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Algebraic Equations of All Involucre Conics in the Configuration of the c-Inscribed Equilateral Triangles of a Triangle

Blas Herrera

Abstract. Let $\triangle ABC$ be a triangle with side length $c = AB$; here we present the determination of the existence and quantity m of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (i.e. $\mathbb{T}_j = \triangle A_j B_j C_j$ with $A_j \in \overleftrightarrow{BC}$, $B_j \in \overleftrightarrow{CA}$, $C_j \in \overleftrightarrow{AB}$, $c = A_j B_j$) of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, and from an algebraic point of view. We give the algebraic equations of all involucre conics –circles $\mathbb{N}_o$, $\mathbb{N}_i$; parabola $\mathcal{P}_i$; ellipses $\mathcal{H}_i$, $\mathcal{H}_o$– in the configuration.

1. Introduction

Many configurations linking conics and equilateral triangles with the triangle have been described by different geometers in the past; here we give a new one. Let $\triangle ABC$ be a triangle with side length $c = AB$; in this work we want to present the determination of the existence and quantity of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (i.e. $\mathbb{T}_j = \triangle A_j B_j C_j$ with $A_j \in \overleftrightarrow{BC}$, $B_j \in \overleftrightarrow{CA}$, $C_j \in \overleftrightarrow{AB}$, $c = A_j B_j$) of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, from an algebraic point of view. We give the algebraic equations of all involucre conics –circles $\mathbb{N}_o$, $\mathbb{N}_i$; parabola $\mathcal{P}_i$; ellipses $\mathcal{H}_i$, $\mathcal{H}_o$– in the configuration.

Readers can find the construction of the c-inscribed equilateral triangles [3]. And from the kinematic point of view we are considering a well known result of planar kinematics: we consider the motion of the point X of an equilateral triangle $\triangle PQX$, where P and Q slide along straight (non-parallel) lines. It is well known that, in the general case, the trajectory of X is an ellipse (for each of the two possible orientations of $\triangle PQX$). Therefore, we consider nothing else than a special case of the well known elliptic motion or Cardan motion [1], [2]. Nevertheless, in this work, through long but straightforward calculations, we present not the well known kinematic point of view, but the algebraic equations of the special case of all the conics which are linked with the c-equilateral triangles which are sliding on a triangle $\triangle ABC$. More precisely, let $\triangle ABC$ be a triangle with side length $c = AB$, let be their equilateral triangles, of side length c, which are sliding on the straight lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$. In the next section we present the algebraic equations

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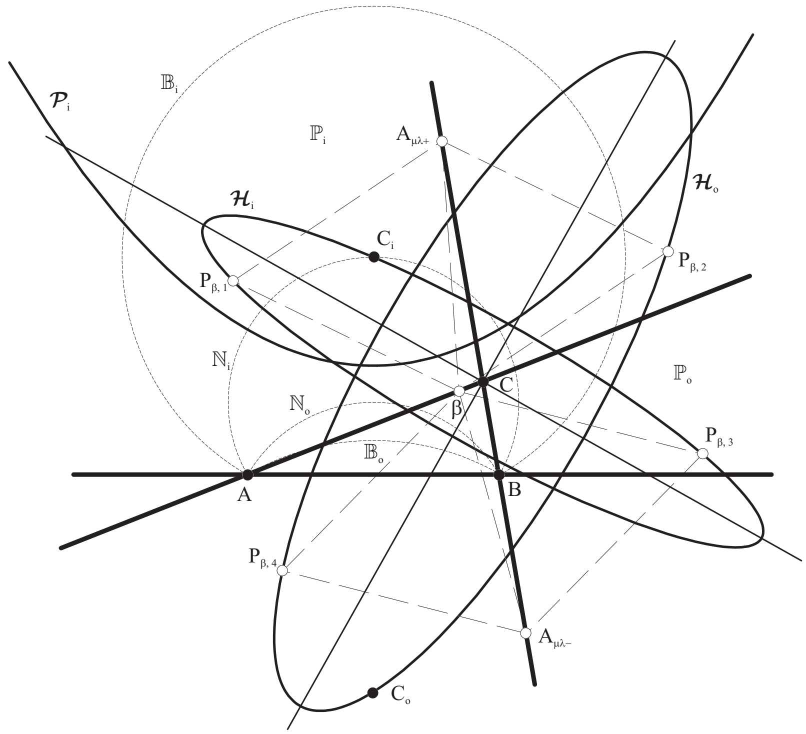

Figure 1. The conics linked with the c-equilateral triangles which are sliding on a triangle.

of all the conics which are linked with these c-equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ (see Figure 1). And with these configuration equations, we present the determination of the existence and quantity m of the c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ of $\triangle ABC$ in function of the position of vertex C respect to a separatrix parabola $\mathcal{P}_i$, from an algebraic point of view. (see Figures 2, 3).

Of course, the triangle $\triangle ABC$ has its other two similar configurations for its other two sides $b = AC$, $a = BC$.

1.1. *Elements of the configuration.* In the following we fix, with precision, the notation and the elements involved with the configuration (for the case of the side $\overline{AB}$).

Lemma 1. *Let $\triangle ABC$ be a triangle in the affine euclidean plane $\mathbb{A}^2$, with its length side $c = AB$, and (see Figures 1, 2, 3):*

1.- Let $\{T_{\beta,k} = \triangle P_{\beta,k} A_{\beta,k} \beta\}_{k=1}^{k=4}$ be its four β-sliding equilateral triangles: i.e. β is an arbitrary point with $\beta \in \overleftrightarrow{CA}$, $A_{\beta,k} \in \overleftrightarrow{BC}$, and $T_{\beta,k}$ has its length sides equal to $c = AB$ (see Figure 1, and the proof of Lemma 2) [this is a special case of the well known elliptic motion].

2.- Let $\mathbb{N}_o$ and $\mathbb{N}_i$ be the circumcircle of $\triangle ABC_o$ and $\triangle ABC_i$, the outer equilateral triangle and the inner equilateral triangle of $\overline{AB}$, respectively. Let $\mathbb{B}_o$ and

$\mathbb{B}_i$ *be the circumference of center* C_o *with radius* $\overline{C_oA}$ *and the circumference of center* C_i *with radius* $\overline{C_iA}$*, respectively.*

3.- Let $\mathcal{P}_i$ *be the parabola such that* $\overleftrightarrow{AB}$ *and* C_i *are its: directrix and focus, respectively. Let* $\mathbb{P}_i$ *and* $\mathbb{P}_o$ *be the convex arc-connected region and the non-convex arc-connected region, respectively, of* $\mathbb{S} \setminus \mathcal{P}_i$*, where* $\mathbb{S}$ *is the semiplane, with side* $\overleftrightarrow{AB}$*, containing* C_i *and* $\triangle ABC$*.*

4.- Let $\mathcal{H}_k$ *be the locus of the points* $P_{\beta,k}$ *when* $T_{\beta,k}$ *is being sliding on the pair of the straight lines* $\overleftrightarrow{BC}$*,* $\overleftrightarrow{CA}$*: i.e. when* β *is being sliding on* $\overleftrightarrow{CA}$ *(see Figure 1, and Corollary 6) [this is a special case of the well known elliptic motion].*

5.- Let $\{\mathbb{T}_j\}_{j=1}^{j=m}$ *be the* c*-inscribed equilateral triangles of* $\triangle ABC$*: i.e.* $\mathbb{T}_j = \triangle A_jB_jC_j$ *with* $A_j \in \overleftrightarrow{BC}$*,* $B_j \in \overleftrightarrow{CA}$*,* $C_j \in \overleftrightarrow{AB}$*, and* $c = A_jB_j$ *(see Figures 2, 3).*

The number m will be determined in the Corollary 8.

Without loss generality we can assume, along of this paper, that $AB = c = 1$; also we can consider a Cartesian system of coordinates (x, y) such that $A = (0, 0)$, $B = (1, 0)$, $C = (a, b)$ with $b > 0$.

2. Results

Lemma 2. *Equations (1), (2), (3), (4) are algebraic formulae of* $\{P_{\beta,k}\}_{k=1}^{k=4}$*.*

Proof. Let $\beta_\lambda = (\lambda a, \lambda b)$, $\lambda \in \mathbb{R}$ with $\beta_\lambda = \beta$, be an arbitrary point on $\overleftrightarrow{CA}$. Let $A_\mu = (1 + \mu a - \mu, \mu b)$, $\mu \in \mathbb{R}$, be an arbitrary point on $\overleftrightarrow{BC}$.

With an easy calculation, we can observe that there are two points A_μ with $\beta_\lambda A_\mu = 1$, which depend on λ, and with $\mu = \mu_{\lambda\pm}$ where:

$$\mu_{\lambda\pm} = \tfrac{1}{(a-1)^2+b^2}(\lambda \left(a^2 - a + b^2\right) - a + 1 \pm \sqrt{(a-1)^2 + (2-\lambda)\lambda b^2}).$$

We will put: $\Phi = \sqrt{(a-1)^2 + b^2} = BC$, $\Psi_\lambda = \sqrt{(a-1)^2 + (2-\lambda)\lambda b^2}$ and $\Lambda_\lambda = \lambda \left(a^2 - a + b^2\right)$.

