An introductory course on geometric field theories: the multisymplectic setting

First lecture: Introductory remarks. Previous geometric background

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XXXIV International Fall Workshop on Geometry and Physics

Universitat Rovira i Virgili (URV)

September 1–4, 2026

Course Programme

1. **First lecture: Introductory remarks. Previous geometric background.**
2. **Second lecture: Multisymplectic Lagrangian field theory.**
3. **Third lecture: Multisymplectic Hamiltonian field theory.**
4. **Fourth lecture: Symmetries. Gauge symmetries. Conservation laws.**

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Introductory remarks

Origin and basic ideas on field theories

From the historical point of view, the notion of field was introduced in the 19*th* century, to overcome the idea of action at a distance in the theories of *gravitational*, *electric* and *magnetic forces*.

In that context, a field is a scalar or a vector or tensor function of several variables (usually space-time coordinates), representing the space-time distribution of a physical magnitude; (i.e., a measurable physical property in a neighbourhood of every point of space at each moment of time). The field is originated by physical entities (particles) that have certain characteristics and acts producing a force on those same type of entities.

Later, its meaning was expanded to describe a wide variety of phenomena and problems in physics and mathematics (such as, *thermodynamics*, *fluid dynamics*, *wave propagation*, *theoretical physics*, or *geometry*).

All of them are described by partial differential equations which, in most cases, can be derived from a variational principle, using multiple variational calculus. Then, the field theory is of variational type.

Some examples of classical field theories

Name	Equation	Lagrangian
Heat	$\dfrac{\partial T}{\partial t} - k\nabla^2 T = 0$ T *Temperature*, k *thermal diffusivity*.	It is not variational
Navier–Stokes	$\rho\left(\dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla \mathrm{p} - \mu\nabla^2\mathbf{v} = \rho\,\mathbf{f}$ $\mathbf{v}$ *fluid velocity*, p *pressure*, ρ *density*, μ *viscosity*, $\mathbf{f}$ *external force*.	It is not variational
Wave (D'Alembert)	$\dfrac{1}{v^2}\dfrac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = 0$ ψ *wave displacement*, v *velocity prop.*	$\dfrac{1}{2}\left(\dfrac{\partial\psi}{\partial t}\right)^2 - \Vert\nabla\psi\Vert^2$
Maxwell	$\nabla\cdot\mathbf{E} = \dfrac{\rho}{\epsilon_o}\ ,\ \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t}$ $\nabla\cdot\mathbf{B} = 0\ ,\ \nabla\times\mathbf{B} = \mu_o\mathbf{j} + \mu_o\epsilon_o\dfrac{\partial\mathbf{E}}{\partial t}$ $\mathbf{E}$ *electric field*, $\mathbf{B}$ *magnetic field*, ρ *charge density*, $\mathbf{j}$ *current density*, ϵ_o *elec. constant*, μ_o *magn. constant*.	$\dfrac{1}{2}\left(\epsilon_o E^2 - \dfrac{B^2}{\mu_o}\right)$ $-\Phi\rho + \mathbf{A}\cdot\mathbf{j}$ Φ *scalar potential*, $\mathbf{A}$ *vector potential*.

Name	Equation	Lagrangian
Klein-Gordon	$\dfrac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - m^2\psi$ ψ *scalar field* .	$\dfrac{1}{2}\left(\dfrac{\partial\psi}{\partial t}\right)^2 - \dfrac{1}{2}\left((\nabla\psi)^2 + m^2\psi^2\right)$
Sine-Gordon	$\dfrac{\partial^2\psi}{\partial x\partial t} = \sin\psi$ ψ *scalar field* .	$\dfrac{1}{2}\dfrac{\partial\psi}{\partial x}\dfrac{\partial\psi}{\partial t} - \cos\,\psi$
Dirac	$\gamma^i\dfrac{\partial\psi}{\partial x^i} = m\mathrm{I}\psi$ ψ *scalar field* . γ^i *Dirac matrices*, I *identity matrix*	$\bar\psi\gamma^i\dfrac{\partial\psi}{\partial x^i} - \bar\psi\psi$
Laplace	$\nabla^2\psi = 0$ ψ *scalar field* .	$\dfrac{1}{2}\lvert\nabla\psi\rvert^2$
Minimal surfaces	$(1 + f_y{}^2)f_{xx} - 2f_xf_yf_{xy}$ $+(1 + f_x{}^2)f_{yy} = 0$ $f = f(x, y),\ f_x \equiv \dfrac{\partial f}{\partial x}\ ,\ \ f_y \equiv \dfrac{\partial f}{\partial y}$	$\sqrt{1 + f_x^2 + f_y^2}$

