Geometry of infinite-dimensional manifolds and applications to integrable systems

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V.G Kac classifies Lie groups and Lie algebras into 4 classes (with non trivial intersections)

1. groups of diffeomorphisms of a manifold and associated Lie algebras of vector fields
2. groups (algebras) of smooth applications from a manifold into a finite dimensional Lie group (Lie algebra)
3. classical Lie groups and Lie algebras of operators acting on a Hilbert or a Banach space
4. Kac-Moody algebras.

Outline

Outline

Lecture 1: The fundamental building blocks of infinite-dim. Geometry

1. Manifolds
2. Tangent bundles and their relatives
3. Tools from Functional Analysis
4. Geometric structures

Why infinite-dimensional geometry?

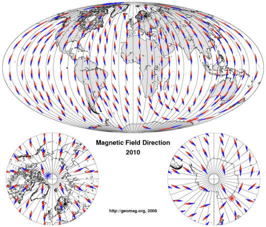
At the backstage of finite-dimensional geometry

- existence of geodesics on a finite-dimensional manifold is an infinite-dimensional phenomenon
  - initial value problem or shooting : geodesic is a solution of a Cauchy problem, i.e. a fixed point of a contraction in an appropriate infinite-dimensional space of curves
  - 2 boundary value problem: geodesic is a curve minimising an energy functional on a infinite-dimensional space of curves

Why infinite-dimensional geometry?

At the backstage of finite-dimensional geometry

- natural objects on a finite-dimensional manifold are elements of an infinite-dimensional space (vector fields, Riemannian metrics, mesures...)
- Each time one want to vary the geometry of a finite-dimensional manifold, one ends up with a infinite-dimensional manifold (of Riemannian metric, of connexions, of symplectic forms....)



Why infinite-dimensional geometry?

- because most shape spaces are infinite-dimensional

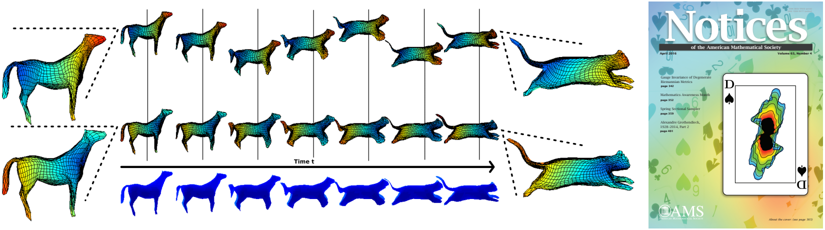
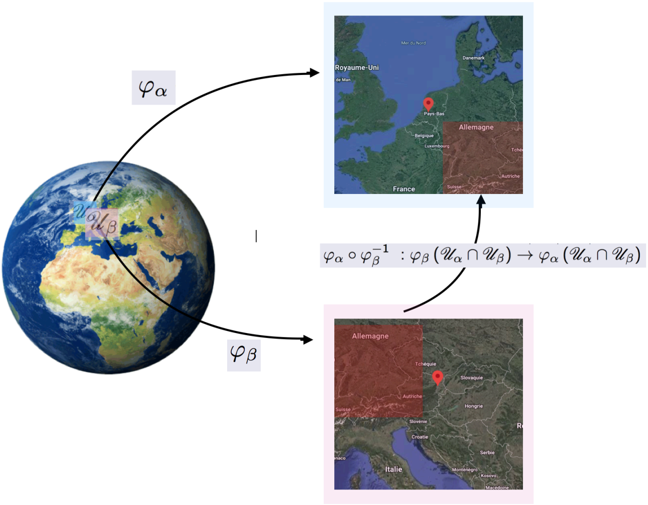
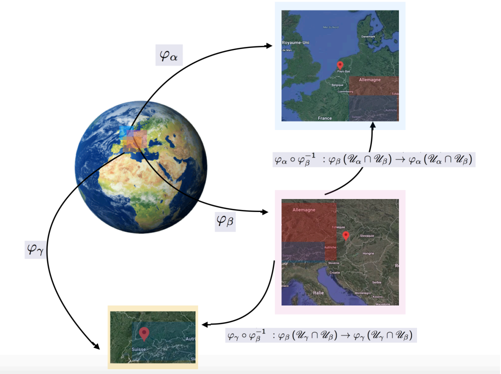


Figure: **Image taken from Tumpach et al: *Gauge Invariant Framework for Shape Analysis of Surfaces*, TPAMI 2016 and Notices of AMS**

Definition of an infinite-dimensional manifold



Definition of an infinite-dimensional manifold



The notion of manifold is build out from the notion of SMOOTH maps (or $\mathscr{C}^k$, or $\mathscr{C}^w$) between MODEL SPACES, the crucial condition on the set of smooth maps is the CHAIN RULE.

