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n-dimensional generalizations of a Thébault conjecture

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Abstract—This paper presents some generalizations to Thébault's conjecture, provides an analogy of Thébault's conjecture for the n-simplex, and also solves a conjecture in [6, Herrera and Tran (2022)] by using linear algebra.

Key words: *Thébault's conjecture*, *n-simplex*, *Monge points*, *n-dimensional Euclidean space*

1. INTRODUCTION

The geometry of n-dimensional simplices remains a current topic of research with numerous recent publications. Several results are inspired by geometric results for triangles and tetrahedrons, such as [7, Samet (2021)], [8, Ding Y (2008)], [9, Buba-Brzozowa(2005)],[10, Buba-Brzozowa (2004)], [11, Hajja (2005)], among others. Using classical techniques of linear algebra, this article presents new geometric results for the n-dimensional simplices. These results are inspired by a conjecture of the famous French problemist Victor Thébault (1882 - 1960) and by an analogous result of [6, Herrera and Tran (2022)].

Victor Thébault conjectured in [13, Thébault (1953)] the following geometric fact linking the radical center of four spheres with other elements of a tetrahedron:

Theorem 1. *Let AA', BB', CC', and DD' be the altitudes of a tetrahedron $ABCD$ with feet A', B', C' and D', respectively. Let P be the radical center of the spheres with centers A, B, C, and D and radii AA', BB', CC', and DD' respectively. Then each plane passing through the midpoint of the segment $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, and $D'C'$ which is orthogonal to the segment BC, CA, AB, DA, DB, and DC, respectively, contains the point P.*

This conjecture remained open since 1953, but was proved in 2015 in [5, Herrera (2015)].

In [6, Herrera and Tran (2022)] a result was proved which is similar to the result of Thébault, but linking the radical center of four spheres with the insphere and the Monge point of a tetrahedron (the Monge point of a tetrahedron is the concurrence point of six planes through the midpoints of the edges of a tetrahedron and perpendicular to the opposite edges). The result is:

Theorem 2. *Let ω be the insphere of a tetrahedron $ABCD$. This insphere ω, with its center at point I, touches the faces (BCD), (CDA), (DAB), and (ABC) at points A', B', C', and D', respectively. The spheres having centers A, B, C, and D and radii AA', BB', CC', and DD' are called ω_a, ω_b, ω_c, and ω_d, respectively. Let M be the Monge point of the tetrahedron $A'B'C'D'$, and let P be the reflection of I with respect to M. Then point P is the radical center of the spheres ω_a, ω_b, ω_c, and ω_d.*

In paper [6], authors conjecture a generalization of Theorem 2 for n-dimensional Euclidean space. Here, in this paper, the conjecture is proven, and a generalization is obtained. Moreover, in this paper, new properties of the Monge point of the n-dimensional simplex are found.