Hybrid quantum-classical field theory: classical gravity with quantum sources.

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Contents

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Hybrid geometrodynamics: a Hamiltonian description of classical gravity coupled to quantum matter

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Geometric flavors of Quantum Field theory on a Cauchy hypersurface. Part I: Gaussian analysis and other mathematical aspects

José Luis Alonso[a,b,c], Carlos Bouthelier-Madre[a,b,c], Jesús Clemente-Gallardo[a,b,c], David Martínez-Crespo[a,c,*]

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Geometric flavours of quantum field theory on a Cauchy hypersurface. Part II: Methods of quantization and evolution

José Luis Alonso[a,b,c], Carlos Bouthelier-Madre[a,b,c], Jesús Clemente-Gallardo[a,b,c], David Martínez-Crespo[a,c,*]

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- QFT on a curved background
- Hybrid geometrodynamics
- Conclusions and outlook

Joint work with:
J. L. Alonso,
C. Bouthelier-Madre
and D. Martínez-Crespo

Classical geometrodynamics

Classical geometrodynamics

Quantization of Einstein's Gravitational Field Equations. II

PDF

F. A. E. Pirani*

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Phys. Rev. **87**, 452 – **Published 1 August, 1952**

DOI: https://doi.org/10.1103/PhysRev.87.452

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Classical geometrodynamics

PROCEEDINGS OF THE ROYAL SOCIETY A | MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Theory of Gravitation in Hamiltonian Form

P. A. M. Dirac

Proc. R. Soc. Lond. A 1958 **246**, 333-343
doi: 10.1098/rspa.1958.0142

Classical geometrodynamics

Gen Relativ Gravit (2008) 40:1997–2027
DOI 10.1007/s10714-008-0661-1

GOLDEN OLDIE

Republication of: The dynamics of general relativity

**Richard Arnowitt · Stanley Deser ·
Charles W. Misner**

Published online: 8 August 2008

Abstract This article—summarizing the authors' then novel formulation of General Relativity—appeared as Chap. 7, pp. 227–264, in *Gravitation: an introduction to current research*, L. Witten, ed. (Wiley, New York, 1962), now long out of print. Intentionally unretouched, this republication as Golden Oldie is intended to provide contemporary accessibility to the flavor of the original ideas. Some typographical corrections have been made: footnote and page numbering have changed–but not section nor equation numbering, etc. Current institutional affiliations are encoded in: arnowitt@physics.tamu.edu, deser@brandeis.edu, misner@umd.edu.

An editorial note to this paper and biographies of the authors can be found via doi:10.1007/s10714-008-0649-x.

Original paper: R. Arnowitt, S. Deser and C. W. Misner, in Gravitation: an introduction to current research (Chap. 7). Edited by Louis Witten. John Wiley & Sons Inc., New York, London, 1962, pp. 227–265. Reprinted with the kind permissions of the editor, Louis Witten, and of all three authors.

Classical geometrodynamics

ANNALS OF PHYSICS 96, 88–135 (1976)

Geometrodynamics Regained*

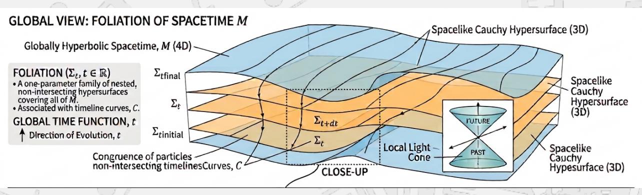
SERGIO A. HOJMAN,[†] KAREL KUCHAŘ,[‡] AND CLAUDIO TEITELBOIM[1]

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

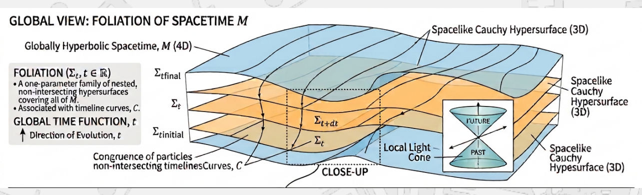
Received September 17, 1974

Einsteinian geometrodynamics is the only (time-reversible) canonical representation of the set of generators of deformations of a spacelike hypersurface embedded in a Riemannian spacetime, if the intrinsic metric of that hypersurface and a conjugate momentum are the sole canonical variables.