Points $A_{\mu_{\lambda\pm}}$ exist if and only if $\Psi_\lambda \geq 0 \Leftrightarrow \lambda \in \left[1 - \frac{\Phi}{b}, 1 + \frac{\Phi}{b}\right]$. Points $A_{\mu_{0\pm}}$exist, with $\lambda = 0$, and $\mu_{0-} = \frac{1}{\Phi^2}\left(-a + 1 - |a-1|\right)$, $\mu_{0+} = \frac{1}{\Phi^2}(-a + 1 + |a-1|)$; and we have:

$$\text{if } a - 1 \geq 0 \Rightarrow\ A_{\mu_{0-}} = \tfrac{1}{\Phi^2}\left(b^2 - (a-1)^2, 2b\,(1-a)\right), A_{\mu_{0+}} = B;$$
$$\text{if } a - 1 < 0 \Rightarrow\ A_{\mu_{0-}} = B, A_{\mu_{0+}} = \tfrac{1}{\Phi^2}\left(b^2 - (a-1)^2, 2b\,(1-a)\right).$$

It happens that $B = A_{\mu_{0-}} = A_{\mu_{0+}} \Leftrightarrow \mu = 0 \Leftrightarrow 1 = a \Leftrightarrow \angle ABC$ is a right angle. Also $A_{\mu_{1\pm}}$ exist with $\lambda = 1 \Rightarrow \mu_{1-} = 1 - \frac{1}{\Phi}$ and $\mu_{1+} = 1 + \frac{1}{\Phi}$, thus $A_{\mu_{1-}} = \left(a - \frac{a-1}{\Phi}, b - \frac{b}{\Phi}\right)$ and $A_{\mu_{1+}} = \left(a + \frac{a-1}{\Phi}, b + \frac{b}{\Phi}\right)$.

In general, to any $\lambda \in \left[1 - \frac{\Phi}{b}, 1 + \frac{\Phi}{b}\right] \setminus \{0\}$, we have that

$$A_{\mu_{\lambda\pm}} = \left(\tfrac{b^2 + (a-1)(\Lambda_\lambda \pm \Psi_\lambda)}{\Phi^2}, b\tfrac{-a+1+\Lambda_\lambda \pm \Psi_\lambda}{\Phi^2}\right).$$

And the two middle points $M_{\mu_{\lambda\pm}}$, of the segments $\overline{A'_{\mu_{\lambda\pm}}B'_{\lambda}}$, are:

$$M_{\mu_{\lambda+}} = \left(\tfrac{(\lambda+1)b^2+(a-1)(2\Lambda_\lambda+\Psi_\lambda)}{2\Phi^2}, b\tfrac{(\lambda+1)(-a+1)+2\Lambda_\lambda+\Psi_\lambda}{2\Phi^2}\right),$$
$$M_{\mu_{\lambda-}} = \left(\tfrac{(\lambda+1)b^2+(a-1)(2\Lambda_\lambda-\Psi_\lambda)}{2\Phi^2}, b\tfrac{(\lambda+1)(-a+1)+2\Lambda_\lambda-\Psi_\lambda}{2\Phi^2}\right).$$

Then, making a calculation, it follows that: $C_{\mu_{\lambda++}} = \left(C_{\mu_{\lambda++x}}, C_{\mu_{\lambda++y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda++x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)+\left(a-1-\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C_{\mu_{\lambda++y}} &= \tfrac{b(\lambda+1)(1-a)-(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(b-\sqrt{3}+\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}; \end{aligned} \tag{1}$$

and also that $C_{\mu_{\lambda+-}} = \left(C_{\mu_{\lambda+-x}}, C_{\mu_{\lambda+-y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda+-x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(-\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)+\left(a-1+\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C_{\mu_{\lambda+-y}} &= \tfrac{b(\lambda+1)(1-a)+(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(b+\sqrt{3}-\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}; \end{aligned} \tag{2}$$

and also that $C_{\mu_{\lambda-+}} = \left(C_{\mu_{\lambda-+x}}, C_{\mu_{\lambda-+y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda-+x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)-\left(a-1-\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C''_{\mu_{\lambda-+y}} &= \tfrac{b(\lambda+1)(1-a)-(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(-b+\sqrt{3}-\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}; \end{aligned} \tag{3}$$

and also that $C_{\mu_{\lambda--}} = \left(C_{\mu_{\lambda--x}}, C_{\mu_{\lambda--y}}\right)$, with

$$\begin{aligned} C_{\mu_{\lambda--x}} &= \tfrac{b^2(\lambda+1)+(a-1)\left(-\sqrt{3}b(1-\lambda)+2\Lambda_\lambda\right)-\left(a-1+\sqrt{3}b\right)\Psi_\lambda}{2\Phi^2}, \\ C_{\mu_{\lambda--y}} &= \tfrac{b(\lambda+1)(1-a)+(\lambda-1)\sqrt{3}b^2+2b\Lambda_\lambda+\left(-b-\sqrt{3}+\sqrt{3}a\right)\Psi_\lambda}{2\Phi^2}. \end{aligned} \tag{4}$$

Where $C_{\mu_{\lambda++}}$, $C_{\mu_{\lambda+-}}$, $C_{\mu_{\lambda-+}}$, $C_{\mu_{\lambda--}}$, are the vertices of the equilateral triangles $\triangle C_{\mu_{\lambda++}}A_{\mu_{\lambda+}}\beta_\lambda$, $\triangle C_{\mu_{\lambda+-}}A_{\mu_{\lambda+}}\beta_\lambda$, $\triangle C_{\mu_{\lambda-+}}A_{\mu_{\lambda-}}\beta_\lambda$, $\triangle C_{\mu_{\lambda--}}A_{\mu_{\lambda-}}\beta_\lambda$, respectively.

In the case of $\lambda = 0$ it happens that: if $a-1>0$ then $C_{\mu_{0++}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C_{\mu_{0+-}} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; and if $a-1<0$ then $C_{\mu_{0-+}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C_{\mu_{0--}} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; and if $a-1=0$ then $C_{\mu_{0++}} = C_{\mu_{0-+}} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $C_{\mu_{0+-}} = C_{\mu_{0--}} = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

Therefore, finally we have: $\beta = \beta_\lambda$, $A_{\beta,1} = A_{\beta,2} = A_{\mu_{\lambda+}}$, $A_{\beta,3} = A_{\beta,4} = A_{\mu_{\lambda-}}$, $P_{\beta,1} = C_{\mu_{\lambda++}}$, $P_{\beta,2} = C_{\mu_{\lambda+-}}$, $P_{\beta,3} = C_{\mu_{\lambda-+}}$, $P_{\beta,4} = C_{\mu_{\lambda--}}$. □

With $g_{P,\theta}$ the rotation with angle of amplitude θ and with center point P: by construction, we have that $g_{P_{\beta,1},-\frac{\pi}{3}}(A_{\beta,1}) = g_{P_{\beta,2},\frac{\pi}{3}}(A_{\beta,2}) = g_{P_{\beta,3},-\frac{\pi}{3}}(A_{\beta,3}) = g_{P_{\beta,4},\frac{\pi}{3}}(A_{\beta,4}) = \beta$; and we have that the equilateral triangles $\triangle P_{\beta,1}A_{\beta,1}\beta$, $\triangle P_{\beta,2}A_{\beta,2}\beta$, $\triangle P_{\beta,3}A_{\beta,3}\beta$, $\triangle P_{\beta,4}A_{\beta,4}\beta$ are not coincident.

Along all the paper we will put $\Phi = \sqrt{(a-1)^2+b^2}$, $\Lambda_\lambda = \lambda\left(a^2-a+b^2\right)$ and $\Psi_\lambda = \sqrt{(a-1)^2+(2-\lambda)\lambda b^2}$.

The following lemma also is a *Definition*:

Lemma 3. *Let $\mathcal{H}_i$, $\mathcal{H}_o$ be the two conics determined by: both conics have the center point C, the outer and inner bisectors of the angle $\angle ACB$ are the axes of the both conics, the two points $C_i = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $C_{\mu_{1++}} = \left(a + \frac{a-1-b\sqrt{3}}{2\Phi}, b+ \frac{b+(a-1)\sqrt{3}}{2\Phi}\right)$ belong to $\mathcal{H}_i$, the two points $C_o = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and $C_{\mu_{1+-}} = \left(a+ \frac{a-1+b\sqrt{3}}{2\Phi}, b + \frac{b+(1-a)\sqrt{3}}{2\Phi}\right)$ belong to $\mathcal{H}_o$.*

Then we have:

$\mathcal{H}_i$ is circumference $\Leftrightarrow C \in \mathbb{B}_i$, in this case $\mathcal{H}_i$ has equation (5);

$\mathcal{H}_o$ is circumference $\Leftrightarrow C \in \mathbb{B}_o$, in this case $\mathcal{H}_o$ has equation (6);

if $a = \frac{1}{2}$, it happens that $b \neq \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_i$ is ellipse, in this case $\mathcal{H}_i$ has equation (7);

if $a = \frac{1}{2}$, it happens that $b = 1 + \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_i$ is circumference, in this case $\mathcal{H}_i$ has equation (8);

if $a = \frac{1}{2}$, it happens that $b = \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_i$ is a pair of coincident straight lines parallel to $\overleftrightarrow{AB}$, in this case $\mathcal{H}_i$ has equation (9);

if $a = \frac{1}{2}$, it happens that $b \neq \frac{\sqrt{3}}{6} \Leftrightarrow \mathcal{H}_o$ is ellipse, in this case $\mathcal{H}_o$ has equation (10);

if $a = \frac{1}{2}$, it happens that $b = 1 - \frac{\sqrt{3}}{2} \Leftrightarrow \mathcal{H}_o$ is circumference, in this case $\mathcal{H}_o$ has equation (11);

if $a = \frac{1}{2}$, it happens that $b = \frac{\sqrt{3}}{6} \Leftrightarrow \mathcal{H}_o$ is a pair of coincident straight lines orthogonal to $\overleftrightarrow{AB}$, in this case $\mathcal{H}_i$ has equation (12);

if $a \neq \frac{1}{2}$, it happens that $C \notin \mathbb{B}_i \Leftrightarrow \mathcal{H}_i$ is not circumference, in this case $\mathcal{H}_i$ has equation (13);

if $a \neq \frac{1}{2}$, it happens that $C \notin \mathbb{B}_o \Leftrightarrow \mathcal{H}_o$ is not circumference, in this case $\mathcal{H}_o$ has equation (14).