Name	Equation	Lagrangian
Einstein	$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathtt{R}\,\mathbf{g}_{\mu\nu} + \Lambda\,\mathbf{g}_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu}$ $\mathbf{g}$ *metric*, R *Ricci tensor*, $\mathtt{R}$ *scalar curv.*, T *SEM tensor*, Λ *cosmological ctn.*, G *gravitational ctn.*, c *light velocity*.	$\frac{c^4}{16\pi G}\sqrt{-g}(\mathtt{R}-2\Lambda)$ $+L_{SEM}$ $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\partial L_{SEM}}{\partial g^{\mu\nu}}$ $g = \det\,\mathbf{g}_{\mu\nu}$
Korteweg– de Vries	$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$ $u = \frac{\partial\phi}{\partial x}$ *wave amplitude*	$\frac{1}{2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t} - \left(\frac{\partial\phi}{\partial x}\right)^3$ $-\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}$

Others:

Generalized gravitational theories, Chern–Simons theory, string and branes theories, Monge–Ampère equation, Carrollian models, Ricci flow,

……

Covariant and multisymplectic formulations

Covariant formulations of field theories are relevant in theoretical physics.

Covariance means that the theory is invariant by arbitrary changes of reference systems or generic transformations of coordinates. This means that the field equations are of tensorial type.

For field theories of variational type, this property is manifest in the Lagrangian formalism. On them, in general, the fields are functions on spacetime and the spacetime coordinates are indistinguishable independent variables.

The ***multisymplectic formulation*** provides the most general and complete geometric framework for a covariant, finite-dimensional description of these theories (and also for non-covariant ones).

Other alternative geometrical finite-dimensional covariant models for classical field theories are the ***polysymplectic, k-symplectic*** and ***k-cosymplectic*** formulations.

All these structures are generalization of ***symplectic*** geometry.

Polysymplectic and k-(co)symplectic structures:

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Remark: There are also other infinite dimensional (geometric) formulations of classical field theories (covariant and non-covariant).

On them, the configuration space of the theory is the *space of solutions* of field equations and the formalism is usually developed in the tangent or the cotangent bundles of this infinite dimensional manifold.

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Researchers Cast

(Everyone who is here belongs here,
but not everyone who belongs here is here).

"USA school": H. Goldschmidt, M.J. Gotay, J. Isenberg, J.E. Marsden, R. Montgomery, S. Shkoller, J. Sniatycki, S.Z. Sternberg, *et al*

"Italian school": M. Ferraris, M. Francaviglia, G. Giachetta, L. Mangiarotti, M. Modugno, G. Sardanashvily, L. Vitagliano, *et al*.

"German school": E. Binz, H.R. Fischer, M. Forger, C. Gunther, C. Paufler, H. Roemer, L. Ryvkin, *et al*.

"French school": A. Awane, P.J. Dedecker, J. Kouneiher, F. Hélein, D. Vey, T. Wurzbacher, *et al*.

"Polish school": J. Kijowski, W.M. Tulczyjew, J. Grabowski, K. Grabowska,, W. Szczyrba, *et al*.

"British school": M. Crampin, J.K. Lawson, G. Martin, L.K. Norris, D.J. Saunders, *et al*.

