From Local to Multilocal Observables via Configuration Spaces

34th International Workshop on Geometric Field Theory (Tarragona 2026)

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Institut Camille Jordan (Lyon) under the supervision of Alessandra Frabetti & Leonid Ryvkin

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Equivariant Poisson 2-Algebra Bundles over Configuration Spaces
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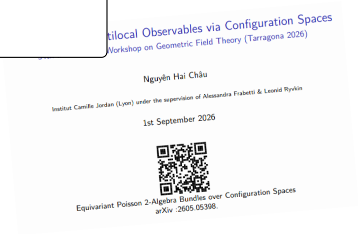
Universal property of the last bite

Nobody really wants it.



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- **Cons :** Bracket is **not** Poisson (no internal multiplication) $\rightsquigarrow$ no "multilocal" structure.

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Our approach to $\mathcal{C}^\infty\big(\Gamma(M,E),\mathbb{R}\big)$.

$\boxed{\text{In a nutshell : Functionals are represented by } \textbf{integral kernels}.}$

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▷ For instance, for a field $\varphi \in \Gamma(M,E)$,

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$\rightsquigarrow$ symmetrization of $\boxtimes^{\mathrm{ext}}$: $f \boxtimes g = f \boxtimes^{\mathrm{ext}} g + g \boxtimes^{\mathrm{ext}} f$.

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▷ **Recap.** Multilocal observables have the general form

$$F^{\mathrm{multi}}_{f\otimes\mu} : \varphi \in \Gamma(E) \longmapsto \frac{1}{n!}\int_{M^n} \left\langle f(x_1,\cdots,x_n), \varphi^{\otimes p_1} \boxtimes \cdots \boxtimes \varphi^{\otimes p_n}(x_1,\ldots,x_n) \right\rangle \mathrm{d}\mu(x_1,\cdots,x_n)$$

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My work. Restoring the diagonals in the previous framework by working with equivariant structures via functorial approaches.

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$\{$Vector bundles over $M\}$ = full **subcategory** of $\mathcal{VB}_{\mathfrak{S}_\bullet}(M^\bullet)$:

$$V \in \mathcal{VB}(M) \longmapsto (0, V, 0, 0, \dots) \in \mathcal{VB}_{\mathfrak{S}_\bullet}(M^\bullet)$$

$\rightsquigarrow$ *local* vector bundles ($\rightsquigarrow$ local observables).

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Interpretation $\rightsquigarrow$ product over the same configuration ($\rightsquigarrow$ local polynomial observables).

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Take two (ordered) configurations

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Cauchy tensor product $\boxtimes$

Combines configurations of different sizes

$$(\mathbf{V} \boxtimes \mathbf{W})_n = \bigoplus_{p+q=n} \mathrm{Ind}_{\mathfrak{S}_p \times \mathfrak{S}_q}^{\mathfrak{S}_n} \big(V_p \boxtimes^{\mathrm{ext}} W_q \big)$$

Interpretation $\rightsquigarrow$ concatenation of configuration and symmetrization.

More about the Cauchy product

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▷ **Monoidal structure.** $\left(\mathcal{VB}_{\mathfrak{S}_\bullet}(M^\bullet), \boxtimes\right) =$ **symmetric** monoidal category with unit $\mathbf{I}_\boxtimes = (\mathbb{R}, 0, 0, \dots)$.

▷ $\boxtimes$**-algebra structure :** $\mathbf{A} \in \mathcal{VB}_{\mathfrak{S}_\bullet}(M^\bullet)$ with morphisms

$$m^\boxtimes : \mathbf{A} \boxtimes \mathbf{A} \longrightarrow \mathbf{A} \qquad \text{and} \qquad u^\boxtimes : \mathbf{I}_\boxtimes \longrightarrow \mathbf{A}$$

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Summary

Hadamard product $\otimes \rightsquigarrow$ local multiplication over the **same** configuration.
Cauchy product $\boxtimes \rightsquigarrow$ multilocal multiplication over **different** configurations.

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$\otimes$ and $\boxtimes$ are **compatible** via an **interchange law**

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- a compatibility law between these two structures (Eckmann-Hilton like) :

$$m_{p,q}^\boxtimes \circ \big(m_p^\otimes \boxtimes m_q^\otimes\big) = m_{p+q}^\otimes \circ \big(m_{p,q}^\boxtimes \otimes m_{p,q}^\boxtimes\big), \qquad m_{p,q}^\boxtimes\big(\mathbb{1}_p^\otimes, \mathbb{1}_q^\otimes\big) = \mathbb{1}_{p+q}^\otimes \qquad \text{and} \qquad \mathbb{1}_0^\otimes = \mathbb{1}_0^\boxtimes$$

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Example. $M = \mathbb{R}$ and $V = \mathbb{R} \times Q$ rank r trivial bundle. $Q = \mathrm{Vect}(\xi^1, \ldots, \xi^r)$:

$$S^{\otimes}(V)_t = \mathbb{R}\big[\xi^1_t, \ldots, \xi^r_t\big] \quad \rightsquigarrow \quad \text{local observables}$$

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- $\otimes$-Leibniz rule (seen as $\otimes$-biderivation) : $\big\{a, b \odot c\big\} = \big\{a, b\big\} \odot \big(\mathbb{1}^{\otimes} \boxdot c\big) + \big(\mathbb{1}^{\otimes} \boxdot b\big) \odot \big\{a, c\big\}.$

Diagonals already matter !

Let $(x_1, \dots, x_n) \in M^n$, denote

$$\mathrm{Stab}(x_1, \dots, x_n) = \left\{\sigma \in \mathfrak{S}_n,\ \sigma \cdot (x_1, \dots, x_n) = (x_1, \dots, x_n)\right\}$$

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Geometric Consequence

Diagonals produce a rank-drop phenomenon at singular points.

Example. Trivial rank r. $V = \underbrace{\mathbb{R}}_{=M} \times Q$. With basis $(\xi^1, \dots, \xi^r)$ of Q.

$$\mathbf{S}^{\boxtimes}\big(S^{\otimes}(V)\big)_{(t_1,t_2)} = \mathbb{R}\big[\xi^1_{t_1}, \dots, \xi^r_{t_1}, \xi^1_{t_2}, \dots, \xi^r_{t_2}\big]$$

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Introduce $s^i=\frac{\xi^i_{t_1}+\xi^i_{t_2}}{2}$ and $a^i=\frac{\xi^i_{t_1}-\xi^i_{t_2}}{2}$ so that

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Claim 2. Any skew-symmetric kernel $k : (JE)^* \boxtimes (JE)^* \longrightarrow \mathbf{I}_{\otimes}$ induces a Poisson bracket B on $\mathbf{P}(E)$. For example,

$$\big(k = \text{causal propagator}\big) \implies \big(B = \text{Peierls bracket}\big)$$

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