Stäckel Lifts, Symplectic-Haantjes Structures, and Integrable Systems

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Motivation and goal

- **Eisenhart lift** (1929): geometrizes dynamics – electromagnetic fields encoded in a higher-dimensional metric
- **Stäckel geometry**: classical framework for separation of variables via Stäckel matrices $S_{ij}(q^i)$
- **Symplectic-Haantjes ($\omega\mathscr{H}$) geometry** [Tempesta & Tondo]: characterizes separability on phase space, with a constructive algorithm for separating coordinates

Goal: Study separation of variables of the Eisenhart-lifted system from the point of view of $\omega\mathscr{H}$ geometry, and generalize the lift to a Stäckel lift. **Results:**

1. Stäckel lift: a lifting procedure generalizing the Eisenhart lift;
2. classical Eisenhart lift is a special case of the Stäckel lift (Thm. 1);
3. every Stäckel lift carries a symplectic-Haantjes structure (Main Thm.); if the lift is momentum-dependent, the Haantjes operators are non-projectable.

Symplectic-Haantjes ($\omega\mathscr{H}$) geometry: brief review

Let M be a manifold, $\boldsymbol{K} : TM \to TM$ a $(1,1)$ tensor field.

Definition (Haantjes 1955)

The **Haantjes torsion** of $\boldsymbol{K}$ is

$$\mathcal{H}_{\mathbf{K}}(X,Y) := \mathbf{K}^2\mathcal{T}_{\mathbf{K}}(X,Y) + \mathcal{T}_{\mathbf{K}}(\mathbf{K}X,\mathbf{K}Y) - \mathbf{K}\Big(\mathcal{T}_{\mathbf{K}}(X,\mathbf{K}Y) + \mathcal{T}_{\mathbf{K}}(\mathbf{K}X,Y)\Big),$$

where $\mathcal{T}_{\boldsymbol{K}}$ is the Nijenhuis torsion. $\boldsymbol{K}$ is a **Haantjes operator** if $\mathcal{H}_{\boldsymbol{K}} \equiv 0$ (weaker than Nijenhuis).

Definition (Haantjes algebra of rank m)

$(M,\mathscr{H})$, $\mathscr{H}$ a set of Haantjes operators forming a **free module of rank** m over $C^\infty(M)$ and a **ring** under composition. **Abelian** if all $\boldsymbol{K} \in \mathscr{H}$ commute.

Definition (Tempesta–Tondo, $\omega\mathscr{H}$ manifold)

$(M,\omega,\mathscr{H})$ of rank m: (M,ω) symplectic, $\mathscr{H}$ an **Abelian** Haantjes algebra of rank m, algebraically compatible with ω: $\omega(X,\boldsymbol{K}Y) = \omega(\boldsymbol{K}X,Y)$, $\forall \boldsymbol{K} \in \mathscr{H}$.

Haantjes chains and separability

Definition (Haantjes chain)

Let $(M, \mathscr{H})$ be a Haantjes algebra of rank m. A smooth function H generates a **Haantjes chain** of length m if there is a basis $\{\tilde{\boldsymbol{K}}_1, \ldots, \tilde{\boldsymbol{K}}_m\}$ of $\mathscr{H}$ such that $\tilde{\boldsymbol{K}}_\alpha^T \mathrm{d}H =: \mathrm{d}H_\alpha$, $\alpha = 1, \ldots, m$, are locally exact.

Theorem (Reyes, Tempesta & Tondo 2022)

Under mild technical conditions, complete **Liouville integrability** $+$ **separability** *of the Hamilton–Jacobi equations for* $\{H_1, \ldots, H_n\} \Leftrightarrow$ *an* **$\omega\mathscr{H}$ manifold** *of rank n with a Haantjes chain generated by H_1, with operators*

$$\boldsymbol{K}_\alpha = \sum_{i=1}^{n} \frac{\partial H_\alpha / \partial p_i}{\partial H_1 / \partial p_i} \left(\frac{\partial}{\partial q^i} \otimes \mathrm{d}q^i + \frac{\partial}{\partial p_i} \otimes \mathrm{d}p_i \right), \quad \alpha = 1, \ldots, n.$$

Coordinates $(\boldsymbol{q}, \boldsymbol{p})$ diagonalizing $\{\boldsymbol{K}_\alpha\}$: **Darboux–Haantjes (DH) coordinates**. A constructive algorithm exists via eigen-distributions of $\boldsymbol{K}_\alpha$ [Tempesta & Tondo 2021].