"Belgium school": F. Cantrijn, W. Sarlet, J. Vankerschaver, *et al*.

"Czech school": M. Janyska, D. Krupka, O. Krupkova, *et al*.

"Romanian school": R.M. Miron, T.S. Ratiu, *et al*.

In other countries: P. Michor, I.V. Kanatchikov, *et al*.

"Spanish school": V. Aldaya, J.A. de Azcárraga, F. Cariñena, M. Castrillón, M. de León, A. Echeverría-Enríquez, P.L. García, X. Gràcia, L.A. Ibort, D. Martín de Diego, J.C. Marrero, M.C. Muñoz-Lecanda, J. Muñoz-Masqué, N. Román-Roy, M. Salgado, S. Vilariño, *et al*.

Geometric (multisymplectic) field theories

Geometric (multisymplectic) framework for classical field theories:

- An m-dimensional orientable manifold M, with volume form $\omega_{_M} \in \Omega^m(M)$; (typically *spacetime*).
- A fiber bundle $\pi \colon E \longrightarrow M$, with $\dim E = m + n$, which is the ***covariant configuration bundle*** of the theory.
  A ***field*** is a (local) section $\phi \colon U \subset M \to E$ of the projection π.
- A fiber bundle $\bar{\varrho} \colon \mathcal{F} \xrightarrow{\varrho} E \xrightarrow{\pi} M$, which is the ***(multi)phase space*** of the theory.
  A ***state of a field*** ϕ is a (local) section $\psi \colon U \subset M \to \mathcal{F}$ of the projection $\bar{\varrho}$ such that $\phi = \varrho \circ \psi$.
- $\mathcal{F}$ is endowed with an exact $(m+1)$-(pre)multisymplectic form $\Omega \in \Omega^{m+1}(\mathcal{F})$ (or with other alternative structures: polysymplectic or k-(co)symplectic forms).
- The field equations are expressed geometrically in several equivalent ways using these structures.

Note: The case $m = 1$ corresponds to **time-dependent mechanics**.

Remark: A very relevant particular case are ***Yang–Mills theories***, whose geometric description is made using principal bundles.

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Multisymplectic treatments for Yang–Mills theories have been developed:

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Statement and aim of the course

This course provides an introduction to the multisymplectic formulation of first-order variational classical field theories (those where the Lagrangians depend, at most, on first derivatives of the fields).

The objectives of the course are the following:

- To provide a concise introduction to classical field theories and their geometric formulation.
- To introduce the fundamental concepts of multisymplectic geometry and other additional geometric tools that will be used in the course.
- To study the geometry of first-order jet bundles and develop the Lagrangian formalism for first-order classical field theories, covering both regular and singular cases.
- To describe bundles of forms and construct the Hamiltonian formalism for regular and singular (almost-regular) field theories.
- To review the basic concepts of symmetries, gauge symmetries, and conservation laws in the multisymplectic setting.
- To illustrate the theory through some examples and applications drawn from modern physics.

Geometric background

Multisymplectic manifolds

Definition 2.1

Let $\mathcal{M}$ be a differentiable manifold, $\dim \mathcal{M} = n$, and $\Omega \in \varOmega^k(\mathcal{M})$, $k \leq n$

- *The form Ω is* ***1-nondegenerate*** *if, for every $p \in \mathcal{M}$ and $X_p \in \mathrm{T}_p\mathcal{M}$, then $i(X_p)\Omega_p = 0 \iff X_p = 0$.*
- *The form Ω is a* ***multisymplectic form*** *if it is closed and 1-nondegenerate.*

If Ω is exact then it is called an ***exact multisymplectic form****.*
If Ω is only closed then it is called a ***premultisymplectic form****.*
If Ω is only 1-nondegenerate then it is an ***almost-multisymplectic form****.*

- *A* ***(exact, pre, almost) multisymplectic manifold*** *(of* ***degree k****) is a couple $(\mathcal{M}, \Omega)$, where $\Omega \in \varOmega^k(\mathcal{M})$ is a (exact, pre, almost) multisymplectic form.*

