From Broad–integrability to Euler–Jacobi Theorem and back

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Based on a joint work with D. Murari

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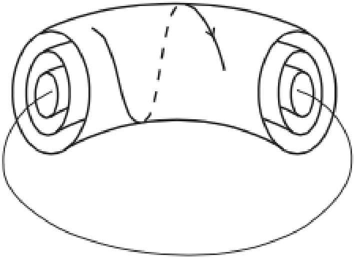
Integrability ...?



Outlines of the Talk

1. Hamiltonian integrable systems
   - Liouville integrability
   - Noncommutative integrability
2. Motivations for investigating non-Hamiltonian integrable systems
3. Broad–integrability
4. Euler-Jacobi Theorem
5. From Broad integrability to Euler–Jacobi and back
   - Broad implies Euler–Jacobi
   - Euler–Jacobi implies broad if
6. Final comments

Liouville–Arnol'd Theorem

Let (M, ω) be a $2d$–dimensional symplectic manifold and let $F : (f_1, \dots, f_d) : M \longrightarrow B \subseteq \mathbb{R}^d$

- be a submersion;
- have compact and connected level sets;
- have components pairwise in involutions $\{f_i, f_j\} = 0$ for all $i, j = 1, \dots, d$.

Let A be the set of regular values of F. Then for each $x \in A$

1. $F^{-1}(x) \cong \mathbb{T}^d$;
2. there exists an open neighborhood U_x of x in A and a diffeomorphism

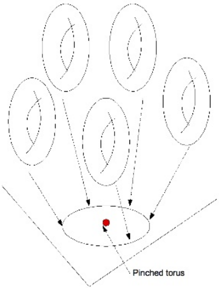
   $$(\boldsymbol{I}, \varphi) : F^{-1}(U_x) \longrightarrow V \times \mathbb{T}^d$$

   with $V \subseteq \mathbb{R}^d$ open, such that $\boldsymbol{I} = (I_1, \dots, I_d) = \kappa \circ F$ for some diffeomorphism $\kappa : F(U_x) \longrightarrow V$;
3. the coordinates $(\boldsymbol{I}, \varphi)$ on M are Darboux coordinates $\omega = d\boldsymbol{I} \wedge d\varphi$.

Geometrical and dynamical meaning of Liouville–Arnol'd Theorem

Geometrical properties

- M is $\mathbb{T}^d$–bundle with Lagrangian fibre;
- $\mathbb{T}^d \longrightarrow M \longrightarrow A$ is a "local" principal toric bundle, with structure group $\mathbb{T}^d$;
- action–angle coordinates $(\boldsymbol{I}, \varphi)$ are bundle (symplectic) coordinates.



Dynamical properties

If $h : M \longrightarrow \mathbb{R}$ is a *hamiltonian* such that $\{h, f_i\} = 0$ for each $i = 1, \ldots, d$, then

the flow of X_h is quasi–periodic on d–dimansional tori $F = const$, with frequencies depending on the action alone

Superintegrability or Noncommutative Integrability

Nekhoroshev–Mishenko–Fomenko Theorem

Let (M, ω) be a $2d$–dimensional symplectic manifold and let $F : (f_1, \ldots, f_{2d-n}) : M \longrightarrow B \subseteq \mathbb{R}^{2d-n}$

- be a submersion;
- have compact and connected level sets.

Let $\mathcal{B}$ be the set of regular values of F and assume that

MF1. there exists a function $P_{ij} : \mathcal{B} \longrightarrow \mathbb{R}$ such that

$$\{f_i, f_j\} = P_{ij} \circ f, \qquad i, j = 1, \ldots, 2d - n.$$

MF2. The matrix P with entries P_{ij} has (constant) rank $2d - 2n$ at all points of $\mathcal{B}$.

Nekhoroshev–Mishenko–Fomenko Theorem, 1972 and 1978

Then for each $x \in \mathcal{B}$

1. $F^{-1}(x) \cong \mathbb{T}^n$;
2. there exists an open neighborhood U_x of x in $\mathcal{B}$ and a diffeomorphism

$$(\boldsymbol{b}, \varphi) : F^{-1}(U_x) \longrightarrow \mathcal{B} \times \mathbb{T}^n, \quad \text{with } \mathcal{B} = \boldsymbol{b}(U_x) \subset \mathbb{R}^{2d-n};$$

   such that the level sets of F coincide with the level sets of $\boldsymbol{b}$, and writing $\boldsymbol{b} = (p_1, \ldots, p_{d-n}, q_1, \ldots, q_{d-n}, I_1, \ldots, I_n)$,

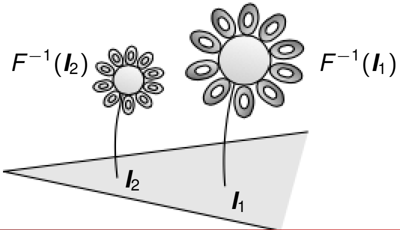
$$\omega_{U_x} = \sum_{k=1}^{d-n} dp_k \wedge dq_k + \sum_{j=1}^{n} dI_j \wedge d\varphi_j \ .$$

   The coordinates $(p, q, \boldsymbol{I}, \varphi)$ are called *generalized action–angles coordinates*.

