clutter Λ is LD-vectorial if the clutter Λ is the set of *circuits of a representable matroid* $\mathcal{M}$ with ground set Ω. (The reader is referred to [4,7] for general references on matroid theory). Observe that since the binary relation $\leq$ is a partial order on the set of clutters of Ω, it is also a partial order on the set of LD-vectorial clutters, and therefore $\leq$ is a partial order on the set of representable matroids. In matroid theory, this is equivalent to the *weak order* (see [4, Proposition 7.3.11]).

There are clutters on a finite set Ω that are not LD-vectorial (in fact, there are matroids that are not representable matroids). So, a natural question that arises at this point is to determine how to complete a clutter Λ to obtain an LD-vectorial clutter. In order to look for LD-vectorial completions, it is important to take into account the binary relation $\leq$ rather than the inclusion $\subseteq$. This is due to the fact that, as the following example shows, there exist clutters Λ such that $\Lambda \not\subseteq \Lambda'$ for any LD-vectorial clutter Λ'.

Example 1. Let us consider the clutter $\Lambda = \{\{1,2\},\{1,3\},\{2,3,4\}\}$ on the finite set $\Omega = \{1,2,3,4\}$. Observe that $\big(\{1,2\} \cup \{1,3\}\big) \setminus \{1\} = \{2,3\} \subset \{2,3,4\}$. Hence it follows that Λ is not an LD-vectorial clutter and, moreover, $\Lambda \not\subseteq \Lambda'$ for any LD-vectorial clutter Λ'. However, we have that $\Lambda \leq \Lambda'$, where Λ' is the LD-vectorial clutter $\Lambda' = \{\{1\},\{2,3,4\}\}$ (an LD-vectorial realization of Λ' is given by the set of vectors $\{v_1, v_2, v_3, v_4\}$ where $v_1 = (0,0,0)$, $v_2 = (1,0,0)$, $v_3 = (0,1,0)$ and $v_4 = (0,0,1)$). Futhermore, if Λ'' is the clutter on Ω defined by $\Lambda'' = \{\{1,2\},\{1,3\},\{2,3\}\}$, then we have that $\Lambda \leq \Lambda''$ and that the clutter Λ'' is also an LD-vectorial clutter (an LD-vectorial realization of Λ'' is given by the set of vectors $\{w_1, w_2, w_3, w_4\}$ where $w_1 = (1,1)$, $w_2 = (1,1)$, $w_3 = (1,1)$ and $w_4 = (0,1)$). Notice that now the clutter Λ can be obtained from the LD-vectorial clutters Λ' and Λ''. Indeed, it is easy to check that $\Lambda = \min\big\{A' \cup A''$ where $A' \in \Lambda'$ and $A'' \in \Lambda''\}$. Therefore, the clutters Λ' and Λ'' in some way provide a decomposition of Λ.

The above example leads us to the following definition. Let Λ be a clutter on a finite set Ω. An *LD-vectorial completion* of the clutter Λ is an LD-vectorial clutter Λ' on the finite set Ω such that $\Lambda \leq \Lambda'$.

The set of all the LD-vectorial completions of a clutter Λ is denoted by LD-Vect(Λ). Observe that if $\emptyset \in \Lambda$, then $\Lambda = \{\emptyset\}$, and thus LD-Vect(Λ) $= \emptyset$. So, from now on, throughout the paper we assume that $\emptyset \notin \Lambda$ if Λ is a clutter. As shown in the next section, this assumption guarantees that LD-Vect(Λ) $\neq \emptyset$ for all clutters and, in addition, we demonstrate that suitable clutters in the non-empty set LD-Vect(Λ) provide a decomposition of the clutter Λ in the same way as in Example 1.

3 Two results on LD-vectorial decompositions

The aim of this section is to present two theoretical results concerning the "decomposition" of a clutter Λ into LD-vectorial clutters $\Lambda_1, \ldots, \Lambda_r$, (Theorem 1 and Theorem 2). The general case is considered in the first theorem, while the second deals with those "decompositions" of Λ whose LD-vectorial components $\Lambda_1, \ldots, \Lambda_r$ admit vectorial realizations over a fixed field $\mathbb{K}$.

Let Λ be a clutter on a finite set Ω. Our first result, Theorem 1, states that the set LD-Vect (Λ) of its LD-vectorial completions is a non-empty set and that its minimal elements provide a decomposition of Λ (in the sense that the elements A of the clutter Λ can be obtained from the elements A_i of its minimal LD-vectorial completions $\Lambda_1, \ldots, \Lambda_r$).

Theorem 1. *Let Λ be a clutter on a finite set Ω. Then, LD-Vect $(\Lambda) \neq \emptyset$ and $\Lambda = \min\left\{A_1 \cup \cdots \cup A_r \text{ where } A_i \in \Lambda_i\right\}$ where $\Lambda_1, \ldots, \Lambda_r$ are the minimal elements of the poset $\left(\text{LD-Vect}\,(\Lambda), \leq\right)$ of the LD-vectorial completions of Λ. In particular, the clutter Λ has a unique minimal LD-vectorial completion if, and only if, Λ is an LD-vectorial clutter.*

Observe that the previous theorem, Theorem 1, deals with LD-vectorial completions and decompositions in the case where no field restrictions are assumed. The next theorem, Theorem 2, states that a similar result occurs if we consider only the case in which the vector spaces of the LD-vectorial completions are over a fixed field $\mathbb{K}$. Before stating the theorem, we introduce some notations.

Let Λ be a clutter on a finite set Ω and let $\mathbb{K}$ be a field. Let us denote by LD-Vect $_{\mathbb{K}}(\Lambda)$ the set whose elements are the LD-vectorial completions of Λ over $\mathbb{K}$; that is, the elements of LD-Vect $_{\mathbb{K}}(\Lambda)$ are the LD-vectorial clutters Λ' over $\mathbb{K}$ with $\Lambda \leq \Lambda'$. Therefore, LD-Vect $(\Lambda) = \bigcup_{\mathbb{K}}$ LD-Vect $_{\mathbb{K}}(\Lambda)$.

The next theorem states that the set LD-Vect $_{\mathbb{K}}(\Lambda)$ is a non-empty set and that its minimal elements provides a decomposition of the clutter Λ (in the sense that the elements of Λ can be obtained from the elements of its minimal LD-vectorial completions over $\mathbb{K}$).

Theorem 2. *Let Λ be a clutter on a finite set Ω and let $\mathbb{K}$ be a field. Then, LD-Vect$_{\mathbb{K}}(\Lambda) \neq \emptyset$ and $\Lambda = \min\left\{A_1 \cup \cdots \cup A_r \text{ where } A_i \in \Lambda_i\right\}$ where $\Lambda_1, \ldots, \Lambda_r$ are the minimal elements of the poset $\left(\text{LD-Vect}_{\mathbb{K}}(\Lambda), \leq\right)$. In particular, the clutter Λ has a unique minimal LD-vectorial completion over $\mathbb{K}$ if, and only if, Λ is an LD-vectorial clutter over $\mathbb{K}$.*

4 Computing LD-vectorial decompositions

This section is devoted to the computation of the LD-vectorial decomposition of clutters. First we present a partial result (Proposition 1). After this, several examples are given in order to illustrate this proposition (Example 2 and Example 3). Finally, an example where the proposition cannot be applied is analyzed (Example 4).

Our result, Proposition 1, provides a complete description of the minimal LD-vectorial completion of a clutter Λ on a finite set Ω of size at most seven. In order to state this proposition, we need to introduce two transformations of clutters, the *I-transformation* and the *$\mathcal{T}$-transformation* (see [2]).

Let Λ be a clutter on a finite set Ω. For a subset $X \subseteq \Omega$, we denote by $I_\Lambda(X)$ the intersection of the subsets A in Λ contained in X, that is, $I_\Lambda(X) = \bigcap_A A$ where $A \in \Lambda$ and $A \subseteq X$, (this intersection is the one involved in the characterization of the set of circuits in connected matroids, see [4, Theorem 4.3.2]). We say that a clutter Λ' is an *I-transformation* of the clutter Λ if $\Lambda' = \min\left(\Lambda \cup \{A_1 \cap A_2\}\right)$ where $A_1, A_2 \in \Lambda$ are two different subsets with $I_\Lambda(A_1 \cup A_2) \neq \emptyset$.

The definition of *$\mathcal{T}$-transformation* is more involved. Let Λ be a clutter. We define the elementary transformations $\mathcal{T}^{(1)}(\Lambda)$ and $\mathcal{T}^{(2)}(\Lambda)$ of the clutter Λ as the clutters $\mathcal{T}^{(1)}(\Lambda) = \min\left(\Lambda \cup \left\{(A_1 \cup A_2) \setminus \{x\}\right.\right.$, where $A_1, A_2 \in \Lambda$ are different and $\left.\left. x \in A_1 \cap A_2\right\}\right)$ and $\mathcal{T}^{(2)}(\Lambda) = \min\left(\Lambda \cup \left\{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2)\right.\right.$, where $\left.\left. A_1, A_2 \in \Lambda \text{ are different}\right\}\right)$. Since $\mathcal{T}^{(1)}(\Lambda)$ and $\mathcal{T}^{(2)}(\Lambda)$ are clutters, we can apply the elementary transformations again. Hence, for $(i_1, i_2) \in \{1,2\} \times \{1,2\}$ we can consider the clutter $\mathcal{T}^{(i_2)}(\mathcal{T}^{(i_1)}(\Lambda))$. At this point we proceed in a recursive way. Let $r \geq 2$ be a non-negative integer and let $(i_1, \ldots, i_r) \in \{1,2\}^r$ be an r-tuple. Then we define the clutter $\mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda)$ by the recursion formula $\mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda) = \mathcal{T}^{(i_r)}(\mathcal{T}^{(i_1,\ldots,i_{r-1})}(\Lambda))$; that is, $\mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda)$ is the i_r elementary transformation of $\mathcal{T}^{(i_1,\ldots,i_{r-1})}(\Lambda)$. We say that a clutter Λ' is a *$\mathcal{T}$-transformation* of the clutter Λ if it is obtained from Λ in this way, that is, if $\Lambda' = \mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda)$ for some r-tuple $(i_1, \ldots, i_r)$.

Proposition 1. *Let Λ be a non-LD-vectorial clutter on a finite set Ω of size $|\Omega| = n \leq 7$. Let Λ' be a clutter such that $\Lambda \leq \Lambda'$. Then the following statements hold:*

1. *The clutter Λ' is an LD-vectorial completion of Λ if, and only if, Λ' is the unique clutter which can be obtained from Λ' by applying I-transformations or $\mathcal{T}$-transformations.*
2. *If the clutter Λ' is a minimal element of the poset $\big(LD\text{-}Vect\,(\Lambda)\,,\leq\big)$, then there is a monotone increasing sequence of clutters $\Lambda=\Lambda_0<\Lambda_1<\ldots<\Lambda_r=\Lambda'$ such that for $i\geq 1$, either Λ_i is an I-transformation of Λ_{i-1}, or Λ_i is a $\mathcal{T}$-transformation of Λ_{i-1}.*

We now give two examples to illustrate the above proposition.

Example 2. First, let us consider the clutter $\Lambda=\{\{1,2\},\{1,3\},\{2,3,4\}\}$ on the finite set $\Omega=\{1,2,3,4\}$. In this case only two clutters are obtained by using or by combining I-transformations and $\mathcal{T}$-transformations; namely, the clutters $\Lambda_1=\{\{1\},\{2,3,4\}\}$ and $\Lambda_2=\{\{1,2\},\{1,3\},\{2,3\}\}$. Therefore, from Proposition 1 it follows that the minimal LD-vectorial completions of the clutter Λ are the minimal elements of $\{\Lambda_1,\Lambda_2\}$. In this case, $\Lambda_1\not\leq\Lambda_2$ and $\Lambda_2\not\leq\Lambda_1$, and so $\min\big(\text{LD-Vect}\,(\Lambda)\big)=\{\Lambda_1,\Lambda_2\}$. Observe that now the LD-vectorial decomposition of Λ given in Example 1 can be stated by applying Theorem 1.

Example 3. Now, on the finite set $\Omega=\{1,2,3,4,5\}$, we consider the clutter $\Lambda=\{\{1,2,3\},\{1,2,4\},\{1,5\},\{4,5\}\}$. In such a case, it is a straightforward calculation to check that by using or by combining I-transformations and $\mathcal{T}$-transformations, eleven clutters $\Lambda_1,\ldots,\Lambda_{11}$ can be obtained. Specifically, by using only I-transformations we obtain the clutters $\Lambda_1=\{\{5\},\{1,2\}\}$ and $\Lambda_2=\{\{1\},\{4,5\}\}$. The clutters obtained by using only $\mathcal{T}$-transformations are the clutters $\Lambda_3=\{\{1,4\},\{1,5\},\{4,5\},\{1,2,3\},\{2,3,4\},\{2,3,5\}\}$ and $\Lambda_4=\{\{1,3\},\{1,4\},\{1,5\},\{3,4\},\{3,5\},\{4,5\}\}$, whereas the clutters obtained by combining the I-transformations and the $\mathcal{T}$-transformations are the clutters $\Lambda_5=\{\{1\},\{5\}\}$, $\Lambda_6=\{\{1\},\{2,4\},\{2,5\},\{4,5\}\}$, $\Lambda_7=\{\{5\},\{1,2\},\{1,4\},\{2,4\}\}$, $\Lambda_8=\{\{1,2\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{4,5\}\}$, $\Lambda_9=\{\{5\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$, $\Lambda_{10}=\{\{5\},\{3,4\},\{1,2,3\},\{1,2,4\}\}$, and $\Lambda_{11}=\{\{4\},\{1,5\},\{1,2,3\},\{2,3,5\}\}$. Therefore, by applying Proposition 1 we obtain that the set of the minimal LD-vectorial completions of the clutter Λ is $\min\big(\text{LD-Vect}\,(\Lambda)\big)=\min\{\Lambda_1,\ldots,\Lambda_{11}\}=\{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_9\}$. So, from Theorem 1 we conclude that Λ admits an LD-vectorial decomposition with four components.

To conclude we give an example where Proposition 1 cannot be applied. In addition, the clutter in this example reveals the difference between Theorem 1 and Theorem 2.

Example 4. On the finite set $\Omega=\{1,2,3,4\}$ of four points, we consider the clutter $\Lambda=\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$. It is easy to prove that the clutter Λ is an LD-vectorial clutter over any field $\mathbb{K}\neq\mathbb{Z}/(2)$. Therefore, by

applying Theorem 1 and Theorem 2 we obtain that $\min\left(\text{LD-Vect}\left(\Lambda\right)\right) = \{\Lambda\}$; that $\min\left(\text{LD-Vect}_{\mathbb{K}}\left(\Lambda\right)\right) = \{\Lambda\}$ if $\mathbb{K} \neq \mathbb{Z}/(2)$, and that $\min\left(\text{LD-Vect}_{\mathbb{Z}/(2)}\left(\Lambda\right)\right)$ has at least two elements. In this example, by an extensive exploration it is not hard to show that $\min\left(\text{LD-Vect}_{\mathbb{Z}/(2)}\left(\Lambda\right)\right)$ has six elements; namely $\min\left(\text{LD-Vect}_{\mathbb{Z}/(2)}\left(\Lambda\right)\right) = \{\Lambda_{1,2}, \Lambda_{1,3}, \Lambda_{1,4}, \Lambda_{2,3}, \Lambda_{2,4}, \Lambda_{3,4}\}$, where if $1 \leq i_1 < i_2 \leq 4$ and if $\{i_3, i_4\} = \{1,2,3,4\} \setminus \{i_1, i_2\}$, then $\Lambda_{i_1,i_2} = \{\{i_1, i_2\}, \{i_1, i_3, i_4\}, \{i_2, i_3, i_4\}\}$. Now, from Theorem 2 it follows that the clutter Λ admits an LD-vectorial decomposition over $\mathbb{Z}/(2)$ with six components.

References

[1] Herzog, J., and T. Hibi, "Monomial Ideals," Grad. Texts in Math. 260, Springer, London, 2010.

[2] Martí-Farré, J., *From clutters to matroids*, Electronic Journal of Combinatorics, **21(1)** (2014), P1.11 (14 pag.).

[3] Martí-Farré, J., and C. Padró, *On secret-sharing schemes, matroids and polymatroids*, Journal of Mathematical Cryptology, **4** (2010), 95–120.

[4] Oxley, J.G., "Matroid Theory," Oxford Graduate Text in Mathematics. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

[5] Seymour, P.D., *On secret-sharing matroids*, J. Combin. Theory Ser. B, **56** (1992), 69–73.

[6] Stanley, R.P., "Combinatorics and Commutative Algebra," Progress in Mathematics 41. Second Edition. Birkhäuser, 1995.

[7] Welsh, D.J.A., "Matroid Theory," Academic Press, London, 1976.

Sobre la enumeración de diseños sobresaturados óptimos con dos niveles

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Resumen. En este trabajo los diseños sobresaturados $E(s^2)$-óptimos y minimax-óptimos k-circulantes con $n = 2p$ $(3 \leq p \leq 14)$ observaciones, $m = k(n-1)$ factores y $s_{\max} \in \{2, 4, 6\}$ son enumerados por medio de una búsqueda computacional.

Palabras clave: Diseños sobresaturados, $E(s^2)$-optimalidad, minimax-optimalidad, diseño de bloques incompletos balanceados resolubles, búsqueda computacional

1 Introducción

Los diseños sobresaturados (DSSs) son diseños factoriales en los cuales el número de factores es mayor que el número de corridas experimentales (observaciones). Los DSSs son usados en la etapa inicial de un experimento industrial o científico para la detección de factores activos. Son extremadamente útiles cuando hay un número grande de factores bajo investigación y sólo un número muy limitado de corridas experimentales son disponibles. Los diseños sobresaturados con dos niveles fueron introducidos por Satterthwaite [11]. Booth y Cox [1] fueron los primeros en examinarlos sistemáticamente y en sugerir los criterios minimax y $E(s^2)$-óptimos. Desde entonces, varios métodos ha sido propuestos para la construcción de estos diseños, ver por ejemplo, [3,6,7,10]. Bajo el criterio minimax, el objetivo es minimizar $s_{\max}$ donde $s_{\max} = \max_{i<j}|s_{ij}|$ y s_{ij}/n es la correlación entre los factores i y j. Ryan y Bulutoglu [10] demostraron que la $E(s^2)$-optimalidad es una condición necesaria para que la optimalidad mimimax $s_{\max} \in \{2, 4, 6\}$.

En este trabajo, usando una búsqueda computacional, enumeramos todos los DSSs $E(s^2)$-óptimos y minimax-óptimos k-circulantes con $n = 6$, 8, 10, 12, 14, 16, 18, 20 22, 24, 26 y 28 observaciones, $m = k(n-1)$ factores y $s_{\max} \in \{2, 4, 6\}$. Nuestro metódo de enumeración usa la correspondencia, dada en [5], entre los DSSs $E(s^2)$-óptimos y minimax-óptimos k-circulantes con n observaciones y $m = k(n-1)$ factores y los diseños de bloques incompletos balanceados resolubles con parámetros $(n, 2k(n-1), k(n-1), n/2, k(n/2-1))$

construidos cíclicamente al desarrollar un conjunto de bloques iniciales, los cuales satisfacen ciertas propiedades. Algunas propiedades combinatorias de estos diseños resolubles son usadas para restringir el espacio de búsqueda. Finalmente, reducimos el problema de enumeración de estos diseños resolubles a un problema de encontrar cliques en una gráfica.

2 Criterios de optimalidad para DSSs

Un diseño sobresaturado con dos niveles, n observaciones y m factores es representado por una matriz de orden $n \times m$, $D = D(n, m)$. Cada entrada de la matriz D es ± 1. Las frecuencias de $+1$ y -1 son las mismas en cada columna. La matriz D no tiene dos columnas d_i y d_j tales que $d_i = \pm d_j$. Se puede suponer que la última fila del diseño es una fila de $+1$'s.

Dos DSSs son isomorfos si uno puede ser obtenido del otro por medio de permutaciones de filas y columnas, y una multiplicación de un subconjunto de columnas por -1.

Un criterio comúnmente usado para comparar DSSs del mismo tamaño es la minimización de la función

$$E(s^2) = \sum_{i<j} \frac{s_{ij}^2}{\binom{m}{2}}$$

donde s_{ij} es la (i, j)-ésima entrada de la matriz $D^T D$. El término s_{ij} mide el grado de no ortogonalidad entre los factores i y j. Ngyuen [9] demostró que

$$E(s^2) \geq \frac{n^2(m - n + 1)}{(m - 1)(n - 1)}. \tag{1}$$

Esta cota puede ser alcanzada únicamente si $m = q(m - 1)$ y $n \equiv 0 \pmod 4$, ó si $m = 2q(m - 1)$ y $n \equiv 2 \pmod 4$ para algún entero positivo q. Así n es un número par. Un diseño es llamado $E(s^2)$-óptimo cuando esta cota es alcanzada.

Otro criterio para construir y comparar DSSs con el mismo valor $E(s^2)$ es la minimización de $r_{\max} = s_{\max}/n$, donde $s_{\max} = \max_{i<j}|s_{ij}|$. Este criterio es llamado el criterio minimax. En un DSS, $D(n, m)$, con las propiedades anteriores, se tiene que: (1) Si $n \equiv 0 \pmod 4$ entonces $|s_{ij}|$ pertenece al conjunto $\{0, 4, 8, \ldots\}$. (2) Si $n \equiv 2 \pmod 4$ entonces $|s_{ij}|$ pertenece al conjunto $\{2, 6, 10, \ldots\}$.

Definición 1. *Un DSS es llamado* minimax óptimo *si no hay otro DSS con el mismo tamaño que tenga menor* $s_{\max}$ *o el mismo* $s_{\max}$ *pero con menor frecuencia de* $s_{\max}$.

Teorema 1. ([10]) *Sea* $D(n, m)$ *un DSS* $E(s^2)$*-óptimo con* n *obsevaciones y* $m = k(n - 1)$ *factores.*

(a) *Si* $n \equiv 0$ *(mod* 4*) y* $s_{\max} = 4$*, entonces* $D(n, m)$ *es minimax-óptimo.*
(b) *Si* $n \equiv 2$ *(mod* 4*) y* $s_{\max} \in \{2, 6\}$*, entonces* $D(n, m)$ *es minimax-óptimo.*

3 RBIBDs 1-rotacionales

Un diseño de bloques incompletos balanceados (RBIBDS, por sus siglas en inglés) con (v, b, r, h, λ), denotado por BIBD(v, b, r, h, λ), es una pareja $(V, \mathcal{B})$ donde V es un conjunto de puntos de tamaño v, y $\mathcal{B}$ es una colección de b h-subconjuntos (llamados *bloques*) de V, $h < v$, tal que cada punto está contenido en exactamente r bloques, y cada pareja de puntos de V ocurren juntos en exactamente λ bloques. Claramente, los parámetros satisfacen

$$vr = bh, \quad r(h-1) = \lambda(v-1). \tag{2}$$

Un BIBD(v, b, r, h, λ) es llamado *resoluble* (RBIBD), si sus bloques pueden ser particionados en r conjuntos de bloques, llamados *clases paralelas*, cada una de las cuales particiona el conjunto de puntos.

Dos BIBDs son *isomorfos* si existe una biyección entre los puntos que mapea bloques sobre bloques; tal biyección es llamada un *isomorfismo*. Un *automorfismo* de un diseño es un isomorfismo del diseño sobre el mismo.

Un BIBD, $(V, \mathcal{B})$, es llamado un 1-rotacional si $V = Z_{v-1} \cup \{\infty\}$, y el mapeo ϕ de i a $i + 1 \pmod{v-1}$, que fija ∞, es un automorfismo.

Definición 2. *Sea* $\{S_1 \ldots, S_s, T_1, \ldots, T_t\}$ *una colección de* h*-subconjuntos de* $V = Z_p \cup \{\infty\}$ *tales que*

(a) *Cada* $S_1, \ldots, S_s$ *contiene únicamente elementos de* Z_p*, y cada* $T_1, \ldots, T_t$ *contiene* ∞ *y elementos de* Z_p*.*
(b) *Sea* $S_i = \{s_{i1}, \ldots, s_{ih}\}$ $(1 \le i \le s)$ *y* $T_j = \{t_{j1}, \ldots, t_{jh-1}, \infty\}$ $(1 \le j \le t)$*. Entonces, todo elemento no cero de* Z_p *ocurre* λ *veces entre las diferencias* $s_{ix} - s_{iy}$ $(i = 1, \ldots, s;\ x, y = 1, \ldots, h)$ *y* $t_{jz} - t_{jw}$ $(j = 1, \ldots, t;\ z, w = 1, \ldots, h-1)$*.*

Entonces, $\{S_1, \ldots, S_s, T_1, \ldots, T_t\}$ *es llamada una familia de diferencias* 1*-rotacional con parámetros* (p, s, t, h, λ)*. Los conjuntos* $S_1, \ldots, S_s, T_1, \ldots, T_t$ *son llamados* bloques iniciales*.*

Una familia de diferencias 1-rotacional es llamada *resoluble* si la colección de los $(s + t)$ bloques iniciales pueden ser particionados sobre t subconjuntos de $(s + t)/t$ bloques, es decir $\{S_{11}, \ldots, S_{1q}, T_1\}, \ldots, \{S_{t1}, \ldots, S_{tq}, T_t\}$ (*clases paralelas*), cada una de ellas particiona $V = Z_p \cup \{\infty\}$. Aquí $q = s/t$.

Sea $D = \{d_1, \ldots, d_h\}$ un h-subconjunto de Z_p. Entonces para cualquier $g \in Z_p$, el conjunto $D + g = \{d_1 + g, \ldots, d_h + g\} \pmod{p}$ es llamado una *traslación* de D.

Teorema 2. ([2]) *Sea $S_1, \ldots, S_s, T_1, \ldots, T_t$ una familia de diferencias resoluble 1-rotacional con parámetros (p, s, t, h, λ). Entonces los conjuntos $S_i + g$ $(i = 1, \ldots, s;\ g = 0, \ldots, p-1)$ y $T_j + g$ $(j = 1, \ldots, t;\ g = 0, \ldots, p-1)$, con la convención que $\infty + g = \infty$, forman un RBIBD$(p+1, p(s+t), pt, h, \lambda)$ 1-rotacional.*

4 Diseños sobresaturados a partir de RBIBDs

La estructura cíclica que consideramos para los DSSs está basada sobre k generadores que producen de manera cíclica el resto de las columnas del diseño, ver [5]. Así, un diseño con $n-1$ factores es llamado 1-circulante si es de la forma

$$D = \begin{pmatrix} g_1 & g_{n-1} & \cdots & g_2 \\ g_2 & g_1 & \cdots & g_3 \\ \vdots & \vdots & & \vdots \\ g_{n-2} & g_{n-3} & \cdots & g_{n-1} \\ g_{n-1} & g_{n-2} & \cdots & g_1 \\ 1 & 1 & \ldots & 1 \end{pmatrix}$$

donde $(g_1, g_2 \ldots, g_{n-1})^T$ es la columna generadora que contiene $n/2$ elementos iguales a -1, y $n/2-1$ elementos iguales a $+1$. Entonces, el diseño k-circulante con $m = k(n-1)$ factores es obtenido por la yuxtaposición de las columnas de esos k diseños 1-circulantes.

