Generating Infinite Links as Periodic Tilings of the da Vinci–Dürer Knots

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The Austrian art historian Moritz Thausing (1838–1884) wrote:

> There are six woodcuts of Albert Dürer's black discs, upon which a symmetrical and concentrically arranged arabesque scrollwork of ribbons or festoons stands out in relief. These wonderful ornamentations are commonly called Dürer's "Patterns for Embroidery"; in his Netherlands diary he himself calls them "Die sechs Knoten"—the Six Knots—. By adding four corner ornaments to the discs, Dürer gave these woodcuts an oblong shape. His monogram was not placed on them, in the centre, till the second impression. The very same patterns are to be found in some old Italian copper engravings, but on a white ground, and bearing in the centre the curious inscription "ACADEMIA LEONARDI VINCI."
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> [Thausing, 1882, p. 362]

Other authors directly claim that the Six Knots are by da Vinci, and that Dürer copied them [Richter, 1970, p. 297]. In the catalogue of the Herzog Anton Ulrich Museum, for instance, these works of art are registered as "da Vinci, Leonardo (Leonardo da Vinci) (Inventor); Dürer, Albrecht (publisher, draftsman)" [Virtuelles Kupferstichkabinett]. In this article we do not attempt to determine who the real author is, so the "Six Knots" will be referred as: da Vinci–Dürer Six Knots.

We remark that our only reasons for considering the da Vinci–Dürer knots in this article are their beauty and because they are known in the history of art; and our only reason for converting these knots to infinite links is, again, their beauty.

However, the da Vinci–Dürer Six Knots are not really knots, because they are not one-component links. In actual fact, they are multiknot links, so they should be called the da Vinci–Dürer Six Links. Figure 1 shows the Six Links in the version of the six woodcuts of Albert Dürer's black discs. Apart from the four corner ornaments, there is a link inside each of the six discs. There is a total of six finite links, each having a finite number of knots.

Next we describe two of these six links—more specifically, the fourth and sixth links. We choose these two links because from them we can generate two new infinite links—having an infinite number of knots—using the p6m periodic tiling technique. This is so because, as seen in Figure 1, in the centers of the fourth and sixth da Vinci–Dürer Six Knots there is a hexagonal lattice core that is invariant by reflection symmetry and by $\frac{\pi}{3}$ radian rotations. We will show how the knots of these two initial da Vinci–Dürer finite links transform into the knots of the two new infinite links. However, as can also be seen in Figure 1, the remaining four da Vinci–Dürer links do not have a lattice core belonging to any of the 17 plane symmetry groups. For these reasons, the four remaining links do not offer the possibility of generating new links using the periodic tiling technique. Those readers who are not familiar with the basic theory of periodic tilings may read, for instance, Grünbaum and Shephard [2016], which is accessible, for the most part, to high-schoolers. There they will learn about the