- The theorem reduces to the LA Theorem for $n = d$.

Geometrical meaning of noncommutative integrability

- The fibration by the invariant tori is isotropic (condition *MF2.*) and has a polar (foliation) fibration.
- The base $\mathcal{B}$ of the fibration by the invariant tori is a Poisson manifold.
- The coordinates (p, q) are symplectic coordinates on the leaves of $\mathcal{B}$.



- The local actions $I_1, \ldots, I_n$ are (local) Casimirs of the Poisson structure of $\mathcal{B}$.
- The local actions $I_1, \ldots, I_n$ are local coordinates on the base manifold $\mathcal{A}$ of the polar. $\mathcal{A} =$ action manifold.

Dynamical meaning of noncommutative integrability

- Dynamical meaning of the isotropic fibration $\longrightarrow$ **if all the integrals of motion have been taken into consideration, they form the finest invariant fibration of (some part of) the phase space.**
- Dynamical meaning of the coisotropic fibration $\longrightarrow$ **all motions inside the same coisotropic fiber have the same frequencies.**

If $h : M \longrightarrow \mathbb{R}$ is a *hamiltonian* such that $\{h, f_i\} = 0$ for each $i = 1, \ldots, 2d - n$, then h is conjugated to a new hamiltonian $k = k(\boldsymbol{I})$ and

$$\dot{\varphi} = \frac{\partial k}{\partial \boldsymbol{I}}(\boldsymbol{I})$$

the flow is quasi–periodic on the invariant *n*–dimensional tori $F = const$, with frequencies depending on the actions alone.

Examples

- Harmonic oscillators in resonance;
- Euler–Poinsot system;
- Kepler problem.

Original Motivation

WHAT IS A COMPLETELY INTEGRABLE NONHOLONOMIC DYNAMICAL SYSTEM?

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A simple paradigmatic example: the nonholonomic oscillator

- $Q = S^1 \times \mathbb{R} \times S^1$ configuration space
- $H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \frac{1}{2}y^2$ Hamiltonian
- Constraint $p_z = y\, p_x$
- $\mathcal{D}_q = \mathrm{span}_{\mathbb{R}}\{\partial_y,\ \partial_x + y\partial_z\}$ constraint distribution
- $G = \mathbb{T}^2$ translation along x and y
- $\mathcal{G}_q = \mathrm{span}_{\mathbb{R}}\{\partial_x, \partial_z\}$,
- Dynamical vector field

$$\dot{x} = p_x\,, \quad \dot{y} = p_y\,, \quad \dot{p}_x = -\frac{y}{1+y^2} p_x p_y\,, \quad \dot{p}_y = -y$$

- p_x and p_z are not first integrals

A simple paradigmatic example: the nonholonomic oscillator

- $\mathcal{S} = (\mathcal{G} \cap \mathcal{D})_q = \mathrm{span}_{\mathbb{R}}\{\partial_x + y\partial_z\}$
- Consider
  - $f = f(y)$ G-invariant on Q
  - $\xi_Q = f(\partial_x + y\partial_z) \in \Gamma(\mathcal{S})$
  - ξ^{T^*Q} cotangent lift
- Impose

$$\xi^{T^*Q}(H)|_M = 0$$

- obtaining

$$(1 + y^2)f' + yf = 0 \quad \Rightarrow \quad J_{\xi_Q} = \xi_Q \cdot p|_D = p_x\sqrt{1 + y^2}$$

is a G-invariant **first integral** of the system.

The motions are quasi-periodic on $\mathbb{T}^3$:

- two independent first integrals
- 3 dynamical symmetries tangent to the level sets of the first integrals

A simple paradigmatic example: nonholonomic oscillator

The dynamics is G-invariant and descends to the quotient space $Q/G \cong \mathbb{R}^3$

$\Downarrow$

J_{ξ_Q} and $E_{\rm nf}$ descend to the quotient

$\Downarrow$

the reduced dynamics is periodic

X is Hamiltonian with respect to the rescaled nonholonomic-symplectic form $f\Omega_{\rm nh}$ and Hamiltonian $H_{\rm nh}$