Ahora, daremos la correspondencia entre los diseños sobresaturados y los RBIBDs. Supongamos que $(S_1+i, T_1+i), \ldots, (S_k+i, T_k+i)$ $(i = 0, \ldots, n-2)$ son las clases paralelas del RBIBD$(n, 2k(n-1), k(n-1), n/2, k(n/2-1))$ generado por la familia de diferencias $(S_1, T_1), \ldots, (S_k, T_k)$. Entonces para cada clase paralela $(S_j + (i-1), T_j + (i-1))$ $(1 \le j \le k;\ 1 \le i \le n-1)$, se define el vector columna d_{ij} como sigue:

$$d_{ij\ell} = \begin{cases} 1 & \text{si } \ell \in T_j + (i-1) \\ -1 & \text{si } \ell \in S_j + (i-1) \end{cases} \tag{3}$$

para $1 \le \ell < n-1$ y $d_{ijn} = 1$, donde $d_{ij\ell}$ es la ℓ-ésima componente de la columna d_{ij}, aquí d_{ij} denota la i-ésima columna del j-ésimo DSS 1-circulante. No es difícil probar que esta correspondencia es uno a uno, y además respeta isomorfismos, ver [5].

Ejemplo. Sea $(\{2,4,5,6\}, \{0,1,3,\infty\}), (\{2,3,4,6\}, \{0,1,5,\infty\})$ una familia de diferencias con parámetros $(7,2,2,4,6)$. Entonces la correspondencia entre este diseño y un DSS, $D(8,14)$, está dada por

$$\begin{aligned} (\{2,4,5,6\}, \{0,1,3,\infty\}) &\quad \Leftrightarrow \quad (1,1,-1,1,-1,-1,-1)^T \\ (\{2,3,4,6\}, \{0,1,5,\infty\}) &\quad \Leftrightarrow \quad (1,1,-1,-1,-1,1,-1)^T. \end{aligned}$$

5 El espacio de búsqueda

Para simplificar la notación, si $n \equiv 0 \pmod 4$ denotamos $n = 4a$; mientras si $n \equiv 2 \pmod 4$, asumimos que $n = 2c$ con c impar. Suponemos que $k \geq 2$.

Definimos la *matriz de intersección de clases paralelas* (MICP) de dos diferentes clases paralelas $Q = (B_1, B_2)$ y $Q' = (B_1', B_2')$ sobre n puntos, como la matriz (a_{ij}) de 2×2 con entradas definidas por $a_{ij} = |B_i \cap B_j'|$ (ver [8]).

Dado que un RBIBD$(n, 2k(n-1), k(n-1), n/2, k(n/2-1))$ tiene $k(n-1)$ clases paralelas para cada clase paralela Q hay $k(n-1)-1$ MICPs con respecto a las otras clases paralelas. Claramente, las sumas de las columnas y de filas de cualquier MICP son iguales a $n/2$. Entonces, re-etiquetando los bloques en las clases paralelas, si es necesario, cada una de las $k(n-1)-1$ matrices asociadas con Q debe de ser alguna de la siguientes

$$\mathcal{S}_0 = \begin{pmatrix} 0 & 2a \\ 2a & 0 \end{pmatrix},\ \mathcal{S}_1 = \begin{pmatrix} 1 & 2a-1 \\ 2a-1 & 1 \end{pmatrix}, \ldots, \mathcal{S}_a = \begin{pmatrix} a & a \\ a & a \end{pmatrix} \text{ para } n = 4a, \tag{4}$$

$$\mathcal{T}_0 = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix},\ \mathcal{T}_1 = \begin{pmatrix} 1 & c-1 \\ c-1 & 1 \end{pmatrix}, \ldots, \mathcal{T}_{\frac{c-1}{2}} = \begin{pmatrix} \frac{c-1}{2} & \frac{c+1}{2} \\ \frac{c+1}{2} & \frac{c-1}{2} \end{pmatrix} \text{ para } n = 2c. \tag{5}$$

Sea Q una clase paralela fija de un RBIBD$(n, 2k(n-1), k(n-1), n/2, k(n/2-1))$. Supongamos que $n = 4a$, y sean $n_0,\ n_1, \ldots, n_a$ el número de matrices $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_a$, respectivamente asociada con Q (ver (4)). Claramente

$$\sum_{i=0}^{a} n_i = k(4a-1) - 1. \tag{6}$$

Por un argumento de conteo de parejas de puntos ocurriendo en un bloque de Q podemos ver que

$$\sum_{i=0}^{a} \left[\binom{i}{2} + \binom{2a-i}{2} \right] n_i = \binom{2a}{2} [k(2a-1) - 1]. \tag{7}$$

De forma similar, para $n = 2c$ con c impar, podemos concluir que

$$\sum_{i=0}^{\frac{c-1}{2}} n_i = k(2c-1) - 1 \tag{8}$$

$$\sum_{i=0}^{\frac{c-1}{2}} \left[\binom{i}{2} + \binom{c-i}{2} \right] n_i = \binom{c}{2} [k(c-1) - 1], \tag{9}$$

donde $n_0, n_1, \ldots, n_{\frac{c-1}{2}}$ son los números de matrices $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{\frac{c-1}{2}}$, respectivamente asociadas con Q (ver (5)).

Por simple algebra tenemos:

Teorema 3. *Sea $n \geq 6$ un entero par.*

(a) *Supongamos que $n = 4a$. Entonces no hay soluciones enteras con $n_0 = \cdots = n_{a-1} = 0$ y $n_a > 0$ para las ecuaciones (6) y (7). Además,*

$$n_0 = \cdots = n_{a-2} = 0, \quad n_{a-1} = a^2(k-1), \quad n_a = k(4a-1) - 1 - a^2(k-1)$$

es una solución entera de (6) y (7).

(b) *Supongamos que $n = 2c$ con c impar. Si k es par, entonces*

$$n_0 = \cdots = n_{\frac{c-5}{2}} = 0, \quad n_{\frac{c-3}{2}} = \frac{(c-1)^2k - (c-1)(c+1)}{8},$$
$$n_{\frac{c-1}{2}} = k(2c-1) - 1 - \frac{(c-1)^2k - (c-1)(c+1)}{8}$$

es una solución entera de las ecuaciones (8) y (9). Además, $n_0 = \cdots = n_{\frac{c-3}{2}} = 0$ y $n_{\frac{c-1}{2}} = k(2c-1) - 1$ si y sólo si $n = 6$ y $k = 2$.

Si $A = (a_{ij})$ es una matriz 2×2, definimos $S(A) = |a_{11} - a_{12} - a_{21} + a_{22}|$. Sea D un diseño sobresaturado $E(s^2)$-óptimo con n observaciones y $m = k(n-1)$ factores. Sean Q_i y Q_j dos clases paralelas del RBIBD definidas por la i-ésima y j-ésima columna de D. Entonces, si A es la MICP de Q_i y Q_j, tenemos que $|s_{ij}| = S(A)$.

Supongamos que $n \equiv 0 \pmod 4$. Claramente, si $s_{\max} = 4$, entonces $n_0 = \cdots = n_{a-2} = 0$. Por tanto, los valores para n_{a-1} y n_a estan dados por el Teorema 3(a). Esto implica que si $s_{\max} = 4$, cada columna de D^TD tiene n_{a-1} veces el valor ± 4, y n_a veces el valor 0. En forma similar, si $n \equiv 2 \pmod 4$, podemos probar $n_0 = \cdots = n_{\frac{c-5}{2}} = 0$, para $s_{\max} = 6$. Por tanto, los valores para $n_{\frac{c-3}{2}}$ y $n_{\frac{c-1}{2}}$ están dados por el Teorema 3(b). También, el Teorema 3(b) implica que $s_{\max} = 2$ si y sólo si $n = 6$ y $k = 2$.

6 La búsqueda

En esta sección, utilizamos la correspondencia entre DSSs $E(s^2)$-óptimos k-circulantes, $D(n, k(n-1))$, y las familias de diferencias 1-rotacionales resolubles con los parámetros $(n-1, k, k, n/2, k(n/2-1))$ dada en la Sección 4, para buscar todos los DSSs $E(s^2)$-óptimos y minimax-óptimos k-circulantes.

Sea $\mathcal{C}$ el conjunto de todas la parejas (S, T) tales que $\{S, T\}$ es una partición de $Z_{n-1} \cup \{\infty\}$ con $|S| = |T| = n/2$ y T contiene ∞. Para cada (S, T) en $\mathcal{C}$, definimos su órbita bajo el grupo Z_{n-1} por $(S, T) + Z_{n-1} = \{(S + x, T + x) \mid x \in Z_{n-1}\}$. Hay $g = \frac{1}{n-1}\binom{n-1}{\frac{n-1}{2}}$ órbitas en $\mathcal{C}$ bajo Z_{n-1}. Sea $\mathcal{R} = \{(R_1, P_1), \ldots, (R_g, P_g)\}$ los representantes de cada una de las g órbitas. Entonces la búsqueda de todas las familias de diferencias 1-rotacionales resolubles con los parámetros $(n-1, k, k, n/2, k(n/2-1))$ es equivalente a encontrar

todos los k-conjuntos en $\mathcal{R}$ satisfaciendo la Definición 2(b) con $\lambda = k(n/2-1)$. Sin embargo, para enumerar todos los DSSs $E(s^2)$-óptimos k-circulantes con $s_{\max}$ dado, es necesario reducir el espacio de búsqueda. Decimos que una órbita $(S,T)+Z_{n-1}$ tiene intersección mínima si $S(A((S,T)+x,(S,T)+y)) \leq s_{\max}$ para cualquier x y y en Z_{n-1} con $x \neq y$. Ahora construimos un grafo con un vértice por cada elemento en $\mathcal{R}$, y una arista conecta a dos vértices (S,T) y (S',T') cuando $S(A((S,T)+x,(S',T')+y)) \leq s_{\max}$ para cualquier x y y en Z_{n-1}. Por tanto, por la Sección 4, la enumeración de todos los DSSs, $D(n,k(n-1))$, $E(s^2)$-óptimos y minimax-óptimos k-circulantes es equivalente a encontrar todos los k-cliques en este grafo, cuyos k vértices satisfacen la Definición 2(b) con $\lambda = k(n/2-1)$.

Ahora, describimos el algoritmo para enumerar todos los DSSs, $D(n,k(n-1))$, $E(s^2)$-óptimos k-circulantes con $s_{\max} = 4$ si $n \equiv 0 \pmod 4$, y $s_{\max} \in \{2,6\}$ si $n \equiv 2 \pmod 4$. Por el Teorema 1, estos DSSs son también minimax-óptimos.

Primero, se construye el grafo asociado al DSS.

Segundo, una vez que el grafo fue construido, para $k = 2,\ldots,max_k$ (aquí max_k es el tamaño del máximo clique), encontramos todos los k-cliques cuyos k vértices satisfacen la Definición 2(b) con $\lambda = k(n/2-1)$. Claramente, el algoritmo permite la posibilidad de generar diseños resolubles, los cuales son isomorfos. Sin embargo, estos diseños isomorfos serán eliminados al final de la búsqueda.

El algoritmo descrito en este trabajo fue implementado en C y ejecutado en una máquina 2.67GHz.

Tabla 1: Los DSSs $E(s^2)$-óptimos k-circulantes y minimax-óptimos con $n \leq 28$ observaciones, $m = k(n-1)$ factores y $s_{\max} \in \{2,4,6\}$ donde n es par.

(n,m)	$\lvert Aut(D)\rvert$	Nd	(n,m)	$\lvert Aut(D)\rvert$	Nd	(n,m)	$\lvert Aut(D)\rvert$	Nd
(6,10)	720	1	(12,22)	11	1	(16,30)	15	2
(8,14)	336	1	(12,33)	11	1	(16,60)	5760	1
(8,21)	336	1	(12,55)	7920	1	(18,34)	17	2
(8,28)	1344	1	(12,55)	1320	1	(18,68)	17	3
(8,35)	40320	1	(12,66)	95040	1	(18,102)	17	3
(10,18)	9	1	(14,26)	39	2	(18,136)	4896	1
(10,36)	9	3	(14,52)	13	3	(20,38)	57	1
(10,54)	9	3	(14,78)	13	3	(22,42)	21	2
(10,72)	9	2	(14,104)	13	2	(22,84)	21	3
(10,90)	9	1	(14,130)	39	1	(26,50)	25	3
(10,90)	27	1						
(10,108)	9	1						
(10,126)	3628800	1						

En la Tabla 1 presentamos el número de DSSs $E(s^2)$-óptimos y minimax-óptimos k-circulantes con n observaciones, $k(n-1)$ factores y $s_{\max} \in \{2, 4, 6\}$, para $n = 6, 8, \ldots, 26, 28$. También los tamaños de sus grupos de automorfismos son presentados en la tabla. La columna "Nd" contiene el número de DSSs no isomorfos. El tiempo de CPU del algoritmo fue alrededor de 281 minutos para generar todos los diseños presentados en este trabajo.

Referencias

[1] K. H. V. Booth and D. R. Cox, Some systematics supersaturated designs, Technometrics 4 (1962), 489–495.

[2] R. C. Bose, On the construction of balanced incomplete block designs, Ann. Eugenics 9 (1939), 353–399.

[3] N. A. Butler, Minimax 16-run supesatirated designs, Statist. Probab. Lett. 73 (2005), 139–145.

[4] K. T. Fang, D. K. J. Lin and M. Q. Liu, Optimal mixed-level supersaturated design, Metrika 58 (2003), 279–291.

[5] S. Gupta, K. Hisano and L. B. Morales, Optimal k-circulant supersaturated designs, J. Statist. Plann. Inference 141 (2011), 782–786.

[6] S. Gupta and L. B. Morales, Constructing $E(s^2)$-optimal and minimax-optimal k-circulant supersaturated designs via multi-objective tabu search, J. Statist. Plann. Inference 142 (2012), 1415–1420.

[7] C. Koukouvinos, K. Mylona and D. E. Simos, $E(s^2)$-optimal and minimax-optimal cyclic supersaturated designs via multi-objective simulated annealing, J. Statist. Plann. Inference 138 (2008), 1639–1646.

[8] L. B. Morales and C. Velarde, Enumeration of resolvable 2-(10,5,16) and 3-(10,5,6) designs, J. Combin. Des. 13 (2005), 108–119.

[9] N.-K. Ngyuen, An algorithmic approach to constructing supersaturated designs, Technometrics 38 (1996), 69–73.

[10] K. J. Ryan and D. A. Bulutoglu, $E(s^2)$-optimal supersaturated designs with good minimax properties, J. Statist. Plann. Inference 137 (2007), 2250–2260.

[11] F. E. Satterthwaite, Random balance experimentation (with discussions), Technometrics 1 (1959), 111–137.

Existence of spanning $\mathcal{F}$-free subgraphs with large minimum degree

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Abstract. Let $\mathcal{F}$ be a family of fixed graphs and let d be large enough. For every d-regular graph G, we study the existence of a spanning $\mathcal{F}$-free subgraph of G with large minimum degree. This problem is well-understood if $\mathcal{F}$ does not contain bipartite graphs. Here we provide asymptotically tight results for many families of bipartite graphs such as cycles or complete bipartite graphs.

Key words: Extremal Graph Theory, Subgraph of d-regular Graphs, Probabilistic Method, Iterative Colorings.

Let $G = (V, E)$ be a d-regular graph on n vertices. It is easy to see that every such graph has a spanning triangle-free (in particular, bipartite) subgraph H, with minimum degree at least $d/2$. Let $\mathcal{F}$ denote a family of fixed graphs. We say that G is $\mathcal{F}$-free if for every $F \in \mathcal{F}$, G does not contain any subgraph isomorphic to F. Then, a natural question is which is the largest minimum degree of a spanning $\mathcal{F}$-free subgraph H of G. Let $\mathrm{ex}(n, \mathcal{F})$ be the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. Since the complete graph on $n = d + 1$ vertices, denoted by K_n, is d-regular, the largest minimum degree a spanning $\mathcal{F}$-free subgraph H of K_n is at most $2\mathrm{ex}(d+1, \mathcal{F})/(d+1)$. In [8], Foucaud, Krivelevich and Perarnau conjectured the following,

Conjecture 1 ([8]). Let $\mathcal{F}$ be a family of fixed graphs and let d be large enough. Then for every d-regular graph G, there exists a spanning $\mathcal{F}$-free subgraph $H \subseteq G$ with minimum degree $\delta(H) = \Omega(\mathrm{ex}(d, \mathcal{F})/d)$.

If the chromatic number of $\mathcal{F}$ is at least 3 (no bipartite graphs in $\mathcal{F}$), it is easy to verify the conjecture. Kun [11] showed that if $\mathcal{F} = \{C_3, \dots, C_{g-1}\}$ then every d-regular graph G admits a spanning subgraph with minimum degree $\Omega(d^{1/g})$. This gives a lower bound on the minimum degree which it is still far from the conjectured one. A better non explicit result is obtained in [8]. In particular, if $\mathcal{F} = \{C_3, C_4, C_5\}$, it is showed that Conjecture 1 holds up to a logarithmic factor. A similar problem has been studied when $\mathcal{F}$ is composed by complete bipartite graphs in [3].

In this paper we prove that the conjecture is true for a large number of families $\mathcal{F}$ with $\chi(\mathcal{F}) = 2$.

For any two graphs F and Γ, we say that $\varphi : V(F) \to V(\Gamma)$ is an *homomorphism* if $uv \in E(F)$ implies $\varphi(u)\varphi(v) \in E(\Gamma)$. We say that φ is *locally injective* if, for every $v \in F$, the restriction of φ onto the neighbors of v in F is injective. Let $\hom^*(F, \Gamma)$ be the number of locally injective homomorphisms from F to Γ. Observe that the condition $\hom^*(F, \Gamma) = 0$, for every $F \in \mathcal{F}$, is stronger than $\mathcal{F}$-freeness. Any copy of F in Γ induces an injective homomorphism from F to Γ which, in particular, is also locally injective. We call a family of fixed graphs $\mathcal{F}$ *closed* if for every graph Γ we have: $\hom^*(F, \Gamma) = 0$ for every $F \in \mathcal{F}$ if and only if Γ is $\mathcal{F}$-free.

Here we prove that Conjecture 1 holds for every closed family $\mathcal{F}$ of graphs.

Theorem 1. *Let $\mathcal{F}$ be a closed family of graphs and let d be large enough. Then for every d-regular graph G, there exists a spanning $\mathcal{F}$-free subgraph $H \subseteq G$ with minimum degree $\delta(H) = \Omega(ex(d, \mathcal{F})/d)$.*

The proof goes as follows. First, we select a large bipartite subgraph of G with stable sets A and B. Then we consider a graph Γ of order αd for some large constant α, such that $\hom^*(F, \Gamma) = 0$, for every $F \in \mathcal{F}$, and $\delta(\Gamma) = \Omega(\mathrm{ex}(d, \mathcal{F})/d)$. We color A in a two steps procedure. First, we randomly assign a color to each vertex $a \in A$ and we either remove the color, or keep it and delete some dangerous edges incident to a from G. We complete the coloring by deterministically assigning colors to the uncolored vertices in A. After that, we color B and remove edges from G iteratively in a similar fashion. During all the procedure, we make sure that all the degrees stay large enough and that no color appears more than once in each neighborhood, that is neighborhoods are rainbowly colored. Finally, we use this coloring to embed the resulting subgraph to Γ. According to this embedding, from all the edges still remaining we only keep the ones whose embedding agrees with the edges in Γ. This final subgraph satisfies the desired properties.

As a corollary, we get explicit results for some important families $\mathcal{F}$ with $\chi(\mathcal{F}) = 2$. The problem of determining $\mathrm{ex}(d, \mathcal{F})$ when $\mathcal{F}$ contains bipartite graphs has attracted a lot of attention (see [9] for a complete survey on the topic). Here we use some well-known constructions in extremal graph theory, to provide explicit corollaries of Theorem 1:

- Let $\mathcal{F} = \{C_3, \ldots, C_{2r+1}\}$. Observe that $\mathcal{F}$ is a closed family. However, the asymptotic order of $\mathrm{ex}(d, \mathcal{F})$ is not known in general. Using the ErdHos-Rényi random graph $G(n, p)$ for some good probability $p = p(n, r)$, it is easy to show the existence of $\mathcal{F}$-free graphs of order d and many edges. Thus, by Theorem 1 we have that every d-regular graph G admits a spanning subgraph with girth $g \geq 2r + 2$ and minimum degree $\Omega(d^{1/(2r-1)}) = \Omega(d^{1/(g-3)})$. This improves the result of Kun [11]. Never-

theless, for any family of graphs $\mathcal{F}$, lower bounds on $\mathrm{ex}(d, \mathcal{F})$ obtained from mimicking random graphs do not seem to be tight.

- Let $\mathcal{F} = \{C_3, C_4, C_5\}$. From the constructions of C_4-free graphs provided by ErdHos, Rényi and Sós [6] and Brown [2] one can obtain an $\mathcal{F}$-free graph of order d and $\Omega(d^{3/2})$ edges. Again, by Theorem 1, for every d-regular graph G we can show the existence of a spanning subgraph with girth at least 6 and minimum degree $\Omega(\sqrt{d})$. Extremal constructions for graphs with girth at least 8 and 12 are also known [13,12]. From them we can obtain tight explicit lower bounds for $\mathcal{F} = \{C_3, \ldots, C_7\}$ and $\mathcal{F} = \{C_3, \ldots, C_{11}\}$.
- Let $\mathcal{F} = \{C_{2q}\}$ where $q > 2$ is a prime. While $\mathcal{F}$ is not close, $\mathcal{F}' = \{C_p, C_{2p}\}$ is. Moreover, since q is odd, $\mathrm{ex}(d, \mathcal{F}) = \Theta(\mathrm{ex}(d, \mathcal{F}'))$. Thus, Conjecture 1 is true for these families.
- Let $\mathcal{F} = \{K_{a,b}\}$ with $a \leq b$. Since $K_{a,b}$ has diameter two, each locally injective homomorphism of $K_{a,b}$ onto a graph Γ, is also injective. Thus, if $\hom^*(K_{a,b}, \Gamma) = 0$, then Γ is $K_{a,b}$-free, and $\mathcal{F}$ is closed. In particular, any family composed by complete bipartite graphs is closed. It is conjectured that $\mathrm{ex}(d, K_{a,b}) = \Theta(n^{2-1/a})$. While the upper bound has been proved for all values of a and b (Kovári, Sós and Turán [10]), the lower bound is still widely open. However, it is known to be true in the following cases: $a = 2$ and $b \geq 2$, $a = 3$ and $b \geq 3$ (Brown [2]), and $b > (a-1)!$ (Alon, Rónyai and Szabó [1]). Using these results we can get tight explicit lower bounds on the existence of subgraphs with large minimum degree and inducing no complete bipartite subgraph of a given size.
- Let $\mathcal{F} = \{Q_3\}$, where Q_s is the s-dimensional hypercube. The family $\mathcal{F}$ is closed: again, every locally injective homomorphisms is also injective. This is not true when we consider $s \geq 4$. It is conjectured in [7] that $\mathrm{ex}(d, Q_3) = \Theta(d^{8/5})$, but no better bound that $\mathrm{ex}(d, Q_3) \geq \mathrm{ex}(d, C_4) = \Omega(d^{3/2})$ is known.

Theorem 1 solves in the affirmative Conjecture 1 for families of graphs satisfying a "local" condition. In order to solve the conjecture for every family of graph $\mathcal{F}$, one needs to extend the idea of local injectivity to injectivity. In terms of colorings, one needs to prove the existence of a spanning subgraph H with large minimum degree and a coloring χ such that all the copies of F in H are rainbow. This is stronger than the rainbow condition we impose here in the neighborhoods of vertices in H.

Finally, Theorem 1 studies the case when G is d-regular. Similar results in terms of the maximum and minimum degree have been given in [8]. We believe that the same techniques used here could be extended to the non regular case.

1 Proof of Theorem 1

For every $v \in V(G)$ we denote by $N_G(v)$ the set of vertices adjacent to v in G, by $d_G(v) = |N_G(v)|$ the degree of v in G and by $N^2_G(v)$ the set of vertices at distance two from v in G. If the graph G is clear from the context, we use $N(v)$, $d(v)$ and $N^2(v)$. We denote by $\Delta(G)$ and by $\delta(G)$ the maximum and the minimum degree of G. We denote by χ a vertex (partial) coloring of a graph G. We use $\chi(G)$ to denote the chromatic number of G, and for every family of graphs $\mathcal{F}$, we also use $\chi(\mathcal{F}) = \min_{F\in\mathcal{F}} \chi(F)$. For every vertex $v \in V$, we denote by $\chi(v)$ its color and for every set $S \subseteq V(G)$, by $\chi(S)$ the set of colors appearing in S. We call $S \subseteq V$ *rainbow* if $\chi(u) \neq \chi(v)$ for every $u, v \in S$, $u \neq v$, that were assigned a color by χ.

First, we may assume that $\chi(\mathcal{F}) = 2$. Otherwise, Theorem 1 can be easily proven (see Proposition 1 in [8] for a stronger version of it).

We use the technique of almost-regularization to obtain an $\mathcal{F}$-free graph Γ where all the degrees are similar. For every $\beta \geq 1$, we say that G is β-*almost regular* if $\Delta(G) \leq \beta\delta(G)$. ErdHos and Simonovits [7] showed the following. For every $\gamma \in (0, 1)$, there exists some $\beta = \beta(\gamma)$ such that if the maximum number of edges of an $\mathcal{F}$-free β-almost regular graph of order m is $O(m^{1+\gamma})$, then $\mathrm{ex}(m, \mathcal{F}) = O(m^{1+\gamma})$. Otherwise stated, for every family $\mathcal{F}$ with $\chi(\mathcal{F}) = 2$, there exists an $\mathcal{F}$-free graph Γ with $\Delta(\Gamma) \leq \beta\delta(\Gamma)$, for some constant $\beta = \beta(\gamma)$. In particular, this implies that $\delta(\Gamma) = \Omega(\mathrm{ex}(m, \mathcal{F})/m)$

Observe that we may also assume that $\mathcal{F}$ does not contain any forest. Suppose that $\mathcal{F}$ contains a forest T. Since $\mathrm{ex}(d, \mathcal{F}) \leq \mathrm{ex}(d, T) = O(d)$, then Theorem 1 is trivially true. By the result of ErdHos in [4,5], if $\mathcal{F}$ does not contain any forest, there exists a constant $\varepsilon_0 = \varepsilon_0(\mathcal{F}) > 0$ such that for every large enough m, $\mathrm{ex}(m, \mathcal{F}) > m^{1+\varepsilon_0}$. Thus, the graph Γ satisfies $\delta(\Gamma) = \Omega(\mathrm{ex}(m, \mathcal{F})/m) > m^{\varepsilon}$, for some $\varepsilon < \varepsilon_0$.

If $\mathcal{F}$ is closed, such Γ also satisfies $\hom^*(F, \Gamma) = 0$, for every $F \in \mathcal{F}$.

Finally, observe that for every constant $\alpha > 0$,

$$\frac{\mathrm{ex}(\alpha d, \mathcal{F})}{\alpha d} = O\left(\frac{\mathrm{ex}(d, \mathcal{F})}{d}\right) .$$

Then, the following proposition implies Theorem 1.

Proposition 1. *Let $\mathcal{F}$ be a family of graphs with $\chi(F) = 2$ that does not contain any forest. Let G be a d-regular graph. Suppose that there exists $\varepsilon > 0$, $\beta > 0$, $\alpha \geq 32\beta$ and a graph Γ on αd vertices such that,*

- *G is β-almost regular,*
- *G has minimum degree $\delta(\Gamma) > d^{\varepsilon}$, and*
- *$\hom^*(F, \Gamma) = 0$, for every $F \in \mathcal{F}$.*

Then there exists a spanning $\mathcal{F}$-free subgraph $H \subseteq G$ with $\delta(H) = \Omega(\delta(\Gamma))$.