For $p \in \mathcal{M}$, a form $\Omega \in \Omega^k(\mathcal{M})$ defines the vector bundle morphism

$$\begin{array}{rccc} \Omega^\flat : & \mathrm{T}_p\mathcal{M} & \to & \bigwedge^{k-1} \mathrm{T}_p^*\mathcal{M} \\ & X_p & \mapsto & \iota(X_p)\Omega_p \end{array} ,$$

and thus the corresponding morphism of $\mathcal{C}^\infty(M)$-modules

$$\begin{array}{rccc} \Omega^\flat : & \mathfrak{X}(\mathcal{M}) & \to & \Omega^{k-1}(\mathcal{M}) \\ & X & \mapsto & \iota(X)\Omega \end{array} .$$

Then, if Ω is closed, it is multisymplectic if, and only if, these morphisms are injective.

Examples:

- Multisymplectic manifolds of degree 2 are symplectic manifolds.
- Multisymplectic manifolds of degree n are orientable manifolds and the n-multisymplectic forms are volume forms.
- Bundles of k-forms (k-multicotangent bundles), $\Lambda^k(\mathrm{T}^*Q)$, with their *canonical* $(k+1)$-*forms* are exact multisymplectic manifolds of degree $k+1$.
- Jet bundles (over m-dimensional manifolds) endowed with the *Poincaré-Cartan* $(m+1)$-*forms* associated with (singular) *Lagrangian densities* are exact (pre)multisymplectic manifolds of degree $m+1$.

Remarks:

- In general, multisymplectic manifolds have no *Darboux coordinates*
For k-multicotangent bundles $\Lambda^k(\mathrm{T}^*Q)$, systems of *natural adapted coordinates* $(x^i, p_{i_1\ldots i_k})$ in $U \subset \Lambda^k(\mathrm{T}^*Q)$ are Darboux coordinates, and their canonical multisymplectic $(k+1)$-forms read as
$$\Omega\mid_U = \mathrm{d}p_{i_1\ldots i_k} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \ .$$
For a generic multisymplectic manifold $(\mathcal{M}, \Omega)$, additional properties are needed in order to assure the existence of Darboux coordinates. In particular, it is necessary they have additional structures (some distributions satisfying certain conditions). These are called special multisymplectic manifolds.
- If $\kappa\colon \mathcal{M} \to M$ is a fiber bundle, a *multisymplectic form* $\Omega \in \varOmega^k(\mathcal{M})$, $k = \dim\, M + 1$, is said to be a ***variational multisymplectic form*** if
$$\iota(X)\iota(Y)\iota(Z)\Omega = 0 \ , \ \text{for every } X, Y, Z \in \mathfrak{X}^{\mathrm{V}(\kappa)}(\mathcal{M}) \ .$$
(These are precisely the kinds of forms that arise in *variational classical field theories*).

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Multivector fields

Let $\mathcal{M}$ be a manifold with $\dim\mathcal{M} = n$.

Definition 2.2

*Sections of the multitangent bundle $\Lambda^m\mathrm{T}\mathcal{M} \to \mathcal{M}$ $(2 \leq m \leq n)$ are called **m-multivector fields** or **multivector fields of degree m** on $\mathcal{M}$. They are contravariant skew-symmetric tensor fields of degree m in $\mathcal{M}$.*

The set of m-multivector fields in $\mathcal{M}$ is denoted $\mathfrak{X}^m(\mathcal{M})$.