$\Downarrow$

reduced-integrability by Liouville-Arnol'd theorem

$\Downarrow$

the complete dynamics is generically quasi–periodic on $\mathbb{T}^3$

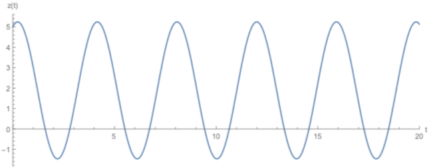
- Lagrangian

$$L = \frac{1}{2}m(\|\dot{x}\|^2) + \frac{1}{2}I_C\|\omega\|^2 - mg\,\psi\,;$$

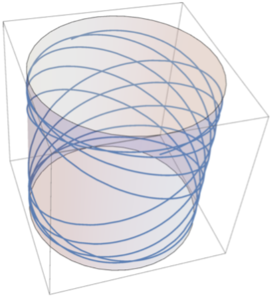
- Phase space

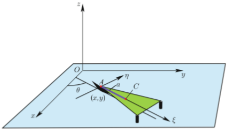
$$M_8 = \mathbb{R}^2 \times \mathrm{SO}(3) \times \mathbb{R}^2 \times \mathbb{R}$$

- Symmetry $G = \mathrm{SO}(2) \times \mathrm{SO}(3)$
- G-reduced space $M_4 \cong M_8/G$

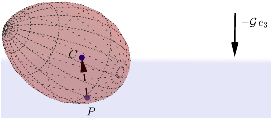


Oscillations of $z(t)$

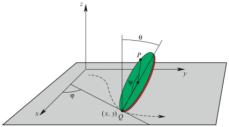




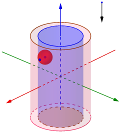
Chaplygin sleigh



Rolling solids

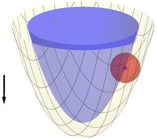


Rolling penny



Ball in a cylinder

Tippe-top



Ball in a cup

Caveat!!!

Nonholonomic mechanical systems:

- mechanical systems with constraints on the velocities;
- defined on a subbundle of TQ or of T^*Q;
- not Hamiltonian!!!
- any link symmetries first integrals ?

$$\downarrow$$

need a 'good' notion of integrability

$$\downarrow$$

quasi-periodic dynamics on invariant tori

What is an integrable nonholonomic system?

Using ideas of Kozlov, Borisov and Mamaev: integrability as reduction to quadratures, then

- *integrability by Lie's Theorem;*
- **integrability by Euler–Jacobi Theorem;**
- Hamiltonization $\rightsquigarrow$ *representation of a (reduced) system in (conformally) Hamiltonian form and then use LA Theorem.*

Do not give linear motions (if the level sets are compact and connected), but after a change of time.

- **Broad–integrability** (see Bogoyavlenskij 1998, Fedorov 1999, Zung 2018).

Guarantees the linearity of the flow (if the level sets are compact and connected).

Bogoyavlenskij's characterization of integrability

Let M be a smooth n-dimensional manifold and X a smooth vector field on M. Assume that for an integer $0 < k \leq n$ there exists a smooth map $F = (f_1, ..., f_{n-k}) : M \to F(M) \subseteq \mathbb{R}^{n-k}$ with compact and connected fibers, such that:

B1. $L_X f_\alpha = 0$, for any $\alpha = 1, \ldots, n-k$.

Let M_{reg} denote the manifold of regular points of F, and assume it is dense in M. Moreover, assume the existence of k everywhere linearly independent smooth vector fields $Y_1, ..., Y_k$ on M such that

B2. $\mathcal{L}_{Y_i} X = [Y_i, X] = 0$, $i = 1, ..., k$,

B3. $[Y_i, Y_j] = 0$, $i, j = 1, ..., k$,

B4. $\mathcal{L}_{Y_i} f_\alpha = 0$, for all $i = 1, ..., k$, $\alpha = 1, ..., n-k$.

Then:

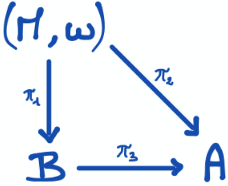
1. for every regular value $c \in F(M)$ of F, the fiber $F^{-1}(\{c\})$ of F is diffeomorphic to the $k-$dimensional torus $\mathbb{T}^k$;
2. F defines a locally trivial fibration of M_{reg} in k-dimensional tori;
3. X is conjugated to a vector field with linear flow in a neighborhood of each invariant torus.

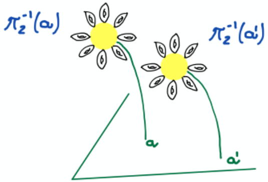
Why broad–integrability?

- extends the classical notions of complete and noncommutative integrability;
- "many" nonholonomic systems fit this criteria;
- many properties of integrable systems do not require the Hamiltonian structure: i.e global properties, "flower–like structure";
- the class of integrable systems enlarges, but the dynamical properties remain the same.

What do we lose?

- Integrability and reduction do not (necessarily) commute anymore;
- Noether's Theorem: in general symmetries do not give origin to first integrals.





