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LETTER TO THE EDITOR

On the short-time dynamics of networks of Hebbian coupled oscillators

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Abstract. We have analysed the collective behaviour of an assembly of coupled oscillators interacting through Hebbian synaptic weights designed to store information about the phase of each element. Our study focuses on the dynamic properties of the population showing the existence of a relevant time-scale, useful for the processing of information.

The study of the collective behaviour of assemblies of coupled oscillators is an interesting topic. Models accounting for the oscillatory properties of individual biological neurons have appeared during the last thirty years but now the studies focus on the behaviour of large populations of them. The interest comes from some experiments performed on the visual cortex of cats [1, 2] which show that an external stimulus may induce synchronization on the temporal activity of neurons. These coherent oscillations could be a mechanism of feature linking [3].

But the mutual synchronization between members of a given population is not a new phenomenon: it appears frequently in biological systems, and has been analysed for years [4]. Synchronous flashing of swarms of fireflies [5] or coherent rhythms between cardiac pacemaker cells are some examples [6]. A possible way to model the features of these systems is to consider the whole population as an assembly of nonlinear oscillators, each running at its natural frequency picked up from a random distribution and interacting with each other through long-ranged couplings. Thus each oscillator tries to run independently at its own frequency while the coupling tends to synchronize it to all others. When the coupling is sufficiently weak, the oscillators run incoherently whereas above a certain threshold collective synchronization appears spontaneously. As a particular case, Kuramoto [7] proposed a mathematically tractable model of phase oscillators whose dynamical evolution is described by the equations

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \gamma_i(t) + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i) \,. \tag{1}$$

Here θ_i and ω_i represent the phase and natural frequency (randomly distributed over the population with a density $g(\omega)$) of the ith oscillator, K_{ij} the coupling strength between

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oscillators, N the size of the population and $\gamma_i(t)$ Gaussian white noise with zero mean and correlation

$$\langle \gamma_i(t)\gamma_j(t')\rangle = 2D\delta_{ij}\delta(t-t') \qquad D \geqslant 0\,. \tag{2}$$

Three elements play a relevant role in this model. The first one concerns the features of the distribution of natural frequencies of the population. It has been shown that the order of the transition from the incoherent state to the synchronized state depends on whether $g(w)$ is unimodal (non-increasing for $w > 0$) [8] or bimodal [9]. The second element is the external noise acting over the system given by the constant coefficient D. Its basic effect is to shift the critical point above which the incoherent state becomes unstable.

Finally, the third element and, perhaps, the most interesting, accounts for the explicit form for the couplings. The simplest choice is to assume infinite-range interactions equal for all members of the population ($K_{ij} = K/N$, $K > 0$), studied originally by Kuramoto and subsequent authors [8, 10]. Randomness in the couplings is another ingredient whose effect on the collective behaviour of the oscillators has been studied as well [11, 12]. If the disorder is sufficiently strong (frustration) new phases may appear whose nature shows clear analogies with the glassy or mixed phases typical of disordered systems such as spin glasses. In another range of interest more sophisticated prescriptions such as stimuli-dependent couplings [13, 14] have successfully reproduced experimentally observed features in visual processing although their analytical study is rather difficult. Following this line of thought, a natural question is to wonder whether large populations of coupled oscillators could store information after a proper choice of the matrix K_{ij}, as in models of attractor neural networks (ANN).

Such a problem has been analysed by several authors. Cook [15] considered a static approach (no frequencies) where each element of the system is modelled as a q-state clock. Furthermore, a Hebbian learning rule was proposed as a coupling. In this case, since the J_{ij} are symmetric it is possible to define a Lyapunov function whose minima coincide with the stationary states of (1). Cook solved the problem by deriving mean-field equations in the replica-symmetric approximation, finding that in the limit $q \to \infty$ and zero temperature, the stationary storage capacity of the network is $\alpha_c = 0.038$, much lower than in the Hopfield model ($q = 2$) for which $\alpha_c = 0.14$, as well as for $q = 3$ where $\alpha_c = 0.22$. A more general analysis was performed by Gerl *et al* [16] who followed a standard formalism derived by Gardner [17] to calculate the volume of solutions satisfying stability conditions. In this way they showed that in the optimal case and for a fixed stability κ the storage capacity decreases as q increases but the information content per synapse grows if κ scales as q^{-1}. Although this final result seems to be promising it has serious limitations given by the size of the network (since $N > q$) and also because the time required to reach the stationary state is proportional to q, as has been corroborated in [18]. Other models with similar features display the same type of behaviour [19].