$\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is ***locally decomposable*** if, for every $\mathrm{p} \in \mathcal{M}$, there exists an open neighborhood $U_\mathrm{p} \subset \mathcal{M}$ and $X_1, \ldots, X_m \in \mathfrak{X}(\mathcal{U}_\mathrm{p})$ such that

$$\mathbf{X}|_{U_\mathrm{p}} = X_1 \wedge \cdots \wedge X_m \equiv \bigwedge_{\mu=1}^{m} X_\mu \ .$$

If $\{y^i\}$ are local coordinates in $\mathcal{M}$, their local expressions are

$$\mathbf{X} = \bigwedge_{\mu=1}^{m} X_\mu = \bigwedge_{\mu=1}^{m} X_\mu^i \frac{\partial}{\partial y^i} \,, \qquad (1)$$

The set of loc. decomp. m-multivector fields in $\mathcal{M}$ is denoted $\mathfrak{X}^m_{ld}(\mathcal{M})$.

Locally decomposable m-multivector fields are locally associated with m-dimensional distributions $\mathcal{D}$ in $\mathcal{M}$, where $\mathcal{D}$ is locally spanned by $\langle X_\mu \rangle$. Multivector fields associated with the same distribution form an equivalence class $\{\mathbf{X}\}$ in $\mathfrak{X}^m_{ld}(\mathcal{M})$, and if $\mathbf{X}, \mathbf{X}' \in \{\mathbf{X}\}$, then $\mathbf{X}' = f\,\mathbf{X}$, for $f \in \mathrm{C}^\infty(\mathcal{M})$.

Definition 2.3

*A locally decomposable multivector field is **integrable** if its associated distribution is integrable; that is, involutive.*

If $\mathbf{X}$ is integrable, a map $\psi \colon \mathbb{R}^m \to \mathcal{M}$, $\psi = \big(\psi^i(x^\mu)\big)$, is an ***integral map*** of $\mathbf{X}$ if, for the local expression (1),

$$\frac{\partial \psi^i}{\partial x^\mu} = X^i_\mu \circ \psi \,.$$

Operations with multivector fields

Definition 2.4

The ***contraction*** *between* $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ *and* $\alpha \in \Omega^k(\mathcal{M})$, $m \leq k \leq n$, *is their total tensorial contraction. For loc. dec, m.v.f.* $\mathbf{X} = \bigwedge_{\mu=1}^m X_\mu$,

$$\iota(\mathbf{X})\alpha := \iota(X_1 \wedge \cdots \wedge X_m)\alpha = \iota(X_m)\cdots\iota(X_1)\alpha\,.$$

The ***Lie derivative*** *of* α *with respect to* $\mathbf{X}$ *is defined by*

$$\mathrm{L}_{\mathbf{X}}\alpha := [\mathrm{d}, \iota(\mathbf{X})]\alpha = \mathrm{d}\iota(\mathbf{X})\alpha - (-1)^m\iota(\mathbf{X})\mathrm{d}\alpha\,.$$

The ***Schouten–Nijenhuis bracket*** *of* $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ *and* $\mathbf{Y} \in \mathfrak{X}^n(\mathcal{M})$ *is the multivector field* $[\mathbf{X}, \mathbf{Y}] \in \mathfrak{X}^{m+n-1}(\mathcal{M})$, *such that,*

$$\mathrm{L}_{[\mathbf{X},\mathbf{Y}]} := [\mathrm{L}_{\mathbf{X}}, \mathrm{L}_{\mathbf{Y}}] = \mathrm{L}_{\mathbf{X}}\mathrm{L}_{\mathbf{Y}} - \mathrm{L}_{\mathbf{Y}}\mathrm{L}_{\mathbf{X}}\,.$$

For locally decomposable m.v.f. $\mathbf{X} = \bigwedge_{\mu=1}^m X_\mu$, $\mathbf{Y} = \bigwedge_{\nu=1}^m Y_\nu$,

$$[\mathbf{X}, \mathbf{Y}] = \sum_{i,j}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge \cdots \widehat{X}_i \cdots \wedge X_m \wedge Y_1 \wedge \cdots \widehat{Y}_j \cdots \wedge Y_n\,.$$

If $\mathbf{X} \in \mathfrak{X}^k(M)$ *and* $Y \in \mathfrak{X}(M)$, *then,*

$$\iota([Y, \mathbf{X}]) = \mathrm{L}_Y\,\iota(\mathbf{X}) - \iota(\mathbf{X})\,\mathrm{L}_Y\,.$$

Multivector fields in fiber bundles

Let $\kappa \colon \mathcal{M} \to M$ be a fiber bundle with $\dim M = m$ and $\dim \mathcal{M} = N + m$.

