Stochastic Euler–Poincaré reduction on Lie groups with Cartan–Schouten connections

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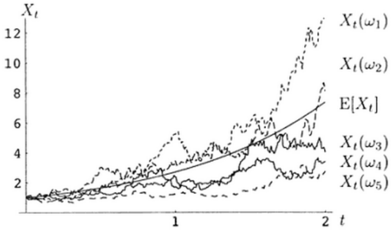
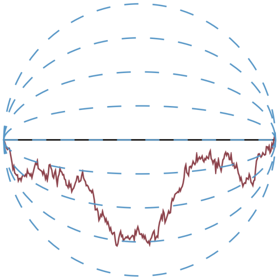
A diffusion needs covariance as well as drift

1 Introduction





For $dX_t^i = b^i(X_t)\,dt + \sigma_r^i(X_t)\,dW_t^r$,

$$\mathbb{E}_t[\Delta X^i] = b^i(X_t)\,\Delta t + o(\Delta t), \quad \mathbb{E}_t[\Delta X^i \Delta X^j] = a^{ij}(X_t)\,\Delta t + o(\Delta t), \qquad a = \sigma\sigma^\top.$$

Thus the conditional covariance survives at the same order Δt as the mean increment.

Itô's formula opens two geometric routes

1 Introduction

Itô formula:

$$\begin{aligned} df(X_t) &= \partial_i f(X_t)\, dX_t^i + \frac{1}{2}\partial_{ij} f(X_t) d[X^i, X^j]_t \\ &= \partial_i f(X_t) \circ dX_t^i. \end{aligned}$$

For a diffusion with $d[X^i, X^j]_t = a^{ij}(X_t)\, dt$, the Hessian term contributes to the drift.

Route I: Stratonovich / first order

- The ordinary chain rule is restored.
- Classical tangent vectors, differential forms, and symplectic structures remain.

The correction moves into the drift.

Route II: Itô / second order

- Quadratic variation becomes intrinsic geometric data.
- Tangent and cotangent structures become second order.

The correction remains in the geometry.

Stochastic geometric mechanics: first-order

Literature: Stratonovich and classical geometric structures

- [Bismut'81]: developed stochastic mechanics on symplectic manifolds

$$\circ dz_t = X_{H_0}(z_t)\, dt + \sum_{r=1}^{k} X_{H_r}(z_t) \circ dW_t^r.$$

- [Lázaro-Camí–Ortega'08, Lázaro-Camí–Ortega'09]: stochastic Hamilton equations on Poisson manifolds, together with stochastic Hamilton–Jacobi theory.
- [Li'08, Wei–Chao–Duan'19]: completely integrable stochastic Hamiltonian system and averaging principle.
- [Wei–Huang–Duan'25]: stochastic Hamilton–Jacobi theory on Jacobi manifold.

Stochastic geometric mechanics: second-order

Literature: Itô calculus and Schwartz–Meyer geometry

- [Nelson'67]: forward and backward mean derivatives, supplying the kinematics for diffusion-based mechanics.
- [Meyer'81, Schwartz'84]: made the diffusion generator an intrinsic second-order tangent vector:

$$\mathcal{A} = b^i \partial_i + \frac{1}{2} a^{ij} (\partial_i \partial_j - \Gamma_{ij}^k \partial_i)$$

- [Nakagomi–Yasue–Zambrini'81]: built variational mechanics from mean derivatives, obtaining stochastic Euler–Lagrange and Noether equations in Euclidean spaces, which are called "Euclidean quantum mechanics".
- [Huang–Zambrini'23]: developed second-order symplectic geometry, connecting stochastic Lagrangian/Hamiltonian mechanics with HJB equations and Schrödinger problems.

Mean derivatives

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and equip it with a filtration $\{\mathcal{P}_t\}_{t\in\mathbb{R}}$. Let M be a manifold with a linear connection ∇.

For an M-valued diffusion X adapted to $(\mathcal{P}_t)$, define

$$(DX)^i(t) := \lim_{\varepsilon\downarrow 0} \mathbb{E}\left[\frac{X^i(t+\varepsilon) - X^i(t)}{\varepsilon}\middle|\mathcal{P}_t\right],$$

$$(QX)^{jk}(t) := \lim_{\varepsilon\downarrow 0} \mathbb{E}\left[\frac{\left(X^j(t+\varepsilon) - X^j(t)\right)\left(X^k(t+\varepsilon) - X^k(t)\right)}{\varepsilon}\middle|\mathcal{P}_t\right].$$

- Due to Itô's formula, $DX(t)$ does not transform as a vector, but the following is:

$$(D_\nabla X)^i(t) = (DX)^i(t) + \frac{1}{2}\Gamma^i_{jk}(X(t))(QX)^{jk}(t).$$

- $(Dx, Qx) = D^i_\nabla x\partial_i + \frac{1}{2}Q^{jk}x\nabla^2_{\partial_j,\partial_k}$ is the second-order vector field.

Stochastic parallel transport

Stochastic parallel displacement

The field V is parallel along X when

$$\nabla_{\circ dX_t} V_t = 0,$$

or, in local coordinates,

$$dV_t^i + \Gamma^i_{jk}(X_t) V_t^j \circ dX_t^k = 0, \qquad V_{t_0} = v.$$

Its solution defines the stochastic parallel transport

$$\mathbf{\Gamma}(X)_{t_0}^t : T_{X_{t_0}} M \longrightarrow T_{X_t} M, \qquad \mathbf{\Gamma}(X)_{t_0}^t v = V_t.$$

Damped parallel displacement

The curvature-corrected field solves

$$dV_t^i + \Gamma^i_{jk}(X_t) V_t^j \circ dX_t^k + \frac{1}{2} R^i{}_{kj\ell}(X_t) V_t^j (QX_t)^{k\ell}\, dt = 0, \qquad V_{t_0} = v.$$

We denote its solution by

$$\overline{\mathbf{\Gamma}}(X)_{t_0}^t v = V_t.$$

The additional drift contracts curvature with the quadratic variation of X.

- For a metric connection, stochastic parallel transport preserves length, while the damped one includes the distortion of noise.
- Covectors are transported by dual equations; the pairing $\eta_t(V_t)$ is preserved under dual parallel transport.

Mean covariant derivatives along diffusion

Definition (Mean covariant derivatives)

Let V be a time-dependent vector field along X_t. The mean covariant derivatives of V with respect to X are a vector field defined by

$$\frac{\mathbf{D}V}{dt}(t) = \lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\frac{\mathbf{\Gamma}(X)^t_{t+\varepsilon}V(t+\varepsilon)-V(t)}{\varepsilon}\,\middle|\,\mathcal{P}_t\right], \quad \frac{\overline{\mathbf{D}}V}{dt}(t) = \lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\frac{\overline{\mathbf{\Gamma}}(X)^t_{t+\varepsilon}V(t+\varepsilon)-V(t)}{\varepsilon}\,\middle|\,\mathcal{P}_t\right]$$

The mean covariant derivatives of covector η along X_t are defined similarly.

Proposition

$$\frac{\overline{\mathbf{D}}V}{dt} = \frac{\mathbf{D}V}{dt} + \frac{1}{2}(QX)^{ij}R(V,\partial_i)\partial_j = \frac{\partial V}{\partial t} + \nabla_{D_\nabla X}V + \frac{1}{2}(QX)^{ij}\left(\nabla^2_{\partial_i,\partial_j}V + R(V,\partial_i)\partial_j\right),$$

$$\frac{\overline{\mathbf{D}}\eta}{dt} = \frac{\mathbf{D}\eta}{dt} - \frac{1}{2}(QX)^{ij}R(\eta,\partial_j)\partial_i = \frac{\partial \eta}{\partial t} + \nabla_{D_\nabla X}\eta + \frac{1}{2}(QX)^{ij}\left(\nabla^2_{\partial_i,\partial_j}\eta - (R(\cdot,\partial_j)\partial_i)^*\eta\right).$$

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Stochastic variational problem

2 Stochastic variational principle on manifold with linear connection

- Lagrangian $L : \mathbb{R} \times TM \to \mathbb{R}$;
- Stochastic action

$$\mathcal{S}[X] := \mathbb{E} \int_0^T L\big(t, X, D_\nabla X\big)\, dt;$$

- Admissible class

$$\mathcal{A}_g([0,T]; q, \mu) = \left\{ X \in I_{0,q}^{T,\mu}(M) : QX(t) = g(t, X(t)),\ \forall t \in [0,T],\ \text{a.s.} \right\};$$

- (Cameron–Martin) variation of diffusion

$$\frac{\partial}{\partial \epsilon} X_\epsilon^v(t) = \mathbf{\Gamma}(X_\epsilon^v)_0^t v(t), \quad X_0^v(t) = X(t),$$

  where $v \in \mathcal{H}_0([0,T]; q)$ and

$$\mathcal{H}_0([0,T]; q) := \left\{ v : [0,T] \times \Omega \to T_q M \text{ is absolutely continuous}, \int_0^T |\dot{v}(t)| dt < \infty, v(0) = v(T) = 0 \right\}.$$

Stochastic Hamilton's principle

- The diffusion $X \in \mathcal{A}_g([0,T];q,\mu)$ is called a stationary (or critical) point of $\mathcal{S}$ if the first variation $\delta\mathcal{S}$ vanishes at X, i.e.,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{S}[X_\varepsilon^v] = 0$$

Stochastic Euler–Lagrange equation

For a regular $L : \mathbb{R} \times TM \to \mathbb{R}$, the diffusion $X \in \mathcal{A}_g([0,T];q,\mu)$ is stationary if and only if

$$\boxed{\frac{\overline{\mathbf{D}}}{dt}\big(d_{\dot{x}}L\big) + d_{\dot{x}}L(i_{D_\nabla X}T) + \frac{1}{2}d_{\dot{x}}L(\operatorname{div}(\nabla T)) = d_x^\nabla L.} \tag{S-EL}$$

- Levi–Civita case

$$\frac{\overline{\mathbf{D}}}{dt}\big(d_{\dot{x}}L\big) = d_x^\nabla L.$$

Velocity variation reveals the geometric corrections

Variation of the mean velocity, $\frac{\partial}{\partial \epsilon} X_\epsilon^v(t) = \mathbf{\Gamma}(X_\epsilon^v)_0^t v(t)$

$$\frac{D}{d\varepsilon}\bigg|_{\varepsilon=0} D_\nabla X_\varepsilon^v = \underbrace{\mathbf{\Gamma}(X)_0^t \dot{v}}_{\text{transported variation}} + \underbrace{T(V, D_\nabla X)}_{\text{torsion}}$$

$$+ \underbrace{\frac{1}{2}(QX)^{ij} R(V, \partial_i)\partial_j}_{\text{curvature}} + \underbrace{\frac{1}{2}(QX)^{ij}(\nabla_{\partial_j} T)(V, \partial_i)}_{\text{torsion gradient}}.$$

The transported derivative is removed by stochastic integration by parts:

$$\mathbb{E}\int_0^T \alpha\big(\Gamma(X)_0^t \dot{v}\big)\, dt = -\mathbb{E}\int_0^T \frac{\mathbf{D}\alpha}{dt}(V)\, dt, \qquad v(0) = v(T) = 0.$$

Remark

The three problems are built from the same stochastic action

$$\mathcal{S}[X] := \mathbb{E} \int_0^T L\big(t, X, D_\nabla X\big)\, dt.$$

Stochastic Hamilton principle	Stochastic optimal control	Stochastic optimal transport
$\delta \mathcal{S}[X] = 0$	$\inf \big\{ \mathcal{S}[X] + \mathbb{E} G(X_T) \big\}$	$\inf\limits_{X_0 \sim \mu_0, X_T \sim \mu_T} \mathcal{S}[X]$
Stochastic EL	HJB	HJB–Fokker–Planck

- Q. Huang, J.-C. Zambrini, *From second-order differential geometry to stochastic geometric mechanics*, J. Nonlinear Sci. **33**, 2023.
- W. H. Fleming, H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd ed., Springer, 2006.
- T. Mikami, *Stochastic optimal transport revisited*, Partial Differ. Equ. Appl. **2**, 2021.

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3 Stochastic Euler–Poincaré reduction

Literature: Euler–Poincaré reduction

- [Marsden–Ratiu'99, Holm–Marsden–Ratiu'98]: gives a modern deterministic account of Euler–Poincaré and Lie–Poisson reduction, momentum maps, reconstruction, and on semidirect product Lie algebras.
- [Simões–Colombo–de León–Marrero–de Diego–Padrón'24, Simões–Colombo–de León–Salgado–Souto'24]: symmetry reduction to contact/Herglotz mechanics: Euler–Poincaré–Herglotz dynamics on $\mathfrak{g} \times \mathbb{R}$; Lagrange–Poincaré–Herglotz reduction and reconstruction.
- [Cipriano–Cruzeiro'07, Arnaudon–Chen–Cruzeiro'14]: Euler–Poincaré reduction for Lie-group-valued diffusions using generalized mean derivatives, with examples on $SO(3)$ and diffeomorphisms group on $\mathbb{T}^2$.
- [Holm'15, Cruzeiro–Holm–Ratiu'18]: consider the advected quantities with a Stratonovich stochastic transport constraint; develop a stochastic Clebsch action in which noise couples through the momentum map, yielding Stratonovich coadjoint motion and Itô double brackets.
- [Chen–Cruzeiro–Ratiu'23] study stochastic semidirect product reduction with advected quantities, dissipative fluid equations, and a stochastic Kelvin–Noether theorem.

Invariant connections on Lie group

Left invariant connection

For a Lie group G, left translation identifies $T_gG \simeq \mathfrak{g}$ and $T_g^*G \simeq \mathfrak{g}^*$. For $u \in \mathfrak{g}$ and $g \in G$, denote $u^{\mathrm{L}}|_g = d(\mathrm{L}_g)_e u$. A left-invariant connection on G is encoded by one bilinear map:

$$\nabla^{\mathrm{L}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad \nabla_{u^{\mathrm{L}}} v^{\mathrm{L}} = \left(\nabla^{\mathrm{L}}_u v\right)^{\mathrm{L}}.$$

Denote the adjoint of ∇^{L} by $\nabla^{\mathrm{L}*} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$; that is, for $\alpha \in \mathfrak{g}^*, v, w \in \mathfrak{g}$,

$$\nabla^{\mathrm{L}*}_v \alpha(w) = \alpha(\nabla^{\mathrm{L}}_v w).$$

Curvature and torsion become algebraic

For every vector field U, V, W on G,

$$(R(U,V)W)_g = d(\mathrm{L}_g)_e \Big(\nabla^{\mathrm{L}}_{g^{-1}U_g} \nabla^{\mathrm{L}}_{g^{-1}V_g} - \nabla^{\mathrm{L}}_{g^{-1}V_g} \nabla^{\mathrm{L}}_{g^{-1}U_g} - \nabla^{\mathrm{L}}_{[g^{-1}U_g, g^{-1}V_g]}\Big)(g^{-1}W_g)$$

$$T(V,W)_g = d(\mathrm{L}_g)_e \Big(\nabla^{\mathrm{L}}_{g^{-1}V_g}(g^{-1}W_g) - \nabla^{\mathrm{L}}_{g^{-1}W_g}(g^{-1}V_g) - [g^{-1}V_g, g^{-1}W_g]_{\mathfrak{g}}\Big).$$

Parallel transport in left trivialization

Parallel transport in left trivialization

For a smooth curve $g(\cdot)$, set $\xi(r) = d(L_{g(r)^{-1}})\dot{g}(r) \in \mathfrak{g}$. Then the parallel transport operator satisfies

$$\Gamma(g)_s^t = d(L_{g(t)})\,\mathcal{T}\exp\Big(-\int_s^t \nabla^{\mathrm{L}}_{\xi(r)}\,dr\Big)\,d(L_{g(s)^{-1}}),$$

where $\mathcal{T}\exp\Big(\int_s^t A(r)\,dr\Big) = \sum_{n=0}^{\infty}\int_{s<r_n<\cdots<r_1<t} A(r_1)\cdots A(r_n)\,dr_n\cdots dr_1$ is the ordered exponential.

Reconstruction equation on Lie group

For vector fields $U(t), \{H_r(t)\}_{r=1}^k$ on G, define $u_t = d(L_{g_t^{-1}})_{g_t}U_t, h_{r,t} = d(L_{g_t^{-1}})_{g_t}H_{r,t}$. A diffusion on G:

$$dg_t = d(L_{g_t})_e\Big(u_t\,dt - \frac{1}{2}\sum_r \nabla^{\mathrm{L}}_{h_{r,t}}h_{r,t}dt + \sum_r h_{r,t}\,dW_t^r\Big), \tag{*}$$

or equivently, $\circ dg_t = d(L_{g_t})_e\Big(u_t\,dt - \frac{1}{2}\sum_r\big(H_r(h_r) + \nabla^{\mathrm{L}}_{h_{r,t}}h_{r,t}\big)dt + \sum_r h_{r,t}\circ dW_t^r\Big)$.

The stochastic parallel transport $\mathbf{\Gamma}(g)_s^t$ can expressed by changing ξ to $d(L_{g(r)^{-1}})_{g(r)}\circ dg(r)$.

Mean covariant derivatives on Lie group

Proposition: Left-trivialized covariant derivatives

Let V be a time-dependent vector field restricted on g, for which we denote $v(t) := d(\mathrm{L}_{g^{-1}})_g V(t)$. Then

$$d(\mathrm{L}_{g(t)^{-1}})_{g(t)} \frac{\overline{\mathbf{D}} V}{dt} = D_t v + \nabla^{\mathrm{L}}_u v + \left(\nabla^{\mathrm{L}}_{h_r} H_r(v) + \frac{1}{2} \sum_{r=1}^k \left(\nabla^{\mathrm{L}}_{h_r} \nabla^{\mathrm{L}}_{h_r} v - \nabla^{\mathrm{L}}_{\nabla^{\mathrm{L}}_{h_r} h_r} v + R(v, h_r) h_r \right) \right),$$

Let η be a time-dependent 1-form restricted on g, for which we denote $\alpha(t) := d(\mathrm{L}_g)^*_e \eta(t)$. Then

$$d(\mathrm{L}_{g(t)})^*_e \frac{\overline{\mathbf{D}} \eta}{dt} = D_t \alpha - \nabla^{\mathrm{L}*}_u \alpha + \sum_{r=1}^k \left(-\nabla^{\mathrm{L}*}_{h_r} H_r(\alpha) + \tfrac{1}{2} \left(\nabla^{\mathrm{L}*}_{h_r} \nabla^{\mathrm{L}*}_{h_r} \alpha + \nabla^{\mathrm{L}*}_{\nabla^{\mathrm{L}}_{h_r} h_r} \alpha - \left(R(\,\cdot\,, h_r) h_r \right)^* \alpha \right) \right),$$

where $\left(R(\,\cdot\,, h)h\right)^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined as $\left(\left(R(\,\cdot\,, h)h\right)^* \alpha \right)(v) = \alpha(R(v, h)h)$.

$$\frac{\mathbf{D} V}{dt}(t) = D_s|_{s=0} \left(\mathbf{\Gamma}(g)^t_{t+s} V_{g(t+s)} \right).$$

Cartan–Schouten connections

- Left-invariant Cartan connections: $\nabla^{\mathrm{L}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric;
- (λ-)Cartan–Schouten connections: $\nabla^{\mathrm{L}}_u v = \lambda[u,v]$, $\lambda \in [0,1]$;
- Curvature and tensor

$$R^{\mathrm{L}}(u,v)w = \lambda(\lambda-1)[[u,v],w], \qquad T^{\mathrm{L}}(u,v) = (2\lambda-1)[u,v];$$

- Special cases:

$\lambda = 0$, $(-)$-connection	$R = 0,\ T(v,w) = -[v,w]$	$\mathbf{\Gamma}(g)^t_s = d\mathrm{L}_{g_t}\, d\mathrm{L}_{g_s^{-1}}$	(left translation)
$\lambda = \frac{1}{2}$, (0)-connection	$R(u,v)w = -\frac{1}{4}[[u,v],w],\ T = 0$	$\mathbf{\Gamma}(g)^t_s = d\mathrm{L}_{g_t}\, \mathcal{T}\exp\left(-\frac{1}{2}\int_s^t \mathrm{ad}_{g_r^{-1}\circ dg_r}\right) d\mathrm{L}_{g_s^{-1}}$	
$\lambda = 1$, $(+)$-connection	$R = 0,\ T(v,w) = [v,w]$	$\mathbf{\Gamma}(g)^t_s = d\mathrm{R}_{g_t}\, d\mathrm{R}_{g_s^{-1}}$	(right translation)

- Bi-invariance of Cartan–Schouten connections: the right bilinear map associated with λ-Catran–Schouten connection is

$$\nabla^{\mathrm{R}}_u v = (\lambda - 1)\,\mathrm{ad}_u\, v;$$

- Reconstruct SDE:

$$\mathbf{d}g_t = d(\mathrm{L}_{g_t})_e\left(u_t dt + \textstyle\sum_{r=1}^k h_{t,r} dW_t^r\right).$$

Theorem

Let G be a Lie group endowed with a Cartan–Schouten connection. Let $L : \mathbb{R} \times TG \to \mathbb{R}$ be a left-invariant Lagrangian and $\ell : \mathbb{R} \times G \times \mathfrak{g} \to \mathbb{R}$ be its restriction. For a diffusion $g(t) \in G$ satisfying (*). The following are equivalent:

(i) *The stochastic variational principle $\delta \mathcal{S}[g] = 0$ holds for variations $\frac{\partial}{\partial \epsilon} g_\epsilon^w(t) = \mathbf{\Gamma}(g_\epsilon^w)_0^t v(t)$;*

(ii) *$g(t)$ satisfies the stochastic Euler–Lagrange equation* (S-EL)*;*

(iii) *The reduced stochastic variational principle $\delta \mathbb{E} \int \ell(t, g(t), u(t)) dt = 0$ holds with variations of the form*

$$\delta u = D_t(d(L_{g(t)^{-1}})_{g(t)} \delta g) + \mathrm{ad}_{u(t)}(d(L_{g(t)^{-1}})_{g(t)} \delta g) - \frac{\lambda}{2} \mathrm{ad}_{h(t)} \mathrm{ad}_{h(t)} \left(d(L_{g(t)^{-1}})_{g(t)} \delta g \right);$$

(iv) *The stochastic Euler–Poincaré equations hold:*

$$D_t(\frac{\delta \ell}{\delta v}) - \mathrm{ad}_u^* \left(\frac{\delta \ell}{\delta v}\right) - \lambda \left(\mathrm{ad}_h^*(H(\frac{\delta \ell}{\delta v})) - \frac{1}{2} \mathrm{ad}_h^* \mathrm{ad}_h^* (\frac{\delta \ell}{\delta v}) \right) = d(L_{g(t)})_e^* (\frac{\delta \ell}{\delta g}). \tag{S-EP-CS}$$

Remark

3 Stochastic Euler–Poincaré reduction

Mechanical case: For a left-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and

$$\ell(t,g,u) = \frac{1}{2}\langle u,u\rangle + \Phi(t,g),$$

the reduced equation becomes,

$$D_t u - \operatorname{ad}_u^\dagger u - \lambda\left(\operatorname{ad}_h^\dagger H(u) - \frac{1}{2}\operatorname{ad}_h^\dagger \operatorname{ad}_h^\dagger u\right) = d(\mathrm{L}_g)_e^* \frac{\delta\Phi}{\delta g},$$

by identifying $\frac{\delta\ell}{\delta u} \in \mathfrak{g}^*$ to $u \in \mathfrak{g}$ and denote $\langle \operatorname{ad}_u^\dagger v, w\rangle := \langle v, \operatorname{ad}_u w\rangle, \forall u, v, w \in \mathfrak{g}$.

- For Cartan–Schouten connection, the curvature and torsion are covariantly constant, i.e., $\nabla R = \nabla T = 0$.
- $h = 0$: $D_t = \partial_t$, so deterministic Euler–Poincaré dynamics is recovered.
- $\lambda = 0$: noise enters through D_t and reconstruction; the Euler–Poincaré operator is classical.
- $H(\frac{\delta\ell}{\delta v}) = 0$: only the double-coadjoint correction remains.

Explicit form of variation for $(+,-)$ connections

Let $w(t)$ be a curve in $\mathcal{H}_0([0,T];e)$. If $e_\varepsilon(t)$ is the maximal integral curve of $\varepsilon w^{\mathrm{L}}(t)$ (resp. $\varepsilon w^{\mathrm{R}}(t)$) with initial condition e, then

$$\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} e_\varepsilon(t) = w(t), \quad \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} e_\varepsilon(t)^{-1} = -w(t).$$

- $\lambda = 0$: The following two definitions of variation for g are equivalent
$$\frac{\partial}{\partial\varepsilon} g_\varepsilon^w(t) = d(\mathrm{L}_{g_\varepsilon^w})_e w(t), g_0^w(t) = g(t) \iff g_\varepsilon(t) = g(t)e_\varepsilon(t).$$
For every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, we have $u_\varepsilon(t) = \dot{e}_\varepsilon(t)^{-1} e_\varepsilon(t) + \mathrm{Ad}_{e_\varepsilon(t)^{-1}} u(t)$.
- $\lambda = 1$: The following two definitions of variation for g are equivalent
$$\frac{\partial}{\partial\varepsilon} g_\varepsilon^w(t) = d(\mathrm{R}_{g_\varepsilon^w})_e w(t), g_0^w(t) = g(t) \iff g_\varepsilon(t) = e_\varepsilon(t)g(t).$$
For every $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, we have $u_\varepsilon(t) = u(t) + \mathrm{Ad}_{g_\varepsilon(t)^{-1}}(\dot{e}_\varepsilon(t)e_\varepsilon(t)^{-1})$.

Theorem

Suppose that G is a Lie group endowed with a left-invariant metric and a left-invariant Levi-Civita connection ∇, whose left bilinear map is $\nabla^{\mathrm{L}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Let $L : \mathbb{R} \times TG \to \mathbb{R}$ be a left-invariant Lagrangian and $\ell : \mathbb{R} \times G \times \mathfrak{g} \to \mathbb{R}$ be its restriction.

(i) *A diffusion g satisfying (*) is a stationary point of*

$$\delta \mathbb{E} \int_0^T \ell(t, g(t), u(t)) dt = 0,$$

with variations of the form

$$\delta u = D_t(d(\mathrm{L}_{g^{-1}})_g \delta g) + \mathrm{ad}_{\tilde{u}(t)}(d(\mathrm{L}_{g^{-1}})_g \delta g) - \sum_{r=1}^k \nabla^{\mathrm{L}}_{h_r} \nabla^{\mathrm{L}}_{h_r}(d(\mathrm{L}_{g^{-1}})_g \delta g) + \bar{K}(d(\mathrm{L}_{g^{-1}})_g \delta g),$$

where $\tilde{u} = u - \frac{1}{2} \sum_{r=1}^k \nabla^{\mathrm{L}}_{h_r} h_r$ the operator $\bar{K} : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\bar{K}(v) := \frac{1}{2} \sum_{r=1}^k \left(\nabla^{\mathrm{L}}_{h_r}[h_r, v] + \nabla^{\mathrm{L}}_{[h_r, v]} h_r \right). \tag{1}$$

Stochastic E–P reduction: Levi–Civita case

Theorem

(ii) *The following stochastic Euler–Poincaré equation holds:*

$$D_t\left(\frac{\delta \ell}{\delta v}\right) - \mathrm{ad}^*_{\tilde{u}}\left(\frac{\delta \ell}{\delta v}\right) + \sum_{r=1}^{k}\left(-\nabla^{\mathrm{L}*}_{h_r} H_r\left(\frac{\delta \ell}{\delta v}\right) + \nabla^{\mathrm{L}*}_{\nabla^{\mathrm{L}}_{h_r} h_r}\left(\frac{\delta \ell}{\delta v}\right)\right) + K\left(\frac{\delta \ell}{\delta v}\right) = d(\mathrm{L}_{g(t)})^*_e \frac{\delta \ell}{\delta g}, \quad \text{(S-EP-LC)}$$

where the operator $K : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by

$$K(\alpha)v := \alpha(\bar{K}(v)) = \textstyle\sum_{r=1}^{k} \frac{1}{2}\alpha\left(\nabla^{\mathrm{L}}_{h_r}([h_r, v]) + \nabla^{\mathrm{L}}_{[h_r,v]} h_r\right). \tag{2}$$

- If u and ℓ are independent with g, then the stochastic Euler–Poincaré equation (S-EP-LC) coincides with the one obtained in [Arnaudon–Chen–Cruzeiro'14].
- By the Koszul formula, the operator can be expressed as
$$K(\alpha)v = -\textstyle\sum_{r=1}^{k} \frac{1}{2}\alpha\left(\mathrm{ad}^{\dagger}_{[h_r,v]} h_r + \mathrm{ad}^{\dagger}_{h_r}([h_r, v])\right).$$
- If $\{H_r\}_{r=1}^{k}$ is an orthonormal basis of $\mathfrak{g}$, then $K(v) = -\frac{1}{2}(\Delta v + \mathrm{Ric}(v))$ by metric dual.

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4 Examples

Example 1: Heisenberg group H_3

The three-dimensional Heisenberg group

$$H_3 = \left\{ g(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}.$$

Heisenberg algebra $\mathfrak{h}_3$: canonical basis

$$E_1 = \frac{\partial}{\partial a}\Big|_e,\ E_2 = \frac{\partial}{\partial b}\Big|_e,\ E_3 = \frac{\partial}{\partial c}\Big|_e.$$

Commutator relation

$$[E_1, E_2] = E_3,\ [E_1, E_3] = [E_2, E_3] = 0.$$

Dual basis

$$\theta^1 = da\big|_e,\ \theta^2 = db\big|_e, \theta^3 = dc\big|_e.$$

For $v = v^i E_i$ and $\alpha = \alpha_i \theta^i$ with their component vectors

$$\operatorname{ad}_v w = (0, 0, v^1 w^2 - v^2 w^1),$$

$$\operatorname{ad}_v^* \alpha = (-\alpha_3 v^2, \alpha_3 v^1, 0).$$

The two-step nilpotency

$$\operatorname{ad}_h^* \operatorname{ad}_h^* u = 0.$$

Example 1: Harmonic oscillator on H_3

Consider a diffusion $g_t = g(a_t, b_t, c_t)$ and constant vector $h \in \mathfrak{h}_3$,

$$\circ dg_t = d(\mathrm{L}_{g_t})_e \big(u_t\, dt + h \circ dW_t\big),$$

whose coordinate form is

$$\circ da_t = u_t^1 dt + h^1 \circ dW_t,$$
$$\circ db_t = u_t^2 dt + h^2 \circ dW_t,$$
$$\circ dc_t = \big(u_t^3 + a_t u_t^2\big) dt + \big(h^3 + a_t h^2\big) \circ dW_t.$$

The harmonic oscillator reduced Lagrangian

$$\ell(g, v) = \frac{1}{2}\big((v^1)^2 + (v^2)^2 + (v^3)^2\big) - \frac{1}{2}(a^2 + b^2 + c^2).$$

Using the Euclidean inner product $\langle \cdot, \cdot \rangle$ to identify $\mathfrak{h}_3^*$ with $\mathfrak{h}_3$.

Harmonic-oscillator S–EP system

$$\begin{cases} D_t u^1 + u^2 u^3 + \lambda h^2 \mathcal{H}(u^3) = -a, \\ D_t u^2 - u^1 u^3 - \lambda h^1 \mathcal{H}(u^3) = -b - ac, \\ D_t u^3 = -c. \end{cases}$$

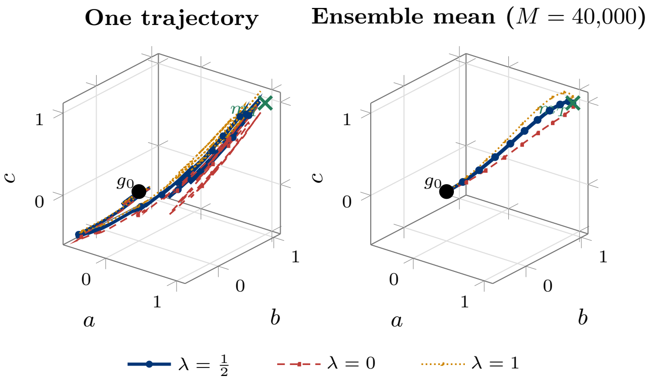
- The mean derivative has the form

$$D_t u^i = \frac{\partial u^i}{\partial t} + \mathcal{U}(u^i) + \frac{1}{2}\mathcal{H}(\mathcal{H}(u^i)),$$

with the operator

$$\mathcal{H} = h^1 \partial_a + h^2 \partial_b + (h^3 + a h^2)\partial_c,$$
$$\mathcal{U} = u^1 \partial_a + u^2 \partial_b + (u^3 + a u^2)\partial_c.$$

An numerical implementation



- Initial point $g_0 = (0, 0, 0)$ and terminal distribution $\rho_1^* = \mathcal{N}\big((1, 1, 1), 0.25\mathbb{I}_3\big)$;
- Noise intensity $h = (1, 1, 1)$;

Example 2: rotation group $SO(3)$

- Identification $\mathfrak{so}(3)$ with $\mathbb{R}^3$ by $\widehat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$

$$v = (v^1, v^2, v^3) \mapsto \widehat{v} = \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix}.$$

  Identificated $\mathfrak{so}(3)^*$ to $\mathbb{R}^3$ by $\breve{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)^*$

$$\breve{\mu}(\hat{v}) = \mu \cdot v, \ \ \mu, v \in \mathbb{R}^3.$$

- For the standard basis e_i of $\mathbb{R}^3$, set $E_i = \widehat{e_i}$

$$[E_1, E_2] = E_3, \ \ [E_2, E_3] = E_1, \ \ [E_3, E_1] = E_2.$$

- The adjoint operators

$$\mathrm{ad}_u\, v = u \times v, \qquad \mathrm{ad}_u^*\, \mu = -u \times \mu.$$

Left-invariant vector fields

The field generated by E_i is $\mathcal{X}_i(g) = d(\mathrm{L}_g)_e E_i$. For $g = (g_{ab}) \in SO(3)$, the coordinate expressions

$$\mathcal{X}_1 = \sum_{a=1}^{3} \left(g_{a3} \frac{\partial}{\partial g_{a2}} - g_{a2} \frac{\partial}{\partial g_{a3}} \right),$$

$$\mathcal{X}_2 = \sum_{a=1}^{3} \left(-g_{a3} \frac{\partial}{\partial g_{a1}} + g_{a1} \frac{\partial}{\partial g_{a3}} \right),$$

$$\mathcal{X}_3 = \sum_{a=1}^{3} \left(g_{a2} \frac{\partial}{\partial g_{a1}} - g_{a1} \frac{\partial}{\partial g_{a2}} \right).$$

For $F \in C^\infty(SO(3))$,

$$(\mathcal{X}_i F)(g) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F\big(g \exp(\varepsilon E_i)\big) = dF_g(gE_i).$$

Example 2: free rigid body

For $SO(3)$ endowed with Cartan–Schouten connection $\nabla_u v = \lambda[u,v]$ and a left-invariant inertia metric

$$\langle v, w\rangle_{\mathbb{I}} = \sum_{i=1}^{3} I_i v_i w_i,\ \ v = v_i E_i, w = w_i E_i,$$

Consider $\ell(g,v) = \frac{1}{2} v\cdot \mathbb{I}v$ and a diffusion satisfying

$$\circ dg_t = g_t\left(\widehat{u(t,g_t)}\,dt + \sum_{i=1}^{3} \frac{1}{\sqrt{I_i}}\widehat{e}_i \circ dW_t^i\right).$$

The total mean derivative

$$D_t\mu = \left(\partial_t + \mathcal{U} + \tfrac{1}{2}\sum_{i=1}^{3}\tfrac{1}{I_i}\mathcal{X}_i^2\right)\mu =: \mathcal{A}\mu,$$

where $\mathcal{U} = u^j \mathcal{X}_j$.

S–EP equation

$$D_t\mu + u\times\mu + \lambda\sum_i I_i^{-1} e_i \times \mathcal{X}_i\mu + \frac{\lambda}{2}\sum_i I_i^{-1} e_i\times(e_i\times\mu) = 0, \qquad \mu = \mathbb{I}u.$$

$$\begin{cases} I_1\mathcal{A}u^1 + (I_3 - I_2)u^2u^3 + \lambda\left(I_3 I_2^{-1}\mathcal{X}_2 u^3 - I_2 I_3^{-1}\mathcal{X}_3 u^2\right) \\ \qquad - \dfrac{\lambda I_1}{2}\left(I_2^{-1} + I_3^{-1}\right)u^1 = 0, \\ I_2\mathcal{A}u^2 + (I_1 - I_3)u^1u^3 + \lambda\left(I_1 I_3^{-1}\mathcal{X}_3 u^1 - I_3 I_1^{-1}\mathcal{X}_1 u^3\right) \\ \qquad - \dfrac{\lambda I_2}{2}\left(I_1^{-1} + I_3^{-1}\right)u^2 = 0, \\ I_3\mathcal{A}u^3 + (I_2 - I_1)u^1u^2 + \lambda\left(I_2 I_1^{-1}\mathcal{X}_1 u^2 - I_1 I_2^{-1}\mathcal{X}_2 u^1\right) \\ \qquad - \dfrac{\lambda I_3}{2}\left(I_1^{-1} + I_2^{-1}\right)u^3 = 0. \end{cases}$$

Example 2: free rigid body

For $SO(3)$ endowed with Levi–Civita connection $\nabla^{\mathbb{I}}$ and a left-invariant inertia metric

$$\langle v, w\rangle_{\mathbb{I}} = \sum_{i=1}^{3} I_i v_i w_i, \;\; v = v_i E_i, w = w_i E_i,$$

Consider $\ell(g, v) = \frac{1}{2} v \cdot \mathbb{I} v$ and a diffusion satisfying

$$\circ dg_t = g_t \left(\widehat{u(t, g_t)}\, dt + \sum_{i=1}^{3} \frac{1}{\sqrt{I_i}} \widehat{e_i} \circ dW_t^i \right).$$

The mean derivative

$$D_t \mu = \left(\partial_t + \mathcal{U} + \frac{1}{2} \sum_{i=1}^{3} \frac{1}{I_i} \mathcal{X}_i^2 \right) \mu =: \mathcal{A}\mu$$

where $\mathcal{U} = u^j \mathcal{X}_j$.

S–EP equation

$$D_t \mu + u \times \mu - \sum_{i=1}^{3} \left(\nabla^{\mathbb{I}}_{h_i}\right)^* (\mathcal{H}_i \mu) + \mathcal{K}_{\mathbb{I}} \mu = 0, \;\; \mu = \mathbb{I} u.$$

$$\begin{cases} I_1 \mathcal{A} u^1 + (I_3 - I_2) u^2 u^3 + \dfrac{I_1 + I_3 - I_2}{2 I_2} \mathcal{X}_2 u^3 \\ \qquad - \dfrac{I_1 + I_2 - I_3}{2 I_3} \mathcal{X}_3 u^2 + \dfrac{(I_2 - I_3)^2}{2 I_2 I_3} u^1 = 0, \\ I_2 \mathcal{A} u^2 + (I_1 - I_3) u^1 u^3 - \dfrac{I_2 + I_3 - I_1}{2 I_1} \mathcal{X}_1 u^3 \\ \qquad + \dfrac{I_1 + I_2 - I_3}{2 I_3} \mathcal{X}_3 u^1 + \dfrac{(I_3 - I_1)^2}{2 I_1 I_3} u^2 = 0, \\ I_3 \mathcal{A} u^3 + (I_2 - I_1) u^1 u^2 + \dfrac{I_2 + I_3 - I_1}{2 I_1} \mathcal{X}_1 u^2 \\ \qquad - \dfrac{I_1 + I_3 - I_2}{2 I_2} \mathcal{X}_2 u^1 + \dfrac{(I_1 - I_2)^2}{2 I_1 I_2} u^3 = 0. \end{cases}$$

Outlook

- Infinite-dimensional framework
- Semidirect product and contact reduction
- Solvability and numerical method

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Conclusion

- Stochastic Euler–Lagrange equation on manifold with linear connections
- Stochastic Euler–Poincaré reduction
- Cartan–Schouten & Levi–Civita connections

Thanks for your attention!