All these outcomes seem to indicate that networks of analogue neurons (with clock symmetry) are not suitable to store information. However, in all these studies only the long-time behaviour is considered. We want to show that for these systems the most interesting properties are associated to the transient not only from a dynamical standpoint but also from a computational point of view. This will be proven analytically and by simulations. Centring our discussion on the ability of the system to work as an associative memory, we will show the existence of a time-scale that might be useful for the processing of information. Our main result is that the system approaches a given memory in a short time t_m for any storage capacity. This is a dynamical effect, in the sense that after t_m the system goes to a steady state with an overlap with the memory that usually is much smaller than that reached at t_m.

Let us consider that an assembly of N oscillators which can carry information about their phase. As usual we want to store p sets of random patterns (phases) $\{\xi\}$ and a simple way to do this task is to assume that the synaptic efficacies (couplings) are given by

$$K_{ij} = \frac{K}{N}\sum_{\mu=1}^{p}\cos(\xi_j^\mu - \xi_i^\mu) \tag{3}$$

which preserves the basic idea of the Hebb rule but now adapted to the structure of our problem. The specific learning rule showed in (3) plus the dynamics given by (1) define the model. To analyse it, it is convenient to define the following order parameters:

$$c_\pm^\mu e^{i\phi_\pm^\mu} = \frac{1}{N}\sum_k e^{i(\theta_k \pm \xi_k^\mu)} \tag{4}$$

where $c_\pm^\mu$ measures the correlation between the state of the system and the pattern μ, and $\phi_\pm^\mu$ give information about the mean phase. Equation (1) can be written in terms of c and ϕ as

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{2}\sum_{\mu=1}^{p}\left[c_-^\mu \sin(\phi_-^\mu - \theta_i + \xi_i^\mu) + c_+^\mu \sin(\phi_+^\mu - \theta_i - \xi_i^\mu)\right] + \gamma_i(t) \tag{5}$$

which turns out to be the dynamics implemented in our simulations. Implicit under this expression is the additional assumption of a parallel updating of θ which will remain present in all our studies.

The first result we want to show is the dynamical behaviour of the state of the system starting from an initial configuration correlated with a given pattern μ. As usual, this information can be written in terms of the overlap $m^\mu = c_\mu \cos\phi_\mu$. By now, we will consider that all the oscillators are identical. The effect of a distribution of frequencies will be discussed later. The typical time evolution of m^μ is shown in figure 1(a) in the absence of noise ($D = 0$), and for a low loading ($\alpha = 0.02$). In contrast, with a conventional ANN where we should expect a monotonic increase of the overlap up to the stable state (remember that the storage capacity is around 0.04 and therefore we are working below α_c), here the overlap increases for very short times, reaches a maximum value (unstable) in a few iterations and finally relaxes to a stationary (stable) state several seconds latter. This behaviour is characteristic of the model, it appears in a wide range of situations showing a complex structure of dynamical attractors.

To study this phenomenon analytically it is necessary to apply a formalism which is able to compute the dynamics of the network for all time. In general, only for very diluted systems where the connectivity between elements is of the order of $\ln N$ is it possible to solve the problem completely [20, 21]. When the connectivity is larger the situation is more complex because correlations at different times, described in terms of order parameters whose numbers increase exponentially with time, play an important role and must be taken into account [22]. As a consequence, the usual description given in terms of a finite number of order parameters can only give information about the first time steps or in the other limit, the stationary state. Even for the short-time regime (first step) the time evolution of the network of oscillators display a quite different behaviour compared with standard ANN. In general, the dynamic evolution of the overlap in the first time step, m_1^μ, depends on the initial overlap $m_0^\mu = m^\mu(t=0)$, and on the storage capacity α, usually as a function of the quotient $m_0^\mu/\sqrt{\alpha}$, which means that above a certain critical dynamic capacity α_d (larger than the stationary observed for $t \to \infty$) the systems go away from the attractor systematically. As an example, for the Hopfield model such $\alpha_d \approx 0.64$ [22]. Interestingly, this is not