Proof. Let $\mathcal{H}$ be a conic, its equation is $\mathcal{H} \equiv \xi x^2 + \zeta y^2 + Dxy + Ex + Fy + G = 0$; and $M = \begin{pmatrix} G & \frac{1}{2}E & \frac{1}{2}F \\ \frac{1}{2}E & \xi & \frac{1}{2}D \\ \frac{1}{2}F & \frac{1}{2}D & \zeta \end{pmatrix}$ is its associated matrix with its algebraic parameters: $T = trace\,(M_{00})$, $U = \det M_{11} + \det M_{22}$, where $M_{00} = \begin{pmatrix} \beta & \frac{1}{2}D \\ \frac{1}{2}D & \zeta \end{pmatrix}$, $M_{11} = \begin{pmatrix} G & \frac{1}{2}F \\ \frac{1}{2}F & \zeta \end{pmatrix}$ and $M_{22} = \begin{pmatrix} G & \frac{1}{2}E \\ \frac{1}{2}E & \xi \end{pmatrix}$.

From now, we consider that the inner and outer bisectors of the angle $\angle ACB$ are the axes of $\mathcal{H}$.

By requiring that C is the center $\mathcal{H}$, we have two possibilities: the bisectors of $\angle ACB$ are parallel to the coordinate axes, or they are not; and these two possibilities are equivalent to $a = \frac{1}{2}$ or $a \neq \frac{1}{2}$. Therefore, if $a = \frac{1}{2}$ then the axes of $\mathcal{H}$ are parallel to the coordinate axes; but if $a \neq \frac{1}{2}$ then the axes of $\mathcal{H}$ are not parallel to the coordinate axes unless $\mathcal{H}$ is circumference.

Now, whatever the value of a, we first consider that $\mathcal{H}$ is circumference with center point C. By imposing the circumference $\mathcal{H}$ passes through the point $C_i =$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we have

$$\mathcal{H}^i \equiv x^2 + y^2 - 2ax - 2by - 1 + a + b\sqrt{3} = 0. \tag{5}$$

By imposing the circumference $\mathcal{H}$ passes through the point $C_o = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ we have

$$\mathcal{H}^o \equiv x^2 + y^2 - 2ax - 2by - 1 + a - b\sqrt{3} = 0. \tag{6}$$

And if $\mathcal{H}^i = \mathcal{H}_i$ passes through the point $C_{\mu_{1++}}$ then $a^2 + b^2 - a - b\sqrt{3} = 0$; also if $\mathcal{H}^o = \mathcal{H}_o$ passes through the point $C''_{\mu_{1+-}}$ then $a^2 + b^2 - a + b\sqrt{3} = 0$.

The equations of $\mathbb{B}_i$ and $\mathbb{B}_o$ are $\mathbb{B}_i \equiv x^2 + y^2 - x - y\sqrt{3} = 0$ and $\mathbb{B}_0 \equiv x^2 + y^2 - x + y\sqrt{3} = 0$ respectively. Accordingly, if the conics $\mathcal{H}_i$, $\mathcal{H}_o$ are circumferences then $C \in \mathbb{B}_i$, $C \in \mathbb{B}_o$ respectively.

The inverse assertion is true, i.e. if the conics $\mathcal{H}_i$ and $\mathcal{H}_o$ have their centers $C \in \mathbb{B}_i$ and $C \in \mathbb{B}_o$, respectively, then they are circumferences; we will prove later this inverse assertion.

Now, regardless if $\mathcal{H}$ is a circumference or not, we consider the case of $a = \frac{1}{2}$. Then we have that the axes of $\mathcal{H}$ are parallel to the coordinate axes; which implies that the matrix A_{00} has the eigenvectors $\{(1,0), (0,1)\}$, which algebraically imposes that $D = 0$. Then $\mathcal{H} \equiv \xi x^2 + \zeta y^2 + Ex + Fy + G = 0$. But, $\mathcal{H}$ is conic with center, then $\mathcal{H}$ is not a parabola; and then, algebraically, the value 0 is discarded as eigenvalue of A_{00}, so $\xi \neq 0$ and $\zeta \neq 0$. Therefore, without loss generality, we can consider that $\zeta = 1$, and we have that $\mathcal{H} \equiv \xi x^2 + y^2 + Ex + Fy + G = 0$. Moreover C is center point of $\mathcal{H}$; then, with $a = \frac{1}{2}$, analytically we have that $\begin{cases} \frac{1}{2}\xi + \frac{1}{2}E = 0 \\ b + \frac{1}{2}F = 0 \end{cases}$, which implies that $\mathcal{H} \equiv -Ex^2 + y^2 + Ex - 2by + G = 0$. If $\mathcal{H}$ also passes through the point C_i then we denote the conic as $\mathcal{H}^i$, and has the following equation: $\mathcal{H}^i \equiv -\left(-3 + 4b\sqrt{3} - 4G\right)x^2 + y^2 + \left(-3 + 4b\sqrt{3} - 4G\right)x - 2by + G = 0$. Similarly, if $\mathcal{H}$ also passes through the point C_o then we denote the conic as $\mathcal{H}^o$, and has the following equation: $\mathcal{H}^o \equiv -\left(-3 - 4b\sqrt{3} - 4G\right)x^2 + y^2 + \left(-3 - 4b\sqrt{3} - 4G\right)x - 2by + G = 0$.

We impose now that $\mathcal{H}^i$ passes through the point $C''_{\mu_{1++}}$, with $a = \frac{1}{2}$ and $b \neq \frac{1}{2}\sqrt{3}$, then we have that $\mathcal{H}^i = \mathcal{H}_i$ with

$$\begin{aligned} &\mathcal{H}_i \equiv 48b^2\left(2b - \sqrt{3}\right)^2 x^2 + 4\left(6b + \sqrt{3}\right)^2 y^2 - 48b^2\left(2b - \sqrt{3}\right)^2 x \\ &-8b\left(6b + \sqrt{3}\right)^2 y + 3\left(8b^3 + 12b^2\sqrt{3} - 6b - \sqrt{3}\right)\left(2b + \sqrt{3}\right) = 0. \end{aligned} \tag{7}$$

With $b \neq \frac{1}{2}\sqrt{3}$, the conic $\mathcal{H}_i$ has the algebraic parameters: $\det A = -192b^2(2b-\sqrt{3})^4(6b+\sqrt{3})^4 \neq 0$, $\det A_{00} = 192b^2\left(2b - \sqrt{3}\right)^2\left(6b + \sqrt{3}\right)^2 > 0$, $T = 48b^2(2b- \sqrt{3})^2 + 4(6b + \sqrt{3})^2$ and $T \det A < 0$; therefore $\mathcal{H}_i$ is a real non-degenerate ellipse. If $b = 1 + \frac{1}{2}\sqrt{3}$, i.e. $C \in \mathbb{B}_i$, then

$$\mathcal{H}_i \equiv x^2 + y^2 - x - \left(2 + \sqrt{3}\right)y + 1 + \sqrt{3} = 0, \tag{8}$$

and $\mathcal{H}_i$ is a circumference of radius 1. If $b = \frac{1}{2}\sqrt{3}$then the conic $\mathcal{H}_i$ degenerates to

$$\mathcal{H}_i \equiv y^2 - \sqrt{3}y + \frac{3}{4} = 0, \tag{9}$$

with $\xi = 0$, whose algebraic parameters are: $\det A = 0$, $\det A_{00} = 0$, $U = 0$; therefore $\mathcal{H}_i$ is the pair of coincident straight lines $\left\{y = \frac{1}{2}\sqrt{3}\right\}$, $\left\{y = \frac{1}{2}\sqrt{3}\right\}$.

We impose now that $\mathcal{H}^o$ passes through the point $C''_{\mu_{1+-}}$,with $a = \frac{1}{2}$ and $b \neq \frac{1}{6}\sqrt{3}$, then we have that $\mathcal{H}^o = \mathcal{H}_o$ with

$$\begin{aligned}\mathcal{H}_o \equiv\, & 48b^2\left(2b+\sqrt{3}\right)^2 x^2 + 4\left(6b-\sqrt{3}\right)^2 y^2 - 48b^2\left(2b+\sqrt{3}\right)^2 x \\ & -8b\left(6b-\sqrt{3}\right)^2 y + +3\left(8b^3 - 12b^2\sqrt{3} - 6b + \sqrt{3}\right)\left(2b-\sqrt{3}\right) = 0.\end{aligned} \tag{10}$$

With $b \neq \frac{1}{6}\sqrt{3}$, the conic $\mathcal{H}_o$ has the algebraic parameters: $\det A = -192b^2(2b+\sqrt{3})^4\ (6b-\sqrt{3})^4 \neq 0$, $\det A_{00} = 192b^2\left(2b+\sqrt{3}\right)^2\left(6b-\sqrt{3}\right)^2 > 0$, $T = 48b^2(2b+\sqrt{3})^2 + 4(6b-\sqrt{3})^2$ and $T\det A < 0$; therefore $\mathcal{H}_o$ is a real non-degenerate ellipse. If $b = 1 - \frac{1}{2}\sqrt{3}$, i.e. $C \in \mathbb{B}_o$, then

$$\mathcal{H}_o \equiv x^2 + y^2 - x - \left(2-\sqrt{3}\right) y + 1 - \sqrt{3} = 0, \tag{11}$$

is a circumference of radius 1. If $b = \frac{1}{6}\sqrt{3}$ then the conic $\mathcal{H}_o$ degenerates to

$$\mathcal{H}_o \equiv x^2 - x + \frac{1}{4} = 0, \tag{12}$$

with $\zeta = 0$, whose algebraic parameters are: $\det A = 0$, $\det A_{00} = 0$, $U = 0$; therefore $\mathcal{H}_o$ is the pair of coincident straight lines $\left\{x = \frac{1}{2}\right\}$, $\left\{x = \frac{1}{2}\right\}$.

We have just seen that if $a = \frac{1}{2}$ and $C \in \mathbb{B}_i$, $C \in \mathbb{B}_o$ then $\mathcal{H}_i$, $\mathcal{H}_o$ are circumferences, respectively. Let's see what is also true with $a \neq \frac{1}{2}$. If $C \in \mathbb{B}_o$, then is easy to calculate that C is at the same distance from C_o to $C''_{\mu_{1+-}}$. But the three points C_o, $C''_{\mu_{1+-}}$ and $C \in \mathbb{B}_o$, are not aligned. To prove the above assertion: if we impose that $\left(\left(a + \frac{a-1+b\sqrt{3}}{2\Phi}, b + \frac{b+(1-a)\sqrt{3}}{2\Phi}\right) - (a,b)\right) - \alpha\left(\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) - (a,b)\right) = (0,0)$, then this implies that $\alpha = -\frac{a-1+b\sqrt{3}}{\Phi(2a-1)}$ and $3a^2 + 3b^2 - 3a + b\sqrt{3} = 0$. But moreover $a^2 + b^2 - a + b\sqrt{3} = 0$, then necessarily $C = (0,0) = A$ or $C = (1,0) = B$, in contradiction. Therefore: C is at the same distance from C_o to $C''_{\mu_{1+-}}$, they are three not aligned points, and C is center of symmetry of the conic $\mathcal{H}_o$ which passes through C_o and $C''_{\mu_{1+-}}$; then $\mathcal{H}_o$ is a circumference. Similarly: if $C \in \mathbb{B}_i$, then is easy to calculate that C is at the same distance from C_i to $C''_{\mu_{1++}}$. But the three points C_i, $C''_{\mu_{1++}}$ and $C \in \mathbb{B}_i$, are not aligned. To prove the above assertion: if we impose that $\left(\left(a + \frac{a-1-b\sqrt{3}}{2\Phi}, b + \frac{b+(a-1)\sqrt{3}}{2\Phi}\right) - (a,b)\right) - \alpha\left(\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) - (a,b)\right) = (0,0)$, then this implies that $\alpha = \frac{-a+1+b\sqrt{3}}{\Phi(2a-1)}$ and $3a^2 + 3b^2 - 3a - b\sqrt{3} = 0$. But moreover $a^2 + b^2 - a - b\sqrt{3} = 0$, then necessarily $C = (0,0) = A$ or $C = (1,0) = B$, in contradiction. Therefore: C is at the same distance from C_i to $C''_{\mu_{1++}}$, they are three not aligned points, and C is center of symmetry of the conic $\mathcal{H}_i$ which passes through C_i and $C''_{\mu_{1++}}$; then $\mathcal{H}_i$ is a circumference.

Now we consider the case of $a \neq \frac{1}{2}$.

The axes of $\mathcal{H}$ are not parallel to the coordinate axes unless that any straight line through C is axis of $\mathcal{H}$, and then $\mathcal{H}$ is circumference. Algebraically, $\mathcal{H}$ is not circumference if and only if the vector $\{(1,0),(0,1)\}$ are not eigenvectors of M_{00}, then $\mathcal{H}$ is not circumference if and only if $D \neq 0$.

We consider that $\mathcal{H}$ is not circumference (if $\mathcal{H}$ is circumference then is already a studied case) and we can assume that $D = 1$, so that $\mathcal{H} \equiv \xi x^2 + \zeta y^2 + xy + Ex + Fy + G = 0$. The vectors $\overrightarrow{u}_+ = \left(\beta - \gamma + \sqrt{(\beta-\gamma)^2+1}, 1\right)$, $\overrightarrow{u}_- = \left(\beta - \gamma - \sqrt{(\beta-\gamma)^2+1}, 1\right)$ are eigenvectors with eigenvalues $\frac{1}{2}\beta + \frac{1}{2}\gamma + \frac{1}{2}\sqrt{(\beta-\gamma)^2+1}$, $\frac{1}{2}\beta + \frac{1}{2}\gamma - \frac{1}{2}\sqrt{(\beta-\gamma)^2+1}$, of M_{00}, respectively. Then $\overrightarrow{u}_+$, $\overrightarrow{u}_-$, are direction vectors of the axes of $\mathcal{H}$. The inner and outer bisectors of the angle $\angle ACB$ are directed by $\overrightarrow{v}_+ = (a,b)\frac{1}{\sqrt{a^2+b^2}} + (a-1,b)\frac{1}{\sqrt{(a-1)^2+b^2}}$, $\overrightarrow{v}_- = (a,b)\frac{1}{\sqrt{a^2+b^2}} - (a-1,b)\frac{1}{\sqrt{(a-1)^2+b^2}}$. Then we have the proportionality $\overrightarrow{u}_+ = \epsilon \overrightarrow{v}_+$, $\overrightarrow{u}_- = \varepsilon \overrightarrow{v}_-$, which implies, with $a \neq \frac{1}{2}$, that $\beta = \frac{a^2-b^2-a}{b(2a-1)} + \zeta$; so: $\mathcal{H} \equiv \left(\frac{a^2-b^2-a}{b(2a-1)} + \zeta\right) u^2 + \zeta y^2 + xy + Ex + Fy + G = 0$. By imposing that (a,b) is the center $\mathcal{H}$ we have: $\zeta = -\frac{2a^3-2a^2-b^2}{2ab(2a-1)} - \frac{E}{2a}$, $F = -\frac{a^2+b^2}{a(2a-1)} + \frac{Eb}{a}$, which implies that $\xi = -\frac{1}{2}\frac{b+E}{a}$. Therefore:

$$\mathcal{H} \equiv -\frac{1}{2}\frac{b+E}{a}x^2 - \frac{2a^3-2a^2-b^2+Eb(2a-1)}{2ab(2a-1)}y^2 + xy + Ex + \left(-\frac{a^2+b^2}{a(2a-1)} + \frac{Eb}{a}\right)y + G = 0.$$

And by imposing that $\mathcal{H} = \mathcal{H}^o$ passes through the point $C_o = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and by computing we obtain

$$G_o = \frac{3a^3-3a^2+ab^2-2b^2-\sqrt{3}ab-2\sqrt{3}b^3+Eb\left(6a-2-4a^2+4\sqrt{3}ab-2\sqrt{3}b\right)}{4ab(2a-1)},$$

for the conic $\mathcal{H}^o$.

Also, by imposing that $\mathcal{H} = \mathcal{H}^i$ passes through the point $C_i = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and by computing we obtain

$$G_i = \frac{3a^3-3a^2+ab^2-2b^2+\sqrt{3}ab+2\sqrt{3}b^3+Eb\left(6a-2-4a^2-4\sqrt{3}ab+2\sqrt{3}b\right)}{4ab(2a-1)},$$

for the conic $\mathcal{H}^i$.

Now, considering that $a \neq \frac{1}{2}$, and also that $a^2 + b^2 - a - b\sqrt{3} \neq 0$ because $C \notin \mathbb{B}_i$ -$\mathcal{H}$ is not circumference-; by imposing that $\mathcal{H}^i = \mathcal{H}_i$ passes through the point $C_{\mu_{1++}}$ we have $E_i = b\frac{a^2+b^2+a-b\sqrt{3}}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}$. The above expression of E_i

implies that:

$$\begin{array}{l} G_i = \frac{\left(3a^3-6a^2-a^2\sqrt{3}b+3a+3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+b\sqrt{3}\right)}{12b\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}, \\ \mathcal{H}_i \equiv -b\frac{a^2+b^2-a-\sqrt{3}b+1}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}x^2 - \frac{a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b-ab^2+\sqrt{3}ab+b^2}{b\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}y^2 \\ +xy + b\frac{a^2+b^2+a-\sqrt{3}b}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}x - \frac{a^3-a^2+ab^2-\sqrt{3}ab-2b^2}{\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)}y + G_i = 0. \end{array} \tag{13}$$

Similarly, considering that $a \neq \frac{1}{2}$, and also that $a^2+b^2-a+b\sqrt{3} \neq 0$ because $C \notin \mathbb{B}_0$ -$\mathcal{H}$ is not circumference- by imposing that $\mathcal{H}^o = \mathcal{H}_o$ passes through the point $C_{\mu_{1+-}}$ we have $E_o = b\frac{a^2+b^2+a+\sqrt{3}b}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}$. The above expression of E_o implies that:

$$\begin{array}{l} G_o = \frac{\left(3a^3-6a^2+a^2\sqrt{3}b+3a+3ab^2-6b^2-\sqrt{3}b+\sqrt{3}b^3\right)\left(3a-\sqrt{3}b\right)}{12b\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}, \\ \mathcal{H}_o \equiv -b\frac{a^2+b^2-a+\sqrt{3}b+1}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}x^2 - \frac{a^4-2a^3+a^2b^2+a^2+a^2\sqrt{3}b-ab^2-\sqrt{3}ab+b^2}{b\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}y^2 \\ +xy + b\frac{a^2+b^2+a+\sqrt{3}b}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}x - \frac{a^3-a^2+ab^2+\sqrt{3}ab-2b^2}{\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)}y + G_o = 0. \end{array} \tag{14}$$

□

And, as a result of the above lemmas, in short we have the following algebraic equations

Theorem 4. *The conics $\mathcal{H}_i$, $\mathcal{H}_o$ have the equations:*

$$\begin{array}{l} \mathcal{H}_i \equiv -b^2(a^2+b^2-a-\sqrt{3}b+1)x^2 - (a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b \\ -ab^2+\sqrt{3}ab+b^2)y^2 + b(a^2+b^2-a-b\sqrt{3})\,(2a-1)\,xy + b^2(a^2+b^2+a \\ -\sqrt{3}b)x - b(a^3-a^2+ab^2-\sqrt{3}ab-2b^2)y + \frac{1}{12}(3a^3-6a^2-a^2\sqrt{3}b+3a \\ +3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3)(3a+b\sqrt{3}) = 0; \end{array} \tag{15}$$

$$\begin{array}{l} \mathcal{H}_o \equiv -b^2(a^2+b^2-a+\sqrt{3}b+1)x^2 - (a^4-2a^3+a^2b^2+a^2+a^2\sqrt{3}b \\ -ab^2-\sqrt{3}ab+b^2)y^2 + b(a^2+b^2-a+\sqrt{3}b)\,(2a-1)\,xy + b^2(a^2+b^2+a \\ +\sqrt{3}b)x - b(a^3-a^2+ab^2+\sqrt{3}ab-2b^2)y + \frac{1}{12}(3a^3-6a^2+a^2\sqrt{3}b+3a \\ +3ab^2-6b^2-\sqrt{3}b+\sqrt{3}b^3)(3a-\sqrt{3}b) = 0. \end{array} \tag{16}$$

Now, the following Proposition 5 and the Corollary 6 are consequence of a special case of the well known elliptic motion; but we can get these results with algebraic arguments using the above Theorem 4:

Proposition 5. *The two conics $\mathcal{H}_i$, $\mathcal{H}_o$:*

1.- are ellipses if and only if $C \notin \mathbb{N}_i$ and $C \notin \mathbb{N}_o$, respectively.

2.- are a pair of coincident straight lines if and only if $C \in \mathbb{N}_i$ and $C \in \mathbb{N}_o$, respectively; and they are outer and the inner bisectors of the angle $\angle ACB$, respectively.

3.- are circumferences if and only if $C \in \mathbb{B}_i$ and $C \in \mathbb{B}_o$, respectively.

Proof. The affirmation 3 has previously been shown in the proof of the previous theorem. Then here we consider the affirmations 1 and 2 in the case $C \notin \mathbb{B}_i$ and $C \notin \mathbb{B}_o$.

The equations of $\mathbb{N}_i$ and $\mathbb{N}_o$ are: $\mathbb{N}_i \equiv x^2+y^2-x-\frac{1}{3}\sqrt{3}y=0$ and $\mathbb{N}_0 \equiv x^2+y^2-x+\frac{1}{3}\sqrt{3}y=0$.

If $a=\frac{1}{2}$ then the affirmations 1 and 2 have previously been shown in the proof of the previous theorem.

Let us consider then $a \neq \frac{1}{2}$.

The algebraic parameters of $\mathcal{H}_o$ are:

$$\det A=\frac{\left(3a^2+3b^2-3a+\sqrt{3}b\right)^4}{144b\left(a^2+b^2-a+\sqrt{3}b\right)^3(2a-1)^3},\ \det A_{00}=\frac{\left(3a^2+3b^2-3a+\sqrt{3}b\right)^2}{12\left(a^2+b^2-a+\sqrt{3}b\right)^2(2a-1)^2},$$
$$T=-\frac{2a^2b^2-2ab^2+2b^2+b^4+\sqrt{3}b^3+a^4-2a^3+a^2+a^2\sqrt{3}b-\sqrt{3}ab}{b\left(a^2+b^2-a+\sqrt{3}b\right)(2a-1)};$$

where $b\neq 0$, $2a-1\neq 0$ and $a^2+b^2-a+\sqrt{3}b\neq 0$ because $C\notin\mathbb{B}_o$. Moreover if $C\notin\mathbb{N}_0$ then $\det A\neq 0$, $\det A_{00}>0$, and $T\det A=-\frac{1}{144}\frac{\varphi(a,b)\left(3a^2+3b^2-3a+\sqrt{3}b\right)^4}{b^2\left(a^2+b^2-a+\sqrt{3}b\right)^4(2a-1)^4}$ with $\varphi(a,b)=2a^2b^2-2ab^2+2b^2+b^4+\sqrt{3}b^3+a^4-2a^3+a^2+a^2\sqrt{3}b-\sqrt{3}ab$. We have that $\varphi(1,1)>0$ and the equation $\varphi(a,b)=0$ has the four roots: $a=\frac{1}{2}+\frac{1}{2}\sqrt{1+2ib\sqrt{5}-4b^2-2\sqrt{3}b}$, $a=\frac{1}{2}-\frac{1}{2}\sqrt{1+2ib\sqrt{5}-4b^2-2\sqrt{3}b}$, $a=\frac{1}{2}+\frac{1}{2}\sqrt{1-2ib\sqrt{5}-4b^2-2\sqrt{3}b}$, $a=\frac{1}{2}-\frac{1}{2}\sqrt{1-2ib\sqrt{5}-4b^2-2\sqrt{3}b}$, which are real roots only if $b=0$. Therefore $T\det A<0$ and then $\mathcal{H}_o$ is a real non-degenerate ellipse. But, if $C\in\mathbb{N}_0$ then $\det A=0$, $\det A_{00}=0$, and

$$U=\frac{-\sqrt{3}\left(3b^3+2\sqrt{3}b^2+a^2b^2\sqrt{3}-2\sqrt{3}ab^2+3a^2b-3ab+a^2\sqrt{3}+a^4\sqrt{3}-2a^3\sqrt{3}\right)\left(3a^2+3b^2-3a+\sqrt{3}b\right)^2}{36b^2\left(a^2+b^2-a+\sqrt{3}b\right)^2(2a-1)^2}$$

with $U=0$. Therefore if $C\in\mathbb{N}_0$ then $\mathcal{H}_o$ are two coincident straight lines; and, by construction of $\mathcal{H}_o$, they are the inner bisectrix of the angle $\angle ACB$.

The algebraic parameters of $\mathcal{H}_i$ are:

$$\det A=\frac{\left(3a^2+3b^2-3a-\sqrt{3}b\right)^4}{144b\left(a^2+b^2-a-\sqrt{3}b\right)^3(2a-1)^3},\ \det A_{00}=\frac{\left(3a^2+3b^2-3a-\sqrt{3}b\right)^2}{12\left(a^2+b^2-a-\sqrt{3}b\right)^2(2a-1)^2},$$
$$T=-\frac{2a^2b^2-2ab^2+2b^2+b^4-\sqrt{3}b^3+a^4-2a^3+a^2-a^2\sqrt{3}b+\sqrt{3}ab}{b\left(a^2+b^2-a-\sqrt{3}b\right)(2a-1)}.$$

where $b\neq 0$, $2a-1\neq 0$ and $a^2+b^2-a-\sqrt{3}b\neq 0$ because $C\notin\mathbb{B}_i$. Moreover if $C\notin\mathbb{N}_i$ then $\det A\neq 0$, $\det A_{00}>0$, and $T\det A=-\frac{1}{144}\frac{\psi(a,b)\left(3a^2+3b^2-3a-\sqrt{3}b\right)^4}{b^2\left(a^2+b^2-a-\sqrt{3}b\right)^4(2a-1)^4}$ with $\psi(a,b)=2a^2b^2-2ab^2+2b^2+b^4-\sqrt{3}b^3+a^4-2a^3+a^2-a^2\sqrt{3}b+\sqrt{3}ab$. We have that $\psi(1,1)>0$ and the equation $\psi(a,b)=0$ has the four roots: $a=\frac{1}{2}+\frac{1}{2}\sqrt{1+2ib\sqrt{5}-4b^2+2\sqrt{3}b}$, $a=\frac{1}{2}-\frac{1}{2}\sqrt{1+2ib\sqrt{5}-4b^2+2\sqrt{3}b}$, $a=\frac{1}{2}+\frac{1}{2}\sqrt{1-2ib\sqrt{5}-4b^2+2\sqrt{3}b}$, $a=\frac{1}{2}-\frac{1}{2}\sqrt{1-2ib\sqrt{5}-4b^2+2\sqrt{3}b}$, which are real roots only if $b=0$. Therefore $T\det A<0$ and then $\mathcal{H}_i$ is a real non-degenerate ellipse. But, if $C\in\mathbb{N}_i$ then $\det A=0$, $\det A_{00}=0$, and

$$U=\frac{\sqrt{3}\left(3b^3-2\sqrt{3}b^2-a^2b^2\sqrt{3}+2\sqrt{3}ab^2+3a^2b-3ab-a^2\sqrt{3}-a^4\sqrt{3}+2a^3\sqrt{3}\right)\left(3a^2+3b^2-3a-\sqrt{3}b\right)^2}{36b^2\left(a^2+b^2-a-\sqrt{3}b\right)^2(2a-1)^2}$$

with $U = 0$. Therefore if $C \in \mathbb{N}_i$ then $\mathcal{H}_i$ are two coincident straight lines; and, by construction of $\mathcal{H}_i$, they are the outer bisectrix of the angle $\angle ACB$. $\square$

With the Lemma 2 and the Theorem 4, we can prove algebraically the following:

Corollary 6. *Let* $\{T_{\beta,k} = \triangle P_{\beta,k} A_{\beta,k} \beta\}_{k=1}^{k=4}$*, then:*
1.- $\mathcal{H}_i = \mathcal{H}_1 \cup \mathcal{H}_3$ *is the geometrical locus of the vertices* $P_{\beta,1}$ *and* $P_{\beta,3}$*.*
2.- $\mathcal{H}_o = \mathcal{H}_2 \cup \mathcal{H}_4$ *is the geometrical locus of the vertices* $P_{\beta,2}$ *and* $P_{\beta,4}$*.*

Proof. With a very lengthy and straightforward calculation we can check the result in all its parts and implications.

Let us see a case, let us see, for example, the explicit calculations that show that $C_{\mu_{\lambda++}} = P_{\beta,1} \in \mathcal{H}_i$. Using the Equations (1), (15) we must to prove that

$$\begin{aligned}
&-4\Phi^4 b \tfrac{a^2+b^2-a-\sqrt{3}b+1}{1} \left(C_{\mu_{\lambda++x}}\right)^2 \\
&-4\Phi^4 \tfrac{\left(a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b-ab^2+\sqrt{3}ab+b^2\right)}{b} \left(C_{\mu_{\lambda++y}}\right)^2 \\
&+4\Phi^4 \left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)\, C_{\mu_{\lambda++x}} C_{\mu_{\lambda++y}} \\
&+4\Phi^4 b \tfrac{a^2+b^2+a-\sqrt{3}b}{1} C_{\mu_{\lambda++x}} - 4\Phi^4 \tfrac{a^3-a^2+ab^2-\sqrt{3}ab-2b^2}{1} C_{\mu_{\lambda++y}} \\
&+4\Phi^4 \tfrac{\left(3a^3-6a^2-a^2\sqrt{3}b+3a+3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+b\sqrt{3}\right)}{12b} = 0.
\end{aligned}$$

Then: with a very lengthy and straightforward calculation we have

$$\begin{aligned}
\Psi_1 = &-4\Phi^4 b \tfrac{a^2+b^2-a-\sqrt{3}b+1}{1} \left(C_{\mu_{\lambda++x}}\right)^2 \\
&-4\Phi^4 \tfrac{\left(a^4-2a^3+a^2b^2+a^2-a^2\sqrt{3}b-ab^2+\sqrt{3}ab+b^2\right)}{b} \left(C_{\mu_{\lambda++y}}\right)^2,
\end{aligned}$$

where

$$\begin{aligned}
\Psi_1 = &-4b - 3\sqrt{3}a - 4\lambda b^3 + \sqrt{3}b^2 - 80ba^4 + 100ba^3 + 4ab^3 - 70ba^2 + 26ba \\
&-11a^2b^3 + 4b^4\sqrt{3} + 6\Psi_\lambda b^3 - 6ba^6 + 34ba^5 + 24b^5\lambda - 18b^5\lambda^2 + 22a^3b^3 \\
&-12a^4b^3 - 10a^2b^5 + 22b^5a + 3b^6\sqrt{3} - 10b^5\Psi_\lambda - 4b^7\lambda - 2b^7\lambda^2 - 56b^5a\lambda \\
&-158\lambda^2a^4b^3 - 78a^2\lambda^2b^5 - 18\Psi_\lambda ab^3 + 8a^2\lambda b^3 - 94a^2\lambda^2b^3 + 20a\lambda^2b^3 \\
&-4\sqrt{3}a^3b^2 - 10\Psi_\lambda\lambda b^3 - 8\sqrt{3}b^4\lambda + 6\sqrt{3}b^4\lambda^2 + 2\Psi_\lambda\sqrt{3}b^4 - 3a^4\sqrt{3}b^2 \\
&+3a^2b^4\sqrt{3} + 12a^2\Psi_\lambda b^3 + 88a^5\lambda^2b^3 + 56a^3\lambda^2b^5 + 4a^5b^3\lambda - 4a^3b^5\lambda \\
&+10a^2\sqrt{3}b^2 - 6\sqrt{3}ab^2 + 12a\lambda b^3 + 168\lambda^2b^3a^3 + 64\lambda^2b^5a - 9b^4\sqrt{3}a \\
&-32b^3\lambda a^3 - 4a\sqrt{3}b^2\lambda - 8\sqrt{3}a^4b^2\lambda - 16\sqrt{3}a^3b^2\lambda - 48\sqrt{3}a^2b^4\lambda \\
&+16\sqrt{3}a^2b^2\lambda + 86a^4\sqrt{3}b^2\lambda^2 - 58a^3\sqrt{3}b^2\lambda^2 + 76a^2\sqrt{3}b^4\lambda^2 + 18a^2\sqrt{3}b^2\lambda^2 \\
&-2\sqrt{3}b^2\lambda^2a - 10\Psi_\lambda a^2\sqrt{3}b^2 - 52\Psi_\lambda a^2\lambda b^3 + 4\Psi_\lambda a\sqrt{3}b^2 + 32\sqrt{3}ab^4\lambda
\end{aligned}$$

$$
\begin{aligned}
&-34\sqrt{3}ab^4\lambda^2+4\Psi_\lambda\sqrt{3}b^2\lambda+2\Psi_\lambda\sqrt{3}b^4\lambda-8a^6\sqrt{3}b^2\lambda+20a^5\sqrt{3}b^2\lambda\\
&-16a^4\sqrt{3}b^4\lambda+16a^6\sqrt{3}b^2\lambda^2-60a^5\sqrt{3}b^2\lambda^2+32a^4\sqrt{3}b^4\lambda^2-2a^4\Psi_\lambda\sqrt{3}b^2\\
&-16a^4\Psi_\lambda\lambda b^3+8a^3\Psi_\lambda\sqrt{3}b^2+40a^3\sqrt{3}b^4\lambda-80a^3\sqrt{3}b^4\lambda^2-8b^6\sqrt{3}a^2\lambda\\
&+16b^6a^2\sqrt{3}\lambda^2+40b^3\Psi_\lambda a^3\lambda-8b^5\Psi_\lambda a^2\lambda+20b^6\sqrt{3}a\lambda-20b^6\sqrt{3}a\lambda^2\\
&-2b^6\Psi_\lambda\sqrt{3}\lambda-8b^5\Psi_\lambda a\lambda-106b\Psi_\lambda a^4\lambda+110b\Psi_\lambda a^3\lambda-54b\Psi_\lambda a^2\lambda\\
&+10b\Psi_\lambda\lambda a-8ba^6\Psi_\lambda\lambda+48ba^5\Psi_\lambda\lambda-3b^3-16b^5-b^7+54\Psi_\lambda a^2\sqrt{3}b^2\lambda\\
&-24\Psi_\lambda a\sqrt{3}b^2\lambda-24a^6\lambda^2b^3-24a^4\lambda^2b^5+12a^4\lambda b^3+2a^5\sqrt{3}b^2+32a^2b^5\lambda\\
&+8b^7\lambda^2a-2b^6\sqrt{3}a-4b^7a\lambda-8b^7a^2\lambda^2+4b^5\Psi_\lambda a+6b^5\Psi_\lambda\lambda-8b^6\sqrt{3}\lambda\\
&+6b^6\sqrt{3}\lambda^2+2b^6\Psi_\lambda\sqrt{3}-82b\lambda^2a^6+88b\lambda^2a^5-52b\lambda^2a^4+16b\lambda^2a^3\\
&-2b\lambda^2a^2-8ba^8\lambda^2+40ba^7\lambda^2-56\Psi_\lambda\sqrt{3}b^2\lambda a^3-8\Psi_\lambda\sqrt{3}b^4a\lambda\\
&+6a^2\Psi_\lambda\sqrt{3}b^4\lambda+26a^4\Psi_\lambda\sqrt{3}b^2\lambda-4a^5\Psi_\lambda\sqrt{3}b^2\lambda+4a^3\Psi_\lambda\sqrt{3}b^4\lambda\\
&+4b^6\Psi_\lambda\sqrt{3}a\lambda-8b\lambda a^2-\tfrac{45}{b}a^4+\tfrac{18}{b}a^3-\tfrac{3}{b}a^2-\tfrac{45}{b}a^6+\tfrac{60}{b}a^5-\tfrac{3}{b}a^8\\
&+\tfrac{18}{b}a^7+2\Psi_\lambda a^2\sqrt{3}\lambda-12\Psi_\lambda\sqrt{3}\lambda a^3+28a^4\Psi_\lambda\sqrt{3}\lambda-32a^5\Psi_\lambda\sqrt{3}\lambda\\
&-8a^5\Psi_\lambda\sqrt{3}+2\Psi_\lambda a^2\sqrt{3}+12a^4\Psi_\lambda\sqrt{3}-8a^3\Psi_\lambda\sqrt{3}+38b^3\Psi_\lambda a\lambda+2a^6\Psi_\lambda\sqrt{3}\\
&+17a^2\sqrt{3}+36b\lambda a^3+6b\Psi_\lambda a-40\sqrt{3}a^3+50a^4\sqrt{3}-14ba^2\Psi_\lambda+56ba^5\lambda\\
&+6ba^3\Psi_\lambda-64ba^4\lambda-35a^5\sqrt{3}+13a^6\sqrt{3}+6ba^4\Psi_\lambda+4ba^7\lambda-4ba^5\Psi_\lambda\\
&-24ba^6\lambda-2a^7\sqrt{3}-2b^4a^2\Psi_\lambda\sqrt{3}+18a^6\Psi_\lambda\sqrt{3}\lambda-4a^7\Psi_\lambda\sqrt{3}\lambda+2b^4a^3\sqrt{3},
\end{aligned}
$$

and we have

$$\Psi_2=4\Phi^4\left(a^2+b^2-a-b\sqrt{3}\right)(2a-1)\,C_{\mu_{\lambda++x}}C_{\mu_{\lambda++y}},$$

where

$$
\begin{aligned}
\Psi_2=\;&3b+\sqrt{3}a+6\lambda b^3+\sqrt{3}b^2-5ba^4-20ba^3+32ab^3+30ba^2-16ba\\
&-40a^2b^3-14\Psi_\lambda b^3-4ba^6+12ba^5-28b^5\lambda+18b^5\lambda^2+16a^3b^3+4a^2b^5\\
&-12b^5a-b^6\sqrt{3}+2b^5\Psi_\lambda-2b^7\lambda+2b^7\lambda^2+80b^5a\lambda+158\lambda^2a^4b^3\\
&+78a^2\lambda^2b^5+42\Psi_\lambda ab^3-48a^2\lambda b^3+94a^2\lambda^2b^3-20a\lambda^2b^3-16\sqrt{3}a^3b^2\\
&+10\Psi_\lambda\lambda b^3+8\sqrt{3}b^4\lambda-6\sqrt{3}b^4\lambda^2+2\Psi_\lambda\sqrt{3}b^4+9a^4\sqrt{3}b^2+3a^2b^4\sqrt{3}\\
&-28a^2\Psi_\lambda b^3-88a^5\lambda^2b^3-56a^3\lambda^2b^5-4a^5b^3\lambda+4a^3b^5\lambda+14a^2\sqrt{3}b^2\\
&-6\sqrt{3}ab^2-4a\lambda b^3+2\Psi_\lambda\sqrt{3}b^2-168\lambda^2b^3a^3-64\lambda^2b^5a-b^4\sqrt{3}a\\
&+80b^3\lambda a^3+4a\sqrt{3}b^2\lambda+8\sqrt{3}a^4b^2\lambda+16\sqrt{3}a^3b^2\lambda+48\sqrt{3}a^2b^4\lambda\\
&-16\sqrt{3}a^2b^2\lambda-86a^4\sqrt{3}b^2\lambda^2+58a^3\sqrt{3}b^2\lambda^2-76a^2\sqrt{3}b^4\lambda^2\\
&-18a^2\sqrt{3}b^2\lambda^2+2\sqrt{3}b^2\lambda^2a+26\Psi_\lambda a^2\sqrt{3}b^2+52\Psi_\lambda a^2\lambda b^3-12\Psi_\lambda a\sqrt{3}b^2\\
&-32\sqrt{3}ab^4\lambda+34\sqrt{3}ab^4\lambda^2-4\Psi_\lambda\sqrt{3}b^2\lambda-2\Psi_\lambda\sqrt{3}b^4\lambda+8a^6\sqrt{3}b^2\lambda\\
&-20a^5\sqrt{3}b^2\lambda+16a^4\sqrt{3}b^4\lambda-16a^6\sqrt{3}b^2\lambda^2+60a^5\sqrt{3}b^2\lambda^2-32a^4\sqrt{3}b^4\lambda^2\\
&+8a^4\Psi_\lambda\sqrt{3}b^2+16a^4\Psi_\lambda\lambda b^3-24a^3\Psi_\lambda\sqrt{3}b^2-40a^3\sqrt{3}b^4\lambda+80a^3\sqrt{3}b^4\lambda^2\\
&+8b^6\sqrt{3}a^2\lambda-16b^6a^2\sqrt{3}\lambda^2-40b^3\Psi_\lambda a^3\lambda+8b^5\Psi_\lambda a^2\lambda-8b^4\Psi_\lambda a\sqrt{3}\\
&-20b^6\sqrt{3}a\lambda+20b^6\sqrt{3}a\lambda^2+2b^6\Psi_\lambda\sqrt{3}\lambda+8b^5\Psi_\lambda a\lambda+106b\Psi_\lambda a^4\lambda\\
&-110b\Psi_\lambda a^3\lambda+54b\Psi_\lambda a^2\lambda-10b\Psi_\lambda\lambda a+8ba^6\Psi_\lambda\lambda-48ba^5\Psi_\lambda\lambda-8b^3+5b^5\\
&-54\Psi_\lambda a^2\sqrt{3}b^2\lambda+24\Psi_\lambda a\sqrt{3}b^2\lambda+24a^6\lambda^2b^3+24a^4\lambda^2b^5-30a^4\lambda b^3\\
&-2a^5\sqrt{3}b^2-50a^2b^5\lambda-8b^7\lambda^2a+2b^6\sqrt{3}a+4b^7a\lambda+8b^7a^2\lambda^2-4b^5\Psi_\lambda a\\
&-6b^5\Psi_\lambda\lambda+8b^6\sqrt{3}\lambda-6b^6\sqrt{3}\lambda^2+82b\lambda^2a^6-88b\lambda^2a^5+52b\lambda^2a^4-16b\lambda^2a^3
\end{aligned}
$$