Charts and complete Atlas

Definition of a chart

Definition of an atlas

$\mathscr{C}^k$ equivalent atlases

Manifold = Hausdorff topological space with an equivalence class of $\mathscr{C}^k$ atlases

Charts and complete Atlas

Definition of a chart

A **chart** on a topological space $\mathscr{M}$ is a triple $(\mathscr{U}, \varphi, \mathscr{F}_\alpha)$ where $\mathscr{U}$ is an open set in $\mathscr{M}$ and ϕ an homeomorphism from $\mathscr{U}$ to an open set in a model topological vector space $\mathscr{F}_\alpha$.

Definition of an atlas

An **atlas** on a topological space $\mathscr{M}$ is a collection of charts $(\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha)_{\alpha\in\mathscr{I}}$ such that $\cup_{\alpha\in\mathscr{I}}\mathscr{U}_\alpha = \mathscr{M}$

Atlas of class $\mathscr{C}^k$

An **atlas** $\mathscr{A} = (\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha)_{\alpha\in\mathscr{I}}$ on $\mathscr{M}$ is of class $\mathscr{C}^k$ if all transition maps are $\mathscr{C}^k$-maps between the model topological vector spaces :
$\forall(\mathscr{U}_\alpha, \varphi_\alpha, \mathscr{F}_\alpha) \in \mathscr{A}$ and $(\mathscr{U}_\beta, \varphi_\beta, \mathscr{F}_\beta) \in \mathscr{A}$ such that $\mathscr{U}_\alpha \cap \mathscr{U}_\beta \neq \emptyset$
$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \to \varphi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$ is of class $\mathscr{C}^k$

Charts and complete Atlas

$\mathscr{C}^k$ equivalent atlases

Two atlases on a topological space $\mathscr{M}$ are said to be $\mathscr{C}^k$ **equivalent** if their union is of class $\mathscr{C}^k$

Manifolds of class $\mathscr{C}^k$

A **manifold of class** $\mathscr{C}^k$ $(k \geq 0)$ is an Hausdorff topological space endowed with an equivalence class of $\mathscr{C}^k$-atlases.

Hausdorff space

A topological space $\mathscr{M}$ is said to be **Hausdorff** if for any pair of distinct points $f_0 \neq f_1$ in $\mathscr{M}$ one can find two disjoints open sets $\mathscr{U}_0 \cap \mathscr{U}_1 = \emptyset$ in $\mathscr{M}$ such that $f_0 \in \mathscr{U}_0$ and $f_1 \in \mathscr{U}_1$

Remarks

On the global level

WE WILL NOT ASSUME that a manifold $\mathscr{M}$ is

- **paracompact** (every open cover has an open refinement that is locally finite)
- **admits smooth partitions of unity**
- **second countability** (the topology has a countable base)
- **separability** (there exists a countable dense subset)
- **Lindelöf** (every open cover has a countable subcover)

On the local level

WE WILL NOT ASSUME existence of **smooth bump functions**
BUT the manifolds will be **first-countable** (every point has a countable neighbourhood basis) because our model spaces will be metrizable

Complete metric spaces

Metric space

A metric space is a space $\mathscr{M}$ endowed with a **distance function** $d\ : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$

- $d(f_0, f_1) = 0 \Leftrightarrow f_0 = f_1$
- $d(f_0, f_1) = d(f_1, f_0)$
- $d(f_0, f_1) \leq d(f_0, f_2) + d(f_2, f_1)$

Cauchy sequence

A sequence $\{f_k\}_{k\in\mathbb{N}}$ in a metric space $\mathscr{M}$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N > 0$ suth that $d(f_n, f_m) < \varepsilon$ for all $n, m > N$

Complete metric space

A metric space $\mathscr{M}$ is said to be **complete** if any Cauchy sequence of elements in $\mathscr{M}$ converges

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ Locally Convex Spaces

Hilbert space H = **complete** vector space for the distance given by an inner product $= \langle \cdot, \cdot \rangle \ : H \times H \to \mathbb{R}^+$

- symmetric :$\langle x, y \rangle = \langle y, x \rangle$
- bilinear : $\langle x, y + \lambda z \rangle = \langle x, y \rangle + \lambda \langle x, z \rangle$
- non-negative : $\langle x, x \rangle \geq 0$
- definite : $\langle x, x \rangle = 0 \Rightarrow x = 0$

$H^* = H$ (Riesz Theorem).

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ **Banach** $\subset$ Fréchet $\subset$ Locally Convex spaces

Banach space B = complete vector space for the distance given by a norm $= \|\cdot\| \ : B \to \mathbb{R}^+$

- triangle inequality : $\|x + y\| \leq \|x\| + \|y\|$
- absolute homogeneity : $\|\lambda x\| = |\lambda| \|x\|$.
- non-negative : $\|x\| \geq 0$
- definite : $\|x\| = 0 \Rightarrow x = 0$.