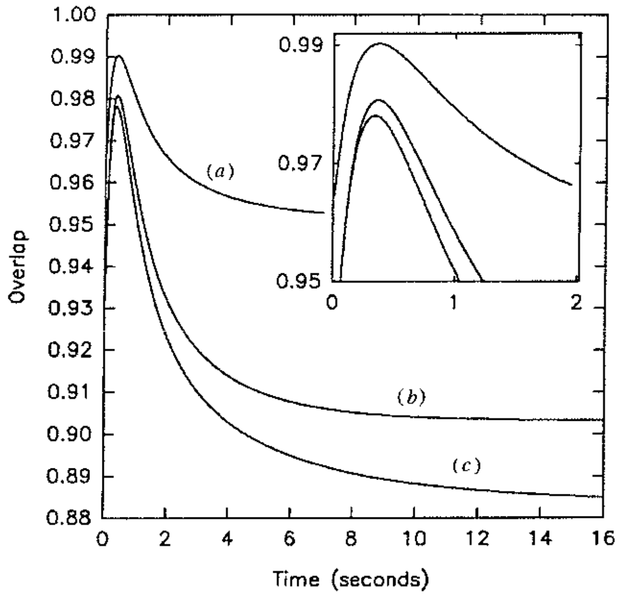

Figure 1. Typical evolution of the overlaps defined in the text for a system of 200 oscillators when $D = 0$, $K = 8$ and $\alpha = 0.02$. Our results have been obtained by integrating (5) with the Euler method taking as a time step $\Delta t = 0.001$. In our scale one second is equivalent to one thousand time steps. Three distributions of frequencies have been considered: (*a*) a static situation given by $w_i = \delta(w)\ \forall i$, and two dynamic situations characterized by (*b*) $w_i = 0.2 + \Delta_i$ where Δ_i is a random variable uniformly distributed in $[-0.2, 0.2]$ and (*c*) $w_i = 0.2 + \Delta_i$, $\Delta_i \in [-0.4, 0.4]$. The overlaps increase from an arbitrary initial value to a maximum one reached at a characteristic time t_m with almost no sensitivity to the distribution of frequencies.

the case for the model studied in this letter, since the network always evolves towards the attractor and this fact does not depend on the storage capacity α. This apparently surprising result can be proven easily by analysing m_0^μ and its first derivative. For simplicity, we will just consider identical oscillators and the zero temperature limit ($D = 0$), but it can be generalized following the same ideas.

Let us calculate the value of the derivative of the order parameter at $t = 0$. From the definition of m^μ

$$m^\mu = q_\mu^- \cos\phi_\mu^- = \left\langle \int_0^{2\pi} \mathrm{d}\theta\, \rho(\theta, t) \cos(\xi^\mu - \theta) \right\rangle_{\xi^\mu} = \langle\langle \cos(\xi^\mu - \theta) \rangle\rangle \tag{6}$$

where the symbol $\langle\langle \cdots \rangle\rangle$ means the integration over θ and ξ_μ. This equation has been written in terms of the one oscillator probability density $\rho(\theta, t, \xi)$ solution of the Fokker–Planck equation

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}(\rho v) \tag{7}$$

where the drift velocity is given by

$$v(\theta, t, \xi, \eta) = \left[\omega + \frac{K}{2} \sum_{\mu=1}^{p} \left[m_+^\mu \sin(\phi_+^\mu - \theta - \xi^\mu) + m_-^\mu \sin(\phi_-^\mu - \theta + \xi^\mu) \right] \right]. \tag{8}$$

