Collective perturbations of Hamiltonian systems

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XXXIV International Fall Workshop on Geometry and Physics, 1 September 2026

Joint work with Archishman Saha and Cristina Stoica

Outline

Perturb a G-symmetric Hamiltonian system by collective Hamiltonians — possibly stochastically. What structure survives?

- Collective Hamiltonians and collective perturbations
- Stochastic perturbations: deterministic reduction, stochastic reconstruction
- Mirror collective systems on Lie groups
- Diffeomorphic image registration and stochastic pulsons

A. Saha, T. Schmah, C. Stoica, Stochastic Collective Hamiltonian Systems (In preparation)

Collective Hamiltonians

Setting: $(P, \{\cdot,\cdot\})$ a Poisson manifold.
G a Lie group acting freely, properly and canonically on P, with equivariant momentum map $J : P \to \mathfrak{g}^*$.

Definition A collective function is one of the form $f \circ J$ for some $f : \mathfrak{g}^* \to \mathbb{R}$.

A Hamiltonian of the form $H = f \circ J$ is called a collective Hamiltonian.

Collective Hamiltonian Theorem

$$X_{f\circ J}(z) = \left(\frac{\delta f}{\delta \mu}\right)_P (z),$$

where $(\cdot)_P(z)$ denotes the infinitesimal action of $\mathfrak{g}$ on P.

V. Guillemin and S. Sternberg, The moment map and collective motion, 1980

Symmetric and collective functions Poisson commute

Lemma *For any G-invariant $h \in \mathcal{C}^\infty(P)$, and any $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$,*

$$\{h, f \circ J\} = 0.$$

Proof.

$$\{h, f \circ J\} = dh(z) \cdot X_{f \circ J}(z) = dh(z) \cdot \left(\frac{\delta f}{\delta \mu}\right)_P (z) = 0.$$

$\square$

This reflects a **dual pair** structure (with $\pi : P \to P/G$):

$$\begin{tikzcd}
& (P,\ \{\cdot,\cdot\}) \arrow[dl, "\pi"'] \arrow[dr, "J"] & \\
\left(P/G,\ \{\cdot,\cdot\}^{\mathrm{red}}\right) & & \left(\mathfrak{g}^*,\ \{\cdot,\cdot\}_+\right)
\end{tikzcd}$$

Both legs are Poisson; $\ker T\pi \perp \ker TJ$ symplectically (leafwise).

A. Weinstein, The local structure of Poisson manifolds, 1983

Collective perturbations of Hamiltonian systems

In the same setting, let H_0 be a G-invariant Hamiltonian on P.

A collective perturbation of H_0 is a Hamiltonian of the form

$$H = H_0 + f \circ J,$$

for some $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$. *(The "perturbation" need not be small.)*

Energy H_0 is conserved by the Hamiltonian flow of H:

$$\begin{aligned} \dot{H}_0 &= \{H_0, H_0 + f \circ J\} \\ &= \{H_0, H_0\} + \{H_0, f \circ J\} = 0. \end{aligned}$$

But symmetry is broken (generically): H is not G-invariant.

What is conserved?

J itself is not conserved, but some functions of it are:

Proposition If $K \circ J$ is G-invariant, for $K \in \mathcal{C}^\infty(\mathfrak{g}^*)$, then $K \circ J$ is conserved by the perturbed system.

In particular this holds whenever K is Ad^*-invariant, i.e. a Casimir of the Lie-Poisson bracket.

Example (rotational symmetry): $G = SO(3)$ acting by (lifted) rotations, e.g. on

- n-body problems: $P = T^*(\mathbb{R}^3)^n$,
- the rigid body: $P = T^*SO(3)$.

J is angular momentum, and $K(x) = \|x\|$ is rotationally invariant.

The perturbed system **need not conserve** J.
But it **always conserves** $\|J\|$.

Stochastic collective perturbations

Now perturb by <u>noise</u> instead: let $f_1, \ldots, f_k \in \mathcal{C}^\infty(\mathfrak{g}^*)$, and consider the stochastic Hamiltonian system (in Stratonovich form)

$$\bullet d\Gamma = X_{H_0}(\Gamma)\, dt \;+\; \sum_{i=1}^{k} X_{f_i \circ J}(\Gamma) \bullet dY^i,$$

driven by arbitrary semimartingales $Y^1, \ldots, Y^k$.

