Vorticity and curvature at a general material surface

Blas Herrera[a)]
Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda Països Catalans 26, Tarragona 43007, Spain

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If S is a general material surface in a flow in the space, we show the relationship between the three components of the vorticity field on S and the curvatures of the streamlines of the tangential component of the velocity field (geodesic torsion, normal curvature, and geodesic curvature); and we show the relationship for the case of a general free surface. We present geometric formulae relating vorticity flux, from S, and curvatures of S (Gaussian curvature, mean curvature, curvatures of the streamlines of the tangential component of the velocity field); and we show the formulae for the case of a general free surface. © *2010 American Institute of Physics*. [doi:10.1063/1.3407653]

I. INTRODUCTION

The first part of this introduction, before Eq. (1.7), is dedicated to fix the notation. We consider a general material surface S in a flow $\mathcal{F}$ in $\mathbb{R}^3$; that is, let S be a $\mathcal{C}^\infty$-smooth surface of $\mathbb{R}^3$ (oriented Euclidean three-dimensional space) tangent or not to the $\mathcal{C}^\infty$-smooth velocity vector field $\vec{u}=\vec{u}(p,t)$ of the flow at any fixed time $t=t_0$.

Then we can consider the trio $(\vec{u},\vec{\omega},D)$ formed by the smooth velocity vector field of $\mathcal{F}$ at a given time t_0, its vorticity field: curl $(\vec{u})$, and its 2-covariant rate-of-strain tensor field, respectively. We denote the vectors and the tensor, at the point p, by $\vec{u}(p)$, $\vec{\omega}(p)$, and D_p, respectively. Let $\vec{x}$ be a parametrization of the smooth surface S:

$$\vec{x}:U\subset\mathbb{R}^2\to\mathbb{R}^3,(\theta_1,\theta_2)\mapsto\vec{x}(\theta_1,\theta_2) = [x(\theta_1,\theta_2),y(\theta_1,\theta_2),z(\theta_1,\theta_2)]. \quad (1.1)$$

Let $\vec{N}(\theta_1,\theta_2)$ be the normal vector given by the parametrization $\vec{N}=\frac{\partial\vec{x}}{\partial\theta_1}\times\frac{\partial\vec{x}}{\partial\theta_2}/\|\frac{\partial\vec{x}}{\partial\theta_1}\times\frac{\partial\vec{x}}{\partial\theta_2}\|$.

For $p\in S$ we know that exist an enough small three-dimensional neighborhood Ω in the space such that $\forall q\in\Omega$,

$$\vec{u}(q)=\vec{u}_t(q)+u_3\vec{N}(q),$$

$$\vec{u}_t(q)=u_1(q)\frac{\partial\vec{x}}{\partial\theta_1}(q_s)+u_2(q)\frac{\partial\vec{x}}{\partial\theta_2}(q_s), \quad (1.2)$$

$$u_3\vec{N}(q)=u_3(q)\vec{N}(q_s),$$

where $q_s\in S\cap\Omega$, $q=p+\lambda\vec{N}(q_s)$ $\lambda\in(-\varepsilon,\varepsilon)$, $\vec{u}_t(q)$ is the tangent component vector field of the velocity in Ω to the surface S, that is $\vec{u}_t(q)\cdot\vec{N}(q_s)=0$ (where $\cdot$ denotes the scalar product), and $u_3(q)$ is the scalar vertical component of the velocity in Ω to S.

Let $\vec{v}=\vec{u}_t(q_s)$ be the tangent velocity vector field on $S\cap\Omega\ni q_s$; let $\|\vec{v}\|=\|\vec{u}_t(q_s)\|=v$ be the tangent velocity scalar field on $S\cap\Omega$. Let $\vec{v}^\perp=\vec{u}_t^\perp(q_s)$ be the tangent vector field on $S\cap\Omega\ni q_s$ such that $\{\vec{v}/v,\vec{v}^\perp/v,\vec{N}\}$ is an orthonormal, direct basis (positively oriented), where the tangent velocity scalar $v=\|\vec{v}\|\neq 0$. Then, we also have with $q=p+\lambda\vec{N}(q_s)\in\Omega$ and $q_s\in S\cap\Omega$ that

$$\vec{u}_t(q)=\vec{u}_\parallel(q)\frac{\vec{v}}{v}(q_s)+\vec{u}_\perp(q)\frac{\vec{v}^\perp}{v}(q_s). \quad (1.3)$$

Let $u_3(q_s)=n$ be the vertical velocity scalar field on $S\cap\Omega\ni q_s$.

Similarly to Eq. (1.2), in $\Omega\ni q$ we have that

$$\vec{\omega}(q)=\vec{\omega}_t(q)+\omega_3\vec{N}(q) = \omega_1(q)\frac{\partial\vec{x}}{\partial\theta_1}(q_s)+\omega_2(q)\frac{\partial\vec{x}}{\partial\theta_2}(q_s)+\omega_3(q)\vec{N}(q_s), \quad (1.4)$$

where $q_s\in S\cap\Omega$, $q=p+\lambda\vec{N}(q_s)$ $\lambda\in(-\varepsilon,\varepsilon)$, $\vec{\omega}_t(q)$ is the tangent component vector field of the vorticity in Ω to the surface S, that is $\vec{\omega}_t(q)\cdot\vec{N}(q_s)=0$, and $\omega_3(q)$ is the scalar vertical component of the vorticity in Ω to S. If $\vec{v}^\perp$ exists then we also have

$$\vec{\omega}_t(q)=\omega_\parallel(q)\frac{\vec{v}}{v}(q_s)+\omega_\perp(q)\frac{\vec{v}^\perp}{v}(q_s). \quad (1.5)$$

We will denote the vectorial derivative of any $\mathcal{C}^\infty$-smooth function f on Ω with any vector $\vec{w}=w_\parallel(\vec{v}/v)+w_\perp(\vec{v}^\perp/v)+w_3\vec{N}\in T_p(\mathbb{R}^3)$ at $p\in S$ with

$$D_{\vec{w}}f=w_\parallel D_{\vec{v}/v}f+w_\perp D_{\vec{v}^\perp/v}f+w_3D_{\vec{N}}f = w_\parallel f_{,\parallel}+w_\perp f_{,\perp}+w_3f_{,3}, \quad (1.6)$$

or the vectorial derivative of any $\mathcal{C}^\infty$-smooth function f on $S\cap\Omega$ by any tangent vector $\vec{w}=w_\parallel(\vec{v}/v)+w_\perp(\vec{v}^\perp/v)\in T_p(S)$ at $p\in S$ with

$$D_{\vec{w}}f=w_\parallel D_{\vec{v}/v}f+w_\perp D_{\vec{v}^\perp/v}f=w_\parallel f_{,\parallel}+w_\perp f_{,\perp}. \quad (1.7)$$

Longuet-Higgins[1] shows, if S is a steady and free surface (i.e., $n=0$ and S is a free surface), the relationship between the tangent components, $\omega_\parallel$, $\omega_\perp$, of the vorticity on S and the geodesic torsion, τ_g, and the normal curvature, k_n, of the streamlines on S. Then, Herrera[2] generalizes the formulae of

[a)]Electronic mail: blas.herrera@urv.net.