Stäckel geometry

A large class of separable systems satisfies the **Stäckel equation**

$$\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = S^{-1} \begin{bmatrix} f_1(q^1, p_1) \\ f_2(q^2, p_2) \\ f_3(q^3, p_3) \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11}(q^1) & S_{12}(q^1) & S_{13}(q^1) \\ S_{21}(q^2) & S_{22}(q^2) & S_{23}(q^2) \\ S_{31}(q^3) & S_{32}(q^3) & S_{33}(q^3) \end{bmatrix}$$

Stäckel matrix S: rows depend on separated coordinates; **generalized Stäckel functions** $f_i(q^i, p_i)$ [Arnold, Kozlov & Neishtadt 1997]. Classical case $f_k = \frac{1}{2}p_k^2 + W_k(q^k)$: associated Haantjes operators project to Killing tensors,

$$\boldsymbol{K}_{j-1} := \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \left(\frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r + \frac{\partial}{\partial p_r} \otimes \mathrm{d}p_r \right) \xrightarrow{\pi} \sum_{r=1}^{n} \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r,$$

$\tilde{S}_{jk}$ = cofactor of S_{kj}.

Eisenhart lift: review

Eisenhart lift: EM fields $(\boldsymbol{A}, V)$ encoded in the metric on an extended space.
Riemannian: $M \to \hat{M} = M \times \mathbb{R}$,

$$\hat{H} = \tfrac{1}{2} g^{ij} \left(p_i + A_i(\boldsymbol{q})\, p_u\right) \left(p_j + A_j(\boldsymbol{q})\, p_u\right) + V(\boldsymbol{q})\, p_u^2$$

Lorentzian: $M \to \tilde{M} = M \times \mathbb{R} \times \mathbb{R}$, add $p_u p_t$ to $\hat{H}$.
Properties: integrals become homogeneous in momenta; (super)integrability is preserved; u, t cyclic.

Question: can this lift be produced systematically, and does it carry a Haantjes structure?

Stäckel lift: definition

Definition (Stäckel lift)

Let $S^{(n)}(\boldsymbol{q})$ be an $(n \times n)$ Stäckel matrix. Its **Stäckel lift** is the $(n+1) \times (n+1)$ **lifting matrix**

$$\mathcal{L}^{(n+1)}(\boldsymbol{q}, \boldsymbol{p}) := \left[\begin{array}{c|c} & S_{1,n+1}(q^1, p_1) \\ S^{(n)}(\boldsymbol{q}) & \vdots \\ & S_{n,n+1}(q^n, p_n) \\ \hline 0 \cdots 0 & S_{n+1,n+1}(q^{n+1}, p_{n+1}) \end{array}\right]$$

with $S_{1,n+1}, \ldots, S_{n+1,n+1}$ arbitrary smooth, $S_{n+1,n+1} \neq 0$. Can depend on momenta.

Lifted system on $T^*\tilde{M}$:

$$\begin{bmatrix} H_1 \\ \vdots \\ H_{n+1} \end{bmatrix} = (\mathcal{L}^{(n+1)})^{-1} \begin{bmatrix} f_1(q^1, p_1) \\ \vdots \\ f_{n+1}(q^{n+1}, p_{n+1}) \end{bmatrix}$$

Integrable and separable by construction; can be iterated.

Theorem 1: Eisenhart lift as a special case

Theorem (Kubů & Tempesta 2026)

The Riemannian Eisenhart lift is a **special case** *of the Stäckel lift.*

Construction: given $S^{(n)}$ and Stäckel vector $\boldsymbol{W} = (W_1, \ldots, W_n)^T$, take a momentum-independent lift

$$\mathcal{L}^{(n+1)} = \left[\begin{array}{c|c} S^{(n)}(\boldsymbol{q}) & -\boldsymbol{W}(\boldsymbol{q}) \\ \hline \mathbf{0}^T & 1 \end{array}\right], \quad f_k = \tfrac{1}{2} p_k^2,\ k = 1, \ldots, n, \quad f_{n+1} = p_u^2.$$

Result: the lifted Hamiltonians coincide with the Riemannian Eisenhart lift,

$$H_j = \frac{1}{2} \sum_{k=1}^{n} \left[(S^{(n)})^{-1} \right]_j^k \left(p_k^2 + W_k(q^k)\, p_u^2 \right).$$

Momentum-independence of $-\boldsymbol{W} \Rightarrow$ purely geodesic (Riemannian) Hamiltonians.