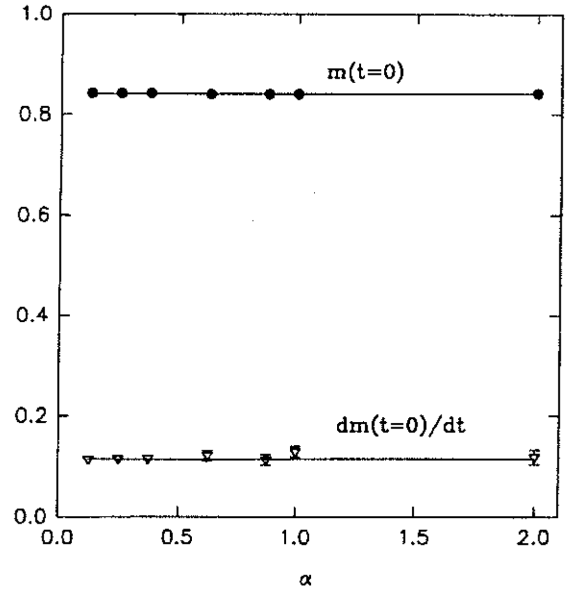

Figure 2. Values of $m^\mu(t=0)$ and for $\dot{m}^\mu(t=0)$ as a function of α for a network of identical oscillators with $N=400$, $K=1$ and $D=0$. The full curves denote the theoretical results while the full circles come from simulations and are an average over 25 samples. When not shown the error bars are of the size of the symbols.

To perform the derivative of (6) we have assumed that the initial configuration is $\theta_i = \xi_i^\mu + z_i$ with z_i a random variable uniformly distributed in the interval $[-d, d]$. The generalization to other probability distributions is straightforward. Now, by using the Fokker–Planck equation (7) for $D=0$, and integrating by parts we lead to

$$\dot{m}^\mu = \left\langle \int_0^{2\pi} \mathrm{d}\theta\, \rho(\theta, t)\dot{\theta}\sin(\xi^\mu - \theta) \right\rangle_{\xi^\mu} = \langle\langle \sin(\xi^\mu - \theta)\dot{\theta} \rangle\rangle \tag{9}$$

yielding

$$m^\mu(t=0) = \frac{\sin d}{d} \tag{10}$$

and

$$\dot{m}^\mu(t=0) = \frac{K \sin d}{4d}\left(1 - \frac{\sin 2d}{2d}\right). \tag{11}$$

These results deserve several comments. First of all, we observe that $\dot{m}$ is independent of the storage capacity α, so the very short time dynamics of the system is always the same no matter how many patterns are stored in the network. Figure 2 shows perfect agreement between theory (full curve) and the simulations for a wide range of loadings. This result has no counterpart in ANN. On the other hand, there is a dependence on the coupling strength K and on the variance of the initial distribution, given in terms of d. The larger the value of K the faster the retrieval process. Figure 3 shows the variation of $\dot{m}^\mu$ in terms of d and the excellent agreement with the simulations. Notice that the negative region does not mean that the state of the system goes away from the memory since it can only be reached when m is negative. In other words, if $m^\mu > 0$ then $\dot{m}^\mu > 0$.

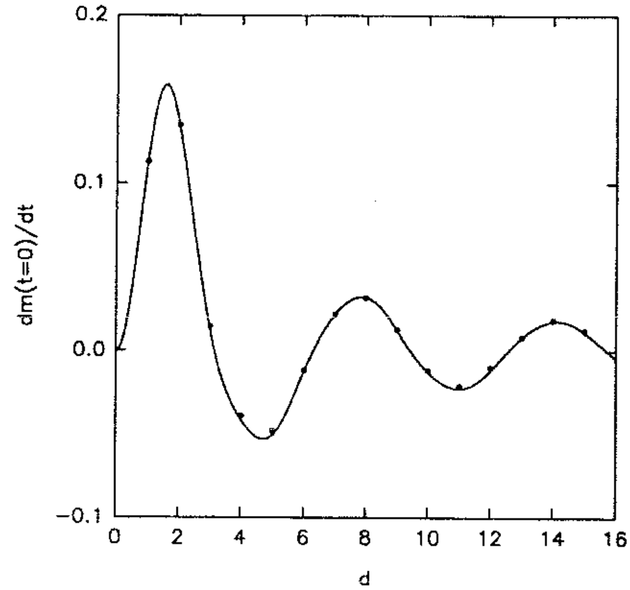

Figure 3. Variation of $\dot{m}^{\mu}(t=0)$ as a function of d for a network of identical oscillators with $N=400$, $K=1$ and $D=0$. The full curve denotes the theoretical result while the full circles come from simulations and are an average over 100 samples. When not shown the error bars are of the size of the symbols.

It is remarkable to realize that these results imply the existence of a characteristic time t_m for which the overlap reaches a maximum. Then, the maximum retrieval of information (understood as the recall of a given pattern) is not associated with the stationary state but to the transient and, therefore, this is evidence of a time-scale that could make the system useful for computational purposes (as an associative memory), and by extension, relevant for the processing of information. Again, we want to stress that this behaviour occurs for any storage capacity. Of course, as α increases the peak becomes less important and is observed at shorter times. Recently, Coolen and Sherrington [23] have developed a formalism that gives the correct dynamical behaviour of the system at short time-scales and also for $t \to \infty$. This technique could give information about the features of t_m. This study is currently in progress.

To ensure that the system works as an associative memory in t_m is not sufficient to prove the existence of the peak. It is also relevant to determine the size of the basins of attraction of the dynamic attractors. Figure 4 shows that for very low capacities the basins are enormous, as in other Hebbian models of ANN, and for short times it is possible to recall a given memory even for very small initial overlaps. As usual, a further increase of α implies a reduction of the basins of attraction.

Up to now, we have considered systems made of identical oscillators. If we consider a distribution of frequencies, the state of the system, described by an N-dimensional vector whose ith component is the phase of the ith oscillator, is changing continuously in time. However, this is not a problem since it is possible to store information as a difference of phases between pairs of oscillators, a quantity described by the two-point correlation $\langle\cos(\theta_i - \xi_i^{\mu} - \theta_j + \xi_j^{\mu})\rangle$, that may remain constant in time. In this way the attractor should

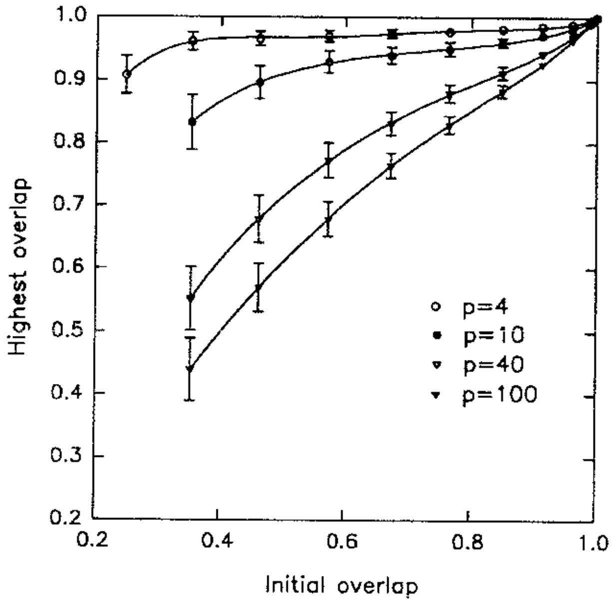

Figure 4. Basins of attraction of the dynamic attractors for different storage capacities $\alpha = p/N$. We have considered a network of $N = 200$ oscillators, $K = 8$, $D = 0$ and a distribution of frequencies given by the situation (a) described in figure 1.

be understood as a kind of phase locking and if the initial state is correlated with one of the embedded patterns, the final state will also have a macroscopic correlation with the same pattern provided p is below a critical value. This special characteristic of the model holds because of the particular form of the learning rule given by (3). The maximum of the overlap is quite robust to a distribution of frequencies as we can see in figure 1(b) and (c). In contrast, the stationary properties of the model depend strongly on $g(w)$ [24].

Finally, we would like to comment on some recent results obtained by Aoyagi [25] in a similar model. This author considers a network of oscillators where not only the phase but also the amplitude evolves in time (in contrast to our model where the amplitudes are fixed). He shows that in this system the overlap evolves monotonously from the initial configuration to the attractor without displaying any sort of peak. However, this is not in contradiction with our results since he works in the limit of sparse coding, that is, in the limit where the majority of oscillators have zero amplitude. In this regime, it is well known that the storage capacity of the network increases dramatically. In particular, for Hopfield-like models it grows as $\alpha \approx 1/a \ln a$ where a is the level of sparseness. We have checked his model far from the sparse coding limit and the behaviour is in perfect agreement with our outcomes (the peak appears).

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