- Energy H_0 is still conserved, and so is $K \circ J$ whenever it is G-invariant.
- But the system is <u>not</u> G-symmetric, so standard reduction of symmetric stochastic systems does not apply.

J.-A. Lázaro-Camí and J.-P. Ortega, Reduction, reconstruction, and skew-product decomposition of symmetric stochastic differential equations, 2009

Deterministic reduction, stochastic reconstruction

Let $z_0 \in P$, with momentum $\mu_0 = J(z_0)$, and let $z^{\mathrm{det}}(t)$ be an integral curve of the *unperturbed* system X_{H_0} with $z^{\mathrm{det}}(0) = z_0$.

Theorem 1 (Saha, Schmah, Stoica) Let $g(t)$ solve the Stratonovich equation in G

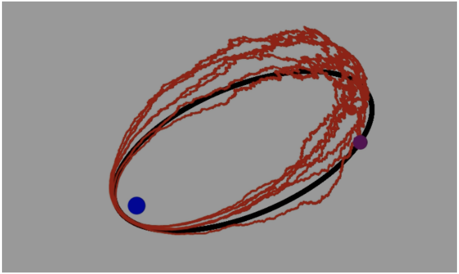
$$\bullet dg = \sum_{i=1}^{k} T_e R_g \left(\left. \frac{\delta f_i}{\delta \mu} \right|_{\mu = \mathrm{Ad}^*_{g^{-1}} \mu_0} \right) \bullet dY^i, \qquad g(0) = e.$$

Then $\Gamma = g \cdot z^{\mathrm{det}}$ solves the stochastic system.

- The projection to P/G is deterministic: it is the reduced dynamics of H_0.
- All the randomness enters through the stochastic phase $g(t)$.

Example: Kepler with a linear stochastic perturbation

$$\bullet d\Gamma = X_{H_0}(\Gamma)dt + \sum_{i=1}^{3} X_{\nu_i J^i}(\Gamma) \bullet dZ^i \,,$$



stochastic precession, $\|J\|$ conserved

deterministic radial dynamics, $\|\mathbf{q}(t)\|$

A. Saha, Moser Regularization of a Stochastically Perturbed Kepler Problem, Celestial Mechanics and Dynamical Astronomy, 2025

Motion on Lie groups: $P = T^*G$

G acts on itself two ways:

$$\begin{aligned}
&\text{(left translation)} \quad g \cdot h := L_g(h) := gh \\
&\text{(right translation)} \quad g \cdot h := R_g(h) := hg
\end{aligned}$$

Both have lifted actions on T^*G and corresponding coadjoint-equivariant momentum maps:

$$\begin{aligned}
&\text{(for left translation)} \quad J_L(g, \alpha) = T^*R_g\,(\alpha) =: \alpha \cdot g^{-1}, \\
&\text{(for right translation)} \quad J_R(g, \alpha) = T^*L_g\,(\alpha) =: g^{-1} \cdot \alpha.
\end{aligned}$$

$$\begin{array}{ccc}
 & (T^*G\ \{\cdot,\cdot\}) & \\
J_R \swarrow & & \searrow J_L \\
\left(\mathfrak{g}^*,\ \{\cdot,\cdot\}_-\right) & & \left(\mathfrak{g}^*,\ \{\cdot,\cdot\}_+\right)
\end{array}$$

Example: in the rigid body,

J_L is spatial momentum π,

J_R is body angular momentum $\Pi = R^{-1}\pi$.

Mirror collective Hamiltonians

$$H = \underbrace{f_L \circ J_R}_{\text{left-invariant}} + \underbrace{f_R \circ J_L}_{\text{right-invariant}} .$$

Theorem 2 (S, S, S) Let $\big(g_L(t), \mu_L(t)\big)$ solve $X_{f_L \circ J_R}$ in left-trivialised coordinates and $\big(g_R(t), \mu_R(t)\big)$ solve $X_{f_R \circ J_L}$ in right-trivialised coordinates, with compatible initial data. Then

$$g(t) = g_R(t)\, g_L(t)$$

is the base component of a solution of X_H.

There is also a stochastic version, with noise on both legs.

