Lecture Notes in Computer Science 686

J. Mira J. Cabestany A. Prieto (Eds.)

New Trends in Neural Computation

International Workshop on Artificial Neural Networks, IWANN '93
Sitges, Spain, June 1993
Proceedings

Springer-Verlag

DYNAMIC ANALYSIS OF NETWORKS OF NEURAL OSCILLATORS

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Abstract

We present here a dynamic formalism which allow to compute analitically the stationary properties of networks of neural oscillators. This technique, derived originally to study situations away from equilibrium, is an alternative to standard methods developed to analyze the behaviour of attractor neural networks.

1 Introduction

The analysis of the dynamical properties of attractor neural networks (ANN) are the focus of important works in the last years. Not only because it can provide information about the short and long time behaviour of networks characterized by symmetric and asymmetric couplings but also because they are the only way to understand the nature of some collective phenomena, such as mutual synchronization in the temporal activity of large assemblies of neurons [1], which are responsible of interesting effects related to the processing of information observed in real experiments performed in the visual cortex of monkeys [2].

The conventional models of ANN characterize the activity of the neurons through binary values, corresponding to the active and non-active state of each neuron [3]. However, in order to reproduce synchronization between members of a population it is convenient to introduce new variables which could provide information about the degree of coherence in the temporal response of active neurons. A possible way to do this, is by associating a phase to each element of the system and consequently to model neurons as oscillators. One of the most common models of phase oscillators is the so called Kuramoto's model [4], whose dynamics is governed by

$$\frac{d\theta_i}{dt} = \omega_i + \gamma_i(t) + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i) \tag{1}$$

where K_{ij} is the coupling matrix, θ_i the phase of the i-th oscillator, ω_i is a random frequency for each oscillator that obeys a certain distribution $g(\omega)$, N the size of the population and $\gamma_i(t)$ independent white noise random processes with zero mean and correlation

$$< \gamma_i(t)\gamma_j(t') >= 2D\delta_{ij}\delta(t-t'), \qquad D \geq 0 \tag{2}$$

An important point in our discussion is the specific form proposed for the couplings since it is the bridge that allows to make the analogy between models of phase oscillators and ANN. After a suitable choice of K_{ij} it is possible to wonder about the ability of the system to work as an associative memory [5].

Let us consider a population of N neurons, active at high rate during a given period of time that can carry information about their phase. As usual in ANN we want to store p sets of random patterns (phases) $\{\xi\}$ and a simple way to do this task is to assume that the synaptic efficacies (couplings) are given by

$$K_{ij} = \frac{k}{N} \sum_{\mu=1}^{p} \cos(\xi_i^\mu - \xi_j^\mu) \tag{3}$$

where k is the intensity of the coupling. This form preserves the basic idea of the Hebb´s rule but now adapted to the symmetry of our problem.

Our goal is to determine the stationary properties of the model described by equations (1) and (3) through a mean-field formalism widely used in the analysis of large populations of coupled oscillators [6], but new in the treatment of the features of ANN. We will show that this technique is an excellent alternative to conventional methods of analysis of associative memories, such as the replica method.

Notice that when the distribution of frequencies vanishes ($g(\omega) = \delta(\omega)$) our neurons are no longer oscillators. In this case our system becomes a Q-state clock model of neural network in the limit $Q \to \infty$, which has been extensively studied by Cook [7] in the replica symmetry approximation. We will show that with our method it is possible to reproduce the results of [7] in a simpler way, emphasizing the relevant influence of $g(\omega)$ on the long time properties of the system.

Description of problem and results

In this preliminar study we have only considered the behaviour of the system in the low loading limit, i.e., in the case where the capacity α=p/N, defined as the ratio between the number of patterns and the number of units of the system, goes to zero. To analyze our model it is convenient to introduce the following order parameters

$$q_{\pm}^{\mu} e^{i\phi_{\pm}^{\mu}} = \frac{1}{N} \sum_{j} e^{i(\theta_j \pm \xi_j^{\mu})} \tag{4}$$

$\phi_{\pm}^{\mu}$ play the role of a mean phase, q_{-}^{μ} measures the correlation between the state of the system and the pattern ξ^{μ}, and q_{+}^{μ} is another correlation not relevant in our study. Then the evolution equation for the phase oscillators is

$$\frac{d\theta_i}{dt} = \omega_i + \frac{k}{2} \sum_{\mu=1}^{p} [q_{-}^{\mu} \sin(\phi_{-}^{\mu} - \theta_i + \xi_i^{\mu}) + q_{+}^{\mu} \sin(\phi_{+}^{\mu} - \theta_i - \xi_i^{\mu})] + \gamma_i(t) \tag{5}$$

In the thermodynamic limit $N \to \infty$, it is possible to derive a non-linear Fokker-Planck equation for the one oscillator probability density $\rho(\theta, t, \omega, \xi)$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}[\mathcal{V}\rho] - D\frac{\partial^2 \rho}{\partial \theta^2} = 0 \tag{6}$$

where $\mathcal{V}$ is the drift velocity term

$$\mathcal{V} = \left[\omega + \frac{k}{2} \sum_{\mu=1}^{p} [q_{+}^{\mu} \sin(\phi_{+}^{\mu} - \theta - \xi^{\mu}) + q_{-}^{\mu} \sin(\phi_{-}^{\mu} - \theta + \xi^{\mu})]\right] \tag{7}$$