$+2b\lambda^2a^2+8ba^8\lambda^2-40ba^7\lambda^2+56\Psi_\lambda\sqrt{3}b^2\lambda a^3+8\Psi_\lambda\sqrt{3}b^4a\lambda-6a^2\Psi_\lambda\sqrt{3}b^4\lambda$
$-26a^4\Psi_\lambda\sqrt{3}b^2\lambda+4a^5\Psi_\lambda\sqrt{3}b^2\lambda-4a^3\Psi_\lambda\sqrt{3}b^4\lambda-4b^6\Psi_\lambda\sqrt{3}a\lambda+2b\lambda a^2$
$-2\Psi_\lambda a^2\sqrt{3}\lambda+12\Psi_\lambda\sqrt{3}\lambda a^3-28a^4\Psi_\lambda\sqrt{3}\lambda+32a^5\Psi_\lambda\sqrt{3}\lambda-38b^3\Psi_\lambda a\lambda$
$-7a^2\sqrt{3}-12b\lambda a^3+2b\Psi_\lambda a+20\sqrt{3}a^3-30a^4\sqrt{3}-10ba^2\Psi_\lambda-32ba^5\lambda$
$+18ba^3\Psi_\lambda+28ba^4\lambda+25a^5\sqrt{3}-11a^6\sqrt{3}-14ba^4\Psi_\lambda-4ba^7\lambda+4ba^5\Psi_\lambda$
$+18ba^6\lambda+2a^7\sqrt{3}+8b^4a^2\Psi_\lambda\sqrt{3}-18a^6\Psi_\lambda\sqrt{3}\lambda+4a^7\Psi_\lambda\sqrt{3}\lambda-2b^4a^3\sqrt{3},$

and we have

$$\Psi_3=4\Phi^4b\frac{a^2+b^2+a-\sqrt{3}b}{1}C_{\mu_{\lambda++x}}-4\Phi^4\frac{a^3-a^2+ab^2-\sqrt{3}ab-2b^2}{1}C_{\mu_{\lambda++y}},$$

where

$$\begin{aligned}\Psi_3=&-2\lambda b^3+12ba^4-8ba^3-24ab^3+2ba^2+24a^2b^3+8\Psi_\lambda b^3\\&+2ba^6-8ba^5+4b^5\lambda-16a^3b^3+6a^4b^3+6a^2b^5-8b^5a+8b^5\Psi_\lambda\\&+6b^7\lambda-24b^5a\lambda-24\Psi_\lambda ab^3+40a^2\lambda b^3-4\Psi_\lambda\sqrt{3}b^4+16a^2\Psi_\lambda b^3\\&-8a\lambda b^3-2\Psi_\lambda\sqrt{3}b^2-48b^3\lambda a^3-16\Psi_\lambda a^2\sqrt{3}b^2+8\Psi_\lambda a\sqrt{3}b^2\\&-6a^4\Psi_\lambda\sqrt{3}b^2+16a^3\Psi_\lambda\sqrt{3}b^2+8b^4\Psi_\lambda a\sqrt{3}+10b^3+12b^5+2b^7\\&+18a^4\lambda b^3+18a^2b^5\lambda-2b^6\Psi_\lambda\sqrt{3}+6b\lambda a^2+8a^5\Psi_\lambda\sqrt{3}-2\Psi_\lambda a^2\sqrt{3}\\&-12a^4\Psi_\lambda\sqrt{3}+8a^3\Psi_\lambda\sqrt{3}-2a^6\Psi_\lambda\sqrt{3}-24b\lambda a^3-8b\Psi_\lambda a+24ba^2\Psi_\lambda\\&-24ba^5\lambda-24ba^3\Psi_\lambda+36ba^4\lambda+8ba^4\Psi_\lambda+6ba^6\lambda-6b^4a^2\Psi_\lambda\sqrt{3}.\end{aligned}$$

And with all these, simplifying, we arrive to

$$\Psi_1+\Psi_2+\Psi_3=-\frac{1}{3}\frac{\left(3a^3-6a^2-a^2\sqrt{3}b+3ab^2+3a-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+\sqrt{3}b\right)\left(a^2-2a+1+b^2\right)^2}{b},$$

and then $\Psi_1+\Psi_2+\Psi_3+4\Phi^4\frac{\left(3a^3-6a^2-a^2\sqrt{3}b+3a+3ab^2-6b^2+\sqrt{3}b-\sqrt{3}b^3\right)\left(3a+b\sqrt{3}\right)}{12b}=0$, finishing the calculation. □

Now, in the following, with the Theorem 4, we present the determination and the construction with ruler and compass, with Equations (15), (16), of $\mathbb{T}_j=\triangle A_jB_jC_j$ the c-inscribed equilateral triangles of $\triangle ABC$ (Figures 2, 3). Of course the following proposition 7 also is consequence of a special case of the well known elliptic motion; but, with our approach, we give the algebraic formulae (17), (18):

Proposition 7. *The conic $\mathcal{H}_o$:*

1.- with $C\in\mathbb{N}_o$, is a pair of coincident straight lines which intersect in one point with $\overleftrightarrow{AB}$. (Figure 3c)

2.- with $C\notin\mathbb{N}_o$, is an ellipse which intersect in two points with $\overleftrightarrow{AB}$. (Figures 2, 3a, 3b, 3d)

The algebraic formula of the above intersections is (17).

The conic $\mathcal{H}_i$:

3.- with $C=C_i$, is a pair of coincident straight lines parallel to $\overleftrightarrow{AB}$, and if $C\in\mathbb{N}_i\setminus C_i$ then is a pair of coincident straight lines which intersect in one point with $\overleftrightarrow{AB}$. (Figure 3d)

4.- with $C\in\mathbb{P}_o\setminus\mathbb{N}_i$, $C\in\mathcal{P}_i\backslash\mathbb{N}_i$, $C\in\mathbb{P}_i\setminus\mathbb{N}_i$, is an ellipse which: intersect in two points (Figures 2, 3c), is tangent (Figure 3b), not intersect (Figure 3a), respectively with $\overleftrightarrow{AB}$.

The algebraic formula of the above intersections is (18).

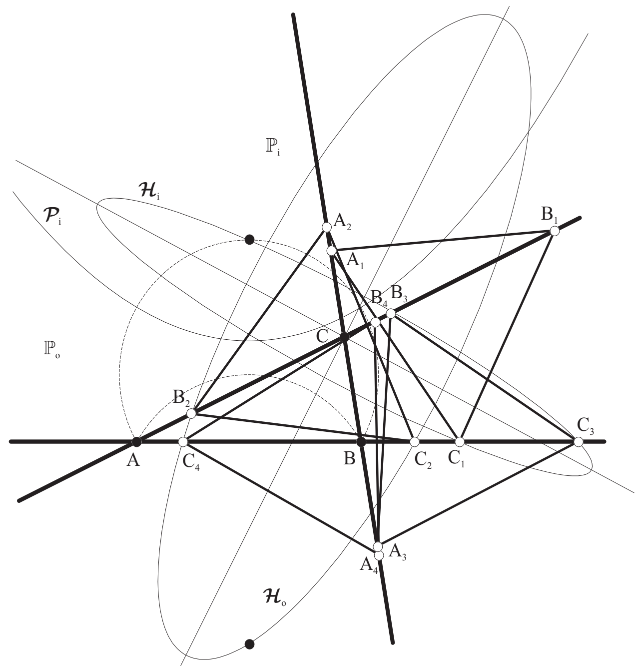

Figure 2. $\mathcal{H}_o$, $\mathcal{H}_i$ and $\mathbb{T}_j = \triangle A_jB_jC_j$ the c-inscribed equilateral triangles of $\triangle ABC$.

Proof. The equation of $\mathcal{P}_i$ is $x^2 - x + 1 = \sqrt{3}y$, the equation of $\mathbb{P}_i$ is $x^2 - x + 1 < \sqrt{3}y$, and the equation of $\mathbb{P}_o$ is $x^2 - x + 1 > \sqrt{3}y$ with $y > 0$.