1.1 Sketch of the proof of Proposition 1

First of all, observe that we can consider G to be bipartite. If this is not the case, we can always find a bipartite spanning subgraph where every vertex has at least half of its degree in G. Thus we obtain a new graph which is bipartite and where every degree is asymptotically the same as in G. Let A and B be the two stable sets of G.

We construct a spanning subgraph H and an αd coloring χ of H in two phases. By the construction of H and χ we will deduce that H is a spanning subgraph of G satisfying the desired properties, thus proving Proposition 1.

On Phase I we will color the vertices in A. First, we will assign a random color to every vertex $a \in A$. If there are too many vertices $b \in N(a)$ such that the color of a appears more that once in $N(b)$, we will uncolor a. Otherwise it retains its color and we delete the dangerous edges incident to a. After this random partial coloring of A we will deterministically assign a color to all the uncolored vertices in A and delete the corresponding dangerous edges. At the end of Phase I, with positive probability, we will obtain a subgraph with large minimum degree where there are no repeated colors in $N(b)$, for every $b \in B$

On Phase II we will extend the partial coloring to B. In the first part, we will iteratively color the vertices in B until no vertex in A has too many uncolored neighbors. At each iteration a partial coloring is obtained using the same idea used in Phase I: we randomly color all the uncolored vertices, uncolor a vertex $b \in B$ that was assigned a color in this iteration if the previous condition holds and otherwise retain the color in b and delete all the dangerous edges. In this case we also delete an edge ab if $\chi(a)\chi(b) \notin E(\Gamma)$. We will prove that after each iteration, a large proportion of the uncolored vertices in B retain the color they were assigned. Once there are not many uncolored vertices in each neighborhood of $a \in A$, we finish this phase by deterministically assigning every uncolored vertex a color that minimizes the number of conflict edges.

Proposition 2. *After Phase II, the subgraph H and the coloring χ obtained satisfy with positive probability,*

- *for every $v \in V(H)$, $d_H(v) = \Omega(\delta(\Gamma))$,*
- *for every $v \in V(H)$, $N_H(v)$ is rainbow, and*
- *for every $uv \in E(H)$, $\chi(u)\chi(v) \in E(\Gamma)$.*

The proof of this proposition is done by analyzing the coloring procedure described above in a similar fashion than in Part V of [14].

From Proposition 2, it is easy to deduce that the subgraph H satisfies the statement of Proposition 1.

References

[1] N. Alon, L. Rónyai, and T. Szabó, Norm-graphs: variations and applications. *Journal of Combinatorial Theory, Series B*, vol. 76:280–290, 1999.

[2] W. G. Brown, On graphs that do not contain a Thomsen graph. *Canadian Mathematical Bulletin*, vol. 9(2):1–2, 1966.

[3] D. Conlon, J. Fox, and B. Sudakov, Large subgraphs without complete bipartite graphs. *arXiv:1401.6711*, 2014.

[4] P. ErdHos, Graph theory and probability II. *Canad. J. Math*, vol. 11:34–38, 1959.

[5] P. ErdHos, Graph theory and probability I. *Canad. J. Math*, vol. 13:346–352, 1961.

[6] P. ErdHos, A. Rényi, and V. T. Sós, On a problem of graph theory. *Studia Scientiarum Mathematicarum Hungarica*, vol. 1:215–235, 1966.

[7] P. ErdHos and M. Simonovits, *Some extremal problems in graph theory*, Combinatorial theory and its applications, I (Proc. Colloq., Balatonfüred, 1969), 377–390, North-Holland, Amsterdam, 1970.

[8] F. Foucaud, M. Krivelevich, and G. Perarnau, Large subgraphs without short cycles. *arXiv:1401.4928*, 2014.

[9] Z. Füredi, and M. Simonovits, *The history of degenerate (bipartite) extremal graph problems*, ErdHos Centennial, 169–264, Springer, 2013.

[10] T. Kovári, V. T. Sós, and P. Turán, On a problem of K. Zarankiewicz. *Colloquium Mathematicae*, vol. 3(1):50–57, 1954.

[11] G. Kun, Expanders have a spanning Lipschitz subgraph with large girth. *arXiv:1303.4982*, 2013.

[12] T. Lam, and J. Verstraëte, A note on graphs without short even cycles. *Electronic Journal of Combinatorics*, vol. 12(1), Paper 5, 2005.

[13] F. Lazebnik, V. A. Ustimenko, and A. J. Woldar, Polarities and $2k$-cycle-free graphs. *Discrete Mathematics*, vol. 197:503–513, 1999.

[14] M. Molloy, and B. Reed, Graph colouring and the probabilistic method, vol. 23, Springer, 2002. Springer, 2013.

Simultaneous resolvability in graph families

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Abstract. A set $S \subseteq V$ is said to be a simultaneous metric generator for a graph family $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$, defined on a common vertex set, if it is a generator for every graph of the family. A minimum simultaneous metric generator is called a simultaneous metric basis, and its cardinality the simultaneous metric dimension of $\mathcal{G}$. We study the properties of simultaneous metric generators and simultaneous metric bases, and calculate closed formulae or tight bounds for the simultaneous metric dimension of several graph families.

Key words: simultaneous metric generator, simultaneous metric basis, simultaneous metric dimension.

1 Introduction

A generator of a metric space is a set S of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of S. Given a simple and connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{N}$, where $d_G(x, y)$ is the length of a shortest path between u and v and $\mathbb{N}$ is the set of non-negative integers. Clearly, (V, d_G) is a metric space. A vertex $v \in V$ is said to *distinguish* two vertices x and y if $d_G(v, x) \neq d_G(v, y)$. A set $S \subseteq V$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. A minimum metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

The concept of metric dimension of a graph was introduced by Slater in [8], where metric generators were called *locating sets*, and, independently, by Harary and Melter in [3], where metric generators were called *resolving sets*. Applications of the metric dimension to the navigation of robots in networks are discussed in [7] and applications to chemistry in [5,6].

Now we present the navigation problem proposed in [7] where navigation was studied in a graph-structured framework in which the navigating agent

(which was assumed to be a point robot) moves from node to node of a "graph space". The robot can locate itself by the presence of distinctively labeled "landmark" nodes in the graph space. On a graph, there is neither the concept of direction nor that of visibility. Instead, it was assumed in [7] that a robot navigating on a graph can sense the distances to a set of landmarks. Evidently, if the robot knows its distances to a sufficiently large set of landmarks, its position on the graph is uniquely determined. This suggests the following problem: given a graph G, what is the smallest number of landmarks needed, and where should they be located, so that the distances to the landmarks uniquely determine the robot's position on G? The solution of this problem requires us to determine the metric dimension and a metric basis of G. In this article we consider the following extension of this problem. Suppose that the topology of the navigation network may change within a range of possible graphs, say $G_1, G_2, ..., G_k$. In this case, the above mentioned problem becomes that of determining the minimum cardinality of a set S which is a metric generator for each graph G_i, $i \in \{1, ..., k\}$. So, if S is a solution for this problem, then the position of each robot can be uniquely determined by the distance to the elements of S, regardless of the graph G_i that models the network at each moment.

Given a family $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ of (not necessarily edge-disjoint) connected graphs $G_i = (V, E_i)$ with common vertex set V (the union of whose edge sets is not necessarily the complete graph), we define a *simultaneous metric generator* for $\mathcal{G}$ as a set $S \subseteq V$ such that S is simultaneously a metric generator for each G_i. We say that a minimum simultaneous metric generator for $\mathcal{G}$ is a *simultaneous metric basis* of $\mathcal{G}$, and its cardinality the *simultaneous metric dimension* of $\mathcal{G}$, denoted by $\mathrm{Sd}(\mathcal{G})$ or explicitly by $\mathrm{Sd}(G_1, G_2, ..., G_k)$. An example is shown in Figure 1 where $\{c, d\}$ is a simultaneous metric basis of $\{G_1, G_2, G_3\}$.

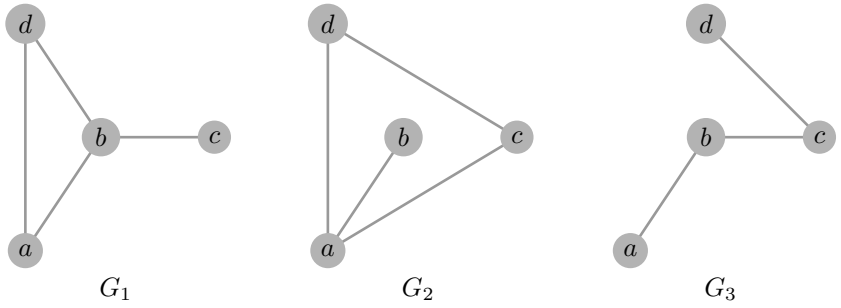

Fig. 1: The set $\{c, d\}$ is a simultaneous metric basis of $\{G_1, G_2, G_3\}$. Thus, $\mathrm{Sd}(G_1, G_2, G_3) = 2$.

The study of simultaneous parameters in graphs was introduced by Brigham and Dutton in [1], where they studied simultaneous domination. This should not be confused with studies on families sharing a constant value on a parameter, for instance the study presented in [4], where several graph families all of whose members have a constant metric dimension are studied, enforcing no constraints regarding whether all members share a metric basis or not.

2 Some basic results

Remark 1. For any family $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ of connected graphs with common vertex set V and any subfamily $\mathcal{H}$ of $\mathcal{G}$ it holds $\mathrm{Sd}(\mathcal{H}) \leq \mathrm{Sd}(\mathcal{G}) \leq |V|-1$. In particular, $\max_{i\in\{1,...,k\}}\{\dim(G_i)\} \leq \mathrm{Sd}(\mathcal{G})$.

The above inequalities are tight. For instance, for the family of graphs shown in Figure 1 we have $\mathrm{Sd}(G_1, G_2, G_3) = 2 = dim(G_1) = \dim(G_2) = \max_{i\in\{1,2,3\}}\{\dim(G_i)\}$, while the upper bound is reached for any family containing the complete graph.

Corollary 1. *Let $\mathcal{G}$ be a family of connected graphs with the same vertex set. If $K_n \in \mathcal{G}$, then*

$$\mathrm{Sd}(\mathcal{G}) = n - 1.$$

As shown in Figure 2, the converse of Corollay 1 does not hold.

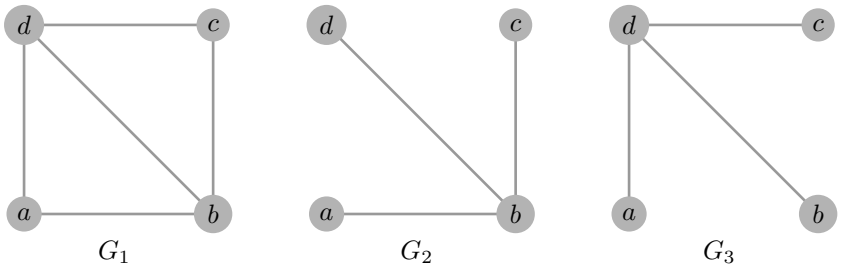

Fig. 2: The set $\{b, c, d\}$ is a simultaneous metric basis of $\{G_1, G_2, G_3\}$. Thus, $\mathrm{Sd}(G_1, G_2, G_3) = 3 = n - 1$.

Given a graph $G = (V, E)$ and a vertex $v \in V$, the set $N_G(v) = \{u \in V : u \sim v\}$ is the *open neighborhood* of v and the set $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v. Two vertices $x, y \in V(G)$ are *twin* in G if $N_G(x) = N_G(y)$ or $N_G[x] = N_G[y]$.

Theorem 1. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set V. Then $\mathrm{Sd}(\mathcal{G}) = |V|-1$ if and only if for every $u, v \in V$, there exists a graph $G_{uv} \in \mathcal{G}$ such that u and v are twin in G_{uv}.*

Corollary 2. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set V. If $\mathcal{G}$ contains three star graphs having different centers, then* $\mathrm{Sd}(\mathcal{G}) = |V| - 1$.

The following two results concern the limit cases of the left hand side inequalities in Remark 1.

Theorem 2. *Let $\mathcal{G}$ be a family of connected graphs on a common vertex set. The following assertions hold:*

1. $\mathrm{Sd}(\mathcal{G}) = 1$ *if and only if $\mathcal{G}$ is a family of paths all of which have a common leaf.*
2. *If $\mathcal{G}$ is a family of paths, then* $1 \leq \mathrm{Sd}(\mathcal{G}) \leq 2$.

Theorem 3. *Let $\mathcal{G}$ be a family of graphs on a common vertex set V. Let $\mathcal{H}$ be a non-empty subfamily of $\mathcal{G}$ consisting of all the graphs of $\mathcal{G}$ which are different from a path. Then* $\mathrm{Sd}(\mathcal{G}) = \mathrm{Sd}(\mathcal{H})$.

Theorem 4. *Let $\mathcal{G} = \{G_1, G_2, ..., G_k\}$ be a family of cycle graphs on a common vertex set V. Then the following assertions hold:*

1. *If $|V|$ is odd, then* $\mathrm{Sd}(\mathcal{G}) = 2$.
2. *If $|V|$ is even, then* $2 \leq \mathrm{Sd}(\mathcal{G}) \leq 3$. *Moreover, for $|V|$ even,* $\mathrm{Sd}(\mathcal{G}) = 2$ *if and only if there are two vertices $u, v \in V$ which are not mutually antipodal in G_i for all $i \in \{1, ..., k\}$.*

A vertex of degree at least 2 in a graph G is called an interior vertex. The set of interior vertices of graph G is represented by $\mathcal{I}(G)$. A vertex of degree at least 3 is called a *major vertex* of G. Any end-vertex (a vertex of degree 1) u of G is said to be a terminal vertex of a major vertex v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The terminal degree $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. The set of exterior major vertices of graph G is represented by $\mathcal{M}(G)$.

Proposition 1. *Let $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ be a family of trees on a common vertex set V that are not paths, and let $S_{\mathcal{I}} = \bigcap_{i=1}^{k} \mathcal{I}(T_i)$ be the set of vertices that are interior vertices of every tree $T_i \in \mathcal{T}$. Then*

$$\mathrm{Sd}(\mathcal{T}) \leq |V| - |S_{\mathcal{I}}| - 1.$$

It was shown in [2] that for any connected graph G of order n and diameter $D(G)$, $\dim(G) \leq n - D(G)$. Our next result is an extension of this bound to the case of the simultaneous metric dimension.

Theorem 5. *Let $\mathcal{G}$ be a family of graphs with common vertex set V having a shortest path of length d in common. Then* $\mathrm{Sd}(\mathcal{G}) \leq |V| - d$.

For example, let G_1 be a graph obtained from a complete graph $K_r = (V', E')$, $r \geq 3$, and a path graph P_d, $d \geq 2$, by identifying leaf a of P_d, with a vertex c of K_r, and let G_2 be the graph obtained by identifying a leaf a of P_d with the center c of a star $K_{1,r-1} = (V', E'')$. In this case, G_1 and G_2 have the same vertex set $V = V(P) \cup V'$ (where $a = c \in V(P) \cap V'$ and so $|V| = d + r - 1$). Also, if b is the leaf of P_d different from a, then for any $v \in V' - \{c\}$ we have $d_{G_1}(b, v) = d_{G_2}(b, v) = d$ and vP_d is a shortest path of length d in both graphs G_1 and G_2. Moreover, $W = V' - \{c\}$ is a simultaneous metric basis of $\{G_1, G_2\}$ and so $\mathrm{Sd}(G_1, G_2) = |V| - d$. Therefore, the above bound is tight.

3 Families of interrelated graphs

Next we analyze some special cases of families composed by graphs that may be obtained by applying some operations or transformations on another graph in the family. The first case concerns a family of graphs obtained as a result of transformations on a cycle graph.

Theorem 6. *Let $\mathcal{G} = \{C_n, \overline{C}_n, W_{1,n-1}, F_{1,n-1}\}$ be a family consisting of a cycle of order $n \geq 5$, its complement, the wheel graph $W_{1,n-1} = \langle v_c \rangle + C_{n-1}$ and the fan graph $F_{1,n-1} = \langle v_c \rangle + P_{n-1}$, where $v_c \in V(C_n)$, v_{c-1} and v_{c+1} are the vertices adjacent to v_c in C_n, $V(C_{n-1}) = V(P_{n-1}) = V(C_n) - \{v_c\}$, $E(C_{n-1}) = \{v_{c-1}v_{c+1}\} \cup E(C_n) - \{v_{c-1}v_c, v_cv_{c+1}\}$ and $E(P_{n-1}) = E(C_n) - \{v_{c-1}v_c, v_cv_{c+1}\}$. Then, for any subfamily $\mathcal{H} \subseteq \mathcal{G}$ such that $\overline{C}_n \in \mathcal{H}$,*

$$\mathrm{Sd}(\mathcal{H}) = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Our next result concerns a second graph family which is also the result of a number of transformations on a cycle graph.

Theorem 7. *Let $\mathcal{G} = \{C_n, \overline{C}_n, P_n, \overline{P}_n, W_{1,n-1}\}$, be a family consisting of a cycle C_n of order $n \geq 5$, its complement, the path graph P_n, its complement, and the wheel graph $\langle v_c \rangle + C_{n-1}$, where $v_c \in V(C_n)$, v_{c-1} and v_{c+1} are the vertices adjacent to v_c in C_n, $V(P_n) = V(C_n)$, $E(P_n) = E(C_n) - \{v_cv_{c+1}\}$, $V(C_{n-1}) = V(C_n) - \{v_c\}$, and $E(C_{n-1}) = \{v_{c-1}v_{c+1}\} \cup E(C_n) - \{v_{c-1}v_c, v_cv_{c+1}\}$. Then, for any subfamily $\mathcal{H} \subseteq \mathcal{G}$ such that $\overline{C}_n \in \mathcal{H}$,*

$$\mathrm{Sd}(\mathcal{H}) = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

4 Families of corona product graphs

Let G and H be two graphs of order n and n', respectively. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G

and n copies of H and joining by an edge each vertex from the i-th copy of H with the i-th vertex of G. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$.

Theorem 8. *Let G_1 and G_2 be two connected non-trivial graphs with common vertex set and let H be a non-trivial graph. Then any metric basis for $G_1 \odot H$ is a metric basis for $G_2 \odot H$.*

Corollary 3. *Let $\{G_1, G_2, ..., G_k\}$ be a family of connected non-trivial graphs with common vertex set V and let H be a non-trivial graph. Then for any $i \in \{1, \ldots, k\}$, $\mathrm{Sd}(G_1 \odot H, G_2 \odot H, \ldots, G_k \odot H) = \dim(G_i \odot H)$.*

Results from [9] allow us to deduce the exact simultaneous metric dimension, or establish bounds, for several special families of corona product graphs.

Theorem 9. *Let $\{G_1, G_2, \ldots, G_k\}$ be a family of connected non-trivial graphs with common vertex set V and let H be a non-trivial graph. The following statements hold:*

1. $|V| \cdot \dim(H) \leq \mathrm{Sd}(G_1 \odot H, G_2 \odot H, \ldots, G_k \odot H) \leq |V| \cdot \dim(K_1 + H)$.
2. *If H has diameter $D(H) \leq 2$, then* $\mathrm{Sd}(G_1 \odot H, G_2 \odot H, \ldots, G_k \odot H) = |V| \cdot \dim(H)$.
3. *If H has diameter $D(H) \geq 6$ or it is a cycle graph of order at least 7, then* $\mathrm{Sd}(G_1 \odot H, G_2 \odot H, \ldots, G_k \odot H) = |V| \cdot \dim(K_1 + H)$.
4. *For any integer $n \geq 7$,*
   a) $\mathrm{Sd}(G_1 \odot C_n, G_2 \odot C_n, \ldots, G_k \odot C_n) = |V| \cdot \left\lfloor \frac{2n+2}{5} \right\rfloor$
   b) $\mathrm{Sd}(G_1 \odot P_n, G_2 \odot P_n, \ldots, G_k \odot P_n) = |V| \cdot \left\lfloor \frac{2n+2}{5} \right\rfloor$
   c) $\mathrm{Sd}(G_1 \odot \overline{C}_n, G_2 \odot \overline{C}_n, \ldots, G_k \odot \overline{C}_n) = |V| \cdot \left\lfloor \frac{2n+2}{5} \right\rfloor$.

Theorem 10. *Let $\{G_1, G_2, \ldots, G_k\}$ be a family of connected non-trivial graphs with common vertex set V and let H be a graph of order $n' \geq 2$. Let α be the number of connected components of H of order greater than 1 and let β be the number of isolated vertices of H. Then*

$$\mathrm{Sd}(G_1 \odot H, G_2 \odot H, \ldots, G_k \odot H) \leq \begin{cases} |V| \cdot (n' - \alpha - 1) & \text{for } \alpha \geq 1 \text{ and } \beta \geq 1 \\ |V| \cdot (n' - \alpha) & \text{for } \alpha \geq 1 \text{ and } \beta = 0 \\ |V| \cdot (n' - 1) & \text{for } \alpha = 0. \end{cases}$$

In a similar way, we may derive other results analogously to the remaining results obtained in [9].

Theorem 11. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a family of non-trivial connected graphs on a common vertex set. Then* $\mathrm{Sd}(G \odot H_1, G \odot H_2, \ldots, G \odot H_k) \geq n \cdot \mathrm{Sd}(\mathcal{H})$.

The next results cover special cases involving graphs of diameter two.

Theorem 12. *Let G be a connected graph of order $n \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a family of non-trivial graphs on a common vertex set where every H_i, $i \in \{1, \ldots, k\}$, has diameter $D(H_i) \leq 2$. Then* $\mathrm{Sd}(G \odot H_1, G \odot H_2, \ldots, G \odot H_k) = n \cdot \mathrm{Sd}(\mathcal{H})$.

Theorem 13. *Let G be a connected graph of order $n \geq 2$. For any family of non-trivial graphs $\{H_1, ..., H_k\}$ with common vertex set,*

$$n \cdot \max_{i \in \{1,\ldots,k\}} \{\dim(H_i)\} \leq \mathrm{Sd}(G \odot H_1, \ldots, G \odot H_k) \leq n \cdot \mathrm{Sd}(K_1 + H_1, \ldots, K_1 + H_k).$$

References

[1] Brigham, R. C. and R. D. Dutton, *Factor domination in graphs*, Discrete Mathematics **86** (1990), pp. 127–136.

[2] Chartrand, G., L. Eroh, M. A. Johnson and O. R. Oellermann, *Resolvability in graphs and the metric dimension of a graph*, Discrete Applied Mathematics **105** (2000), pp. 99–113.

[3] Harary, F. and R. A. Melter, *On the metric dimension of a graph*, Ars Combinatoria **2** (1976), pp. 191–195.

[4] Imran, M., S. A. ul Haq Bokhary, A. Ahmad and A. Semanicová-Fenovcíková, *On classes of regular graphs with constant metric dimension*, Acta Mathematica Scientia **33** (2013), pp. 187–206.

[5] Johnson, M., *Structure-activity maps for visualizing the graph variables arising in drug design*, Journal of Biopharmaceutical Statistics **3** (1993), pp. 203–236, pMID: 8220404.

[6] Johnson, M. A., *Browsable structure-activity datasets*, in: R. Carbó-Dorca and P. Mezey, editors, *Advances in Molecular Similarity*, JAI Press Inc, Stamford, Connecticut, 1998 pp. 153–170.

[7] Khuller, S., B. Raghavachari and A. Rosenfeld, *Landmarks in graphs*, Discrete Applied Mathematics **70** (1996), pp. 217–229.

[8] Slater, P. J., *Leaves of trees*, Congressus Numerantium **14** (1975), pp. 549–559.

[9] Yero, I. G., D. Kuziak and J. A. Rodríquez-Velázquez, *On the metric dimension of corona product graphs*, Computers & Mathematics with Applications **61** (2011), pp. 2793–2798.

Avoiding subgraphs in Series-Parallel graphs

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Abstract. We get asymptotic estimates for the number of (labeled) series-parallel graphs with n vertices without cycles of length 3. As a consequence of our method, we are also able to study the limiting distribution of the number of triangles in a uniformly at random series-parallel graph with n vertices. Further research is also discussed.

Key words: Analytic combinatorics, graph enumeration, random graphs, limit law

1 Introduction

Series-parallel graphs, shortly called SP graphs hereafter, form a very natural family of graphs. There are at least 5 ways of defining them. The first definition is in terms of primitive operations. A graph is SP if and only if its blocks (maximal 2-connected subgraphs) are obtained from a single edge by recursive subdivision of edges (series operation) and duplication of edges (parallel operation). Equivalently, a graph is SP if and only if its tree-width is at most 2. In the language of graph minor theory, a graph is SP if and only if excludes K_4 as a minor. Another equivalent definition is that a graph is SP if and only if its connected components are subgraphs of a 2-tree (namely, a graph obtained starting from and edge and then repeatedly adding vertices in such a way that each added vertex has exactly 2 neighbors that form a triangle).

Finally, the last way to define SP graphs is via a classical decomposition of a graph into 3-connected components, formalised by Tutte [14]: SP graphs are exactly the graphs with no 3-connected component for Tutte's decomposition. This is precisely the definition exploited by Bodirsky, Giménez, Kang and Noy in [2] and by and by Drmota, Fusy, Kang, Kraus and Rué in [9] in order to

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get asymptotic estimates for the number of SP graphs with n labelled and unlabelled vertices, respectively. The main ideas developed in [2] have been extended in several directions, and nowadays many parameters of a uniformly at random SP graph with n vertices are well understood. This includes the degree distribution [7], the maximum degree [6], the size of the maximal block [13] and the diameter [8], among others.

Concerning the number of pending copies of a given subgraph, very precise estimates are known for planar graphs [11] and for graphs defined by 3-connected components [12] (see also [3] for minor-free closed families of graphs). However, little is known about the enumeration of SP graphs where a certain subgraph is avoided. The present extended abstract is the first attempt to systematically study these questions in SP graphs. In particular, we are able to asymptotically enumerate the number of (labelled) SP graphs with n vertices without cycles of length 3 (see Subsection 3.2). As a consequence of our techniques, we are also able to get limiting distributions for the number of triangles in a uniformly at random SP graph with n vertices (see Subsection 3).

The plan of this extended abstract is the following: in Section 2 we introduce the basics on Tutte's tree decomposition and the Dissymetry Theorem for trees. In Section 3.1 we compute the generating function $G(x, y, u)$ of SP graphs where x marks vertices, y marks edges and u marks triangles. In Section 3.2 we write $u = 0$ and $y = 1$ in order to avoid triangles in the counting formulas, getting the asymptotic enumeration of the number of SP graphs with girth at least 4. Finally in Section 3.3 we study the generating function $G(x, 1, u)$ in order to compute the limit law of the number of triangles in a uniformly at random SP graph.

2 Background and definitions

All the generating functions that we are considering are exponential with respect to vertices and ordinary with respect to edges and triangles, respectively. We use variables x, y and u to mark vertices, edges and triangles. The language we use in order to encode enumerative problems is the *Symbolic Method* (see [10]), which provides a direct way to translate algebraic relations between combinatorial classes into equations between the associated generating functions.

The strategy we use to study SP graphs without triangles is based on connectivity decomposition. Specially important in our context is how to get the generating function for 2-connected graphs in the family. For this purpose, let us briefly recall Tutte's decomposition and the definition of RMT-trees (see [4]). Following Tutte's decomposition Theorem (see [14]) every 2-connected graph can be decomposed into 3-connected components using *split operations*. The resulting final structure after all possible split operations have

been performed is a collection of graphs called *bricks*. The resulting bricks are independent of the order in which the split operations are performed.