B^* = Banach space.

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ **Fréchet** $\subset$ Locally Convex spaces

Fréchet space F = complete Hausdorff vector space for the distance $d\ : F \times F \to \mathbb{R}^+$ given by a countable family of semi-norms $\|\cdot\|_n$:

$$d(x, y) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}$$

$F^* \neq$ Fréchet space if F not Banach, but locally convex
$F^{**} =$ Fréchet space.

What are the Model spaces of infinite-dim. geometry?

Hilbert $\subset$ Banach $\subset$ Fréchet $\subset$ **Locally Convex spaces**

Locally Convex spaces = Hausdorff topological vector space whose topology is given by a (possibly not countable) family of semi-norms.

References :

- **An introduction to infinite-dimensional differential geometry**, Schmeding
- **Infinite-dimensional Lie groups**, Neeb, Glöckner
- **The convenient setting of global analysis**, Kriegl, Michor, Cabau, Pelletier
- **Diffeological spaces**, Souriau
- **Frölicher spaces**, Frölicher
- **Ringed spaces**, Egeileh, Michel, and Wurzbacher
- **Comparative smootheologies**, Stacey

What are the smooth maps between the model spaces?

Differentiable function on $\mathbb{R}^n$

For a function $f : \mathscr{U} \subset \mathbb{R} \to \mathbb{R}$, there are 3 equivalent notions of been **differentiable** at $x \in \mathscr{U}$

- $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists and is finite
- $\exists L$ such that $\lim_{h \to 0} \frac{f(x+h)-f(x)-Lh}{h} = 0$
- there exists a function $g : \mathscr{U} \subset \mathbb{R} \to \mathbb{R}$, such that $f(x+h) = f(x) + f'(x)h + g(h)$ and $\lim_{h \to 0} \frac{g(h)}{h} = 0$

Remark

On $\mathbb{R}$, a differentiable function is automatically continuous

In the Banach context

Fréchet differentiability in the Banach context

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be two Banach spaces. A map $P\ :\mathscr{U}\subset\mathscr{B}_1\to\mathscr{B}_2$ is **Fréchet differentiable** at $f_0\in\mathscr{B}_1$ if there exists a **continuous linear operator** $DP(f_0)\ :\mathscr{B}_1\to\mathscr{B}_2$ such that

$$P(f_0+h)=P(f_0)+DP(f_0)(h)+\|h\|_1\cdot\varepsilon(h)\quad\text{with}\quad\lim_{h\to 0}\|\varepsilon(h)\|_2=0$$

Remark

No continuity is assumed in the definition of Fréchet differentiability, but **Fréchet differentiable at** $f_0\Rightarrow$ **continuous at** f_0

In the Banach context

$\mathscr{C}^1$ in the Banach context

A map $P : \mathscr{U} \subset \mathscr{B}_1 \to \mathscr{B}_2$ between Banach spaces is $\mathscr{C}^1$ if it is Fréchet differentiable on $\mathscr{U}$ and the derivative DP is continuous as a map from $\mathscr{U}$ into the Banach space $L_c(\mathscr{B}_1, \mathscr{B}_2)$ of continuous linear operators from $\mathscr{B}_1$ to $\mathscr{B}_2$

Smooth maps between Banach spaces

By induction one defines the notion of smooth maps on Banach spaces.

In the Fréchet context

Directional derivative

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two Fréchet spaces and $P : \mathscr{U} \subset \mathscr{F}_1 \to \mathscr{F}_2$ a **continuous** non-linear map. P admits a **derivative at** f_0 **in the direction** of $h \in \mathscr{F}_1$ if the following limit exists

$$DP(f_0)(h) = \lim_{t \to 0} \frac{P(f_0 + th) - P(f_0)}{t}$$

One says that P is differentiable at f_0 if it admits directional derivatives in every direction $h \in \mathscr{F}_1$

$\mathscr{C}^1$ in the Fréchet context

A map $P : \mathscr{U} \subset \mathscr{F}_1 \to \mathscr{F}_2$ between Fréchet spaces is $\mathscr{C}^1$ if it is differentiable in $\mathscr{U}$ and the derivative DP is continuous as a map from $\mathscr{U} \times \mathscr{F}_1$ into $\mathscr{F}_2$

Comparaison of the two notions of $\mathscr{C}^1$-maps

Remarks

no linearity assumed but if P is $\mathscr{C}^1$ then $DP(f)h$ is always linear in h