Haantjes geometry and Stäckel lifts

Theorem (Kubů & Tempesta 2026 – Main Theorem)

Let $(H_1, \ldots, H_{n+1})$ *be a Stäckel-lifted system with lift matrix* $\mathcal{L}^{(n+1)}(\boldsymbol{q}, \boldsymbol{p})$ *as above* ($S_{n+1,n+1} = S_{n+1,n+1}(q^{n+1})$)*. There exists an associated* $\omega\mathscr{H}$ *manifold, whose Haantjes algebra* $\mathscr{H}$ *is generated by the semisimple operators*

$$\boldsymbol{K}_j := \sum_{r=1}^{n+1} \mu_{jr} \left(\frac{\partial}{\partial q^r} \otimes \mathrm{d}q^r + \frac{\partial}{\partial p_r} \otimes \mathrm{d}p_r \right), \quad \mu_{jr} := \frac{\tilde{\mathcal{L}}_{jr}}{\tilde{\mathcal{L}}_{1r}},$$

$\tilde{\mathcal{L}}_{jr}$ = *cofactor of* $\mathcal{L}_{jr}$ *in the lifting matrix.*

Crucial difference

If $\mathcal{L}^{(n+1)}$ is **momentum-dependent**, the operators $\boldsymbol{K}_j$ are **non-projectable** along the fibers of $T^*\tilde{M} \Rightarrow$ they do not reduce to Killing tensors on the base. Killing/Stäckel geometry cannot see this structure – Haantjes geometry is necessary to describe it.

A concrete illustration: an integrable gravitational wave

Iterating the Stäckel lift twice on a 2D free particle, with lifting matrix

$$\mathcal{L}^{(4)} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1 & -1/2 & -\frac{1}{(x^2)^2} & 0 \\ 0 & 0 & 1 & -p_u \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

yields the **Platonic wave** metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 + 2(x^2)^2\,\mathrm{d}t\,\mathrm{d}u - (x^2)^2\,\mathrm{d}t^2,$$

with completely separable Hamiltonian and integrals

$$H_1 = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{(x^2)^2}\left(\tfrac{1}{2}p_u^2 + p_t p_u\right), \quad H_2 = \tfrac{1}{2}p_1^2, \quad H_3 = \tfrac{1}{2}p_u^2 + p_t p_u, \quad H_4 = \tfrac{1}{2}p_t^2.$$

- wave fronts ($u, t =$const) are flat, yet $\mathcal{R} = -2/(x^2)^2$: curved, non-Einstein, still completely integrable and separable – rare for a Kundt spacetime;
- geodesics with $p_u = m$, $p_t = E$ project to a particle on $\mathbb{R}^2$ in the inverse-square potential $\left(\frac{1}{2}m^2 + Em\right)/(x^2)^2$;
- a candidate solution in $f(R)$-gravity, though not (in general) of Einstein's equations.

Further applications

- **Magnetic systems in cylindrical coordinates** [cf. Kubů, Reyes, Tempesta & Tondo 2024, *Proc. R. Soc. A*]: of the 3 classes (depending on which of ϕ, z is cyclic), only the simplest admits a Stäckel formulation without altering the dynamics; the other two require a Lamé-coefficient correction $\frac{1}{r^2}$ that modifies the Hamiltonian. The Haantjes approach handles all three uniformly.
- **Hamilton and Finsler geometries**: momentum-dependent Stäckel lifts define momentum-dependent metrics on $T^*\tilde{M}$ (Hamilton spaces), dual to Finsler/Lagrange spaces on $T\tilde{M}$ – relevant to Zermelo navigation, deformed relativity, quantum-gravity phenomenology.

Beyond the lift form – the Zernike hierarchy

Most Haantjes operators found in the literature are of **lift form** (block structure $B = 0$), corresponding to separation reachable by an extended point transformation (EPT). The non-projectable operators of the Main Theorem are the exception.

[Kubů & Latini 2026, arXiv:2605.23748] on the generalized Zernike hierarchy $H_N = p_1^2 + p_2^2 + \sum_{n=1}^{N} \gamma_n (q_1 p_1 + q_2 p_2)^n$:

- $N = 2$ (classical Zernike system) – solved: all Haantjes operators are of lift form, giving DH coordinates for all 4 separation types (incl. elliptic, Heun-class radial ODE) via an EPT;
- $N \geq 3$ – obstruction proved: no lift-form operator separates the higher integral from angular momentum, so separation needs genuinely momentum-dependent transformations;
- **Open:** the explicit momentum-dependent construction for $N \geq 3$.

Summary

- **Stäckel lift**: unified lifting procedure extending the Eisenhart lift, generating integrable and separable systems by construction; can be iterated to build higher-dimensional models.
- **Theorem 1**: the Riemannian Eisenhart lift is a special (momentum-independent) case of the Stäckel lift.
- **Main Theorem**: every Stäckel lift carries a symplectic-Haantjes structure; momentum-dependent lifts produce non-projectable Haantjes operators – a new class, not reducible to Killing tensors.
- **Applications**: cylindrical magnetic systems, Platonic/gravitational waves, Hamilton–Finsler geometries.

Thank you!

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