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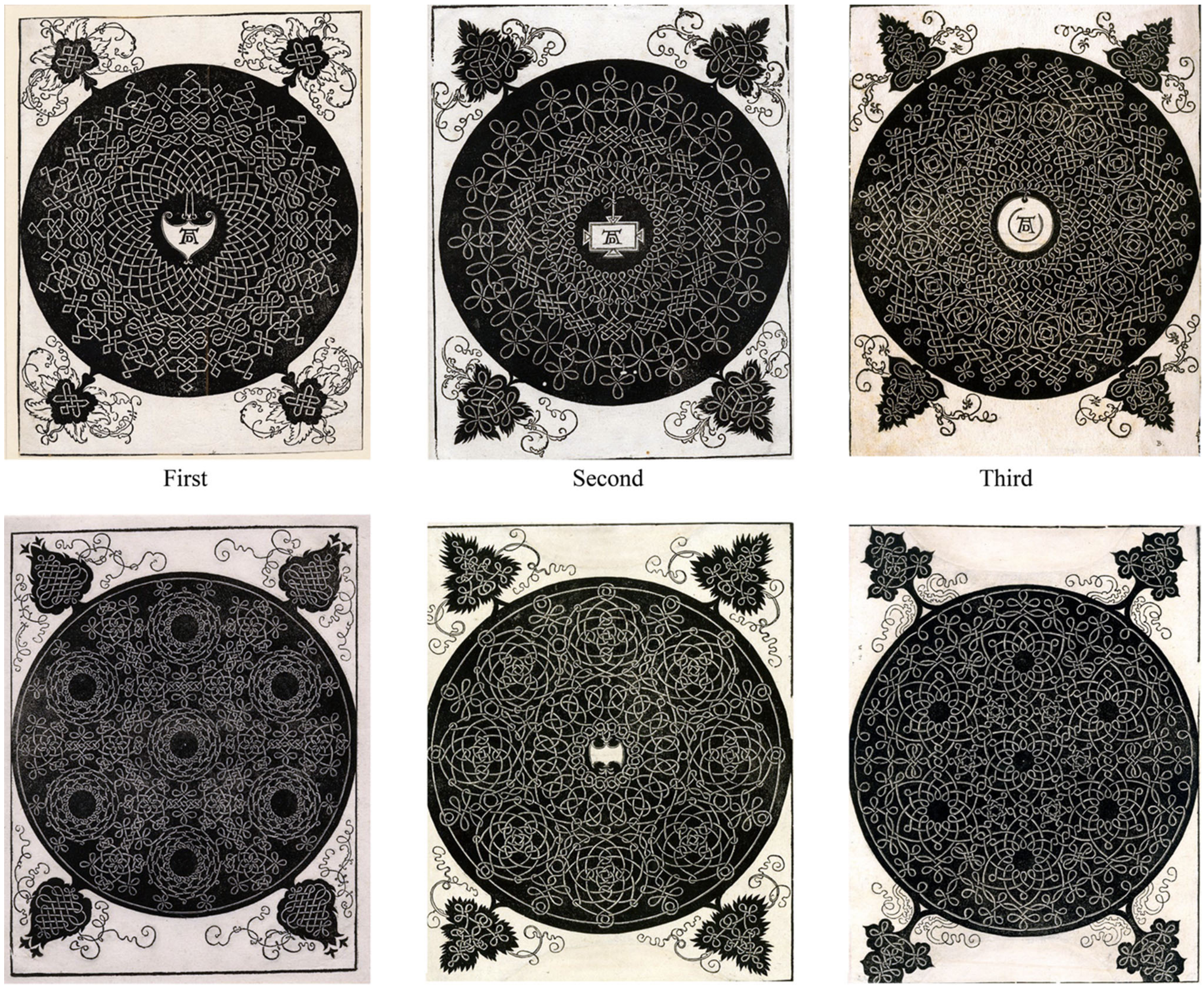

Figure 1. "Die sechs Knoten"—the Six Knots—. The da Vinci–Dürer's Six Knots.

periodic tiling, fundamental region, primitive cell, and lattice and they will find a description of the 17 plane symmetry groups. The letter *p* of "p6m" means that the tiling has a parallelogram that is a primitive cell, in other words, a minimal region which is repeated by the lattice translations, and the tiling is described with respect to the primitive cell axes. The lattice of a p6m tiling is hexagonal and the primitive cell consists of two equilateral triangles. The number *6* of "p6m" indicates the highest order of rotational symmetry, which means there is a rotation symmetry that is 1/6 of a revolution. The letter *m* of "p6m" indicates the existence of a mirror reflection.

Throughout the article, we always first refer to the sixth link and then to the fourth link; the reason is that the sixth link is visually simpler than the fourth link. In fact, this differentiation is not only visual, since, from the point of view of the knot design, the fourth link is more complex than the sixth link.

Description of the Sixth and Fourth da Vinci–Dürer Links

Figure 2 shows the embossed coloured redrawing of the sixth da Vinci–Dürer Link on a white background, hereinafter referred to as the 6-dVD-Link.

Figure 3 shows a simplified image of the six prime knots, which will be mentioned in this paper. Those readers who are not familiar with the basic theory of knots may read, for instance, Hoste, et al. [1998] and Adams [1994]. There, they will learn about the concepts of Dowker-Thistlethwaite code, connected sum (or composition of knots), and prime knot. Also, The Rolfsen Knot Table can be found in the knot database *The Knot Atlas* [Knot Atlas] and also in Adams [1994].

Figure 4 shows the 18 knots that make up the 6-dVD-Link, each knot being located in its corresponding position within the circle of Figure 2. The knots are as follows: