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n-Dimensional Generalizations of a Thébault Conjecture*

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Abstract—This paper presents some generalizations of a Thébault conjecture, provides an analog of the Thébault conjecture for the n-simplex, and also solves a conjecture in a 2022 paper by the authors by using linear algebra.

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1. INTRODUCTION

The geometry of n-dimensional simplices remains a current research topic and is a subject of numerous recent publications. Several results are inspired by geometric results for triangles and tetrahedrons such as [1] (Samet, 2021), [2] (Ding, 2008), [3] (Buba-Brzozowa, 2005), [4] (Buba-Brzozowa, 2004), [5] (Hajja ,2005), et al. Using classical techniques of linear algebra, this article presents new geometric results for n-dimensional simplices. These results are inspired by a conjecture put forward by the famous French problemist Victor Thébault (1882–1960) and by a similar result by the authors [6].

In 1953, Thébault [7] conjectured the following geometric fact linking the radical center of four spheres with other elements of a tetrahedron:

Theorem 1. *Let AA', BB', CC', and DD' be the altitudes of a tetrahedron $ABCD$ with feet A', B', C' and D', respectively. Let P be the radical center of the spheres with centers A, B, C, and D and with respective radii AA', BB', CC', and DD'. Then each plane passing through the midpoint of the segment $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, or $D'C'$ and orthogonal to the respective segment BC, CA, AB, DA, DB, or DC contains the point P.*

This conjecture had remained open since 1953 but was proved in 2015 by Herrera [8].

In [6], a result similar to Thébault's was proved which links the radical center of four spheres with the insphere and the Monge point of a tetrahedron. (The Monge point of a tetrahedron is the concurrence point of the six planes passing through the midpoints of the edges and perpendicular to the opposite edges.) The result is as follows.

Theorem 2. *Let ω be the insphere of a tetrahedron $ABCD$. We denote its center by I and the points where it touches the faces (BCD), (CDA), (DAB), and (ABC) by A', B', C', and D', respectively. Let ω_a, ω_b, ω_c, and ω_d be the spheres with respective centers A, B, C, and D and radii AA', BB', CC', and DD'. Further, let M be the Monge point of the tetrahedron $A'B'C'D'$, and let P be the reflection of I with respect to M. Then the point P is the radical center of the spheres ω_a, ω_b, ω_c, and ω_d.*

In the paper [6], the authors conjectured a generalization of Theorem 2 to the n-dimensional Euclidean space. In the present paper, we prove this conjecture and also generalize and prove Theorem 1 for the n-simplex. Moreover, we find some new properties of the Monge point of the n-dimensional simplex.

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