Furthermore, Tutte showed that a brick graph has to be of one of the following types: multi-edge graph (M-brick), ring-graph (R-brick), or 3-connected graph with at least 4 vertices (T-brick). In the case of SP graphs there does not exist T-bricks, and consequently the vertices of the brick tree are of two types.

One can translate now Tutte's decomposition into a graph structure. For a graph G let the *RMT-tree* be the graph with vertex set equal to the set of bricks of G and edge set corresponding to the virtual edges of the Tutte's decomposition of G. The resulting graph is a tree and by the maximality of the decomposition two R-bricks as well as two M-bricks are never adjacent.

Following [4], and by applying the Dissymmetry Theorem for trees, the generating function for 2-connected SP graphs B can be written as $B = B_R + B_M - B_{RM}$, where B_M, B_R and B_{RM} are the counting formulas associated to the class of graphs in $\mathcal{B}$ such that the RMT-tree has either a distinguished vertex (of type M and R, respectively) or a distinguished edge.

3 Triangles in SP graphs

3.1 Equations

Since there are no 3-connected SP graphs, we start by computing the generating functions $D(x, y, u)$ of networks, where x, y marks vertices and edges, respectively. We add an additional parameter u which will be used in order to encode triangles.

Recall that a network is obtained from a 2-connected SP graph by choosing and orienting an edge. It might not occur in the graph, and the vertices incident to it, the poles, are not labelled, but instead one of them is consider to be 0, and the other one is ∞. For convenience we split both series and parallel generating functions as follows. We define $P_0 := P_0(x, y, u)$ as the generating function of parallel networks that do not contain an edge between the poles, whereas $P_1 := P_1(x, y, u)$ is the generating function of parallel networks where there is an edge connecting the poles. For convenience, we include the network consisting of a single edge in P_1.

We define $S_2 := S_2(x, y, u)$ as the generating function of series networks where there is a path of length exactly 2 between the poles, or equivalently where there exists a single cut vertex. We also define $S_3 := S_3(x, y, u)$ as the remaining series networks. Namely, the ones where the graph distance between the poles is at least 3. The generating function $D := D(x, y, u)$ can be expressed then as the solution of the following system of equations:

$$\begin{aligned} D &= P_0 + P_1 + S_2 + S_3 \\ P_0 &= \exp_{\geq 2}(S_2 + S_3) \\ P_1 &= y \exp(uS_2 + S_3) \\ S_2 &= xP_1^2 \\ S_3 &= xDP_0 + xP_1(P_0 + S_2 + S_3). \end{aligned} \tag{1}$$

From these equations we deduce that triangles can only come up from parallel constructions where the poles are connected by an edge and at least one path of length 2. The generating function $D(x, y, u)$ cannot be expressed in terms of elementary functions, but we can obtain the first terms of its expansion near 0:

$$D(x,y,u) = y + \left(y^2 + uy^3\right) x + \left(2y^3 + 3y^4 + 4uy^4 + 5u^2y^5\right) \frac{x^2}{2!} + \\ + \left(6y^4 + 30y^5 + 7y^6 + 18uy^5 + 48uy^6 + 36u^2y^6 + 49u^3y^7\right) \frac{x^3}{3!} + O(x^4).$$

Now that we know D and the auxiliar functions P_0, P_1, S_2 and S_3, we can use the Dissymmetry Theorem for trees in order to obtain the generating function $B(x, y, u)$ of 2-connected series-parallel graphs, where x, y, u marks vertices, edges and triangles, respectively. In order to apply the Dissymmetry Theorem we need to root a network in either a cycle or a multiple edge, obtaining the following expressions:

$$B_R(x,y,u) = \mathrm{Cyc}(x(P_0 + P_1)) + (u-1)\frac{x^3 P_1^3}{6}$$

$$B_M(x,y,u) = \frac{x^2}{2}\left(y \exp_{\geq 2}(uS_2 + S_3) + \exp_{\geq 3}(S_2 + S_3)\right)$$

$$B_{MR}(x,y,u) = \frac{x^2}{2}\left((S_2 + S_3)(P_0 + P_1 - y) + (u-1)(P_1 - y)S_2\right),$$

where $\mathrm{Cyc}(x) = -\log(1-x)/2 - x/2 - x^2/4$ is the exponential generating function of cycles consisting of at least 3 vertices. Note that this is not a system of equations provided that we know the values of P_0, P_1, S_2 and S_3. Finally, following [4], the generating function $B(x, y, u)$ is obtained as

$$B(x,y,u) = B_R(x,y,u) + B_M(x,y,u) - B_{MR}(x,y,u).$$

Once that we know $B(x, y, u)$, we can use the standard decomposition of a class of graphs into its 2-connected components

$$xC_x(x,y,u) = x \exp(B_x(xC_x(x,y,u), y, u)),$$

where the subindex means that we are considering the derivative of the function with respect to x. This equation determines unequivocally $C(x, y, u)$ by a standard integration. In the last step we just compute the generating function $G(x, y, u)$ as the set of its connected components, which is encoded as

$$G(x, y, u) = \exp(C(x, y, u)).$$

This last equation finally determines the generating function $G(x, y, u)$ of SP graphs where x counts vertices, y counts edges and u counts triangles.

3.2 Enumeration of triangle-free SP graphs

Now that we have the generating function $G(x, y, u)$, if now we impose $u = 0$, we get the generating function $G(x, 1, 0)$ of SP graphs with girth greater or equal than 4. The singularity of this function provides information about its asymptotic growth:

Theorem 1. *The number of 2-connected SP graphs with n vertices without cycles of length 3 is asymptotically equal to*

$$b \cdot n^{-5/2} \cdot \gamma^n \, n!,$$

where $b \approx 1.5195 \cdot 10^{-3}$ and $\gamma \approx 5.0929$. Furthermore, the number of connected and general SP graphs with n vertices without cycles of length 3 is asymptotically equal to

$$c \cdot n^{-5/2} \cdot \rho^n \, n!, \quad g \cdot n^{-5/2} \cdot \rho^n \, n!,$$

where $c \approx 2.8588 \cdot 10^{-3}$, $g \approx 3.3398 \cdot 10^{-3}$ and $\rho \approx 6.2816$.

The proof of Theorem 1 is based on singularity analysis (see [10, Chapter VI]). We briefly sketch the ideas behind the proof of this theorem. We start studying the system of equations defining the networks by means of the so-called Drmota-Lalley-Woods Theorem (see [5]).

Once the singular behaviour of $D(x, y, u)$ (for $y = 1$, $u = 0$) is known, we conclude that the dominant singularity of $B(x, 1, 0)$ is determined by the dominant nature of $D(x, 1, 0)$. Let us mention that the singular nature of $B(x, 1, 0)$ is of type $3/2$ instead of $1/2$. This can be understood as the unrooting step which is required when relating networks with 2-connected graphs in the family.

Going from the 2-connected to the connected level the singularity is moved due to the existence of a branch point of the equation relating $B(x, 1, 0)$ and $C(x, 1, 0)$. This phenomena is also translated to the general level by means of the set construction. Finally, by applying the classic transfer theorems of singularity analysis [10, Thm VI.2] we deduce the asymptotic behaviour of the coefficients.

3.3 Number of triangles

Now that we have the generating function $G(x, y, u)$, by means of the so-called Quasi-Powers Theorem (see [10]) we can show that the number of triangles tends to a normal law, and obtain its mean and variance:

Theorem 2. *The number of triangles X_n of a SP graph with n vertices is asymptotically Gaussian, with*

$$\mathbb{E}(X_n) \sim \mu n, \qquad Var(X_n) \sim \lambda n,$$

where $\mu \approx 0.3947$ and $\lambda \approx 0.4143$.

One may compare these values with the expected number of pending triangles in a random SP-graph, computed in [12] as approximately $2.2313 \cdot 10^{-3} n$. As expected, the number of pending copies of a triangle in a random SP graph is much smaller than the number of occurrences of the triangle in a random SP graph.

4 Future work

The next step in this program is to study the same problem when avoiding the cycle of length four as a subgraph. By similar arguments it seems feasable to get asymptotic estimates for the number of SP graphs without cycles of length four. Surprisingly, a preliminary analysis shows that the study of the distribution of the number of four cycles on a uniformly at random SP graph with n vertices becomes more involved: as in the case of triangles, the creation of four cycles is essentially due to parallel compositions. However, in this situation is it possible to create a quadratic number of copies. This phenomena is completely different to the one obtained in the study of triangles. In particular, the existence of a limiting distribution will not be based on the application of the Quasi-Powers Theorem.

In this direction, a first study over 2-trees (maximal SP graphs) suggests that the distribution of the number of triangles on a random 2-tree with n vertices has linear expectation and variance, so the distribution is concentrated around the expected value. This is interesting by itself, because it is easy to construct 2-trees with a quadratic number of four cycles. Consequently this situation shows that the order of magnitude of expected value (linear) could be completely different of an object in the family (quadratic) . Indeed, heuristic arguments suggests that the limiting distribution (conveniently reescaled) is in fact a normal distribution, and we also believe that a similar phenomenon would arise in the SP family.

We also plan to study SP graphs with girth greater or equal than a given value g: this would emerge using similar ideas as in the case of triangles, but

excluding at the same time cycles of length less than g. It also seem feasable to study these questions when excluding a 2-connected subgraph and, in general, a connected subgraph.

References

[1] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1998.

[2] M. Bodirsky, O. Giménez, M. Kang, and M. Noy. Enumeration and limit laws of series-parallel graphs. *European Journal of Combinatorics*, 28(8):2091–2105, 2007.

[3] C. McDiarmid. Random Graphs from a Minor-Closed Class *Combinatorics, Probability and Computing*, vol. 18:583–599, 2009.

[4] G. Chapuy, É. Fusy, M. Kang, and B. Shoilekova. A complete grammar for decomposing a family of graphs into 3-connected components. *Electronic Journal of Combinatorics*, 15:Research Paper 148, 39, 2008.

[5] M. Drmota. Systems of functional equations. *Random Structures & Algorithms*, 1-2 (10), 103–124, 1997.

[6] M. Drmota, O. Giménez, and M. Noy. The maximum degree of series-parallel graphs. *Combinatorics, Probability and Computing*, 20(4), 529–570, 2011.

[7] M. Drmota, O. Giménez, and M. Noy. Vertices of given degree in series-parallel graphs. *Random Structures and Algorithms*, 36(3), 273–314, 2010.

[8] M. Drmota and M. Noy. Extremal Parameters in Sub-Critical Graph Classes, Proceedings Analco 2013, pp. 1–7.

[9] M. Drmota, É. Fusy, M. Kang, V. Kraus, and J. Rué, Asymptotic study of subcritical graph classes, *SIAM Journal of Dicrete Mathematics*, 1615–1651, 2011.

[10] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009.

[11] O. Giménez and M. Noy. Asymptotic enumeration and limit laws of planar graphs. *Journal of the American Mathematical Society*, 22, 309–329, 2009.

[12] O. Giménez, M. Noy, and J. Rué. Graph classes with given 3-connected components: Asymptotic enumeration and random graphs. *Random Structures & Algorithms*, 42(4), 438–179, 2013.

[13] A. Steger and K. Panagiotou, Maximal biconnected subgraphs of random planar graphs. *Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '09)*, 432–440, 2009.

[14] W.T. Tutte. *Connectivity in graphs*. Oxford University Press, 1966.

About a class of Hadamard propelinear codes. *

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Abstract. This article aims to explore the algebraic structure of Hadamard propelinear codes, which are not abelian in general but they have good algebraic and combinatorial properties. Concretely, we construct a subclass of Hadamard propelinear codes which enlarges the Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-codes. Several papers have been devoted to the relations between difference sets, t-designs, cocyclic-matrices and Hadamard groups, and we present a link between them and a class of Hadamard propelinear codes, which will be called full propelinear. Finally, as an exemplification, we go over Hadamard codes of length sixteen giving a propelinear structure for all of them.

Key words: Hadamard group, propelinear code, full propelinear codes.

1 Introduction

The discovery of the existence of a quaternary structure in some relevant families of codes with better parameters than any linear code [9] has raised the interest in the study of these codes and more generally on codes with a group structure. Propelinear codes issued from the idea of study the relationship between completely regular codes and regular graphs. Any propelinear code is associated to a group structure, for instance $\mathbb{Z}_2\mathbb{Z}_4$-linear codes ([6,9]) are propelinear codes. An important subclass of propelinear codes are those which are translation invariant, which were charaterized as $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code in [14]. The goal of this article is to study the algebraic properties of a kind of propelinear, which we call full propelinear.

In Section 2, we present the preliminaries of propelinear codes and the connections between the difference sets, t-designs, Hadamard groups, and cocyclic matrices. In Section 3, we construct the subclass of Hadamard full propelinear codes and anaylize some of the algebraic properties of these codes, while concluding that all Hadamard codes of length sixteen are Hadamard propelinear codes.

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2 Preliminaries

We denote by $\mathbb{Z}$, $\mathbb{Z}_r$, $\mathbb{F}_q$, the ring of integers, the ring of integers modulo r and any representation of a finite field with q elements, respectively. Any subset C of $\mathbb{F}_2^n$ is called a binary code. It is denoted by $d_H(\cdot,\cdot)$ and $\mathrm{wt}_H(\cdot)$ the *Hamming distance* and the *Hamming weight* on $\mathbb{F}_q^n$, respectively. We write $d_H(C)$ for the minimum distance of a linear code C, which is equal to its minimum weight, for C a linear subspace. A $[n,k,d]$ linear code C over $\mathbb{F}_q$ is a k-dimensional subspace of $\mathbb{F}_q^n$. The elements of C are called *codewords*. If the code is not linear we will call (n,M,d) a code of length n, cardinality M and minimum distance equal to d. The parameter d determine the error-correcting capability of C which is given by $e = \left\lfloor \frac{d-1}{2} \right\rfloor$. For a word v in $\mathbb{F}_q^n$, the support of v, denoted by $\mathrm{Supp}(v)$, is defined as the set of its nonzero positions.
Let S_n denote the symmetric group of permutations of the set $\{1,\ldots,n\}$. For any $\pi \in S_n$ and any vector $v \in \mathcal{F}_2^n$, $v = (v_1, v_2, \ldots, v_n)$, we write $\pi(v)$ to denote $(v_{\pi^{-1}(1)}, v_{\pi^{-1}(2)}, \ldots, v_{\pi^{-1}(n)})$. The isometries of a code C (distance preserving bijective mappings from C to C) form a group, $\mathrm{Iso}(C)$. We will call $\mathrm{Perm}(C)$ the group of coordinate permutations stabilizing C. Two binary codes C_1, C_2 of length n are said to be *isomorphic* if there is a coordinate permutation $\pi \in S_n$ such that $C_2 = \{\pi(x) : x \in C_1\}$. They are said to be *equivalent* if there is a vector $y \in \mathcal{F}_2^n$ and a $\pi \in S_n$ such $C_2 = \{y + \pi(x) : x \in C_1\}$. The *rank* of a binary code C is the dimension of the linear span of the codewords of C. The *kernel* K of a binary code C is the set of words which keeps the code invariant by translation, so $K(C) = \{z \in \mathcal{F}_2^n : C + z = C\}$. Assuming the all zero vector is in C we have that the kernel is a linear subspace and the dimension of $K(C)$ will be denoted by by $k(C)$ or simply k.

Definition 1. *[15] A binary code C of length n has a **propelinear** structure if for each codeword $x \in C$ there exists $\pi_x \in S_n$ satisfying the following conditions:*

1. *For all $x, y \in C$, $x + \pi_x(y) \in C$,*
2. *For all $x, y \in C$, $\pi_x \pi_y = \pi_z$, where $z = x + \pi_x(y)$.*

For all $x \in C$ and for all $y \in \mathbb{F}_2^n$, denote by $*$ the binary operation such that $x * y = x + \pi_x(y)$. Then, $(C, *)$ acts over $\mathbb{F}_2^n$ and, specifically, it is a group, which is not abelian in general. The vector $\mathbf{0}$ is always a codeword and $\pi_{\mathbf{0}}$ is the identity permutation. Hence, $\mathbf{0}$ is the identity element in C and $x^{-1} = \pi_x^{-1}(x)$, for all $x \in C$ [15]. We call $(C, *)$ a **propelinear code** if it can be provided with a propelinear structure.

Definition 2. *The action of a group, G on a set X is regular if it is both transitive and semiregular. Transitivity requires that for all $x, y \in X$, there is some $g \in G$ such that $gx = y$. Semiregularity requires that the stabilizers of all points be trivial. Obviously, if G acts regularly on X then $|G| = |X|$.*

Proposition 1. *[11] Let $(C, *)$ be a group. C is a propelinear code if and only if* $\mathrm{Iso}(C)$ *contains a regular subgroup acting transitively on C.*

Definition 3. *A Hadamard matrix is an $4n \times 4n$ matrix H containing entries from the set $\{1, -1\}$, with the property that:*

$$HH^T = 4nI,$$

where I means the identity matrix.

Let H be a normalized Hadamard matrix of order $4n$, so a matrix with all the entries in the first row and first column equal to $+1$. let H' be the matrix H after removing the first row and the first column. Let $B = \frac{1}{2}(H' + J)$, where J is the all one matrix. Hadamard matrices of order $4n$ $(n > 1)$, can be used to create an special family of 2-designs.
A set T of vectors $v \in \mathbb{F}^n$ of weight w is a *t-design*, t-(n, w, λ), if for any vector $z \in \mathbb{F}_2^n$ of weight t, $1 \leq t \leq w$, there are precisely λ vectors v_i, $i = 1, \ldots, \lambda$ from T, each of them covering z. A square divisible (n, m, w, λ)-design consists of a set of nm points and a set of nm blocks, where each point is in w blocks and each block consists of w points. Further, the point set is partitioned into n point classes of m points each, such that two points in distinct classes are both contained in precisely λ blocks, and no block contains distinct points in the same class. A 2-(n, w, λ)-design is just a divisible $(n, 1, w, \lambda)$-design.
Thus, note that the before defined matrix B is the incidence matrix of a 2-$(4n - 1, 2n - 1, n - 1)$ design, and we can take it as an alternative definition for a Hadamard matrix [1]. Let H be a $4n \times 4n$ Hadamard matrix, and A the incidence matrix defined by $A = \frac{1}{2}(H + J)$. Write $\bar{A}$ for the complement of A. Then

$$\Phi = \begin{pmatrix} A & \bar{A} \\ \bar{A} & A \end{pmatrix}$$

is the incidence matrix of a divisible $(4n, 2, 4n, 2n)$-design.

Elliott and Butson [5] define a **relative (v, m, k, λ)-difference set in a group** G relative to a normal subgroup N, where $|G| = vm$ and $|N| = m$. This is a subset D of G such that $|D| = k$ and the multiset of quotients $d_1 d_2^{-1}$ of distinct elements $d_1, d_2 \in D$ contains each element of $G \backslash N$ exactly λ times, and contains no elements of N. Thus $k(k-1) = \lambda m(v-1)$ and $v \neq 2k$. Equivalently, $|D \cap xD| = \lambda$, for all $x \in G \backslash N$. Let R be a relative $(4n, 2, 4n, 2n)$-difference set in a group G of order $8n$ relative to a normal subgroup $N \simeq \mathbb{Z}_2$ of G. Such a group is called a **Hadamard group** of order $8n$ [8]. In other words, G is a Hadamard group of order $8n$ and identity element $\mathbf{e}$, if it is a finite group containing a $4n$-subset D and an element $\mathbf{u}$ (called Hadamard subset corresponding to $\mathbf{u}$), such that

- D and $\mathbf{u}D$ are disjoints,

- aD and D intersect exactly in $2n$ elements, for any $a \in G$, $a \neq \mathbf{u}$, $a \neq \mathbf{e}$.
- aD and $\{b, b\mathbf{u}\}$ intersect exactly in one element, for any $a, b \in G$.

Let G be a finite group of order $4n$ and let $\langle -1 \rangle \simeq \mathbb{Z}_2$. A (normalized, binary, two-dimensional) cocycle is a set map $\psi \colon G \times G \to \mathbb{Z}_2$ satisfying $\psi(\mathbf{e}, \mathbf{e}) = 1$ and

$$\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k), \text{for all } g,h,k \in G.$$

A cocycle over G is naturally displayed as a **cocyclic matrix** M; that is, under some fixed ordering of the elements of G which indexes rows, and some (possibly different) fixed ordering of the elements of G which indexes columns, the entry in the (g,h)th position of the cocyclic matrix is $\psi(g,h)$, for all $g, h \in G$.
The connection between cohomology theory and Hadamard matrices afforded by cocyclic matrices was introduced by de Launey and Horadam. Furthermore, in [3] it is stated that the existence of a normal relative $(4n, 2, 4n, 2n)$ difference set is equivalent to the existence of a cocyclic Hadamard matrix of order $4n$. In [7], Flannery proved that the concepts of Hadamard group and cocyclic Hadamard matrix are equivalent.

Definition 4. *Any binary $(2n, 4n, n)$-code is called a Hadamard code. Furhter, C is said to be a Hadamard propelinear code if it is a Hadamard code and also a propelinear code.*

In [12] it was computed all possible values for two structural parameters (rank and dimension of the kernel) of a binary Hadamard code of length a power of two. Our interest is to deal with Hadamard codes with some kind of algebraic structure. The most basic structure is coming from groups of order 8 which, apart from those composed by $\mathbb{Z}_2$ and $\mathbb{Z}_4$, are the cyclic $\mathbb{Z}_8$, the dihedral D_8 and the quaternionic Q_8. The next proposition summarizes the results we obtained.

Proposition 2. *(The propelinear structures for Q_8, D_8, $\mathbb{Z}_8$)*

1. *The minimum lenght n for which a Hadamard propelinear structure exists for $Q_8 = \langle \mathbf{a}, \mathbf{b} : \mathbf{a}^4 = \mathbf{e}, \mathbf{a}^2 = \mathbf{b}^2, \mathbf{a}^{\mathbf{b}} = \mathbf{bab}^{-1} = \mathbf{a}^{-1} \rangle$ is $n = 4$. Furthermore, this structure is unique (up to isomorphism) and is given by:*

$$\mathbf{a} = (0,1,0,1), \mathbf{b} = (0,1,1,0), \pi_{\mathbf{a}} = (1,2)(3,4), \pi_{\mathbf{b}} = (1,3)(2,4).$$

2. *The unique Hadamard propelinear structure of length n for the dihedral $D_8 = \langle a, b : a^4 = \mathbf{e}, b^2 = \mathbf{e}, a^b = bab^{-1} = a^{-1} \rangle$ is given by:*

$$a = (1,1,0,0), b = (0,1,1,0), \pi_a = (1,4)(2,3), \pi_b = (1,4)(2,3).$$

Furthermore, there are only two propelinear structures of length 3 (up to isomorphism) given by:

$$a = (1,1,0), b = (1,0,0), \pi_a = (23), \pi_b = (23).$$
$$a = (1,0,0), b = (0,0,1), \pi_a = (12), \pi_b = (12).$$

3. *There is no any Hadamard propelinear structure for the cyclic group* $\mathbb{Z}_8 = \langle a : a^8 = \mathbf{e} \rangle$. *Although, there is a unique propelinear structure (up to isomorphism) of length 4 given by:*

$$a = (1,1,1,0), \pi_a = (1,2,3,4).$$

3 Hadamard Full Propelinear Codes

In this section we introduce the concept of Hadamard full propelinear code C. These codes have the property that the associated permutation π_x to each $x \in C$ do not have any fix point, except for $x \in \{\mathbf{e}, \mathbf{u}\}$. We show that the above definition is equivalent to the well known concepts of Hadamard group, 2-cocyclic matrices and relative difference sets. The section concludes showing that all binary Hadamard codes of length 16 are full propelinear.

Definition 5. *A Hadamard full propelinear code is a Hadamard propelinear code* C *such that for every* $a \in C$, $a \neq \mathbf{e}, a \neq \mathbf{u}$ *the permutation* π_a *has not any fixed coordinate and* $\pi_{\mathbf{e}} = \pi_{\mathbf{u}} = I$.

Lemma 1. *In a Hadamard full propelinear code* $(C, *)$ *let* $\mathbf{u}$ *be the all one vector. Then vector* $\mathbf{u}$ *is central in* C *and* $\pi_{\mathbf{u}} = I$.

Let C be a Hadamard full propelinear code of length $4n$. Define $D_j \subset C$ the subset of all vectors in C such that the jth coordinate is zero. Vectors in C have $4n$ coordinates and we can associate each one of them to a vector in D_j. Let $x \in D_j$ such that $\pi_x(e_j) = e_x$, where, for $i \in \{1, \ldots, 4n\}$, e_i means the unitary vector with only one nonzero coordinate at the position ith. The position where e_x is nonzero is the associated coordinate to vector x. This association is well defined, for a vector $y \neq x$ the associated coordinate is $e_y \neq e_x$. Let k the position where e_x has the nonzero coordinate. Note that either $D_k = x * D_j$ or $D_k = \mathbf{u} * x * D_j$ depending on the value of the kth coordinate of vector x. Calling $\delta_{x,j} = \mathbf{e}$ when the value of the kth coordinate of vector x is zero and $\delta_{x,j} = \mathbf{u}$ when the value of the kth coordinate of vector x is one, we have $D_k = \delta_{x,j} * x * D_j$, for $x \in D_j$.

Let H be the normalized Hadamard matrix corresponding to C and assume that the columns and the rows of H are indexed by the elements in D_1. The (y, x)-entry of H is zero if vector y belongs to D_k, where $\pi_x(e_1) = e_k$, so

$$(y, x)\text{-entry of } H \text{ is zero if and only if } y * x^{-1} * \delta_{x,1} \in D_1. \tag{1}$$

Proposition 3. *Let $(C, *)$ be a Hadamard propelinear code of length $4n$. Let D_1 the set of codewords with a zero in the first coordinate. Then for any $a \in C$ we have $|D_1 \cap a * D_1| \in \{0, 4n, 2n\}$.*

If C is a Hadamard full propelinear code of length $4n$ then C is a Hadamard group in the sense of [8] and D_1 is a Hadamard set corresponding to $\mathbf{u}$.

Proof.

1. If $a \in D_1$ and π_a does not include the first position then $a * D_1 = D_1$ and $|D_1 \cap a * D_1| = 4n$.
2. If $a \notin D_1$ and π_a does not include the first position then $a * D_1 = \mathbf{u} * D_1$ and $|D_1 \cap a * D_1| = 0$.
3. If π_a includes the first position, say that $\pi_a(e_1) = e_k$ then $D_k = \delta_{a,1} * a * D_1$ and $|D_1 \cap a * D_1| = 2n$.