Nonholonomic systems that fit Broad–integrability

- nonholonomic oscillator;
- snakeboard;
- vertical and falling disk;
- Routh's sphere;
- rolling ellissoid;
- ball on a vertical cylinder;
- heavy ball inside a (an uniformely rotating) surface of revolution;
- ...

Caveat

In the Hamiltonian integrability is ensured from only one set of data – the integrals of motion. This is made possible by the Lie algebra anti-homomorphism between functions and Hamiltonian vector fields

$$[X_f, X_g] = -X_{\{f,g\}}$$

specific to the Hamiltonian case, that provides a mechanism to produce (commuting) dynamical symmetries out of (commuting) integrals of motion.

Theorem (Euler–Jacobi)

Let X be a never-vanishing smooth vector field on an n-dimensional smooth manifold M, and let $F = (f_1, ..., f_{n-2}) : M \to F(M) \subseteq \mathbb{R}^{n-2}$ be a smooth map with compact and connected fibers such that

EJ1. $\mathcal{L}_X f_\alpha = 0$, for $\alpha = 1, \ldots, n-2$.

Let M_{reg} denote the manifold of regular points of F, and assume it is dense in M. Moreover let $\mu \in \Omega^n(M)$ be a volume form on M invariant with respect to X, i.e. $\mathcal{L}_X \mu = 0$. Then:

1. for every regular value $c \in F(M)$ of F, the level set $F^{-1}(\{c\})$ of F is diffeomorphic to the 2-dimensional torus $\mathbb{T}^2$;
2. F defines a locally trivial fibration of M_{reg} in 2-dimensional tori;
3. for every regular value $c \in F(M)$, there exists a neighborhood $U_c \subset F(M)$ of c and a diffeomorphism $\mathcal{C} = (f_1, ..., f_{n-2}, x_{n-1}, x_n) : F^{-1}(U_c) \to U_c \times \mathbb{T}^2$, such that in these coordinates X reads

$$\begin{cases} \dot{f}_\alpha = 0, \quad \alpha = 1, ..., n-2, \\ \dot{x}_a = \lambda_a(f_1, \ldots, f_{n-2})/\Phi, \quad a = n-1, n, \end{cases} \tag{1}$$

   for some function $\Phi \in C^\infty(U_c \times \mathbb{T}^2)$, $\Phi = \Phi(f_1, \ldots, f_{n-2}, x_{n-1}, x_n)$, and $\lambda_{n-1}, \lambda_n \in C^\infty(U_c)$.

Nonholonomic systems that fit Euler–Jacobi integrability

- nonholonomic oscillator;
- vertical and falling disk;
- Routh's sphere;
- rolling ellissoid;
- ball on a vertical cylinder;
- heavy ball inside a (an uniformely rotating) surface of revolution;
- ...
- Chaplygin's ball

Broad implies Euler–Jacobi

Proposition

Let (M, X) be a *B*-integrable system. Then there exists a volume form μ on M for which (M, μ, X) is *EJ*-integrable.

Euler–Jacobi implies broad if

Proposition

An *EJ*-integrable system (M, μ, X), with M an n-dimensional manifold, is *B*-integrable if there exists a one-form α on M constant along the flow of X and one of the two following conditions holds:

1. $df_1 \wedge ... \wedge df_{n-2} \wedge \alpha \in \Omega^{n-1}(M)$ is not proportional to $i_X\mu$,
2. there exists $z \in M$ such that $\alpha_z = 0$ but $(d\alpha)_z \neq 0$.

Proposition

Let $F = (f_1, \ldots, f_{n-2}) : M \to \mathbb{R}^{n-2}$ be a submersion with compact and connected fibers, and X, Y be everywhere linearly independent vector fields on M that are tangent to the fibers of F. X is *B*-integrable in 2-dimensional tori if one of the following conditions is satisfied:

I. there exist $g, h \in \mathcal{C}^\infty(M)$, h never vanishing, such that $[X, Y] = gY$, and $X(h) = -gh$,

II. there exist $g, h \in \mathcal{C}^\infty(M)$, h never vanishing, such that $[X, Y] = gX$, and $X(h) = -g$.

Euler–Jacobi implies broad if

Proposition

Let (M, μ, X) and (M, μ, Y) be two *EJ*-integrable systems with the same invariant fibration, and let X and Y be everywhere linearly independent vector fields. Assume there exists a function $\lambda \in \mathcal{C}^\infty(M)$ such that $[X, Y] = \lambda X$. Then X is *B*-integrable in circles, i.e., it has $n - 1$ functionally independent first integrals with compact fibers.

Final comments

- need for proper weaker notion of integrability;
- which could be the role of almost–symplectic and almost–Poisson structure?
- any new ideas how to couple first integrals and dynamical symmetries?
- weaken the notion of integrability(?) and use weaker notions of first integrals and dynamical symmetries?
- ???

THANK YOU FOR YOUR ATTENTION !!!

Essential bibliography

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