On a Banach space

- $\mathscr{C}^1$ in the Banach context $\Rightarrow$ $\mathscr{C}^1$ in the Fréchet context
- $\mathscr{C}^2$ in the Fréchet context $\Rightarrow$ $\mathscr{C}^1$ in the Banach context [Keller]

Taylor formula

If $P : \mathscr{U} \subseteq \mathscr{F} \to G$ is $\mathscr{C}^2$ and if the path connecting f and $f+h$ lies in $\mathscr{U}$ then

$$P(f+h) = P(f) + DP(f)(h) + \int_0^1 (1-t) D^2 P(f+th)(h,h) dt$$

Examples of infinite-dimensional manifolds

Spheres

The sphere in a Hilbert space is a smooth Hilbert manifold

Remark : The sphere in a Banach space is not smooth in general.

Linear Grassmannians

The projective space of an Hilbert space is a Hilbert manifold
The Grassmannian of p-dimensional subspaces in an Hilbert space is Hilbert manifold (p finite)
The Grassmannian of subspaces in an Hilbert space with infinite dimension and codimension is a Banach manifold

Examples of infinite-dimensional manifolds

Restricted Linear Grassmannians

Let $H = H_+ \oplus H_-$ be a decomposition of an Hilbert space into the sum of two closed infinite-dimensional orthogonal subspaces. The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(H)$ denotes the set of all closed subspaces W of H such that the orthogonal projection $p_- : W \to H_-$ belongs to some given Schatten class (compact operator, Hilbert-Schmidt, trace class operators,...), then one obtain a Grassmannian manifold modelled on the space of operator from H_- to H_+ in this given Schatten class.

Reference

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- A. Tahiri, A.B. Tumpach, *The Restricted Schatten-class Grassmannian as affine coadjoint orbit*
- E. Andruchow, G. Larotonda, *Hopf-Rinow Theorem in the Sato Grassmannian*

Examples of infinite-dimensional manifolds

Manifolds of maps

The space of smooth maps from a compact manifold into a finite-dimensional manifold is a Fréchet manifold

Space of sections

The space of smooth section of a finite-dimensional vector bundle over a compact manifold is a Fréchet manifold

Examples of infinite-dimensional manifolds

Fréchet Lie groups

- The **group of diffeomorphisms** $\mathscr{D}(M)$ of a compact manifold M
- The **group of volume preserving diffeomorphisms** $\mathscr{D}_{\mathrm{vol}}(M)$ of a compact Riemannian manifold M

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- R.S. Hamilton, *The inverse function Theorem of Nash and Moser*
- J. Milnor, *Remarks on infinite-dimensional Lie groups*
- H. Glöckner, K.H. Neeb, *Banach-Lie Quotients, Enlargibility, and Universal Complexifications*
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- M. Bauer, M. Bruveris, P.W. Michor, *Overview of the Geometries of Shape Spaces and Diffeomorphism Groups*

Examples of infinite-dimensional manifolds

Manifolds of curves

The space of H^1 curves from $[0,1]$ into a finite-dimensional Riemannian manifold M is a Hilbert manifold, and the critical points of the energy functional are the geodesics of M

Square Root Velocity Framework

Length one curves can be seen as points on the sphere of an Hilbert space.

Arc-length parameterized curves

The space of arc-length parameterized curves $c \ : [0,1] \to \mathbb{R}^n$ is a Fréchet submanifold of the space of all parameterized curves.

Examples of infinite-dimensional manifolds

References

- W.P.A. Klingenberg, *Riemannian Geometry*
- S. Lahiri, D. Robinson, E. Klassen, *Precise Matching of PL Curves in RN in the Square Root Velocity Framework*
- A. Schmeding, *Manifolds of absolutely continuous curves and the square root velocity framework*
- E. Celledoni, S. Eidnes, A. Schmeding, *Shape analysis on homogeneous spaces: a generalised SRVT framework*
- S. Preston, A.B.Tumpach, *Quotient Elastic Metrics on the manifold of arc-length parameterized plane curves*

Shape spaces are non-linear manifolds

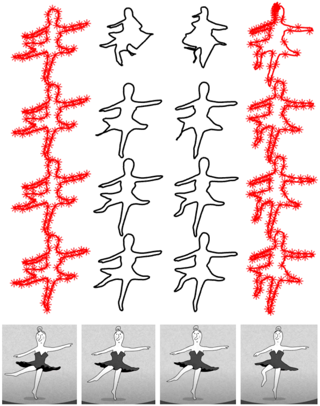
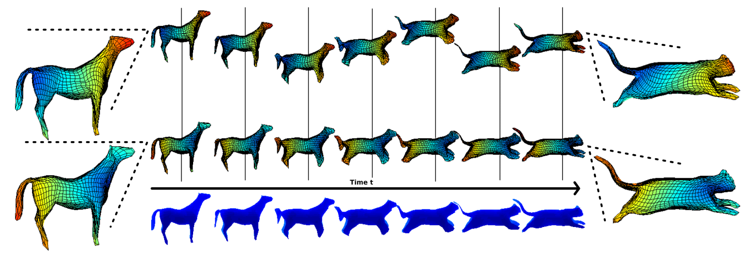


Figure: First line : linear interpolation between some parameterized ballerinas, second line : linear interpolation between arc-length parameterized ballerinas, third line : geodesic on a hilbert sphere, fourth line : improvement of third line, fifth line : ground truth

Examples from Shape Analysis



Pre-shape space $\mathscr{F} := \{f \text{ embedding } : \mathbb{S}^2 \to \mathbb{R}^3\} \subset \mathscr{C}^\infty(\mathbb{S}^2, \mathbb{R}^3)$
Shape space $\mathscr{S} :=$ 2-dimensional submanifolds of $\mathbb{R}^3$

Examples of infinite-dimensional manifolds

Non-linear Grassmannians and non-linear Flags

The space of embeddings $N \hookrightarrow M$ from a compact manifold N into a finite-dimensional manifold M is a Fréchet manifold. More generally the space of flags $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_k \subset M$ is a Fréchet manifold

References

- S. Haller, C. Vizman, *nonlinear Flag manifolds as coadjoint orbits*, *Nonlinear Grassmannians: plain, decorated, augmented*
- I.Ciuclea, C. Vizman, *Manifolds of vortex loops as coadjoint orbits*,
- I. Ciuclea, A.B.T., C. Vizman, *Shape spaces of nonlinear flags*
- F. Gay-Balmaz, C. Vizman, *Vortex sheets in ideal 3D fluids, coadjoint orbits, and characters*

What is the Tangent vector ?

Kinetic tangent vector

A Kinetic tangent vector $X \in T_{f_0}\mathscr{M}$ is an equivalence classes of curves $c(t)$, $c(0) = f_0$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Remark : Kinetic tangent vectors can be identified in a chart with elements of the model space.

Operational tangent vector

An operational tangent vector at $f_0 \in \mathscr{M}$ is a linear map $D : C^{\infty}_{f_0}(\mathscr{M}) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(PQ)(f_0) = DP\ Q(f_0) + P(f_0)\ DQ$$

Remark : Even on a Hilbert space there exists operational tangent vectors which are not Kinetic tangent vectors.

What is a cotangent vector ?

(Kinetic) cotangent vector

A (kinetic) cotangent vector $F \in T^*_{f_0}\mathscr{M}$ is an continuous linear functional on the space of kinetic tangent vectors

Remark : Kinetic cotangent vectors can be identified in a chart with elements of the **continuous dual of the model space**. If the model space is a Fréchet space which is not a Banach space, the continuous dual of the model space is not a Fréchet space

What are the tangent bundle and cotangent bundles?

Tangent bundle

On a Fréchet manifold $\mathscr{M}$ the set $T\mathscr{M}$ of all kinetic tangent vectors has a natural structure of smooth Fréchet manifold with canonical projection $\pi\ : T\mathscr{M} \to \mathscr{M}$, $X \in T_{f_0}\mathscr{M} \mapsto f_0$

Cotangent bundle

On a Banach manifold $\mathscr{M}$ the set $T^*\mathscr{M}$ of all (kinetic) cotangent vectors has a natural structure of smooth Banach manifold with canonical projection $\pi\ : T^*\mathscr{M} \to \mathscr{M}$, $F \in T^*_{f_0}\mathscr{M} \mapsto f_0$

What is the problem with tensor products?

Universal property for algebraic tensor product

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two $\mathbb{K}$-vector spaces. The **algebraic tensor product** $\mathfrak{g}_1 \otimes_a \mathfrak{g}_2$ of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the unique (up to isomorphim of $\mathbb{K}$-vector spaces) $\mathbb{K}$-vector space such that there exists a bilinear mapping

$$B : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{g}_1 \otimes_a \mathfrak{g}_2$$

having the following **universal property:**

If $B_1 : \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{g}$ is any bilinear mapping into a $\mathbb{K}$-vector space $\mathfrak{g}$, then there exists a unique linear mapping $L : \mathfrak{g}_1 \otimes_a \mathfrak{g}_2 \to \mathfrak{g}$ such that $B_1 = L \circ B$. **Remark:** The universal property implies in particular that

the algebraic dual of the algebraic tensor product is the $\mathbb{K}$-vector space of $\mathbb{K}$-valued bilinear maps on $\mathfrak{g}_1 \times \mathfrak{g}_2$.

What is the problem with tensor products?

Grothendiek lists 14 different norms on tensor products of Banach spaces

Which norm on $\mathfrak{g}_1 \otimes_a \mathfrak{g}_2$ to complete it into a Banach space?
the projective cross norm?
the injective cross norm?
or one of the 12 others?

Example:

For an Hilbert space $\mathfrak{h}$ and its continous dual $\mathfrak{h}^*$, the injective tensor product of $\mathfrak{h}^*$ and $\mathfrak{h}$ is the Banach space of compact operators on $\mathfrak{h}$, whereas the projective tensor product is the Banach space of trace class operators on $\mathfrak{h}$.

Continuous Multilinear maps

In the Banach case, continuous multilinear maps forms a Banach space

For Banach spaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ and $\mathfrak{h}$, the space

$$\mathrm{L}^k(\mathfrak{g}_1, \mathfrak{g}_2, \ldots \mathfrak{g}_k; \mathfrak{h})$$

of continuous k**-multilinear maps** from the product Banach space $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ to the Banach space $\mathfrak{h}$ is itself a Banach space

Symmetric Multilinear maps

For any Banach space $\mathfrak{g}$, a multilinear map $\mathbf{t} \in L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$ is said to be **symmetric** if and only if

$$\mathbf{t}(e_1, \ldots, e_k) = \mathbf{t}(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$$

for any $e_1, \ldots e_k$ in $\mathfrak{g}$ and any permutation σ in the group $\mathscr{S}(k)$ of all permutations of $\{1, \ldots, k\}$

Continuous Multilinear maps

Skew-symmetric Multilinear maps

For any Banach space $\mathfrak{g}$, a multilinear map $\mathbf{t} \in L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$ is said to be **skew-symmetric** if and only if

$$\mathbf{t}(e_1, \ldots, e_k) = \mathrm{sign}(\sigma)\mathbf{t}(\mathrm{e}_{\sigma(1)}, \ldots, \mathrm{e}_{\sigma(\mathrm{k})})$$

for any $e_1, \ldots e_k$ in $\mathfrak{g}$ and any permutation σ of $\{1, \ldots, k\}$, where $\mathrm{sign}(\sigma)$ denotes the signature of σ.

The space $S^k\mathfrak{g}^*$ consisting of **symmetric multilinear maps** on a Banach space $\mathfrak{g}$ is a closed subspace of $L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$, hence a Banach space

The space $\Lambda^k\mathfrak{g}^*$ consisting of **skew-symmetric multilinear maps** on $\mathfrak{g}$ is a closed subspace of $L^k(\mathfrak{g}, \ldots, \mathfrak{g}; \mathbb{K})$, hence a Banach space

Continuous Multilinear maps

Symmetric Multilinear maps on a Banach manifold

If $\mathscr{M}$ is a Banach manifold, the space $S^k T^*\mathscr{M}$ of **symmetric multilinear maps** on $T\mathscr{M}$ has a natural structure of Banach manifold, and of vector bundle over $\mathscr{M}$

A section $g \ : \mathscr{M} \to T^*\mathscr{M}$ is called a **symmetric tensor** on $\mathscr{M}$.

Skew-symmetric Multilinear maps on a Banach manifold

If $\mathscr{M}$ is a Banach manifold, the space $\Lambda^k T^*\mathscr{M}$ of **skew-symmetric multilinear maps** on $T\mathscr{M}$ has a natural structure of Banach manifold, and of vector bundle over $\mathscr{M}$

A section $\omega \ : \mathscr{M} \to T^*\mathscr{M}$ is called a **skew-symmetric tensor** on $\mathscr{M}$.

What are the Tools from Functional Analysis?

Banach-Picard fixed point Theorem or Contraction Theorem

(E, d) complete metric space
$f\ : E \to E$ contraction of E : $d(f(x), f(y)) \leq kd(x, y)$ where $k \in (0, 1)$

$$\Rightarrow \begin{cases} \exists\, !\ x \in E, f(x) = x \\ \forall x_0 \in E,\ \textit{the sequence } x_{n+1} = f(x_n) \textit{ converges to } x \end{cases}$$

What are the Tools from Functional Analysis?

Hahn-Banach Theorem

E locally convex space
$A \subset E$ a convex
$x \in E$, $x \notin \overline{A}$

$$\Rightarrow \exists \textit{ continuous functional } \ell \; : E \to \mathbb{R} \textit{ with } \ell(x) \notin \overline{\ell(A)}$$

What are the Tools from Functional Analysis?

Open mapping Theorem

$$\begin{cases} F \ \textit{Fréchet} \\ G \ \textit{Fréchet} \end{cases} \text{or} \begin{cases} F \ \textit{webbed locally convex} \\ G \ \textit{inductive limit of Baire locally convex spaces} \end{cases}$$

$$L : F \to G \text{ continuous, linear, and surjective}$$
$$\Rightarrow L \text{ is open}$$

What are the Tools from Functional Analysis?