If C is a full propelinear code the previous first two items show that $|D_1 \cap a * D_1| = 4n$ if and only if $a = \mathbf{e}$ and $|D_1 \cap a * D_1| = 0$ if and only if $a = \mathbf{u}$. Hence, in this case, C is a Hadamard group. ∎

Proposition 4. *Let G be a Hadamard group with D as a Hadamard subset. Then G is a Hadamard full propelinear code.*

Proof. Let G a Hadamard group of order $8n$ with Hadamard subset D. We can construct an $4n \times 4n$ matrix H, where the rows and columns are indexed by the elements in D. The entry (a, b) of H is 0 or 1, depending on whether $ab^{-1}\delta_{b,1} \in D$, where $\delta_{b,1}$ was defined in (1). Matrix H is a Hadamard matrix and G can be equipped with a full propelinear structure. For any $a \in G$ define $\pi_a(x) = a + xa$, where $x \in G$. The map π_a acts as a permutation on the coordinates. Specifically, coordinate given by b is moved to coordinate ba after π_a. To show this, take two vectors x, y with the same value on the coordinate given by b, so xb^{-1} and yb^{-1} simultaneously belong (respectively, does not belong) to D. Consider the values of $(a + xa)$ and $(a + ya)$ on the coordinate given by ba. This pair of values agrees or does not agree like the values of xa and ya on the same coordinate and these last ones agree or do not agree depending on whether $xa(ba)^{-1}$ and $ya(ba)^{-1}$ simultaneously belong (respectively, do not belong) to D. Thus, we reached the same condition that the starting one. Also we see that $\pi_a(\mathbf{e}) = a + \mathbf{e}a = \mathbf{e}$. Finally, we can define the propelinear structure on G given by $a * b = a + \pi_a(b) = ba$. This proves the statement. ∎

It is well known that there are five inequivalent Hadamard codes of length 16. One of them is linear, another is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code and the other three cannot be realized as $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, [6]. However, one of those can be realized as a $\mathbb{Z}_2\mathbb{Z}_4Q_8$-code, more specifically, as a pure Q_8-code [4]. As an

exemplification of the concepts of the current paper we present the last two as Hadamard full propelinear codes. The group structure of these two propelinear codes correspond to a generalized quaternion group of order 32. This generalized quaternion group is given by $Q_{32} = \langle \mathbf{a}, \mathbf{b} : \mathbf{a}^{16} = \mathbf{e}, \mathbf{a}^8 = \mathbf{b}^2, \mathbf{a}^{\mathbf{b}} = \mathbf{bab}^{-1} = \mathbf{a}^{-1} \rangle$. To construct these propelinear codes take $\mathbf{a}, \mathbf{b} \in \mathbb{F}_2^{16}$ and their corresponding permutations $\pi_{\mathbf{a}}, \pi_{\mathbf{b}} \in S_4$.
The code C with rank equal to 8 and dimension of the kernel 2 is given by:

$$\begin{aligned}
\mathbf{a} &= (1,1,1,1,0,0,1,0,0,1,1,0,0,0,0,1),\\
\mathbf{b} &= (0,0,1,0,1,1,0,0,1,1,1,0,0,1,0,1),\\
\pi_{\mathbf{a}} &= (8,7,6,5,4,3,2,1)(16,15,14,13,12,11,10,9),\\
\pi_{\mathbf{b}} &= (1,9)(2,16)(3,15)(4,14)(5,13)(6,12)(7,11)(8,10).
\end{aligned}$$

and the remainder elements are computed giving the code $C = \langle \mathbf{a}, \mathbf{b} \rangle$.
The code $D = \langle \mathbf{a}, \mathbf{b} \rangle$, with rank equal to 8 and dimension of the kernel 1, is computed taking:

$$\begin{aligned}
\mathbf{a} &= (1,1,1,1,0,0,1,0,1,0,1,1,0,0,0,0),\\
\mathbf{b} &= (0,1,1,1,0,0,0,1,1,0,0,0,1,1,1,0),\\
\pi_{\mathbf{a}} &= (8,7,6,5,4,3,2,1)(9,10,11,12,13,14,15,16),\\
\pi_{\mathbf{b}} &= (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16).
\end{aligned}$$

Note that in both cases, the group $\Pi = \{\pi_x : x \in G\}$, where G is either C or D, is the dihedral group of order 16.

References

[1] E. F. Assmus, Jr. and J. D. Key. *Designs and their Codes. Cambridge University Press,* 1992. Cambridge Tracts in Mathematics, Vol. 103.

[2] W. Burnside, *Theory of Groups of Finite Order, 2nd edition, Cambridge University Press,* 1911, reprinted Dover Publications, New York, 1955.

[3] W. de Launey, D.L. Flannery, K.J. Horadam, *Cocyclic Hadamard matrices and difference sets, Discrete Applied Mathematics.* 102 (2000) 47–61.

[4] Á. del Rio, J. Rifà, *Families of Hadamard $\mathbb{Z}_2\mathbb{Z}_4Q_8$-Codes, IEEE Trans. Inf. Theory,* Vol. 59, (8), August 2013. pp: 5140–5155.

[5] J.E.H. Elliott, A.T. Butson, *Relative difference sets, Illinois J. Math. 10* (1966) 517–531.

[6] C. Fernandez-Cordoba, J. Pujol and J.Rifà, M. Villanueva *$\mathbb{Z}_2\mathbb{Z}_4$-linear codes: Rank and kernel Designs,Codes and Cryptography,* volume 54, number 2, p.167-179, August 2009

[7] D.L. Flannery, *Cocyclic Hadamard matrices and Hadamard groups are equivalent, J. Algebra*192 (1997) 749–779.

[8] N. Ito, *On Hadamard groups, J. Algebra 168* (1994) 981–987.

[9] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, P. Solé, *The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes. IEEE Transactions on Information Theory*, 40(2) (1994) 301–319.

[10] K. J. Haradam, *Hadamard matrices and their applications. Princeton University Press.* Princeton New Jersey. 2007.

[11] K. T. Phelps, J. Rifà, *On Binary* 1*-Perfect Additive Codes: Some Structural Properties, IEEE Trans. Inf. Theory,* Vol. 48, No. 9, September 2002. pp. 2587–2592.

[12] K. T. Phelps, J Rifà, M. Villanueva, *Rank and kernel of binary Hadamard codes, IEEE Trans. Inf. Theory,* Vol. 51 (11), pp: 3931–3937. 2005.

[13] K. T. Phelps, J Rifà, M. Villanueva, *On the Additive ($\mathbb{Z}_4$-Linear and Non-$\mathbb{Z}_4$-Linear) Hadamard Codes: Rank and Kernel. IEEE Trans. Inf. Theory,* Vol. 52 (1), pp: 316–324. 2006.

[14] J. Rifà, J. Pujol, *Translation invariant properlinear codes. IEEE Trans. Inform. Theory,* vol. 43, pp.590-598, 1997.

[15] J. Rifà, J. M. Basart, L. Huguet, *On completely regular propelinear codes, Lecture Notes in Computer Science.* Berlin, Germany: Springer-Verlag, vol. 357, pp. 341–355, 1989.

The numerical semigroup of the integers which are bounded by a submonoid of $\mathbb{N}^2$ *

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Abstract. Let M be a submonoid of $(\mathbb{N}^2, +)$ such that

$$\mathrm{A}(M) = \{n \in \mathbb{N} \mid a < n < b \text{ for some } (a, b) \in M\}$$

is a non-empty set. Then $\mathrm{A}(M) \cup \{0\}$ is a numerical semigroup. We will show that a numerical semigroup S can be obtained in this way if and only if $\{a+b-1, a+b+1\} \subseteq S$ for all $a, b \in S \setminus \{0\}$. We will see that such numerical semigroups form a Frobenius variety and we will study it.

Key words: submonoids, numerical semigroups, non-homogeneous patterns, Frobenius varieties, trees.

1 Introduction

Let $\mathbb{N}$ be the set of nonnegative integers and let M be a submonoid of $(\mathbb{N}^2, +)$ (that is, a subset of $\mathbb{N}^2$ that is closed for the addition and that contains the zero element $(0, 0)$). We will say that a positive integer n is *bounded by* M if there exists $(a, b) \in M$ such that $a < n < b$. We will denote by $\mathrm{A}(M) = \{n \in \mathbb{N} \mid n \text{ is bounded by } M\}$.

A *numerical semigroup* is a submonoid S of $(\mathbb{N}, +)$ such that $\gcd(S) = 1$.

It is easy to check that, if M is a submonoid of $(\mathbb{N}^2, +)$ such that $\mathrm{A}(M)$ is not empty, then $\mathrm{A}(M) \cup \{0\}$ is a numerical semigroup. This fact allows us to give the concept of *numerical $\mathcal{A}$-semigroup*. Indeed, a numerical semigroup S is a *numerical $\mathcal{A}$-semigroup* if there exists M, submonoid of $(\mathbb{N}^2, +)$, such that $S = \mathrm{A}(M) \cup \{0\}$.

The purpose of this work is to study this family of numerical semigroups. First of all, we will show a characterization of them, via the concept of non-homogeneous pattern (see [1]), and how we can obtain all of them.

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We have that, if S is a numerical semigroup, then $\mathbb{N} \setminus S$ is finite. This fact allows us to define the *Frobenius number* of S, denoted by $\mathrm{F}(S)$, as the greatest integer that does not belong to S (see [4]).

A *Frobenius variety* (see [6]) is a non-empty family $\mathcal{V}$ of numerical semigroups that fulfills the following conditions,

1. if $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$;
2. if $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{V}$.

If we denote by $\mathcal{A} = \{S \mid S \text{ is a numerical } \mathcal{A}\text{-semigroup}\}$, then we will show that $\mathcal{A}$ is a Frobenius variety. As a consequence of this result, we will arrange the elements of $\mathcal{A}$ in a tree $\mathrm{G}(\mathcal{A})$ with root $\mathbb{N}$.

Finally, we will see that, if X is a non-empty set of positive integers, then there exists the smallest numerical $\mathcal{A}$-semigroup, $\mathcal{A}(X)$, that contains X. Moreover, we will design an algorithm to compute $\mathcal{A}(X)$ starting from X.

We end this introduction pointing out that this work is a short version of the manuscript [5] that is under review process.

2 Numerical $\mathcal{A}$-semigroups

Let us remember that, if M is a submonoid of $(\mathbb{N}^2, +)$, then

$$\mathrm{A}(M) = \{n \in \mathbb{N} \mid a < n < b \text{ for some } (a, b) \in \mathbb{N}^2\}.$$

Lemma 1. *Let M be a submonoid of $(\mathbb{N}^2, +)$. If $x, y \in \mathrm{A}(M)$, then we have that $\{x + y - 1, x + y, x + y + 1\} \subseteq \mathrm{A}(M)$.*

Remark 1. Let us observe that there exist numerical semigroups that are not numerical $\mathcal{A}$-semigroups. For example, the numerical semigroup $S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ (where the symbol $\rightarrow$ means that every integer greater than 14 belongs to S) is not a numerical $\mathcal{A}$-semigroup because $5 + 7 + 1 \notin S$.

Let X be a non-empty subset of a commutative monoid $(\mathfrak{M}, +)$. The monoid generated by X, denoted by $\langle X \rangle$, is the smallest (with respect to the set inclusion) submonoid of $(\mathfrak{M}, +)$ containing X. It is known (see [7]) that $\langle X \rangle = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid n \in \mathbb{N} \setminus \{0\},\ x_1, \ldots, x_n \in X,\ \lambda_1, \ldots, \lambda_n \in \mathbb{N}\}$.

If $\mathfrak{M} = \langle X \rangle$, then we say that X is a *system of generators* of $\mathfrak{M}$ (or that $\mathfrak{M}$ is generated by X). It is well known (see [8]) that the submonoids of $(\mathbb{N}, +)$ (in particular, the numerical semigroups) are finitely generated.

Lemma 2. *Let S be the numerical semigroup generated by a set of positive integers $\{n_1, \ldots, n_p\}$. Let us assume that $\{x + y - 1, x + y + 1\} \subseteq S$, for every $x, y \in S \setminus \{0\}$. If $\lambda_1, \ldots, \lambda_p, x \in \mathbb{N}$ and $\lambda_1(n_1 - 1) + \cdots + \lambda_p(n_p - 1) < x < \lambda_1(n_1 + 1) + \cdots + \lambda_p(n_p + 1)$, then $x \in S$.*

In the next result (that is a consequence of Lemmas 1 and 2) we give a characterization of numerical $\mathcal{A}$-semigroups.

Theorem 1. *Let S be a numerical semigroup. The following conditions are equivalent.*

1. *S is a numerical $\mathcal{A}$-semigroup.*
2. *If $x, y \in S \setminus \{0\}$, then $\{x + y - 1, x + y + 1\} \subseteq S$.*

The next result guarantees us that, to study numerical $\mathcal{A}$-semigroups, we can focus in finitely generated submonoids of $(\mathbb{N}^2, +)$.

Corollary 1. *Let S be a numerical semigroup. The following conditions are equivalent.*

1. *S is a numerical $\mathcal{A}$-semigroup.*
2. *$S = \mathrm{A}(M) \cup \{0\}$ for some finitely generated submonoid M of $(\mathbb{N}^2, +)$.*
3. *There exist $a_1, b_1, \ldots, a_p, b_p \in \mathbb{N}$ such that $S = \{n \in \mathbb{N} \mid a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p$ for some $x_1, \ldots, x_p \in \mathbb{N}\} \cup \{0\}$.*

From Item 3 of Corollary 1, we can see a numerical $\mathcal{A}$-semigroup as a set that contains the integers n such that the system of inequalities

$$a_1x_1 + \cdots + a_px_p < n < b_1x_1 + \cdots + b_px_p$$

has solution in $\mathbb{N}^p$ (where $\{a_1, b_1, \ldots, a_p, b_p\} \subseteq \mathbb{N}$ is a fixed set).

Let $(\mathfrak{M}, +)$ be a commutative monoid and let X be a system of generators of $\mathfrak{M}$. If $\mathfrak{M} \neq \langle Y \rangle$ for all $Y \subset X$ (with $Y \neq X$), then we say that X is a *minimal system of generators* of $\mathfrak{M}$.

The following result is consequence of [8, Lemma 2.3, Corollary 2.8].

Lemma 3. *Let M be a submonoid of $(\mathbb{N}, +)$. Then the minimal system of generators of M is $X = (M \setminus \{0\}) \setminus ((M \setminus \{0\}) + (M \setminus \{0\}))$. In addition, X is finite and is contained in every system of generators of M.*

Let S be a numerical semigroup and let $\{n_1, \ldots, n_p\} \subseteq \mathbb{N} \setminus \{0\}$ be a system of generators of S. If $s \in S$, then it is defined the *order* of s in S by (see [2]) $\mathrm{ord}(s; S) = \max\{a_1 + \cdots + a_p \mid a_1n_1 + \cdots + a_pn_p = s, \text{ with } a_1, \ldots, a_p \in \mathbb{N}\}$. If no ambiguity is possible, then we write $\mathrm{ord}(s)$.

Remark 2. From Lemma 3, the definition of $\mathrm{ord}(s; S)$ is independent of the considered system of generators of S, that is, only depends on s and S. Thus, we can take the minimal system of generators of S in order to define $\mathrm{ord}(s; S)$.

Lemma 4. *Let S be the numerical semigroup with minimal system of generators $\{n_1, \ldots, n_p\}$ and let $s \in S$.*

1. *If $i \in \{1, \ldots, p\}$ and $s - n_i \in S$, then $\mathrm{ord}(s - n_i) \leq \mathrm{ord}(s) - 1$.*
2. *If $s = a_1n_1 + \cdots + a_pn_p$, with $\mathrm{ord}(s) = a_1 + \cdots + a_p$ and $a_i \neq 0$, then $\mathrm{ord}(s - n_i) = \mathrm{ord}(s) - 1$.*

As a consequence of Theorem 1 and Lemma 4, we have another characterization of numerical $\mathcal{A}$-semigroups.

Proposition 1. *Let S be a numerical semigroup with minimal system of generators given by $\{n_1, \ldots, n_p\}$. The following conditions are equivalent.*

1. *S is a numerical $\mathcal{A}$-semigroup.*
2. *If $i, j \in \{1, \ldots, p\}$, then $\{n_i + n_j - 1, n_i + n_j + 1\} \subseteq S$.*
3. *If $s \in S \setminus \{0, n_1, \ldots, n_p\}$, then $\{s-1, s+1\} \subseteq S$.*
4. *If $s \in S \setminus \{0\}$, then $s + z \in S$ for all $z \in \mathbb{Z}$ such that $|z| < \operatorname{ord}(s)$.*

Remark 3. Item 2 of the above proposition allows us to decide, faster than with Theorem 1, if a numerical semigroup is a numerical $\mathcal{A}$-semigroup.

Remark 4. Let us remember that the gaps of a numerical semigroup S are the elements of the set $\mathbb{N} \setminus S$. As a consequence of Item 3 of Proposition 1, we have that a numerical $\mathcal{A}$-semigroups S can be characterized as a numerical semigroup satisfying that, if $s \in S$ is a maximum or a minimum of an interval of non-gaps, then s is a minimal generator of S or $s = 0$.

Example 1. Let S be the numerical semigroup minimally generated by $\{3, 5, 7\}$. Since $\{3+3\}+\{-1,1\}$, $\{3+5\}+\{-1,1\}$, $\{3+7\}+\{-1,1\}$, $\{5+5\}+\{-1,1\}$, $\{5+7\}+\{-1,1\}$, and $\{7+7\}+\{-1,1\}$ are subsets of S, then we can assert that S is a numerical $\mathcal{A}$-semigroup (remember Remark 3).

On the other hand, we have that $S = \{0, 3, 5, 6, 7, \rightarrow\}$ and, thereby, its intervals of non-gaps are $\{0\}$, $\{3\}$ and $\{5, \rightarrow\}$. Inasmuch as the maximum and the minimum of such a sets are zero or a minimal generator, from Remark 4, we have another way to state that S is a numerical $\mathcal{A}$-semigroup.

Example 2. Let T be the numerical semigroup minimally generated by $\{5, 7, 9\}$. Then $T = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$ and its intervals of non-gaps are $\{0\}$, $\{5\}$, $\{7\}$, $\{9, 10\}$, $\{12\}$, and $\{14, \rightarrow\}$. Since $\max\{9, 10\} = 10$ is different from zero and it is not a minimal generator of T, then T is not a numerical $\mathcal{A}$-semigroup.

3 The Frobenius variety of the numerical $\mathcal{A}$-semigroups

Lemma 5. *Let S, T be numerical semigroups.*

1. *$S \cap T$ is a numerical semigroup.*
2. *If $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\}$ is a numerical semigroup.*

From Lemma 5 and Theorem 1, we have the following result.

Proposition 2. *The set $\mathcal{A} = \{S \mid S$ is a numerical $\mathcal{A}$-semigroup$\}$ is a Frobenius variety.*

A *graph* G is a pair (V, E), where V is a non-empty set and E is a subset of $\{(v, w) \in V \times V \mid v \neq w\}$. The elements of V are called *vertices* of G and the elements of E are called *edges* of G. A *path (of length n)* connecting the vertices x and y of G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$.

We say that a graph G is a *tree* if there exist a vertex v^* (the *root* of G) such that, for every other vertex x of G, there exists a unique path connecting x and v^*. If (x, y) is an edge of the tree, then we say that x is a *child* of y.

We define the graph $\mathrm{G}(\mathcal{A})$ in the following way,

- $\mathcal{A}$ is the set of vertices of $\mathrm{G}(\mathcal{A})$;
- $(S, S') \in \mathcal{A} \times \mathcal{A}$ is an edge of $\mathrm{G}(\mathcal{A})$ if $S' = S \cup \{\mathrm{F}(S)\}$.

If S is a numerical semigroup, then we will denote by $\mathrm{msg}(S)$ the minimal system of generators of S. It is easy to show (see [8, Exercise 2.1]) that, if S is a numerical semigroup and $x \in S$, then $S \setminus \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.

The next result is consequence of [6, Proposition 24, Theorem 27].

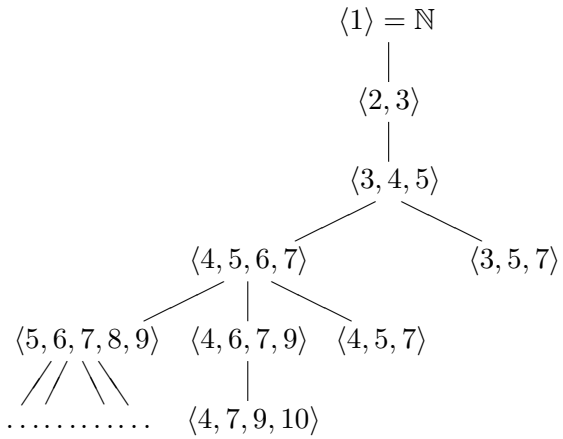
Theorem 2. $\mathrm{G}(\mathcal{A})$ *is a tree with root* $\mathbb{N}$. *Moreover, the set of children of a vertex* $S \in \mathcal{A}$ *is* $\{S \setminus \{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S),\ \text{and } S \setminus \{x\} \in \mathcal{A}\}$.

Now, we will characterize the elements $x \in \mathrm{msg}(S)$ such that $S \setminus \{x\} \in \mathcal{A}$.

Proposition 3. *Let* S *be a numerical* $\mathcal{A}$*-semigroup such that* $S \neq \mathbb{N}$, *and let* $x \in \mathrm{msg}(S)$. *Then* $S \setminus \{x\}$ *is a numerical* $\mathcal{A}$*-semigroup if and only if* $\{x - 1, x + 1\} \subseteq \{0\} \cup (\mathbb{N} \setminus S) \cup \mathrm{msg}(S)$.

Corollary 2. *Let* S *be a numerical* $\mathcal{A}$*-semigroup such that* $S \neq \mathbb{N}$, *and let* x *be a minimal generator of* S *greater than* $\mathrm{F}(S)$. *Then* $S \setminus \{x\}$ *is a numerical* $\mathcal{A}$*-semigroup if and only if* $\{x - 1, x + 1\} \subseteq \mathrm{msg}(S) \cup \{\mathrm{F}(S)\}$.

By applying Theorem 2 and Corollary 2 (see Example 3) we can build recursively the tree $\mathrm{G}(\mathcal{A})$ such as it is shown in the following figure.



Example 3. Let S be the numerical semigroup with minimal system of generators $\{4, 6, 7, 9\}$. Then $S = \{0, 4, 6, \rightarrow\}$ and, therefore, $\mathrm{F}(S) = 5$. From Proposition 1, we deduce that S is a numerical $\mathcal{A}$-semigroup. By applying Theorem 2 and Corollary 2, we get that S has a unique child in $\mathrm{G}(\mathcal{A})$. Namely, $S \setminus \{6\} = \langle 4, 7, 9, 10\rangle$.

Remark 5. Let us observe that, if S' is a child of S in $\mathrm{G}(\mathcal{A})$, then $\mathrm{F}(S') > \mathrm{F}(S)$ and $\mathrm{g}(S') = \mathrm{g}(S) + 1$. Therefore, when we go on along the branches of the tree $\mathrm{G}(\mathcal{A})$, we get numerical semigroups with greater Frobenius number and genus. Thus, we can use this construction in order to obtain all the numerical $\mathcal{A}$-semigroups with a given Frobenius number or genus.

4 The smallest numerical $\mathcal{A}$-semigroup that contains a given set of positive integers

Since $\mathcal{A}$ is a Frobenius variety, we have that the finite intersection of numerical $\mathcal{A}$-semigroups is a numerical $\mathcal{A}$-semigroup. However, the infinite intersection of numerical $\mathcal{A}$-semigroups is not always a numerical $\mathcal{A}$-semigroup. On the other hand, it is clear that the (finite or infinite) intersection of numerical semigroups is always a submonoid of $(\mathbb{N}, +)$.

If M is a submonoid of $(\mathbb{N}, +)$, then we will say that M is an $\mathcal{A}$-monoid if it can be expressed like the intersection of numerical $\mathcal{A}$-semigroups. Being that the intersection of $\mathcal{A}$-monoids is an $\mathcal{A}$-monoid, we can give the following definition.

Definition 1. *Let X be a subset of $\mathbb{N}$. The* $\mathcal{A}$-monoid generated by X *(denoted by $\mathcal{A}(X)$) is the intersection of all $\mathcal{A}$-monoids containing X.*

Let us observe that $\mathcal{A}(X)$ is the smallest $\mathcal{A}$-monoid containing X.

Proposition 4. *If $X \subseteq \mathbb{N}$, then $\mathcal{A}(X)$ is the intersection of all numerical $\mathcal{A}$-semigroups that contain X. Moreover, if $X \neq \emptyset$ and $X \subseteq \mathbb{N} \setminus \{0\}$, then $\mathcal{A}(X)$ is a numerical $\mathcal{A}$-semigroup.*

As a consequence of Proposition 4, we have that M is an $\mathcal{A}$-monoid if and only if M is a numerical $\mathcal{A}$-semigroup or $M = \{0\}$. Moreover, from Lemma 3 and Proposition 4, we can prove the next result.

Theorem 3. $\mathcal{A} = \{\mathcal{A}(X) \mid X$ *is a non-empty finite subset of* $\mathbb{N} \setminus \{0\}\}$.

If M is an $\mathcal{A}$-monoid and X is a subset of $\mathbb{N}$ such that $M = \mathcal{A}(X)$, then we will say that X is an *$\mathcal{A}$-system of generators* of M. In addition, if $M \neq \mathcal{A}(Y)$ for all $Y \subset X$ (with $Y \neq X$), then we will say that X is a *minimal $\mathcal{A}$-system of generators* of M. From [6, Corollary 19], we have the following result.

Proposition 5. *Every $\mathcal{A}$-monoid has a unique minimal $\mathcal{A}$-system of generators, which in addition is finite.*

The next result follows from [6, Proposition 24].

Proposition 6. *Let M be an $\mathcal{A}$-monoid and let $x \in M$. Then $M \setminus \{x\}$ is an $\mathcal{A}$-monoid if and only if x is a minimal $\mathcal{A}$-system generator of M.*

Corollary 3. *Let X be a non-empty subset of $\mathbb{N} \setminus \{0\}$. Then the set given by $\{x \in X \mid \mathcal{A}(X) \setminus \{x\}$ is a numerical $\mathcal{A}$-semigroup$\}$ is the minimal $\mathcal{A}$-system of generators of $\mathcal{A}(X)$.*

Example 4. Beginning in Example 3, we know that $S = \langle 4, 6, 7, 9\rangle$ is a numerical $\mathcal{A}$-semigroup. By applying Proposition 3, we easily deduce that

$$\{x \in \{4, 6, 7, 9\} \mid S \setminus \{x\} \text{ is a numerical } \mathcal{A}\text{-semigroup}\} = \{4, 6\}.$$

Therefore, $S = \mathcal{A}(\{4, 6\})$ and $\{4, 6\}$ is its minimal $\mathcal{A}$-system of generators.

Let $x_1, \ldots, x_t$ be positive integers. We will denote by $S(x_1, \ldots, x_t)$ the set $\{a_1x_1 + \cdots + a_tx_t + z \mid a_1, \ldots, a_t \in \mathbb{N},\ z \in \mathbb{Z},$ and $|z| < a_1 + \cdots + a_t\} \cup \{0\}$.

Theorem 4. *If $x_1, \ldots, x_t$ are positive integers, then $S(x_1, \ldots, x_t)$ is the smallest numerical $\mathcal{A}$-semigroup that contains the set $\{x_1, \ldots, x_t\}$.*

Corollary 4. *If m is a positive integer, then*

$$\mathcal{A}(\{m\}) = \{km + z \mid k \in \mathbb{N} \setminus \{0\} \text{ and } z \in \{-(k-1), \ldots, k-1\}\} \cup \{0\}.$$

Example 5. The smallest numerical $\mathcal{A}$-semigroup containing $m = 10$ is the set $\mathcal{A}(\{10\}) = \{k \cdot 10 + z \mid k \in \mathbb{N} \setminus \{0\}$ and $z \in \{-(k-1), \ldots, k-1\}\} \cup \{0\} = \{0, 10, 19, 20, 21, 28, 29, 30, 31, 32, 37, 38, 39, 40, 41, 42, 43, 46, \rightarrow\}$. By using minimal generators, $\mathcal{A}(\{10\}) = \langle 10, 19, 21, 28, 32, 37, 43, 46, 54, 55\rangle$.