Definition 2.5

*A multivector field $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is **κ-transverse** if the distribution associated with $\{\mathbf{X}\}$ also is.*

If M is an orientable manifold and $\omega_{_M} \in \Omega^m(M)$ is the volume form, $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is κ-transverse if, and only if,

$$\iota(\mathbf{X})(\kappa^*\omega_{_M}) \neq 0\,;$$

and a representative in the class $\{\mathbf{X}\}$ is given by $\iota(\mathbf{X})(\kappa^*\omega_{_M}) = 1$.

Proposition 2.6

If κ-transverse m.v.f. $\mathbf{X} \in \mathfrak{X}^m(\mathcal{M})$ is integrable, then its integral manifolds are locally the images of local sections $\psi \colon U \subset M \to \mathcal{M}$ of κ.

Ehresmann connections

Definition 2.7

An ***Ehresmann connection*** *on a fiber bundle* $\kappa\colon\mathcal{M}\to M$ *is one of the following equivalent objects.*

- *A* κ*-semibasic* 1*-form* ∇ *on* $\mathcal{M}$ *with values in* $\mathrm{T}\mathcal{M}$ *(a* $(1,1)$*-tensor field on* $\mathcal{M}$*) such that* $\iota(\nabla)\alpha=\alpha$*, for every* κ*-semibasic form* $\alpha\in\Omega^1(\mathcal{M})$ *(here* $\iota(\nabla)\alpha$ *denotes the usual tensorial contraction).* ∇ *is called a* ***connection form****.*
- *A subbundle* $\mathrm{H}(\nabla)$ *of* $\mathrm{T}\mathcal{M}$ *such that* $\mathrm{T}\mathcal{M}=\mathrm{V}(\kappa)\oplus\mathrm{H}(\nabla)$*.* $\mathrm{V}(\kappa)$ *and* $\mathrm{H}(\nabla)$ *are called the* ***vertical*** *and* ***horizontal distributions****.*

Proposition 2.8

Classes of locally decomposable and κ*-transverse multivector fields* $\{\boldsymbol{X}\}\subset\mathfrak{X}^m_{ld}(\mathcal{M})$ *are in one-to-one correspondence with Ehresmann connection forms* ∇ *on* $\mathcal{M}\to M$*. This correspondence is characterized by the fact that* $\mathrm{H}(\nabla)$ *is the distribution associated with* $\{\boldsymbol{X}\}$*.*

If $\{x^\mu, z^i\}$ are adapted coordinates in the bundle $\kappa\colon \mathcal{M} \to M$, then:

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + F^i_\mu \frac{\partial}{\partial z^i}\right) ,$$

$$\mathrm{H}(\nabla) = \left\langle \frac{\partial}{\partial x^\mu} + F^i_\mu \frac{\partial}{\partial z^i} \right\rangle ,$$

and a representative of the associated class $\{\mathbf{X}\} \subset \mathfrak{X}^m_{ld}(\mathcal{M})$ is

$$\mathbf{X} = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + F^i_\mu \frac{\partial}{\partial z^i}\right) ;$$

Definition 2.9

*A connection is **integrable** if the horizontal distribution $\mathrm{H}(\nabla)$ is integrable. (The necessary and sufficient condition is that the curvature of ∇ is zero; that is, the connection is flat).*

In the above coordinate expressions, if ∇ and $\mathbf{X}$ are integrable, and $\psi(x^\mu) = \big(x^\mu, \psi^i(x^\mu)\big)$ is an integral section of them, then

$$\frac{\partial \psi^i}{\partial x^\mu} = F^i_\mu \circ \psi \, .$$

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