Cauchy-Lipschitz Theorem in the Banach case

- $\mathscr{I} \subset \mathbb{R}$ be an interval containing 0
- $\mathscr{U}$ open set of a Banach space $\mathscr{B}$
- $P \ : \mathscr{I} \times \mathscr{U} \to \mathscr{B}$

such that

- $\|P(t,f)\| \leq C \quad \forall (t,f) \in \mathscr{I} \times \mathscr{U}$
- $\|P(t,f_1) - P(t,f_0)\| \leq C'\|f_1 - f_0\| \quad \forall t \in \mathscr{I}, \forall f_0, f_1 \in \mathscr{U}$

For any $f_0 \in \mathscr{U}$ we can find a neighboorhood $\tilde{\mathscr{U}}$ of f_0 and an $\varepsilon > 0$ such that for any $f \in \tilde{\mathscr{U}}$ the Cauchy problem

$$\frac{d}{dt}\phi(t,f) = P(t,\phi(t,f))$$

has a unique solution with initial condition $\phi(0,f) = f$ on $[-\varepsilon, \varepsilon]$. Moreover if P is $\mathscr{C}^p$, $t \to \phi(t,x)$ is $\mathscr{C}^p$ for any $f \in \tilde{\mathscr{U}}$

What are the Tools from Functional Analysis?

Cauchy Theorem in the Fréchet case

Let $P : \mathscr{U} \subset \mathscr{F} \to \mathscr{F}$ be a **smooth Banach map**. Then $\forall f_0 \in \mathscr{U}$, $\exists \tilde{\mathscr{U}} \ni f_0$ and $\varepsilon > 0$ s.t. $\forall f \in \tilde{\mathscr{U}}$

$$\frac{d}{dt} f = P(f)$$

has a unique solution with initial condition $f(0) = f$ on $0 \leq t \leq \varepsilon$ depending smoothly on t and f.

We say that P is a smooth Banach map between two Fréchet spaces $\mathscr{F}$ and $\mathscr{G}$ if we can factor $P = Q \circ R$ where $R : U \subset \mathscr{F} \to W \subset B$ and $Q : W \subset B \to V \subset \mathscr{G}$ are smooth maps and B is a Banach space.

Reference: R.S. Hamilton, *The inverse function Theorem of Nash and Moser*

What are the Tools from Functional Analysis?

Inverse function Theorem

Theorem

*Let $f : \mathscr{U} \subset B_1 \to B_2$ be a $\mathscr{C}^1$-map between **Banach** spaces. If $Df(a)$ is invertible at $a \in \mathscr{U}$, then there exists an open neighborhood $\mathscr{V}_a$ of $a \in \mathscr{U}$ and an open neighborhood $\mathscr{V}_{f(a)} \subset B_2$ suth that $f : \mathscr{V}_a \to \mathscr{V}_{f(a)}$ is a $\mathscr{C}^1$-diffeomorphism.*

Counterexample : $\exp : \mathrm{Lie}(\mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Diff}(\mathbb{S}^1)$ not locally onto.

Theorem (Nash-Moser)

*Let $f : \mathscr{U} \subset F_1 \to F_2$ be a smooth tame map between **Fréchet** spaces. Suppose that the equation for the derivative $Df(x)(h) = k$ has a unique solution $h = L(x)k$ for all $x \in \mathscr{U}$ and $\forall k \in F_2$ and that the family of inverses $L : \mathscr{U} \times F_2 \to F_1$ is a smooth tame map. Then f is locally invertible and each local inverse is a smooth tame map.*

What are the Tools from Functional Analysis?

Theorems :	Hilbert	Banach	Fréchet	Locally Convex
Banach-Picard	$\surd$	$\surd$	$\surd$	X
Open Mapping	$\surd$	$\surd$	$\surd$	F webbed G limit of Baire
Hahn-Banach	$\surd$	$\surd$	$\surd$	$\surd$
Cauchy Theorem	$\surd$	$\surd$	Hamilton	X
Inverse function	$\surd$	$\surd$	Nash-Moser	X

Which **Geometric structures** can we consider?

Riemannian $\subset$ Symplectic $\subset$ Poisson Geometry

Riemannian metric = smoothly varying inner product on a manifold M

$$\begin{array}{cccc} g_x : & T_xM \times T_xM & \rightarrow & \mathbb{R} \\ & (U, V) & \mapsto & g_x(U, V) \end{array}$$

strong Riemannian metric = for every $x \in M$, $g_x : T_xM \rightarrow (T_xM)^*$ is an isomorphism

weak Riemannian metric = for every $x \in M$, $g_x : T_xM \rightarrow (T_xM)^*$ is just injective

Levi-Cevita connection may not exist for a weak Riemannian metric.