Let us observe that, in Theorem 4, are described the elements of the smallest numerical $\mathcal{A}$-semigroup containing a given set of positive integers. However, in order to compute such a numerical $\mathcal{A}$-semigroup, we propose the following algorithm that is justified by Proposition 1.

Algorithm 1. *INPUT: A finite set X of positive integers.*

OUTPUT: The minimal system of generators of $\mathcal{A}(X)$.

(1) $Y = \mathrm{msg}(\langle X\rangle)$.

(2) $Z = Y \cup \left(\bigcup_{a,b\in Y}\{a + b - 1, a + b + 1\}\right)$.

(3) If $\mathrm{msg}(\langle Z\rangle) = Y$, *then return* Y.

(4) Set $Y = \mathrm{msg}(\langle Z\rangle)$ *and go to (2).*

Example 6. We are going to compute $\mathcal{A}(\{5,7\})$ applying Algorithm 1.

- $Y = \{5,7\}$.
- $Z = \{5,7,9,11,13,15\}$.
- $\mathrm{msg}(\langle Z\rangle) = \{5,7,9,11,13\}$.
- $Y = \{5,7,9,11,13\}$.
- $Z = \{5,7,9,11,13,15,17,19,21,23,25,27\}$.
- $\mathrm{msg}(\langle Z\rangle) = \{5,7,9,11,13\}$.
- $\mathcal{A}(\{5,7\}) = \langle 5,7,9,11,13\rangle$).

The most complex process in Algorithm 1 is the computation of $\mathrm{msg}(\langle Z\rangle)$, that is, the computation of the minimal system of generators of a numerical semigroup S starting from a system of generators of it. For this purpose, we can use the `GAP` package called `numericalsgps` (see [3]).

References

[1] M. Bras-Amorós, P. A. García-Sánchez and A. Vico-Oton. Nonhomogeneous patterns on numerical semigroups. *Int. J. Algebra Comput.*, 23:1469–1483, 2013.

[2] L. Bryant. Goto numbers of a numerical semigroup ring and the Gorensteiness of associated graded rings. *Comm. Algebra*, 38:2092–2128, 2010.

[3] M. Delgado, P. A. García-Sánchez, and J. Morais. *"numericalsgps": a* GAP *package on numerical semigroups.* `http://www.gap-system.org/Packages/numericalsgps.html`.

[4] J. L. Ramírez Alfonsín. *The Diophantine Frobenius problem.* Oxford Univ. Press, 2005.

[5] A. M. Robles-Pérez and J. C. Rosales. Numerical semigroups in a problem about transport with benefits. Preprint.

[6] J. C. Rosales. Families of numerical semigroups closed under finite intersections and for the Frobenius number. *Houston J. Math.*, 34:339–348, 2008.

[7] J. C. Rosales and P. A. García-Sánchez. *Finitely generated commutative monoids.* Nova Science Publishers, New York, 1999.

[8] J. C. Rosales and P. A. García-Sánchez. *Numerical semigroups.* Developments in Mathematics, vol. 20. Springer, New York, 2009.

Approximating degree sequences with regular graphic sequences

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Abstract. We study conditions to approximate the degree sequence of a given graph by a regular one. We obtain optimal conditions for a few metrics such as the edge rotation distance for graphs, the rectilinear and Euclidean distances over degree sequences. When we require the approximation to have at least k copies of each value in the degree sequence, that is, each value d in the degree sequence appears at least k times, the problem has a direct application in the context of data privacy when k-anonymity is required.

Key words: Edit distance, Edge rotation distance, degree sequences distance, graph distance, k-anonymity.

1 Introduction

For modeling applications, it is very important to know the distance between isomorphism classes of graphs. This is the case in organic chemistry, cf. [1] where the principle of minimal structural change is reflected in the maximum common subgraph, this was defined and proved to be a distance in [10].

In [2], Chartrand, Saba and Zou defined the edge rotation distance, it is also known as the reaction metric. Both concepts are put toghether in [4], where the authors present bounds on the rotation distance of two graphs in terms of their greatest common subgraphs.

In [5] three different metrics derived from edge manipulation are studied. They are the edge move distance, the edge slide distance and the edge rotation

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distance. It is also observed a relation between edge rotation distance between two graphs and the rectilinear distance between its degree sequences.

The edge move distance was first defined by V. Baláz, J.Koca, V. Kvasnícka, and M. Sekanina, cf. [3]. More recently it also is known as edit distance.

The general edit distance problem is to approximate a given graph G by another graph that satisfies an additional property $\mathcal{P}$, i.e. calculate the minimum number of edge modifications (additions or deletions) that have to be done to a graph G in order to guarantee that the modified graph has property $\mathcal{P}$.

In this work we consider the case of the k-anonymity property proceeding from data privacy. In data privacy, a data satisfies *k-anonymity* with respect to a query when for any value in the domain of the result of the query, there are at least k individuals with that value, cf. [8,9]. Applying this concept to the degree sequence of a given graph G means that given k, we are interested in a graph G' such that for any value d in the degree sequence of G', there are at least k vertices in G' with the same degree d. In order to keep some of the structure of G, the graph G' must be the most similar graph to G under a given metric.

We review in Section 2 the main definitions needed in the rest of the paper. In Section 3, we consider the problem of approximating a graph with regular graphs and we obtain the best approximations under the Euclidean and rectilinear metrics between degree sequences and also under the edge rotation distance for graphs. In Section 4, we consider the problem of approximating with degrees of frequencies k, and obtain a result that has an application for k-anonymity of degree sequences. This section contains the main results.

2 Definitions

Let G denote a simple graph, i.e. without loops or multiple edges, $V(G)$ its vertex set and $E(G)$ its edge set.

By considering the degree sequences of graphs with n vertices as vectors in the discrete Euclidean space $\mathbb{N}^n \subset \mathbb{R}^n$, they inherit the *Euclidean distance* $\delta^2(s, s') = \sum_{i=1}^{n}(d_i - d'_i)^2$, where $s = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a graph G and $s' = (d'_1, d'_2, \ldots, d'_n)$ the degree sequence of a graph G'. By measuring the distance coordinate by coordinate, the *rectilinear distance* is obtained $\delta^1(s, s') = \sum_{i=1}^{n} |d_i - d'_i|$.

The distance between two graphs G and G' can also be measured by the number of edge operations that must be done to G in order to obtain the graph G', if the operation is a rotation then we refer to the edge rotation distance δ_{ER}.

Let G and G' be two graphs having the same order $n = |V(G)| = |V(G')|$ and size $q = |E(G)|, q' = |E(G')|$, not necessarily equal. The *edge rotation distance* $\delta_{ER}(G, G')$ is defined as 0 if $G \cong G'$ and, otherwise, as the smallest positive integer r for which there exists a sequence $G_0, G_l, \ldots, G_r$ of graphs such that $G_0 \cong G$, $G_r \cong G'$, where $\cong$ denotes isomorphism, and G_i can be transformed into G_{i+1} by an *edge rotation, deletion or addition*, for $i = 0, 1, ..., r-1$, i.e. $G_{i+1} = G_i - xu + xv$, $G_{i+1} = G_i - xu$ or $G_{i+1} = G_i + xv$, for $xu \in E(G)$ and $xv \notin E(G)$. For graphs G and G' with the same size the following equation holds, cf.[4,5]:

$$\delta_{ER}(G, G') \geq \frac{1}{2} \sum_{i=1}^{n} |d_i - d'_i| \tag{1}$$

We consider that all degree sequences are in non-increasing order, i.e. if $s = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a graph G, then $d_1 \geq d_2 \geq \ldots \geq d_n$, and we will denote by v_i the vertex of degree d_i, that is $d(v_i) = d_i$. By adding a superindex we will denote the *frequency* of the degree, e.g. $(d_1^{n_1}, d_2^{n_2}, \ldots, d_l^{n_l})$ is a sequence in which each degree d_i has n_i repetitions. We will denote $d_{G'}(x)$ as $d'(x)$, and $d_{G''}(x)$ as $d''(x)$

Let $a = \lfloor \frac{1}{n} \sum_{i=1}^{n} d_i \rfloor$ denote the *lower arithmetic mean* of a set of numbers $\{d_1, \ldots, d_n\}$.

Let $C, C' \subset V(G)$ be such that $C \cap C' = \emptyset$, we will denote by $G[C]$ the *graph induced* by the set C, where $uv \in E(G[C])$ if and only if $uv \in E(G)$ and $u, v \in V(C)$. By $G[C, C']$ we will denote the *induced bipartite graph* with vertex set $V(G[C, C']) = V(C) \cup V(C')$, where $uv \in E(G[C, C'])$ if and only if $uv \in E(G), u \in V(C)$ and $v \in V(C')$.

If $X \subset E(G)$ we will denote by $G-X$, the graph with vertex set $V(G-X) = V(G)$ and edge set $E(G - X) = E(G) - X$.

3 Approximating with regular graphs

Lemma 1. *Let $s = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a graph G and $C \subset V(G)$. If $\sum_{v_i \in C} d_i = a|C|$, then the degree sequence s^* with degrees*

$$d_i^* = \begin{cases} d_i & \text{if } v_i \notin C \\ a & \text{if } v_i \in C. \end{cases}$$

is graphic.

Next, we compare the approximation with the median and the lower arithmetic mean of a set of numbers $s = \{d_1, \ldots, d_n\}$, with the edge rotation distance between the corresponding graphs.

For this we need the following definitions. Let $C = \{v_1, v_2, \ldots, v_{|C|}\} \subset V(G)$ and assume that is ordered so that the degrees are in a non-increasing order, i.e. $d_i \geq d_j$, for $i < j$. Let a and m be, repectively, the lower arithmetic mean and the median of the degrees of the vertices in C, from now on assume that $a > m$.

Let $C_1 = \{v_i : d_i \geq m \text{ and } i \leq \frac{|C|}{2}\}$, $C_2 = \{v_i : a > d_i \geq m, i \leq \frac{|C|}{2}\}$. Let X_1 be a maximal set of edges such that $\tilde{G} = G - X_1$ has degrees $\tilde{d}_i \geq m$ for all $\tilde{v}_i \in \tilde{G}[C_1]$.

Theorem 1. *Let s' be the following degree sequence:*

$$d'_i = \begin{cases} d_i & \text{if } v_i \notin C \\ m & \text{if } v_i \in C. \end{cases}$$

If $(a-m)|C|$ is even and there is $X \subset E(\tilde{G}[C_1]) \cup E(\tilde{G}[C_1, G - C_1])$ of cardinality $|X| = \sum_{v_i \in C_1} (\tilde{d}_i - m)$ such that for each $v_i \in X$, there is $v_j \in X$, not necesarilly different, and $u_i v_i, u_j v_j \in E(G)$ such that $u_i \neq u_j$ and $u_i u_j \notin E(G)$. Then, there is a graph G' of degree sequence s'.

Furthermore, $\delta_{ER}(G, G') \leq \delta_{ER}(G, G^*)$ if and only if $\frac{|X|}{2} \leq \sum_{v_i \in C_2} (a - d_i)$, where G^* is the graph from Lemma 1.

In general for every value $t \leq m$, if we supppose that $|E(G[C])| \geq (a - t)\frac{|C|}{2}$, by deleting $(a-t)\frac{|C|}{2}$ edges from $|E(G[C])|$ and applying Lemma 1 we obtain the following simple corollary.

Corollary 1. *Let G and G^* as in Lemma 1, and $\hat{s}$ be the following degree sequence:*

$$\hat{d}_i = \begin{cases} d_i & \text{if } v_i \notin C \\ t & \text{if } v_i \in C. \end{cases}$$

If $(a-t)|C|$ is even and $|E(G[C])| \geq (a-t)\frac{|C|}{2}$, then $\hat{s}$ is graphic.

We can omit the condition $|E(G[C])| \geq (a - t)\frac{|C|}{2}$ and suppose that the graph G is triangle-free, and quadrangle-free and obtain the following corollary.

Corollary 2. *Let G, G^* and $\hat{s}$ as in Corollary 1. Suppose that $(a-t)|C|$ is even and G is triangle-free and quadrangle-free, then $\hat{s}$ is graphic.*

Lemma 2. *Let $s = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a graph G. Then, either the degree sequence $s^* = (a^n)$ or $s^* = ((a+1)^n)$ is graphic.*

From Corollary 1 and Lemma 2 we obtain the following.

Corollary 3. *The degree sequence $s' = (m^n)$ or $s' = ((m+1)^n)$ is graphic.*

Theorem 2. *For any given graph G with degree sequence s, the closest graphs and sequences to G and s, respectively, that are regular are the following under the different metrics:*

i) The sequence $s^* = (a^n)$ or $s^* = ((a+1)^n)$, for the metric δ^2.
ii) The sequence $s' = (m^n)$ or $s' = ((m+1)^n)$, for the metric δ^1.
iii) The graph G' or G^*, for the distance δ_{ER} depending on whether $\frac{1}{2}\sum_{v_i \in C_1}(\tilde{d}_i - m) \leq \sum_{v_i \in C_2}(a - d_i)$ or not.

Where $C_1 = \{v_i : d_i \geq m \text{ and } i \leq \frac{|C|}{2}\}$, $C_2 = \{v_i : a > d_i \geq m, i \leq \frac{|C|}{2}\}$, $\tilde{d}_i$ is the degree of the vertex v_i in $\tilde{G} = G - X_1$ and X_1 is a maximal set of edges such that $\tilde{G} = G - X_1$ has degrees $\tilde{d}_i \geq m$.

4 Approximating with degrees of frequencies k

Let G be a graph and $\mathcal{C}$ be a partition of $V(G)$ such that $\mathcal{C} = \bigcup_{i=1}^{l} \mathcal{C}_i$.

Let a_j be the lower arithmetic mean of $\{d_i : v_i \in C_j\}$, $r_j = \sum_{x \in \mathcal{C}_j} d(x) - a_j|\mathcal{C}_j|$ for all $j \leq l$, and e such that $R = \sum_{1 \leq j \leq l} r_j = e|\mathcal{C}_l| + r'_l$, for $0 \leq r'_l \leq |\mathcal{C}_l|$.

Theorem 3. *Let G be a graph with degree sequence $s = (d_1, d_2, \ldots, d_n)$. Let k be a positive integer and suppose that $n = lk + t$ with $t < k$. Let $\mathcal{C}$ be a partition of $V(G)$, such that $\mathcal{C} = \bigcup_{j=1}^{l} \mathcal{C}_j$, $|\mathcal{C}_j| = k$ for all $j < l$ and $|\mathcal{C}_l| = k + t$. Then the sequence $s^* = (a_1^k, a_2^k, \ldots, a_{l-1}^k, (a_l + e + 1)^{k+t})$ or $s^* = (a_1^k, a_2^k, \ldots, a_{l-1}^k, (a_l + e + 1 \pm 1)^{k+t})$ is graphic.*

Corollary 4. *Let G be a graph with degree sequence $s = (d_1, d_2, \ldots, d_n)$, as in Theorem 3. Let $R = \sum_{1 \leq j \leq l} r_j$ even. If $\Delta(G) < \frac{|E(G \setminus C_l)| - R/2}{2}$, Then $s^* = (a_1^{k_1}, a_2^{k_2}, \ldots, (a_l)^{k_l})$ is graphic.*

In particular we obtain next corollary, since $R < n$

Corollary 5. *Let G be a graph with degree sequence $s = (d_1, d_2, \ldots, d_n)$, as in Theorem 3. Let $R = \sum_{1 \leq j \leq l} r_j$ even. If $\Delta(G) < \frac{|E(G \setminus C_l)| - n/2}{2}$, Then $s^* = (a_1^{k_1}, a_2^{k_2}, \ldots, (a_l)^{k_l})$ is graphic.*

Another good approximation can be given using the following Lemma, cf. [6].

Lemma 3. *[6] Let G be an n-vertex graph with distinguished vertices s and t (not necesarilly distinct), and suppose that the set of vertices adjacent to s is equal to the set of vertices adjacent to t. Suppose also that δ and Δ are natural numbers such that the degrees of all vertices in G other than s and t lie in the range $[\delta, \Delta]$, and such that s and t themselves have degree at least $\delta + 1$. If $(\Delta - \delta + 1)^2 \leq 4\delta(n - \Delta - 1)$ then there exist an edge disjoint alternating path in G that starts at s, ends at t, and has length 1,3,5, or 7.*

Theorem 4. *Let G be a graph with degree sequence $s = (d_1, d_2, \ldots, d_n)$. Let k a positive integer and suppose that $n = lk + t$ with $t < k$. Let $\mathcal{C}$ be a partition of $V(G)$, as in Theorem 3, i.e. such that $\mathcal{C} = \bigcup_{j=1}^{l} \mathcal{C}_j$, $|\mathcal{C}_j| = k$ for all $j < l$ and $|\mathcal{C}_l| = k + t$. Suppose that $R = \sum_{1 \leq j \leq l} r_j$ is even.*

If $(a_1 - a_l + 2)^2 \leq 4a_l(n - a_1)$ then the sequence $s^* = (a_1^k, a_2^k, \ldots, a_{l-1}^k, a_l^{k+t})$ is graphic.

Proof. Let G' be the graph with degree sequence $s' = ((a_1+1)^{r_1}, a_1^{k-r_1}, (a_2+1)^{r_2}, a_2^{k-r_2}, \ldots, (a_l+1)^{r_l} a_l^{k+t-r_l})$, that is obtained from G by rotations.

Note that the maximum degree of G' is $a_1 + 1$ and the minimum degree of G' is a_l. Hence, applying Lemma 3 we know that there is an alternating odd path between any s and t in $V(G')$, by taking the vertices s and t of degrees $a_i + 1$ and $a_h + 1$ respectively, and complementing along such path, we decrease the degrees of s and t in one.

As the sum $\sum_{1 \leq j \leq l} r_j$ is even, and being careful of not taking the vertices of degree $a_1 + 1$ until the last step, we can repeat this operation $\frac{1}{2} \sum_{1 \leq j \leq l} r_j$ times, for all vertices of degree $a_j + 1$ and all $j \leq l$, and in each step the graph fulfills the condition of Lemma 3. Obtaining a graph with degree sequence as desired. ∎

Its interesting for the applications to note that the condition $(a_1 - a_l + 2)^2 \leq 4a_l(n - a_1)$ from Theorem 4, also implies that the degree sequence obtained $s^* = (a_1^k, a_2^k, \ldots, a_{l-1}^k, a_l^{k+t})$ belongs to a P-stable class of sequences, cf. [6] Theorem 8.1. This in turn, implies that the number of graphs with degree sequence $s^* = (a_1^k, a_2^k, \ldots, a_{l-1}^k, (a_l^{k+t})$ can be approximated in polynomial time for any specified level of accuracy, cf. [7].

References

[1] V. Baláz, V. Kvasnícka, J. Pospíchal, Two metrics in a graph theory modeling of organic chemistry, Disc. Appl. Math. 35 (1992) 1-19.

[2] G. Chartrand, F. Saba and H. Zou, Edge rotations and distance between graphs, Casopis Pest. Mat. 110 (1985) 87-91.

[3] V. Baláz, J.Koca, V. Kvasnícka, and M. Sekanina, A measure for graphs, Casopis Pest. Mat. 111 (1986) 431-433.

[4] R. J. Faudree, R. H. Schelp, L. Lesniak, A. Gyárfas and J. Lehel, On the rotation distance of graphs, Discrete Math. 126 (1994), no. 1-3, 121-135.

[5] W. Goddard, H. C. Swart, Distances between graphs under edge operations, Discrete Math. 161 (1996) 121-132.

[6] M. Jerrum, B. D. McKay and A. Sinclair, When is a graphical sequence stable?, Random Graphs Vol. 2 (eds. A. Frieze and T. Luczak; Wiley-Interscience 1992)

[7] M. Jerrum, A. Sinclair, Fast uniform generation of regular graphs, Theoretical Computer Science 73(1) (1990), 91-100.

[8] Samarati, P. Protecting respondents identities in microdata release, IEEE Trans. on Knowledge and Data Engineering, 13:6 (2001)1010-1027.

[9] Sweeney, L. k-anonymity: a model for protecting privacy, Int. J. of Unc., Fuzz. and Knowledge Based Systems 10:5 (2002) 557-570.

[10] B. Zelinka, On a certain distance between isomorphism classes of graphs, Casopis Pest. Mat. 100 (1975) 371-373.

Numerical algorithms for efficient compression of distributed information

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Abstract. The numerical algorithms for distributed information processing are developed. The underlying idea is a reduction of the original information processing task to a special rank-constrained matrix approximation problem. Its solution is derived. The error analysis and a comparison with the existing approach are provided.

Key words: Rank-constrained matrix approximation, large-dimensional covariance matrix estimation, numerical stability.

1 Introduction

1.1 Motivation. We seek to find effective numerical algorithms for an information processing scenario that involves a set of spatially distributed sensors, $\mathsf{q}_1, \ldots, \mathsf{q}_p$, and a fusion center, $\boldsymbol{p}$. The sensors make local noisy observations, $\mathbf{y}_1, \ldots, \mathbf{y}_p$, correlated with a signal of interest, $\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_p^T]^T$. Each sensor transmits *compressed* information about its measurements, $\mathbf{u}_1, \ldots, \mathbf{u}_p$, to the fusion center which should recover the original signal. We show how desired models of the above scenario can be determined from a special rank-constrained matrix approximation problem. A comparison with the existing transform-based approach is provided.

1.2 Contribution. Differences from known technique. The proposed approach differs from the known method [1,2,3,4] in several instances. In [1,2,3,4], the original problem is replaced with two problems of iterative finding of $\mathsf{q}_1, \ldots, \mathsf{q}_p$ and $\boldsymbol{p}$, respectively. Here, the original problem is reduced to a *single* new rank constrained matrix approximation problem. Then, desired $\mathsf{q}_1, \ldots, \mathsf{q}_p$ and $\boldsymbol{p}$ follow from its solution. This procedure is numerically simpler. Second, our method is presented in terms of both the estimates of covariance matrices and also under assumption that the covariance matrices are known. The method [1,2,3,4] has been presented in terms of exact covariance matrices only. A knowledge of exact covariance matrices might be a restrictive condition in some cases. Third, our technique is developed in terms

of pseudo-inverse matrices while the known method [1,2,3,4] is presented in terms of inverse matrices. It is known that in the cases when the matrices are close to singular this may lead to instability and significant increase in the associated error. Moreover, when the matrices are singular, the algorithms [1,2,3,4] may not be applicable. Those drawbacks are avoided in our method. Our solution is numerically stable and always exists since it is obtained in terms of the pseudo-inverses.

1.3 Notation. A formalization of the problem is given in Sections 1.4 and 1.5 below. It requires some specific notation as follows.

Let us write (Ω, Σ, μ) for a probability space[1]. We denote by $\mathbf{x} \in L^2(\Omega, \mathbb{R}^m)$ the signal of interest[2] (a source signal to be estimated) represented as

$$\mathbf{x} = [\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}]^T \quad \text{where } \mathbf{x}^{(j)} \in L^2(\Omega, \mathbb{R}), \text{ for } j = 1, \ldots, m.$$

Further, $\mathbf{y}_1 \in L^2(\Omega, \mathbb{R}^{n_1}), \ldots, \mathbf{y}_p \in L^2(\Omega, \mathbb{R}^{n_p})$ are observations made by the sensors. In this regard, we write

$$\mathbf{y} = [\mathbf{y}_1^T, \ldots, \mathbf{y}_p^T]^T \quad \text{and} \quad \mathbf{y} = [\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}]^T \tag{1}$$

where $\mathbf{y}^{(k)} \in L^2(\Omega, \mathbb{R})$, for $k = 1, \ldots, n$. We would like to emphasize a difference between $\mathbf{y}_j$ and $\mathbf{y}^{(k)}$: in (1), the observation $\mathbf{y}_j$, for $j = 1, \ldots, p$, is a 'piece' of random vector $\mathbf{y}$ (i.e. $\mathbf{y}_j$ is a random vector itself), and $\mathbf{y}^{(k)}$, for $k = 1, \ldots, n$, is an entry of $\mathbf{y}$ (i.e. $\mathbf{y}^{(k)}$ is a random variable). The observations are correlated with $\mathbf{x}$ which is also represented as

$$\mathbf{x} = [\mathbf{x}_1^T, \ldots, \mathbf{x}_p^T]^T, \tag{2}$$

where $\mathbf{x}_j \in L^2(\Omega, \mathbb{R}^{m_j})$, for $j = 1, \ldots, p$, is is a random vector, a 'piece' of $\mathbf{x}$.

For $j = 1, \ldots, p$, let us define a sensor model $\mathsf{q}_j : L^2(\Omega, \mathbb{R}^{n_j}) \to L^2(\Omega, \mathbb{R}^{r_j})$ by the relation $[\mathsf{q}_j(\mathbf{y}_j)](\omega) = Q_j[\mathbf{y}_j(\omega)]$ where $Q_j \in \mathbb{R}^{r_j \times n_j}$,

$$r = r_1 + \ldots + r_p \quad \text{and, for } j = 1, \ldots, p, \quad r_j \leq \min\{m_j, n_j\}. \tag{3}$$

Let us denote $\mathbf{u}_j = \mathsf{q}_j(\mathbf{y}_j)$ and $\mathbf{u} = [\mathbf{u}_1^T, \ldots, \mathbf{u}_p^T]^T$, where for $j = 1, \ldots, p$, $\mathbf{u}_j \in L^2(\Omega, \mathbb{R}^{r_j})$ represents information transmitted by a j-th sensor q_j to the fusion center $\boldsymbol{p}$, and $\mathbf{u} \in L^2(\Omega, \mathbb{R}^r)$.

A fusion center model is defined by $\boldsymbol{p} : L^2(\Omega, \mathbb{R}^r) \to L^2(\Omega, \mathbb{R}^m)$ so that $[\boldsymbol{p}(\mathbf{u})](\omega) = P[\mathbf{u}(\omega)]$. where $P \in \mathbb{R}^{m \times r}$.

We also denote $\|\mathbf{x}\|_\Omega^2 = \int_\Omega \|\mathbf{x}(\omega)\|_2^2 d\mu(\omega)$ where $\|\mathbf{x}(\omega)\|_2$ is the Euclidean norm of $\mathbf{x}(\omega) \in \mathbb{R}^m$.

1.4 Statement of the problem. Find models of the sensors, $\mathsf{q}_1, \ldots, \mathsf{q}_p$, and a model of the fusion center, $\boldsymbol{p}$, that solve

[1] Here, $\Omega = \{\omega\}$ is the set of outcomes, Σ a σ–field of measurable subsets of Ω and $\mu : \Sigma \to [0, 1]$ an associated probability measure on Σ with $\mu(\Omega) = 1$

[2] Here, space $L^2(\Omega, \mathbb{R}^m)$ has to be used because of the norm $\|\mathbf{x}\|_\Omega^2$ introduced below.

$$\min_{\boldsymbol{p},\mathsf{q}_1,\ldots,\mathsf{q}_p} \left\| \mathbf{x} - \boldsymbol{p} \begin{bmatrix} \mathsf{q}_1(\mathbf{y}_1) \\ \vdots \\ \mathsf{q}_p(\mathbf{y}_p) \end{bmatrix} \right\|_{\Omega}^2 . \tag{4}$$

1.5 Reformulation of the original problem. The model of the fusion center, $\boldsymbol{p}$, can be represented as $\boldsymbol{p} = [\boldsymbol{p}_1, \ldots, \boldsymbol{p}_p]$ where for $j = 1, \ldots, p$, $\boldsymbol{p}_j : L^2(\Omega, \mathbb{R}^r) \to L^2(\Omega, \mathbb{R}^m)$. Then, for $\mathsf{f}_i = \boldsymbol{p}_i \mathsf{q}_i$ and $i = 1, \ldots, p$, let us write

$$\begin{aligned} \left\| \mathbf{x} - [\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_p] \begin{bmatrix} \mathsf{q}_1(\mathbf{y}_1) \\ \vdots \\ \mathsf{q}_p(\mathbf{y}_p) \end{bmatrix} \right\|_{\Omega}^2 &= \|\mathbf{x} - [\mathsf{f}_1(\mathbf{y}_1) + \ldots + \mathsf{f}_p(\mathbf{y}_p)]\|_{\Omega}^2 \\ &= \|\mathbf{x} - [\mathsf{f}_1, \ldots, \mathsf{f}_p](\mathbf{y})\|_{\Omega}^2 . \end{aligned} \tag{5}$$

Therefore, on the basis of (5)–(5), the original problem in (4) can equivalently be rewritten as follows. Find $\mathsf{f}_1, \ldots, \mathsf{f}_p$ that solve

$$\min_{\mathsf{f}_1,\ldots,\mathsf{f}_p} \|\mathbf{x} - [\mathsf{f}_1, \ldots, \mathsf{f}_p](\mathbf{y})\|_{\Omega}^2 \ \text{ s. t. } \ \operatorname{rank} \mathsf{f}_1 = r_1, \ldots, \operatorname{rank} \mathsf{f}_p = r_p. \tag{6}$$

2 Main results

2.1 Preliminaries. For $\mathbf{x}$ and $\mathbf{y}$ represented by $\mathbf{x} = [\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}]^T$ and $\mathbf{y} = [\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(n)}]^T$ where $\mathbf{x}^{(j)} \in L^2(\Omega, \mathbb{R})$ and $\mathbf{y}^{(k)} \in L^2(\Omega, \mathbb{R})$, respectively, we write $E[\mathbf{x}\mathbf{y}^T] = E_{xy} = \left\{ \left\langle \mathbf{x}^{(j)}, \mathbf{y}^{(k)} \right\rangle \right\}_{j,k=1}^{m,n} \in \mathbb{R}^{m \times n}$, where $\left\langle \mathbf{x}^{(j)}, \mathbf{y}^{(k)} \right\rangle = \int_{\Omega} \mathbf{x}^{(j)}(\omega) \mathbf{y}^{(k)}(\omega) \, d\mu(\omega)$. The Moore-Penrose generalized inverse for a matrix M is denoted by $M^{\dagger}$. We set $M^{1/2\dagger} = (M^{1/2})^{\dagger}$ where $M^{1/2}$ is a square root of M, i.e. $M = M^{1/2} M^{1/2}$.

For signals $\mathbf{y}, \mathbf{y}_1, \ldots, \mathbf{y}_p$, we denote $E_{y_\ell y_s} = E[\mathbf{y}_\ell \mathbf{y}_s^T]$ where $\ell, s = 1, \ldots, p$. Then $E_{yy} = \begin{bmatrix} E_{y_1 y_1} & \ldots & E_{y_1 y_p} \\ \ldots & \ldots & \ldots \\ E_{y_p y_1} & \ldots & E_{y_p y_p} \end{bmatrix} \in \mathbb{R}^{n \times n}$ and $E_{xy} = \left[E_{xy_1}, \ldots, E_{xy_p} \right] \in \mathbb{R}^{m \times n}$ where $E_{y_\ell y_s}$ is a $n_\ell \times n_s$ block of E_{yy} and E_{xy_s} is a $m \times n_s$ block of E_{xy}. In other words, if E_{yy} and E_{xy} are known then all $E_{y_\ell y_s}$, for $\ell, s = 1, \ldots, p$, E_{xy_s}, for $s = 1, \ldots, p$, are known as well.

We also denote

$$F = [F_1, \ldots, F_p] \quad \text{where } F \in \mathbb{R}^{m \times n} \text{ and } F_j \in \mathbb{R}^{m \times n_j}, \text{ for all } j = 1, \ldots, p,$$

and write $\|\cdot\|$ for the Frobenius norm.

2.2 Assumptions. An assumption in [1,2,3,4] is that E_{xy} and E_{yy} are known. We mainly concern the case when only *estimates* of E_{xy} and E_{yy} are known (e.g. they can be obtained from samples of training signals taken for some different random outcomes ω). At the same time, we provide solution and the error analysis in terms of both matrices E_{xy}, E_{yy} and their estimates.

2.3 Solution of problem (6). The solution of the problem (6) is based on the generalized reduced rank matrix approximation developed in [7,8]. Let the singular value decomposition (SVD) of $A \in \mathbb{R}^{m\times s}$ be given by $A = U_A \Sigma_A V_A^T$, where $U_A \in \mathbb{R}^{m\times m}$ and $V_A \in \mathbb{R}^{s\times s}$ are unitary matrices, $\Sigma_A = \mathrm{diag}(\sigma_1(A),\ldots,\ \sigma_{\min(m,s)}(A)) \in \mathbb{R}^{m\times s}$ is a generalized diagonal matrix. Let $U_A = [u_1\ u_2\ \ldots u_m]$ and $V_A = [v_1\ v_2\ \ldots v_s]$ be the representations of U and V in terms of their m and s columns, respectively. Define

$$[A]_r = \sum_{i=1}^{r} \sigma_i(A) u_i v_i^T = U_{A_r} \Sigma_{A_r} V_{A_r}^T \in \mathbb{R}^{m\times s}, \quad L_A = \sum_{i=1}^{\mathrm{rank}\ A} u_i u_i^T \in \mathbb{R}^{m\times m} \quad (7)$$

for $r = 1,\ldots,\mathrm{rank}\ A$, where $U_{A_r} = [u_1\ u_2\ \ldots u_r]$, $\Sigma_{A_r} = \mathrm{diag}(\sigma_1(A),\ldots,\sigma_r(A))$ and $V_{A_r} = [v_1\ v_2\ \ldots v_r]$. It can be shown that

$$\|\mathbf{x} - [\mathsf{f}_1,\ldots,\mathsf{f}_p](\mathbf{y})\|_\Omega^2 = \|E_{xx}^{1/2}\|^2 - \sum_{j=1}^{p} \left\| E_{xy_j} (E_{y_jy_j}^{1/2})^\dagger \right\|^2 \quad (8)$$

$$+ \sum_{j=1}^{p} \left\| \left(E_{xy_j} - \sum_{\substack{k=1\\k\neq j}}^{p} F_k E_{y_k y_j} \right) (E_{y_jy_j}^\dagger)^{1/2} - F_j E_{y_jy_j}^{1/2} \right\|^2$$

In (8), only the last term depends on $F_1,\ldots,F_p$. It can be written as $\sum_{j=1}^{p} \|A_j - F_j C_j\|^2$ where $A_j = (E_{xy_j} - \sum_{\substack{k=1\\k\neq j}}^{p} F_k E_{y_k y_j})(E_{y_jy_j}^\dagger)^{1/2}$ and $C_j = E_{y_jy_j}^{1/2}$. Therefore, problem (6) is reduced to finding $F_1,\ldots,F_p$ which solve

$$\min_{F_1,\ldots,F_p} \sum_{j=1}^{p} \|A_j - F_j C_j\|^2 \ \text{ s. t. } \mathrm{rank}\ F_j \le r_1, \quad \ldots, \quad \mathrm{rank}\ F_p \le r_p. \quad (9)$$

We observe that $\min_{F_1,\ldots,F_p} \sum_{j=1}^{p} \|A_j - F_j C_j\|^2 = \sum_{j=1}^{p} \min_{F_j} \|A_j - F_j C_j\|^2$. On this basis, we consider the problem of finding F_j, for $j = 1,\ldots,p$, that solves

$$\min_{\mathsf{f}} \|A_j - F_j C_j\|_\Omega^2 \ \text{ s. t. } \mathrm{rank}\ F_j = r_j. \quad (10)$$

Its solution is as follows (also, see [7,8]).

Theorem 1. *For $j = 1,\ldots,p$, let $L_{E_{y_jy_j}^{1/2}}$ be given similar L_A in (7), and let $K_j = N_j(I - L_{E_{y_jy_j}^{1/2}})$, where $N_j \in \mathbb{R}^{s\times s}$ is any matrix. Then*

$$F_j = \left[\left(E_{xy_j} - \sum_{\substack{k=1 \\ k \neq j}}^{p} F_k E_{y_k y_j} \right) (E^{\dagger}_{y_j y_j})^{1/2} \right]_{r_j} (E^{\dagger}_{y_j y_j})^{1/2}(I + K_j) \qquad (11)$$

Proof. The proof is based on [9], pp. 309 – 314.

To find F_j that satisfies the set of matrix equations (11), we consider $F_j^{(q)}$, for $q = 1, 2, \ldots$, given by

$$F_j^{(q)} = \left[\left(E_{xy_j} - \sum_{\substack{k=1 \\ k \neq j}}^{p} F_k^{(q-1)} E_{y_k y_j} \right) (E^{\dagger}_{y_j y_j})^{1/2} \right]_{r_j} (E^{\dagger}_{y_j y_j})^{1/2}(I + K_j). \quad (12)$$

In practice, an exact representation of matrices E_{xy} and E_{yy} might be unknown. Their estimates, $\widetilde{E}_{xy}$ and $\widetilde{E}_{yy}$, respectively, can be obtained, in particular, by the method [6]. In this regard, we represent the procedure in (12) as

$$\widetilde{F}_j^{(q)} = \left[\left(\widetilde{E}_{xy_j} - \sum_{\substack{k=1 \\ k \neq j}}^{p} \widetilde{F}_k^{(q-1)} \widetilde{E}_{y_k y_j} \right) (\widetilde{E}^{\dagger}_{y_j y_j})^{1/2} \right]_{r_j} (\widetilde{E}^{\dagger}_{y_j y_j})^{1/2}(I + K_j). \quad (13)$$

A determination of initial iterations $F_j^{(0)}$ and $\widetilde{F}_j^{(0)}$, for $j = 1, \ldots, p$, is based on the following observation. Let us write

$$\|\mathbf{x} - [\mathsf{f}_1, \ldots, \mathsf{f}_p](\mathbf{y})\|^2_{\Omega} = \|E^{1/2}_{xx}\|^2 - \|E_{xy}(E^{1/2}_{yy})^{\dagger}\|^2 + \|E_{xy}(E^{1/2}_{yy})^{\dagger} - FE^{1/2}_{yy}\|^2.$$

Then F that minimizes $\|\mathbf{x} - [\mathsf{f}_1, \ldots, \mathsf{f}_p](\mathbf{y})\|^2_{\Omega}$ is given by

$$[F_1, \ldots, F_p] = E_{xy}E^{\dagger}_{yy} + M(I - (E^{1/2}_{yy})^{\dagger}E^{1/2}_{yy}) \qquad (14)$$

where M is an arbitrary $m \times n$ matrix. We denote

$$[E_1, \ldots, E_p] = E_{xy}E^{\dagger}_{yy} \quad \text{and} \quad [M_1, \ldots, M_p] = M(I - (E^{1/2}_{yy})^{\dagger}E^{1/2}_{yy}) \quad (15)$$

where for $j = 1, \ldots, p$, E_j and M_j are blocks of matrices $E_{xy}E^{\dagger}_{yy}$ and $M(I - (E^{1/2}_{yy})^{\dagger}E^{1/2}_{yy})$, respectively. Then $F_1, \ldots, F_p$ are given by

$$F_1 = E_1 + M_1, \quad \ldots, \quad F_p = E_p + M_p. \qquad (16)$$

To satisfy the rank-constrained condition, a truncated SVD is applied to each $E_1 + M_1, \ldots, E_P + M_p$ so that $F_1, \ldots, F_p$ are represented in the form

$$F_1^{(0)} = [E_1 + M_1]_{r_1}, \quad \ldots, \quad F_p^{(0)} = [E_p + M_p]_{r_p}. \tag{17}$$

Similarly, we write

$$\widetilde{F}_1^{(0)} = [\widetilde{E}_1 + \widetilde{M}_1]_{r_1}, \quad \ldots, \quad \widetilde{F}_p^{(0)} = [\widetilde{E}_p + \widetilde{M}_p]_{r_p} \tag{18}$$

where $\widetilde{E}_j$ and $\widetilde{M}_j$ are blocks of matrices $\widetilde{E}_{xy}\widetilde{E}_{yy}^{\dagger}$ and $M(I - (\widetilde{E}_{yy}^{1/2})^{\dagger}\widetilde{E}_{yy}^{1/2})$, respectively.

2.4 Compression. Compression of the observations and their simultaneous denoising are realized by the truncated SVD in (12) and (13).

2.5 Error analysis. The proposed method is mainly defined by F_j, $F_j^{(q)}$ and $\widetilde{F}_j^{(0)}$ given by (11), (12) and (13), respectively. First, we establish how close the representation of F_j by $F_j^{(q)}$ and $\widetilde{F}_j^{(q)}$ are. Then the error estimates associated with the proposed method are given. To this end, some additional notation is required as follows. We denote by $Z_j : \mathbb{R}^{m\times n_k} \to \mathbb{R}^{m\times n_j}$ an operator defined by

$$Z(E_j, F_k) = \left[\left(E_{xy_j} - \sum_{\substack{k=1\\k\neq j}}^{p} F_k E_{y_k y_j}\right)(E_{y_j y_j}^{\dagger})^{1/2}\right]_{r_j} (E_{y_j y_j}^{\dagger})^{1/2}(I + K_j) \tag{19}$$

where $E_j = (E_{xy_j}, E_{y_1 y_j}, \ldots, E_{y_{j-1} y_j}, E_{y_{j+1} y_j}, \ldots, E_{y_p y_j})$. Suppose λ and β are Lipschitz constants for $Z(E_j, F_k)$ such that, for any E_j, $\|Z(E_j, F_k) - Z(E_j, F_k^{(q-1)})\|^2 \leq \lambda\|F_k - F_k^{(q-1)}\|^2$,and for any $F_k^{(q-1)}$, $\|Z(E_j, F_k^{(q-1)}) - Z(\widetilde{E}_j, F_k^{(q-1)})\|^2 \leq \beta\|E_j - \widetilde{E}_j\|^2$.

Let us first consider the case when F_j in (11) is approximately determined by $F_j^{(q)}$ in (12), where $q = 0, 1, \ldots$. Let us show that $F_j^{(q)}$ tends to F_j as q increases. Equations (11), (12) and (19) imply

$$\|F_j - F_j^{(q)}\|^2 \leq \lambda\|F_k - F_k^{(q-1)}\|^2, \tag{20}$$

where $j, k = 1, \ldots, p$, $k \neq j$ and $q = 1, 2, \ldots$. It follows from (20) that

$$\|F_j - F_j^{(q)}\|^2 \leq \lambda^q\|F_k - F_k^{(0)}\|^2. \tag{21}$$

Thus, if $\lambda < 1$ then $F_j^{(q)} \longrightarrow F_j \quad \text{as } q \to \infty$.

Let us now consider the case when F_j, for$j = 1, \ldots, p$, is approximately determined by $\widetilde{F}_j^{(q)}$ in (13), for $q = 0, 1, \ldots$. Then

$$\left\|F_j - \widetilde{F}_j^{(q)}\right\|^2 \leq 2\lambda\|F_k - F_k^{(q-1)}\|^2 + \beta\|E_j - \widetilde{E}_j\|^2 \tag{22}$$

where we denote $\left\|E_j - \widetilde{E}_j\right\|^2 = \sum_{\substack{k=0\\k\neq j}}^{p}\|E_{y_k y_j} - \widetilde{E}_{y_k y_j}\|^2$, $\quad E_{y_0 y_j} = E_{xy_j}$ and $\widetilde{E}_{y_0 y_j} = \widetilde{E}_{xy_j}$.Therefore, (22) implies

$$\left\|F_j - \widetilde{F}_j^{(q)}\right\|^2 \leq \lambda^q \|F_k - F_k^{(0)}\|^2 + \frac{1-\lambda^q}{1-\lambda}\beta \sum_{\substack{k=0 \\ k\neq j}}^{p} \|E_{y_k y_j} - \widetilde{E}_{y_k y_j}\|^2. \tag{23}$$

If $\lambda < 1$ then $\widetilde{F}_j^{(q)} \longrightarrow F_j + \frac{\beta}{1-\lambda} \sum_{k=0 k\neq j}^{p} \|E_{y_k y_j} - \widetilde{E}_{y_k y_j}\|^2$ as $q \to \infty$. In other words, a closeness of $\widetilde{F}_j^{(q)}$ to F_j depends on the closeness of the estimate $\widetilde{E}_{y_k y_j}$ to the covariance matrix $E_{y_k y_j}$. The latter issue is not a subject of this paper and we refer to the known related results provided, in particular, in [6].

The error associated with the proposed method, for $F_j^{(q)}$ defined in (12), is represented as

$$\left\|\mathbf{x} - [\mathsf{f}_1^{(q)}, \ldots, \mathsf{f}_p^{(q)}](\mathbf{y})\right\|_{\Omega}^2 = \|E_{xx}^{1/2}\|^2 - \|E_{xy}(E_{yy}^{1/2})^{\dagger}\|^2 + \|E_{xy}(E_{yy}^{1/2})^{\dagger} - F^{(q)}E_{yy}^{1/2}\|^2$$

where $F^{(q)} = [F_1^{(q)}, \ldots, F_p^{(q)}]$. For $\widetilde{F}_j^{(q)}$ defined in (13),

$$\left\|\mathbf{x} - [\widetilde{\mathsf{f}}_1^{(q)}, \ldots, \widetilde{\mathsf{f}}_p^{(q)}](\mathbf{y})\right\|_{\Omega}^2 = \|\widetilde{E}_{xx}^{1/2}\|^2 - \|\widetilde{E}_{xy}(\widetilde{E}_{yy}^{1/2})^{\dagger}\|^2 + \|\widetilde{E}_{xy}(\widetilde{E}_{yy}^{1/2})^{\dagger} - \widetilde{F}^{(q)}\widetilde{E}_{yy}^{1/2}\|^2. \tag{24}$$

3 Numerical experiments

Example 1. Here, we use simulations carried out with some training signals. Estimates of E_{xx}, E_{xy} and E_{yy} are obtained from signal samples [6]. For $p = 2$, observations $\mathbf{y}_1$ and $\mathbf{y}_2$ are assumed to be in the form $\mathbf{y}_1 = A_1\mathbf{x}_1 + \boldsymbol{\xi}_1$ and $\mathbf{y}_2 = A_2\mathbf{x}_2 + \boldsymbol{\xi}_2$, where $\mathbf{x} = [\mathbf{x}_1^T,\ \mathbf{x}_2^T]^T$, $\mathbf{x}_j \in L^2(\Omega, \mathbb{R}^{m_j})$, $m_j = m/2$, $j = 1, 2$ and $A_i \in \mathbb{R}^{m_j \times m_j}$, for each realization of $\mathbf{x}_i$, is represented by a matrix with entries, chosen from a normal distribution with mean zero and variance one, $\mathbf{x}$ has uniformly distributed random entries and $\boldsymbol{\xi}_i$ is a Gaussian random vector with mean zero and variance one. For different m, s, r_1 and r_2 in (4) and (6), the proposed method and known method [1,2,3,4] were applied 1000 times, each time with a different choice of estimates of E_{xx}, E_{xy} and E_{yy} produced by `MATLAB` commands `randn` and `rand`. The method [1,2,3,4] might be set aside since it has only been developed under the assumption of the knowledge of *exact* matrices E_{xx}, E_{xy} and E_{yy}. Nevertheless, it has been used here for the purpose of a numerical comparison with the proposed technique.

Expected errors associated with the proposed method (blue) and the known method [1,2,3,4] (red) versus number of iterations and for some m, s, r_1 and r_2, are given in the Figure 1. The presented results show that the errors associated with the proposed method are smaller than the errors associated with the known method [1,2,3,4]. This is because of the reasons described in Section 1.2: unlike the approach in [1,2,3,4] the proposed numerical scheme is based on the solution of the new *single* matrix approximation problem, its solution is obtained in terms of the pseudo-inverse matrices and, therefore, always exists and is numerically stable.

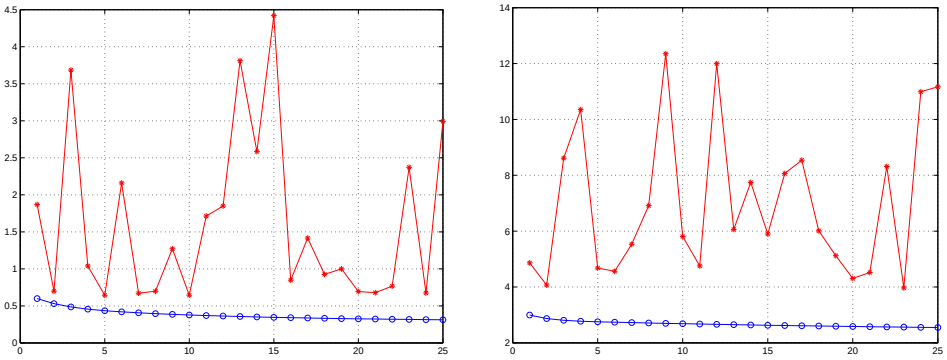

Fig. 1: The left figure: $m = 20$, $s = 10$, $r_1 = 3$, $r_2 = 3$. The right figure: $m = 50$, $s = 20$, $r_1 = 3$, $r_2 = 2$.

References

[1] M. Gastpar, P.L. Dragotti, M. Vetterli, The Distributed Karhunen-Loeve Transform, *IEEE Trans. Information Theory*, 52, 5177–5196, 2006.

[2] O. Roy, M. Vetterli, Dimensionality Reduction for Distributed Estimation in the Infinite Dimensional Regime, *IEEE Trans. Information Theory,* 54 1655-1669 2008.

[3] I. D. Schizas, G. B. Giannakis and Z.-Q. Luo, Distributed estimation using reduced-dimensionality sensor observations, *IEEE Trans. on Signal Processing*, 55 4284–4299, 2007.

[4] A. Amar, A. Leshem, and M. Gastpar, Recursive implementation of the distributed Karhunen-Loéve transform, *IEEE Trans. on Signal Processing*, 58, 5320 – 5330, 2010.

[5] S. Simoens, O. Munoz-Medina, J. Vidal, and A. Del Coso, Compress-and-forward cooperative MIMO relaying with full channel state information, *IEEE Trans. on Signal Processing*, 58 781–791, 2010.

[6] O. Ledoit and M. Wolf, Nonlinear shrinkage estimation of large-dimensional covariance matrices, *Annals of Statistics*, 40, 1024–1060, 2012.

[7] S. Friedland and A. P. Torokhti, Generalized rank-constrained matrix approximations, *SIAM J. Matrix Anal. Appl.,* 29, issue 2, pp. 656-659, 2007.

[8] A.Torokhti, S. Friedland, Towards theory of generic Principal Component Analysis, *J. Multivariate Analysis,* 100, pp. 661-669, 2009.

[9] A. Torokhti and P. Howlett, *Computational Methods for Modelling of Nonlinear Systems,* Elsevier, 2007.

Data estimation under partially missing information

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Abstract. We wish to estimate each random vector $\boldsymbol{x}_\omega(t)$ in the set $\mathcal{K}_x$ from the corresponding large set of noisy observations. The conceptual foundation of the proposed filter is an optimal least squares linear estimate of the incremental change to the p signal pairs, extended by a natural interpolation to an estimated value of each reference signal.

Key words: Rank-constrained matrix approximation, matrix estimation, stability.

1 Introduction

We develop new numerical algorithms for the estimation of noisy data sets.

1.1 Motivation. We write $\mathcal{K}_x = \{\boldsymbol{x}_\omega\}$ for the set of signals of interest and $\mathcal{K}_y = \{\boldsymbol{y}_\omega\}$ for the corresponding noisy observations. Here, $\boldsymbol{x}_\omega = \boldsymbol{x}_\omega(t) = \boldsymbol{x}(\omega, t)$ and $\boldsymbol{y}_\omega = \boldsymbol{y}_\omega(t) = \boldsymbol{y}(\omega, t)$ are represented by stochastic vectors with m and n components, respectively, associated with a random outcome ω and time $t \in T = [a, b] \subset \mathbb{R}$. That is, for each ω, $\boldsymbol{x}_\omega(t)$ and $\boldsymbol{y}_\omega(t)$ are functions of t and for each $t \in T$, $\boldsymbol{x}_\omega(t)$ and $\boldsymbol{y}_\omega(t)$ are stochastic vectors (or stochastic vector valued signals), i.e. functions of ω. In other words, $\boldsymbol{x}_\omega$ and $\boldsymbol{y}_\omega$ can be interpreted as stochastic processes.

In many applications (associated, e.g., with a difficult environment), the sets $\mathcal{K}_x$ and $\mathcal{K}_y$ are large, and *a priori* information (in terms of covariance matrices) on the set $\mathcal{K}_x$, can only be obtained for a few signals $\{\boldsymbol{x}_\omega(t_1), \ldots, \boldsymbol{x}_\omega(t_p)\} \subset \mathcal{K}_x$, where p is a small number, while information on other signals in $\mathcal{K}_x$ is missing. The times $t_1, t_2, \ldots, t_p$ are given and they are such that

$$a = t_1 < t_2 < \cdots < t_{p-1} < t_p = b. \tag{1}$$

Their choice and consequently a choice of signals $\boldsymbol{x}_\omega(t_1), \ldots, \boldsymbol{x}_\omega(t_p)$ might be beyond our control. At the same time, it is required to estimate each reference signal $\boldsymbol{x}_\omega(t)$ in the set $\mathcal{K}_x$ from the corresponding large set of noisy observations $\mathcal{K}_y$.

Thus, all we can exploit to develop an associated filter is observed noisy data $\mathcal{K}_y$ and a sparse information on *reference* signals provided by covariance matrices $E[\boldsymbol{y}_\omega(t_j)\boldsymbol{x}_\omega^T(t_k)]$ and $E[\boldsymbol{x}_\omega(t_j)\boldsymbol{x}_\omega^T(t_k)]$ (or their estimates) formed from $\boldsymbol{y}_\omega(t_j)$ and $\boldsymbol{x}_\omega(t_k)$, for $j,k = 1,\ldots,p$. Methods of covariance matrix estimation were studied in a number of papers (see, for example, [8]) and it is not a subject of our work.

1.1 Formalization of the problem. To formalize the problem, we denote by Ω the set of all experimental outcomes[1], by $\mathcal{K}_x = \{\boldsymbol{x}_\omega \mid \omega \in \Omega\}$ a set of reference stochastic signals and by $\mathcal{K}_y = \{\boldsymbol{y}_\omega \mid \omega \in \Omega\}$ a set of observed signals. Note that, theoretically, $\mathcal{K}_x$ and $\mathcal{K}_y$ are infinite signal sets. In practice, however, sets $\mathcal{K}_x$ and $\mathcal{K}_y$ are finite and large, each with, say, N signals. To each random outcome $\omega \in \Omega$ we associate a unique signal pair $(\boldsymbol{x}_\omega, \boldsymbol{y}_\omega)$ where[2] $\boldsymbol{x}_\omega : T \to \mathcal{C}^{0,1}(T,\mathbb{R}^m)$ and $\boldsymbol{y}_\omega : T \to \mathcal{C}^{0,1}(T,\mathbb{R}^n)$. Write

$$\mathcal{P} = \mathcal{K}_x \times \mathcal{K}_y = \{(\boldsymbol{x}_\omega, \boldsymbol{y}_\omega) \mid \omega \in \Omega\} \tag{2}$$

for the set of all such signal pairs. For each $\omega \in \Omega$, the components $\boldsymbol{x}_\omega = \boldsymbol{x}_\omega(t) = \boldsymbol{x}(\omega,t)$, $\boldsymbol{y}_\omega = \boldsymbol{y}_\omega(t) = \boldsymbol{y}(\omega,t)$ are Lipschitz continuous vector-valued functions on T.

We wish to construct a filter $F^{(p-1)}$ that estimates each reference signal $\boldsymbol{x}_\omega(t)$ in $\boldsymbol{p}$ from related observed input $\boldsymbol{y}_\omega(t)$ under the restriction that *a priori* information on only a *few* reference signals, $\boldsymbol{x}_\omega(t_1)$, …, $\boldsymbol{x}_\omega(t_p)$, is available where $p \ll N$. As has been mentioned above, for each $t_j \in T$, $\boldsymbol{x}_\omega = \boldsymbol{x}_\omega(t_j)$ is a stochastic vector,[3] $\boldsymbol{x}(\cdot,t_j) : \Omega \to \mathbb{R}^m$. In particular, derivations in Section 2 require that $\boldsymbol{x}_\omega \in L^2(\Omega,\mathbb{R}^m)$.

In more detail, this restriction implies the following. Let us denote by $\mathcal{K}_x^{(p)}$ a set of p signals $\boldsymbol{x}_\omega(t_1)$, …, $\boldsymbol{x}_\omega(t_p)$ for which *a priori* information is available. A set of associated observed signals $\boldsymbol{y}_\omega(t_1)$, …, $\boldsymbol{y}_\omega(t_p)$ is denoted by $\mathcal{K}_y^{(p)}$. Then for all $\boldsymbol{y}_\omega(t)$ that do not belong to $\mathcal{K}_y^{(p)}$, filter $F^{(p-1)}$ is said to be the blind filter since no information on $\boldsymbol{x}_\omega(t) \notin \mathcal{K}_x^{(p)}$ is available. If $\boldsymbol{y}_\omega(t) \in \mathcal{K}_y^{(p)}$ then $F^{(p-1)}$ becomes a nonblind filter since information on $\boldsymbol{x}_\omega(t) \in \mathcal{K}_x^{(p)}$ is available. Thus, depending on $\boldsymbol{y}_\omega(t)$, filter $F^{(p-1)}$ is classified differently. Therefore, such a procedure of estimating reference signals in $\mathcal{K}_x$ is here called *the filtering under partially missing information.*

1.3 Differences from known technique. We would like to note that the filtering under partially missing information considered in this paper is different from known methods developed, in particular, in [1]–[7][4]. Indeed, at

[1] We write $\{\Omega,\Sigma,\mu\}$ for a probability space where $\Sigma \subset \Omega$ is a sigma-algebra of measurable sets known as the event space and μ is a non-negative probability measure with $\mu(\Omega) = 1$.

[2] The space $\mathcal{C}^{0,1}(T,\mathbb{R}^p)$ is the set of vector-valued Hölder continuous functions $\boldsymbol{f}$ of order 1 with $\boldsymbol{f}(t) \in \mathbb{R}^p$ and $\|\boldsymbol{f}(s) - \boldsymbol{f}(t)\| \le K|s-t|$.

[3] We would like to emphasize this issue: $\boldsymbol{x}_\omega(t_j)$ is not a sample of $\boldsymbol{x}_\omega$, it is a stochastic signal itself.

[4] The literature on these subjects is very abundant. Here, we listed only some related references.

each particular time $t \in T$, the input of the filter $F^{(p-1)}$ developed below in this paper, is a random vector $\boldsymbol{y}_\omega(t)$. Thus, for different $t \in T$, the input is a different random vector $\boldsymbol{y}_\omega(t)$ but we wish to keep *the same filter* $F^{(p-1)}$ for any $t \in T$, i.e. for any observed signal $\boldsymbol{y}_\omega(t)$ in the set $\mathcal{K}_y$.

By techniques based on the Wiener approach, a filter (here, we choose the united term 'filter' to denote an equalizer or a system) is specifically designed for *each* particular input–output signal pair represented by random vectors. That is, for different inputs (observed signals) $\boldsymbol{y}_\omega(t)$, known techniques require different specified filters and the number of filters should be equal to a number of processed signals. In the case of *large signal sets*, such approaches become inconvenient because the number of signals N can be very large as it is supposed in this paper. Therefore, the restricting condition of using *a priori* information on only p reference signals, with $p \ll N$, is quite significant. To the best of our knowledge, the exception is the method in [7] where for estimation of a set of signals, the single filter is constructed (see also Section 2.4 for a comparison of the results of the numerical experiments).

Much of the papers on piecewise linear filters seems to be *not directly relevant* to the filter proposed here. They are mostly concerned with the problems of approximation by piecewise linear functions which is not the case here.

Further, the semiblind techniques are not applicable to the considered problem because they require a knowledge of some 'parts' of each reference signal in $\mathcal{K}_x$ but it is not the case here. Although the blind techniques allow us to avoid this restriction, it is known that they have difficulties related to accuracy and computational load.

Nonblind filters are not applicable here because they require *a priori* information on each reference signal in $\mathcal{K}_x$ (e.g., a knowledge of covariance matrix $E[\boldsymbol{x}_\omega \boldsymbol{y}_\omega^T]$ where E is the expectation operator). In particular, it is known that there are significant advantages in adaptive or recursive filters (e.g., associated with Kalman filtering approach and it may well be possible to embed our filter into such an environment—but that is not our particular concern here. Furthermore, known filters require that the covariance matrices are invertible. In many cases, it is not true and then known filters are nor applicable or require additional work to overcome ill-conditioning. Our filter is constructed in terms of the Moore-Penrose pseudo-inverse matrix. Therefore it is always well defined.

2 Main results

2.1 Some preliminaries. The conceptual device behind the proposed filter is a linear interpolation of an optimal incremental estimation applied to random signal pairs $(\boldsymbol{x}_\omega(t_j), \boldsymbol{y}_\omega(t_j))$ and $(\boldsymbol{x}_\omega(t_{j+1}), \boldsymbol{y}_\omega(t_{j+1}))$, for $j = 1, \ldots, p-1$, interpreted an extension of the least squares linear (LSL) filter (see, for ex-

ample, [7]). Although this idea may seem reasonable the detailed justification of the new filter is not straightforward and requires careful analysis.

Since the filter proposed below is based on an extension of the LSL filter it is convenient to sketch known related results here. Consider a *single* random signal pair $(\boldsymbol{x}(\omega), \boldsymbol{y}(\omega))$ where $\boldsymbol{x} \in L^2(\Omega, \mathbb{R}^m)$ and $\boldsymbol{y} \in L^2(\Omega, \mathbb{R}^n)$ with zero mean $(E[\boldsymbol{x}], E[\boldsymbol{y}]) = (\boldsymbol{0}, \boldsymbol{0})$, where $\boldsymbol{0}$ is the zero vector. The estimate $\widehat{\boldsymbol{x}}$ of the reference vector $\boldsymbol{x}$ by the optimal least squares linear filter is given by

$$\widehat{\boldsymbol{x}}(\omega) = E_{\boldsymbol{xy}} E_{\boldsymbol{yy}}^{\dagger}\, \boldsymbol{y}(\omega) \tag{3}$$

where $E_{\boldsymbol{xy}} = E[\boldsymbol{x}\boldsymbol{y}^T]$ and $E_{\boldsymbol{yy}} = E[\boldsymbol{y}\boldsymbol{y}^T]$ are known covariance matrices and $E_{\boldsymbol{yy}}^{\dagger}$ is the Moore-Penrose pseudo-inverse of $E_{\boldsymbol{yy}}$. Further, we denote $\boldsymbol{x}(t, \omega) = \boldsymbol{x}_\omega(t)$ and $\boldsymbol{y}(t, \omega) = \boldsymbol{y}_\omega(t)$ so that $\boldsymbol{x}(t, \omega) \in \mathbb{R}^m$ and $\boldsymbol{y}(t, \omega) \in \mathbb{R}^n$.

2.2 The piecewise LSL interpolation filter. For each signal pair (or vector function pair) in the set $\mathcal{P}$, $(\boldsymbol{x}(t, \omega), \boldsymbol{y}(t, \omega))$, we assume that $(E[\boldsymbol{x}(t, \cdot)]$, $E[\boldsymbol{y}(t, \cdot)]) = (\boldsymbol{0}, \boldsymbol{0})$. To begin the estimation process we need to find an initial estimate $\widehat{\boldsymbol{x}}(t_1, \omega)$. It is assumed this can be found by some known method. Further, let us consider a signal estimation procedure at $t_2, \cdots, t_p$. Consider a typical interval $[t_j, t_{j+1}]$ and define incremental random vectors

$$\boldsymbol{v}_j(\omega) = \boldsymbol{x}(t_{j+1}, \omega) - \boldsymbol{x}(t_j, \omega) \in \mathbb{R}^m, \quad \boldsymbol{w}_j(\omega) = \boldsymbol{y}(t_{j+1}, \omega) - \boldsymbol{y}(t_j, \omega) \in \mathbb{R}^n \tag{4}$$

and construct the optimal linear estimate

$$\widehat{\boldsymbol{v}}_j(\omega) = E_{\boldsymbol{v}_j \boldsymbol{w}_j} E_{\boldsymbol{w}_j \boldsymbol{w}_j}^{\dagger}\, \boldsymbol{w}_j(\omega) \tag{5}$$

of the increment $\boldsymbol{v}_j(\omega)$ for each $j = 1, \ldots, p-1$. We will write

$$B_j = E_{\boldsymbol{v}_j \boldsymbol{w}_j} E_{\boldsymbol{w}_j \boldsymbol{w}_j}^{\dagger} \in \mathbb{R}^{m \times n}. \tag{6}$$

We note that matrices $E_{\boldsymbol{v}_j \boldsymbol{w}_j}$ and $E_{\boldsymbol{w}_j \boldsymbol{w}_j}$ can easily be represented in terms of matrices $E_{\boldsymbol{x}_j \boldsymbol{y}_k}$ and $E_{\boldsymbol{y}_j \boldsymbol{y}_k}$ (due to the linearity of the expectation).

Define the estimate at point t_{j+1} by setting $\widehat{\boldsymbol{x}}(t_{j+1}, \omega) = \widehat{\boldsymbol{x}}(t_j, \omega) + \widehat{\boldsymbol{v}}_j(\omega)$. Thus we have

$$\widehat{\boldsymbol{x}}(t_{j+1}, \omega) = \boldsymbol{\epsilon}_j(\omega) + B_j \boldsymbol{y}(t_{j+1}, \omega) \tag{7}$$

where we write

$$\boldsymbol{\epsilon}_j(\omega) = \widehat{\boldsymbol{x}}(t_j, \omega) - B_j \boldsymbol{y}(t_j, \omega).$$

Note that this definition can be rewritten more suggestively as

$$\widehat{\boldsymbol{x}}(t_j, \omega) = \boldsymbol{\epsilon}_j(\omega) + B_j \boldsymbol{y}(t_j, \omega), \quad \text{for each } j = 1, \ldots, p-1. \tag{8}$$

The formula (7) shows that on each subinterval $[t_j, t_{j+1}]$ the estimate of the reference signal at t_{j+1} is obtained from the initial estimate at t_j by adding the optimal LSL estimate of the increment.

Now, consider a signal estimation at any $t \in [a, b]$. The first step is simply to extend the formulæ (7) and (8) to all $t \in [t_j, t_{j+1}]$ by defining

$$\widehat{\boldsymbol{x}}(t, \omega) = \boldsymbol{\epsilon}_j(\omega) + B_j \boldsymbol{y}(t, \omega). \tag{9}$$

We call this procedure as the *LSL piecewise interpolation*. Further, for each $j = 1, 2, \ldots, p-1$, write

$$F_j[\boldsymbol{y}(t, \omega)] = \boldsymbol{\epsilon}_j(\omega) + B_j \boldsymbol{y}(t, \omega) \quad \text{for all } t \in [t_j, t_{j+1}] \tag{10}$$

and hence define the *piecewise LSL interpolation filter* by setting

$$F^{(p-1)}[\boldsymbol{y}(t, \omega)] = \sum_{j=1}^{p-1} F_j[\boldsymbol{y}(t, \omega)][u(t - t_j) - u(t - t_{j+1})] \tag{11}$$

for all $t \in [a, b]$ where $u(t) = \begin{cases} 1 \text{ for } t > 0 \\ 0 \text{ otherwise.} \end{cases}$ is the unit step function. Thus we can now use the estimate

$$\widehat{\boldsymbol{x}}(t, \omega) = F^{(p-1)}[\boldsymbol{y}(t, \omega)] \tag{12}$$

for all $(t, \omega) \in T \times \Omega$.

We note that by (6)-(12), the filter $F^{(p-1)}$ is adaptive to a variation of signals in $\mathcal{K}_x^{(p)}$. A change of signals in $\mathcal{K}_x^{(p)}$ implies a change of determinations of sub-filters B_j in (6) and keep the same structure of the $F^{(p-1)}$.

2.3 Justification of the piecewise LSL interpolation filter. We wish to analyze the error in estimating the reference signals in set $\boldsymbol{p}$. Let us denote $\|\boldsymbol{x}(t, \cdot)\|_\Omega^2 = \displaystyle\int_\Omega \|\boldsymbol{x}(t, \omega)\|^2 d\mu(\omega)$ for each $t \in T$. Assume that for all $t \in T$,

$$\|\boldsymbol{x}(t, \cdot)\|_\Omega^2 < \infty \quad \text{and} \quad \|\boldsymbol{y}(t, \cdot)\|_\Omega^2 < \infty, \tag{13}$$

where $\|\boldsymbol{x}(t, \omega)\|$ and $\|\boldsymbol{y}(t, \omega)\|$ are the Euclidean norms for $\boldsymbol{x}(t, \omega)$ and $\boldsymbol{y}(t, \omega)$ for each $(t, \omega) \in T \times \Omega$, respectively. Thus we write $\boldsymbol{x}(t, \cdot) \in L^2(\Omega, \mathbb{R}^m)$ and $\boldsymbol{y}(t, \cdot) \in L^2(\Omega, \mathbb{R}^n)$ for each fixed $t \in T$.

Let $\mathcal{F} = \{\boldsymbol{f} : T \times \Omega \to \mathbb{R}^m \mid \boldsymbol{f}(t, \cdot) \in L^2(\Omega, \mathbb{R}^m) \text{ for each } t \in T\}$ and define

$$\|\boldsymbol{f}\|_{T,\Omega} = \frac{1}{b-a} \int_{T \times \Omega} \|\boldsymbol{f}(t, \omega)\| \, dt \, d\mu(\omega) = \frac{1}{b-a} \int_T E[\|\boldsymbol{f}(t, \cdot)\|] \, dt$$

for each $\boldsymbol{f} \in \mathcal{F}$ where $\|\boldsymbol{f}(t, \omega)\|$ is the Euclidean norm of $\boldsymbol{f}(t, \omega)$ on $\mathbb{R}^m$ for all $(t, \omega) \in \mathbb{R}^m$. Suppose that for all $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{P}$ there exist constants $\gamma_j, \delta_j > 0$ such that

$$\|\boldsymbol{x}(s, \omega) - \boldsymbol{x}(t, \omega)\| \le \gamma_j |s - t| \quad \text{and} \quad \|\boldsymbol{y}(s, \omega) - \boldsymbol{y}(t, \omega)\| \le \delta_j |s - t| \tag{14}$$

for all $(s,\omega),(t,\omega) \in [t_j, t_{j+1}] \times \Omega$. That is we suppose that the Lipschitz constants in (14) are independent of ω.

The main result is the error bound for the piecewise LSL interpolation filter established in Theorem 1 below. Note that if $\boldsymbol{f} \in \mathcal{F}$ then $\boldsymbol{f}(t,\cdot) \in L^2(\Omega, \mathbb{R}^m)$ for all $t \in T$. Then it follows from the Cauchy-Schwartz inequality that

$$E[\|\boldsymbol{f}(t,\cdot)\|]^2 \le E[\|\boldsymbol{f}(t,\cdot)\|_\Omega^2] < \infty$$

and hence the norm $\|\boldsymbol{f}\|_{T,\Omega}$ is well defined for all $\boldsymbol{f} \in \mathcal{F}$.

Theorem 1. *If condition* (14) *is satisfied where the Lipschitz are independent of* $\omega \in \Omega$ *then the error* $\epsilon_p = \|\boldsymbol{x} - F^{(p-1)}[\boldsymbol{y}]\|_{T,\Omega}$ *associated with the piecewise LSL interpolation filter satisfies the inequality*

$$\epsilon_p \le \max_{j=1,\dots,p-1} \left\{ (\gamma_j + \|B_j\|_2 \delta_j)\Delta t_j + \left[\|E^{1/2}_{\boldsymbol{v}_j,\boldsymbol{v}_j}\|_F^2 - \|E_{\boldsymbol{v}_j\boldsymbol{w}_j}(E^{1/2}_{\boldsymbol{w}_j\boldsymbol{w}_j})^\dagger\|_F^2 \right]^{1/2} \right\} \tag{15}$$

where $\|B_j\|_2$ *denotes the* 2*-norm given by the square root of the largest eigenvalue of* $B_j^T B_j$, $\Delta t_j = t_{j+1} - t_j$ *and* $\|\cdot\|_F$ *denotes the Frobenius norm.*

2.4 Numerical experiments. Suppose we need to filter a set $\mathcal{K}_y$ of $N = 121$ noisy random signals over set $T = [\tau_1, \tau_2, \dots, \tau_N]$ so that each input signal from this set, $\boldsymbol{y}(t,\cdot)$, enters a filter at time $t = \tau_k$ where $\tau_1 = 0$ and $\tau_{k+1} = \tau_k + 0.05$, for $k = 1, \dots, 120$. At each time τ_k, the training reference signal $\mathbf{x}(\tau_k,\cdot)$ is represented by its realizations as a 4×4 matrix

$$X^{(k)} = [\mathbf{x}(\tau_k,\omega_1), \dots, \mathbf{x}(\tau_k,\omega_4)] = \{x^{(k)}_{\ell,r}\}^4_{\ell,r=1}, \quad \text{for } k = 1, \dots, N. \tag{16}$$

A column of matrix $X^{(k)}$, $\mathbf{x}(\tau_k,\omega_i) \in \mathbb{R}^4$, represents the realization of the signal $\mathbf{x}(t,\omega_i)$ at time $t = \tau_k$ generated by the random event ω_i, for each $i = 1, 2, \dots, 4$. Entries of matrix $X^{(k)}$ are simulated in the form

$$x^{(k)}_{1,1} = 0.918\eta^{(k)}_1, \quad x^{(k)}_{1,2} = 1.02\eta^{(k)}_2, \quad x^{(k)}_{1,3} = 1.122\eta^{(k)}_3, \quad x^{(k)}_{1,4} = 0.918\eta^{(k)}_4,$$

where $\eta^{(k)}_1 = -\cos(2k)$, $\eta^{(k)}_2 = \sin(\cos(k))$, $\eta^{(k)}_3 = -\cos(k)$, $\eta^{(k)}_4 = \cos(\sin(k))$. Thus, all training reference signals are simulated as the 4×484 matrix $X = [X^{(1)}, \dots, X^{(121)}]$.

Suppose that an information on the references signals can only be obtained at some times $t_1, \dots, t_p \in T$ where $p \ll N$ in the form of samples given by 4×4 matrices

$$\tilde{X}^{(j)} = \{\tilde{x}^{(k)}_{\ell,r}\}^4_{\ell,r=1} = [\tilde{\mathbf{x}}(t_j,\omega_1), \dots, \tilde{\mathbf{x}}(t_j,\omega_4)], \quad \text{for } j = 1, \dots, p \tag{17}$$

where $\tilde{x}^{(k)}_{\ell,r} = 1.02x^{(k)}_{\ell,r}$, i.e. it is assumed that due to some measurement errors the values of $\tilde{x}^{(k)}_{\ell,r}$ are different from the values of $x^{(k)}_{\ell,r}$. For $p = 6, 11, 16, 31$, four particular choices of points $t_1, \dots, t_p$ are as follows:

for $p = 6$, $t_1 = \tau_1$, $t_{j+1} = \tau_{24j+1}$ where $j = 1, \ldots, 5$,
for $p = 11$, $t_1 = \tau_1$, $t_{j+1} = \tau_{12j+1}$ where $j = 1, \ldots, 10$,
for $p = 16$, $t_1 = \tau_1$, $t_{j+1} = \tau_{8j+1}$ where $j = 1, \ldots, 15$, and
for $p = 31$, $t_1 = \tau_1$, $t_{j+1} = \tau_{4j+1}$ where $j = 1, \ldots, 30$.

Samples of training reference signals, for the choice of $t_1, \ldots, t_p$ with $p = 11$, are given in Fig. 1 (left). Recall the whole set consists of 121 signals.

Further, at time τ_k, the observed signal $\mathbf{y}(\tau_k, \cdot)$, is simulated as 4×4 matrix

$$Y^{(k)} = \{y_{\ell,r}^{(k)}\}_{\ell,r=1}^{4} = [\mathbf{y}(\tau_k, \omega_1), \ldots, \mathbf{y}(\tau_k, \omega_4)], \quad \text{for } k = 1, \ldots, N. \tag{18}$$

Entries of matrix $Y^{(k)}$ are

$$y_{1,1}^{(k)} = 0.9\eta_1^{(k)} + \rho^{(k)}, \quad y_{1,2}^{(k)} = \eta_2^{(k)} + \rho^{(k)}, \tag{19}$$

$$y_{1,3}^{(k)} = 1.1\eta_3^{(k)} - \rho^{(k)}, \quad y_{1,4}^{(k)} = 0.9\eta_4^{(k)} + \rho^{(k)}, \tag{20}$$

where, for each $k = 1, \ldots, N$, $\rho^{(k)} = 0.5$`randn(k)` and `randn(k)` is a normally distributed random variable with mean zero, variance one and standard deviation one. Thus, all observed signals are simulated as the 4×484 matrix $Y = [Y^{(1)}, \ldots, Y^{(121)}]$ represented in Fig. 1 (right).

On the basis of the observed signals $Y^{(k)}$ in (18)–(19) and the samples in (17), four piecewise LSL interpolation filters $F^{(p-1)}$, for $p = 6, 11, 16, 31$, respectively, have been applied to estimate the training reference signals represented by X. The estimated reference signals by the filters $F^{(5)}$ and $F^{(10)}$ obtained from observed signals in Fig. 1 (right), and samples in Fig. 1 (left) are represented in Fig. 2.

The obtained results demonstrate a quite acceptable accuracy of signal estimation and the natural improvement in the accuracy of the proposed filter when the number of points $t_1, \ldots, t_p$ increases. In particular, differences between the estimates obtained by the filter $F^{(30)}$ and the training reference signals X are almost invisible, and therefore, they are not represented here. The errors associated with the filters confirm the above observation.

The averaging polynomial filter [7] has also been applied to the described signals. For the above particular choice of points $t_1, \ldots, t_p$, its associated errors are about 10 times worse than those by the proposed piecewise LSL interpolation filters.

References

[1] S. Avdonin, A. Bulanova and W. Moran, Construction of sampling and interpolating sequences for multi-band signals. The two-band case, *Int. J. Appl. Math. Comput. Sci.,* 17 (2007), no. 2, 101–113, 2007.

[2] L. Stankovic, S. Stankovic, and M. Amin, Missing samples analysis in signals for applications to L–estimation and compressive sensing, *Signal Processing,* vol. 94, pp. 401–408, 2014.

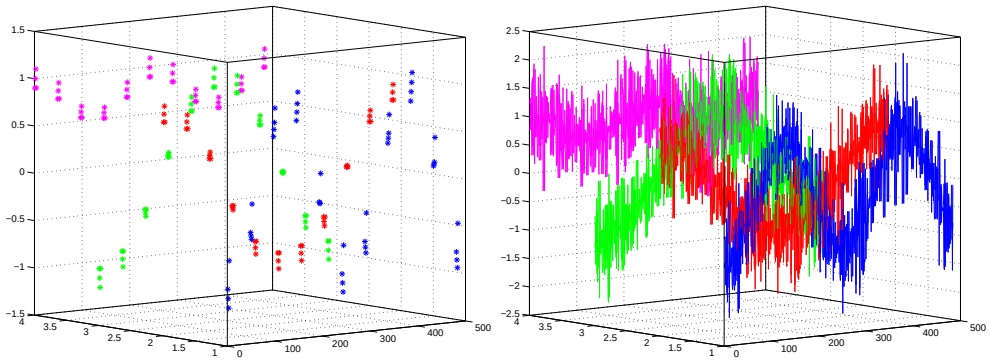

Fig. 1: Samples of signals $\tilde{X}^{(j)}$ at t_j, $j = 1, \ldots, 11$ (left). Observed signals from the set $\mathcal{K}_y$ (right).

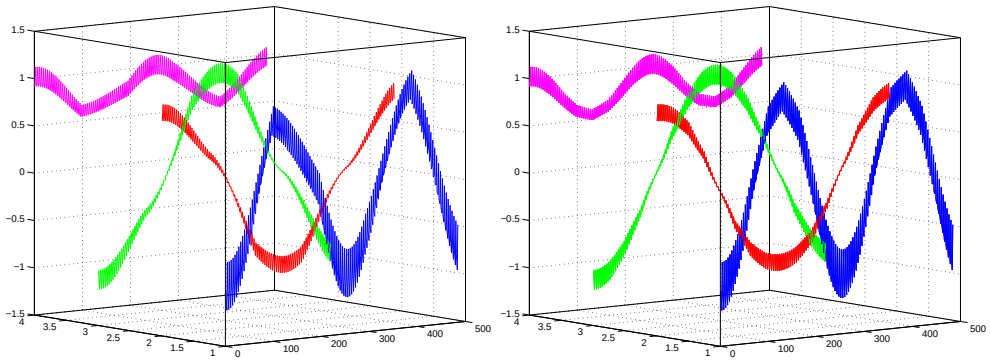

Fig. 2: Estimated signals by filters $F^{(5)}$ (left) and $F^{(10)}$ (right).

[3] P. Oliveira and L. Gomes, Interpolation of signals with missing data using Principal Component Analysis, *Multidim. Syst. Sign. Processing,* 21, pp. 25–43, 2010.

[4] J. Even and K. Sugimoto, An ICA approach to semi-blind identification of strictly proper systems based on interactor polynomial matrix, *Int. J. Robust Nonlinear Control*, 17, pp. 752–768, 2007.

[5] G. Kutz and D. Raphaeli, Maximum-Likelihood Semiblind Equalization of Doubly Selective Channels Using the EM Algorithm, *EURASIP Journal on Advances in Signal Processing*, Springer Open J., 2010.

[6] J.E. Cousseau, J.L. Figueroa, S. Werner, T.I. Laakso, Efficient Nonlinear Wiener Model Identification Using a Complex-Valued Simplicial Canonical Piecewise Linear Filter, *IEEE Trans. on Signal Processing*, 55 5, pp. 1780–1792, 2007.

[7] A. Torokhti and P. Howlett, *Computational Methods for Modelling of Nonlinear Systems*, Elsevier, 2007.

[8] O. Ledoit and M. Wolf, A well-conditioned estimator for large-dimensional covariance matrices, *J. Multivariate Analysis* 88, pp. 365–411, 2004.