If $g(\omega)$ and $f_{\mu}(\xi^{\mu})$ are the frequency and pattern distribution, respectively, the order parameters (4) become

$$q_{\pm}^{\mu} e^{i\phi_{\pm}^{\mu}} = \int ... \int d\omega d\theta e^{i(\theta \pm \xi^{\mu})} \rho(\theta, \omega, \xi^1, ..., \xi^p) g(\omega) \prod_{\mu=1}^{p} f_{\mu}(\xi^{\mu}) d\xi^{\mu} \qquad (8)$$

Since we are interested in the long time behaviour of the system we have solved the equation (6) for the stationary case

$$\rho(\theta, \omega, \xi^{\mu}) = \frac{F(\theta) \int_0^{2\pi} d\eta H(\theta, \eta)}{\mathcal{Z}} \qquad (9)$$

where

$$F(\theta) = exp\left[\frac{k}{2D} \sum_{\mu=1}^{p} [q_{+}^{\mu} \cos(\phi_{+}^{\mu} - \theta - \xi^{\mu}) + q_{-}^{\mu} \cos(\phi_{-}^{\mu} - \theta + \xi^{\mu})]\right] \qquad (10)$$

$$H(\theta, \eta) = exp\left[-\frac{\omega\eta}{D} - \frac{k}{2D} \sum_{\mu=1}^{p} [q_{+}^{\mu} \cos(\phi_{+}^{\mu} - \theta - \eta - \xi^{\mu}) + q_{-}^{\mu} \cos(\phi_{-}^{\mu} - \theta - \eta + \xi^{\mu})]\right] \qquad (11)$$

and

$$\mathcal{Z} = \int_0^{2\pi} d\theta F(\theta) \int_0^{2\pi} H(\theta, \eta) d\eta \qquad (12)$$

Equations (9)-(12) describe the behaviour of the system in the most general case. However since we are interested in the limit of $\alpha \to 0$ we can assume that if the initial state of the system has a macroscopic correlation with a pattern μ, then only the order parameter $q_{-}^{\mu} \equiv q$ will be relevant, what simplifies notably the nature of the problem. The situation with $\alpha \neq 0$ will be considered elsewhere. Notice that q

$$q =< \cos(\phi - \theta - \xi) > \qquad (13)$$

plays the role of the overlap in classical models of ANN, i.e., the projection of the state of the system over the memories except for a mean phase ϕ that goes to zero when the distribution of natural frequencies $g(\omega)$ is even and has zero mean.

To calculate q we can proceed in two different manners, either by solving directly equation (8), what is complex because it means to solve an integral equation impliying to get values of q through numerical integration, or by identifying $\mathcal{Z}$ as a generating functional of the order parameters. This method is more elegant and give algebraic expressions easier to deal with. Let us rewrite $\mathcal{Z}$ as

$$\mathcal{Z} = \int_0^{2\pi} d\theta F(\theta,\sigma) \int_0^{2\pi} H(\theta,\eta) d\eta \tag{14}$$

where

$$F(\theta,\sigma) = exp\left[\sigma \cos(\phi - \theta + \xi)\right] \tag{15}$$

then it is straightforward to see that

$$q = << \frac{\partial}{\partial\sigma} \ln \mathcal{Z} \mid_{\sigma=\frac{kq}{2D}} >> \tag{16}$$

where $<< .. >>$ is an average over ω and ξ. To carry out the calculation we apply the following identity

$$e^{\frac{kq}{2D}\cos(\phi-\theta-\eta+\xi)} = I_0(\frac{kq}{2D}) + 2\sum_{n=1}^{\infty}(-1)^n \cos n(\phi - \theta - \eta + \xi) I_n(\frac{kq}{2D}) \tag{17}$$

Integrating (12), averaging over ξ and evaluating the partial derivate (16) we obtain a self-consistent equation for the q parameter

$$q = < \frac{\frac{D}{\omega} I_0(\beta q) I_1(\beta q) + \sum_1^{\infty} \frac{(-1)^n}{(\omega/D)^2+n^2} I_n(\beta q)(I_{n-1}(\beta q) + I_{n+1}(\beta q))(\frac{\omega}{D})}{\frac{D}{\omega} I_0^2(\beta q) + 2\sum_1^{\infty} \frac{(-1)^n}{(\omega/D)^2+n^2} I_n^2(\beta q)(\frac{\omega}{D})} >_\omega \tag{18}$$

where I_n are the modified Bessel functions of first kind of order n, $\beta = \frac{k}{2D}$ and $<>_\omega$ means an average over the distribution of frequencies. Taking into account the symmetry properties of the modified Bessel functions for n integer ($I_n = I_{-n}$), we can summarize this formula in

$$q = < \frac{\sum_{-\infty}^{\infty} \frac{(-1)^n}{\omega^2+D^2n^2} I_n(\beta q) I_{n-1}(\beta q)}{\sum_{-\infty}^{\infty} \frac{(-1)^n}{\omega^2+D^2n^2} I_n^2(\beta q)} >_\omega \tag{19}$$

In practice the numerical computation of this algebraic expression is not difficult because the maximum contribution to the infinity sum comes from the modified Bessel functions of lower orders.

It is interesting to compare our results with those given by Cook in [7] in the limit of $Q \to \infty$. We observe from (18) that when $\omega \to 0$ (absence of frequencies), the overlap is

$$q = \frac{I_1(\beta q)}{I_0(\beta q)} \tag{20}$$

which is exactly the same expresion reported by Cook. However our result is more general because we have included the effect of a distribution of frequencies. Additionally it is not difficult to deal with more complex situations (e.g. random fields). This shows the power of the formalism developed in this paper.

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