Which **Geometric structures** can we consider?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Symplectic form = smoothly varying skew-symmetric bilinear form

$$\begin{array}{cccc} \omega_x & : & T_xM \times T_xM & \rightarrow & \mathbb{R} \\ & & (U, V) & \mapsto & \omega_x(U, V) \end{array}$$

with $dw = 0$ and $(T_xM)^{\perp_w} = \{0\}$

strong symplectic form = for every $x \in M$, $\omega_x \ : T_xM \rightarrow (T_xM)^*$ is an isomorphism

weak symplectic form = for every $x \in M$, $\omega_x \ : T_xM \rightarrow (T_xM)^*$ is just injective

Darboux Theorem does not hold for a weak symplectic form

Which **Geometric structures** can we consider?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Hamiltonian Mechanics

(M, g) **strong Riemannian manifold**
$\Rightarrow (T^*M, \omega)$ **strong symplectic manifold**

Kinetic energy = Hamiltonian

$$\begin{array}{cccc} H & : & T^*M & \to & \mathbb{R} \\ & & \eta_x & \mapsto & g_x(\eta_x^\sharp, \eta_x^\sharp) \end{array}$$

geodesic flow = flow of Hamiltonian vector field $X_H : dH = \omega(X_H, \cdot)$

Which **Geometric structures** can we consider?

$$\left.\begin{array}{l}\text{Riemannian}\\ \text{Symplectic}\\ \text{Complex}\end{array}\right\} \subset \text{ Kähler} \subset \text{hyperkähler Geometry}$$

Complex structure $=$ smoothly varying endomorphism J of the tangent space s.t. $J^2 = -1$.

Integrable complex structure : s. t. there exists an holomorphic atlas
Formally integrable complex structure : with Nijenhuis tensor $= 0$

Newlander-Nirenberg Theorem is not true in general:
formal integrability does not imply integrability.

Which **Geometric structures** can we consider?

Riemannian $\subset$ Symplectic $\subset$ **Poisson Geometry**

Poisson bracket = family of bilinear maps
$\{\cdot,\cdot\}_U \ : \mathscr{C}^\infty(U) \times \mathscr{C}^\infty(U) \to \mathscr{C}^\infty(U)$, U open in the manifold M with

- skew-symmetry $\{f, g\}_U = -\{g, f\}_U$
- Jacobi identity $\{f, \{g, h\}_U\}_U + \{g, \{h, f\}_U\}_U + \{h, \{f, g\}_U\}_U = 0$
- Leibniz rule $\{f, gh\}_U = \{f, g\}_U h + g\{f, h\}_U$

A strong symplectic form defines a Poisson bracket by
$\{f, g\} = \omega(X_f, X_g)$ where $df = \omega(X_f, \cdot)$ and $dg = \omega(X_g, \cdot)$

A Poisson bracket may not be given by a bivector field

Reference: D. Beltita, T. Golinski, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics.

Possible definitions of **Poisson structures** in ∞-dim

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Reference	[7]	[8]	[26]	[3]	[36]	[24]
Definition number	11	14	18	20	23	27
1. Jacobi for $\mathcal{A}$	√	√	√	√	√	√
2. Jacobi for π on $\mathbb{F}^{\mathcal{A}}$					√	√
3. Localizability					√	√
4. Leibniz for $\mathcal{A}$	√	√	√	√	√	√
5. Separability of $\Gamma(TM)$			√	√	√	√
6. Separability of TM			√		√	√
7. $\mathbb{F}^{\mathcal{A}}$ vector bundle	√	√			√	√
8. Poisson anchor	√	√	√	√		
9. Poisson tensor	√	√			√	√
10. Hamiltonian v.f.	√	√	√	√		

TABLE 1. Summary of the properties of the different definitions of Poisson structures presented in Section 4. A √ sign means that the property is implied by the definition. Otherwise, the property is not necessarily satisfied.

Tomasz Goliński, Praful Rahangdale and A.B.T., *Poisson structures in the Banach setting: comparison of different approaches*

What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is :
"Never believe anything you have not proved yourself!"

- The distance function associated to a Riemannian metric may by the zero function (Example by Michor and Mumford).
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general : geodesic completeness $\neq$ metric completeness
- Darboux Theorem does not apply to weak symplectic forms
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations (even on a Hilbert space)
- a Poisson bracket may not be given by a bivector field (even on a Hilbert space)
- there are Lie algebras that can not be enlarged to Lie groups (Examples by Milnor or Neeb)
- partitions of unity may not exist
- at least 14 differents ways to define tensor products