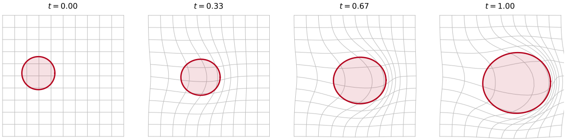
Continuum models

Let $D \subset \mathbb{R}^d$ be a domain, and let $G \subseteq \mathrm{Diff}(D)$ be a group of diffeomorphisms, with Lie algebra $\mathfrak{g}$ a space of smooth vector fields on D.

- Left translation: $L_\psi(\varphi) = \psi \circ \varphi$.
- Right translation (particle relabelling): $R_\psi(\varphi) = \varphi \circ \psi$,

Body and spatial momentum vector fields: $M = J_R$, $m = J_L$.

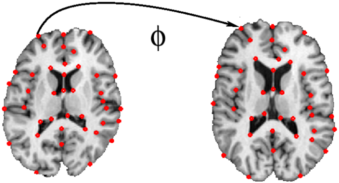
If $f_R(m) = \frac{1}{2}\|m\|_R^2$ for some vector space norm, then the solutions of $X_{f_R \circ J_L}$ are **geodesics** of a right-invariant metric:



The Lie-Poisson equations are in this context called EPDiff.

Diffeomorphic image registration

Given two images I and J with common domain Ω:



The *inexact matching problem*: find $\phi \in \mathrm{Diff}(\Omega)$ minimising

$$\mathcal{A}(\phi) := \underbrace{d\left(I \circ \phi^{-1}, J\right)}_{\text{image matching term}} + \lambda \underbrace{d(\phi, Id)}_{\text{regulariser}}$$

In LDDMM (Trouvé, Beg, Vialard, . . .), the regulariser is geodesic distance for a *right*-invariant metric, from a norm $\|\cdot\|_R = \|\cdot\|_V$ on a space V of vector fields (often RKHS). $H = \frac{1}{2}\|m\|_R^2$.

In left-LDM (Schmah, Risser, Vialard), a *left*-invariant metric instead: spatially-varying regularisation in the frame of the <u>source</u> image.
$H = \frac{1}{2}\|M\|_L^2$

Mirror-LDDMM

Combine both, in a mirror collective Hamiltonian:

$$H = \frac{1}{2}\|M\|_L^2 + \frac{1}{2}\|m\|_R^2.$$

By Theorem 2, the base solution curve is

$$\varphi_R(t) \circ \psi_L(t),$$

a composition of a right-geodesic and a left-geodesic.

For image registration, this represents spatially-varying regularisation in both:

- the frame of the “target image” J (the $\|m\|_R^2$ term, as in LDDMM);
- the frame of the “source image” I (the $\|M\|_L^2$ term, as in left-LDM).

Stochastic pulsons and landmark dynamics

Lie-Poisson equations on $\mathrm{Diff}(D)$ admit singular pulson solutions:

$$m(x,t) = \sum_{a=1}^{N} p_a(t)\,\delta\big(x - q_a(t)\big)$$

(q,p) satisfy canonical Hamilton equations (landmark dynamics)

Proposition The set of pulson $\Sigma_N = \left\{\sum_{a=1}^{N} p_a(t)\,\delta\big(x - q_a(t)\big)\right\}$ is invariant under stochastic Lie-Poisson equations. Dynamics on Σ_N reduces to a $2Nd$-dim'l canonical stochastic Hamiltonian system.

In the RKHS setting this is rigorous: stochastic landmark dynamics.

For smooth kernels, the composed stochastic pulson flows $\varphi_R(t) \circ \psi_L(t)$ solve the full stochastic mirror system.

Linear (SALT) noise: Holm 2015, Holm–Tyranowski 2016. Stochastic landmarks: Trouvé–Vialard 2012, Marsland–Shardlow 2017, Arnaudon–Holm–Sommer 2019.
New here: general collective noise,mirror systems

Summary

For H_0 G-invariant, perturbed by collective Hamiltonians $f_i \circ J$ (deterministically or stochastically):

- Energy H_0 is conserved; so is $K \circ J$ whenever G-invariant.
- The reduced dynamics are deterministic;
  all randomness enters through reconstruction.
- On Lie groups: mirror collective systems, solved by composing left- and right- solutions. Application: mirror-LDDMM.
- Singular (pulson / landmark) solutions persist under noise.

Thank you!