With Equation (16), we have that:

$$\begin{aligned} \mathcal{H}_o \cap \overleftrightarrow{AB} &= \left(\frac{1}{6}\frac{3b^3+3\sqrt{3}b^2+3ba^2+3ba\pm\sqrt{3}\sqrt{\Delta_1}}{\left(a^2+b^2-a+\sqrt{3}b+1\right)b}, 0\right), \\ \Delta_1 &= \left(\sqrt{3}b + a^2 - a + 1\right)\left(3b^2 + \sqrt{3}b - 3a + 3a^2\right)^2, \end{aligned} \tag{17}$$

and moreover $3b^2+3a^2-3a+\sqrt{3}b = 0 \Leftrightarrow C \in \mathbb{N}_o \Rightarrow \mathcal{H}_o \cap \overleftrightarrow{AB} = (\frac{1}{2}\frac{b^2+\sqrt{3}b+a^2+a}{a^2+b^2-a+\sqrt{3}b+1}, 0)$. Note that $a^2 + b^2 - a + \sqrt{3}b + 1 > 0$ and $\sqrt{3}b + a^2 - a + 1 > 0$.

With Equation (15), we have that:

$$\begin{aligned} \mathcal{H}_i \cap \overleftrightarrow{AB} &= \left(\frac{1}{6}\frac{3b^3-3\sqrt{3}b^2+3ba^2+3ba\pm\sqrt{3}\sqrt{\Delta_2}}{\left(a^2+b^2-a-\sqrt{3}b+1\right)b}, 0\right), \\ \Delta_2 &= \left(-\sqrt{3}b + a^2 - a + 1\right)\left(3b^2 - \sqrt{3}b - 3a + 3a^2\right)^2, \end{aligned} \tag{18}$$

and $a^2+b^2-a-\sqrt{3}b+1 = 0 \Leftrightarrow C = C_i = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$; and $3b^2-\sqrt{3}b-3a+3a^2 = 0 \Leftrightarrow C \in \mathbb{N}_i$. And moreover $-\sqrt{3}b+a^2-a+1 = 0 \Leftrightarrow C \in \mathcal{P}_i$; and $-\sqrt{3}b+a^2-a+1 > 0 \Leftrightarrow C \in \mathbb{P}_o$. Also $C \in \mathbb{N}_i \cup \mathcal{P}_i \Rightarrow \mathcal{H}_i \cap \overleftrightarrow{AB} = \left(\frac{1}{2}\frac{b^2-\sqrt{3}b+a^2+a}{a^2+b^2-a-\sqrt{3}b+1}, 0\right)$.

So these calculations together Proposition 5 prove the result. □

Accordingly, through the above we can arrive to the following:

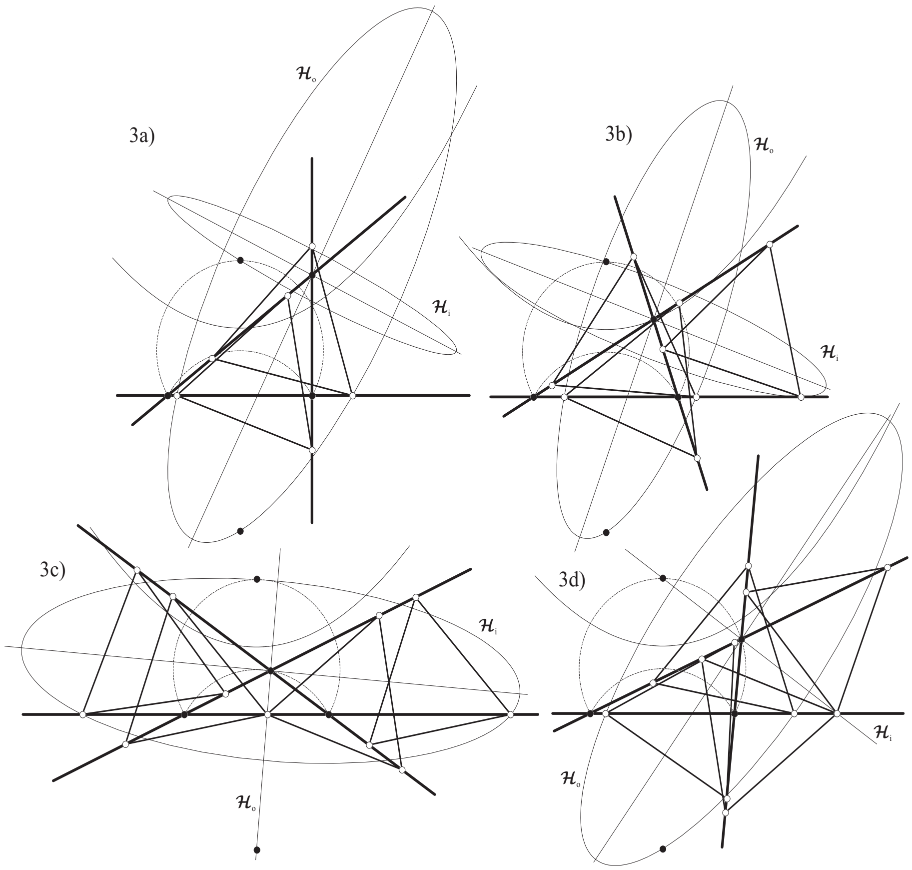

Figure 3. Several cases for $\mathcal{H}_o$, $\mathcal{H}_i$ and $\mathbb{T}_j = \triangle A_j B_j C_j$.

Corollary 8. *On every triangle $\triangle ABC$ exists its c-inscribed equilateral triangles $\{\mathbb{T}_j\}_{j=1}^{j=m}$ with $m = 4$, $m = 3$, $m = 2$, if $C \in \mathbb{P}_o$, $C \in \mathcal{P}_i$, $C \in \mathbb{P}_i$, respectively. (Figures 2, 3)*

Proof. Let the triangle $\mathbb{T}_j = \triangle A_j B_j C_j$, then by construction and with Lemma 1, necessarily $C_j \in \{P_{\beta,k}\}_{k=1}^{k=4}$, and with Corollary 6 we have that $C_j = P_{\beta,j} \in \left(\mathcal{H}_i \cap \overleftrightarrow{AB}\right) \cup \left(\mathcal{H}_o \cap \overleftrightarrow{AB}\right)$.

If $\mathbb{T}_j$ exists then by Lemma 1 $C_1 = P_{\beta,1}$, $C_3 = P_{\beta,3}$, $C_2 = P_{\beta,2}$ and $C_4 = P_{\beta,4}$.

If $C \in \mathbb{P}_i \setminus \mathbb{N}_i$ or $C \in C_i$, then, by Proposition 7, $\mathbb{T}_1 = \triangle A_1 B_1 C_1$and $\mathbb{T}_3 = \triangle A_3 B_3 C_3$ do not exist.

If $C \in \mathbb{P}_i \cup \mathbb{N}_i$ then the triangles $\mathbb{T}_1 = \triangle A_1 B_1 C_1$and $\mathbb{T}_3 = \triangle A_3 B_3 C_3$ do not exist. This claim is true because in this case $C_1 = C_3 = P_{\beta,1} = C_{\mu_{\lambda++}} = P_{\beta,3} = C_{\mu_{\lambda-+}} = \mathcal{H}_i \cap \overleftrightarrow{AB}$, and $\mathcal{H}_i$ is the outer bisectrix of the angle $\angle ACB$ (Proposition 5). And, by continuity, exists two straight lines r', r'' at the both sides of the $\mathcal{H}_i$ which are parallel to $\mathcal{H}_i$ and also they are the outer bisectors of two triangles $\triangle ABC'$ and $\triangle ABC''$, respectively, which have the same $\mathbb{N}_i$, but which have $C' \in \mathbb{P}_i \smallsetminus \mathbb{N}_i$, $C'' \in \mathbb{P}_i \smallsetminus \mathbb{N}_i$. Therefore, by Proposition 7, these bisectors r', r'' intersect to $\overleftrightarrow{AB}$ in two points at distance greater than 1 from the straight line $\overleftrightarrow{AC'}$ and from the straight line $\overleftrightarrow{AC''}$. So, this implies that $\mathcal{H}_i \cap \overleftrightarrow{AB}$ also is a point at distance greater than 1 from the straight line $\overleftrightarrow{AC}$.

If $C \in \mathcal{P}_i \smallsetminus \mathbb{N}_i$ then $\mathcal{H}_i$ is ellipse tangent to $\overleftrightarrow{AB}$, then only one of the two triangles $\mathbb{T}_1$, $\mathbb{T}_3$ exist; then by continuity the same is true the case that $C \in \mathcal{P}_i \cap \mathbb{N}_i$.

If $C \in \mathbb{P}_o$, the two triangles $\mathbb{T}_1$, $\mathbb{T}_3$ exists and, by Lemma 1, they are not coincident.

The triangles $\mathbb{T}_2 = \triangle A_2B_2C_2$, $\mathbb{T}_4 = \triangle A_4B_4C_4$, by Lemma 1, they are not coincident, and by Corollary 1 and Proposition 7, they always exist. □

Remark 9. *If* $\mathbb{T}_j$ *exists then it is constructible with ruler and compass because the points* $\left(\mathcal{H}_i \cap \overleftrightarrow{AB}\right) \cup \left(\mathcal{H}_o \cap \overleftrightarrow{AB}\right)$ *are contructible with ruler and compass; this claim is true because Formulae* (17), (18) *of Proposition 7 are quadratic rationals of the numbers* a, b, *which have been already constructed, they are the coordinates of* C. *The other points* A_j, B_j *are trivially obtained as intersection of the sides* $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ *with the circumference of radius* c *and center point* C_j. *Then, with Formulae* (17), (18) *we can construct* $\mathbb{T}_j$ *with ruler and compass, nevertheless readers can found much more elegant constructions in [3].*

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Blas Herrera: Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda Països Catalans 26, 43007, Tarragona, Spain.

E-mail address: blas.herrera@urv.cat