Globally hyperbolic space-times



Globally hyperbolic space-times



Foliation is seen as a smooth map in the space of hypersurface deformations. We consider a reference hypersurface Σ_0 and define a smooth evolution Σ_τ following a vector field

$$E(x,\tau) = N(x,\tau)\hat{n}(x) + \vec{N}(x,\tau),$$

where $\hat{n}$ is normal to Σ_τ, $\vec{N}(x,\tau) = N^j(x,\tau)\hat{t}_j(x)$ and $\hat{t}$ is tangent to Σ_τ . Hence, the foliation is encoded in the local generators D_n and D_{t_i} transforming Σ_τ, which in space-time coordinates $\{X^\mu\}$ read:

$$D_n := n^\mu(x)\frac{\delta}{\delta X^\mu(x)} \quad \text{and} \quad D_{t_i} := \partial_{x^i} X^\mu(x)\frac{\delta}{\delta X^\mu(x)},$$

Globally hyperbolic space-times II

and satisfy the closing relations

$$[D_n(x), D_n(x')] = h^{ij}(x) D_{t_j}(x)\delta_{,i}(x,x') - (x \leftrightarrow x');$$
$$[D_{t_i}(x), D_n(x')] = D_n(x)\delta_{,i}(x,x');$$
$$[D_{t_i}(x), D_{t_j}(x')] = D_{t_i}(x')\delta_{,j}(x,x') + D_{t_j}(x')\delta_{,i}(x,x').$$

Globally hyperbolic space-times II

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$$[D_n(x), D_n(x')] = h^{ij}(x) D_{t_j}(x) \delta_{,i}(x, x') - (x \leftrightarrow x');$$

$$[D_{t_i}(x), D_n(x')] = D_n(x) \delta_{,i}(x, x');$$

$$[D_{t_i}(x), D_{t_j}(x')] = D_{t_i}(x') \delta_{,j}(x, x') + D_{t_j}(x') \delta_{,i}(x, x').$$

At each hypersurface (parametrized by τ) with metric $h_{jk}(x_\tau) = g_{jk}(\tau, x_\tau)$, space-time coordinates define the lapse and the shift vector fields

$$N = \frac{1}{\sqrt{-g^{00}}}$$

and

$$N_j = g_{0j}.$$

Space-time metric becomes thus:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + g^{jk} N_j N_k & N_k \\ N_j & h_{jk} \end{pmatrix}$$

Classical geometrodynamics

With these elements, we can consider Einstein-Hilbert action

$$S_G = \frac{1}{16\pi G}\int_M dx^4 \sqrt{-|g|}\left(R^{(4)} - \Lambda\right),$$

where $|g|$ is the metric determinant, $R^{(4)}$ is the Ricci scalar and Λ the cosmological constant.

Classical geometrodynamics

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Using the foliation, we can re-write the action in the form

$$S_G = \frac{1}{16\pi G}\int dt \int d^3x N\sqrt{|h|}(K_{jk}K^{jk} - K^2 + R - \Lambda),$$

where $K_{jk} = \mathcal{L}_{\hat{n}} h_{jk} = \frac{1}{2}N^{-1}\left(-\partial_t h_{jk} + \nabla_j N_k + \nabla_k N_j\right)$ is the extrinsic curvature of the leaf Σ_τ, $K = h^{jk}K_{jk}$, R is the Ricci scalar of h and ∇ represents the covariant derivative with the Levi-Civita connection of h.
Hence, we can define the associated field-momentum as

$$\pi^{jk} = \frac{\delta S_G}{\delta h_{jk,0}} = -\frac{\sqrt{|h|}}{16\pi G}(K^{jk} - Kh^{jk}).$$

Classical geometrodynamics II

Now

$$S_G = \int dt \int d^3x(\pi^{jk}\partial_t h_{jk} - N\mathcal{H} - N^j\mathcal{H}_j),$$

where

$$\mathcal{H} = G_{jklm}\pi^{jk}\pi^{lm} - \frac{\sqrt{|h|}}{16\pi G}(R - 2\Lambda)$$

$$\mathcal{H}_j = -\nabla_i(h_{ik}\pi^{jk})$$

are, respectively, the superhamiltonian and the supermomenta of the theory and

$$G_{jklm} = \frac{16\pi G}{\sqrt{|h|}}(h_{jl}h_{km} + h_{jm}h_{kl} - h_{jk}h_{lm}).$$

Classical geometrodynamics III

A similar analysis leads to an expression of the action of a classical scalar matter field as

Classical geometrodynamics III

A similar analysis leads to an expression of the action of a classical scalar matter field as

$$S_M = \int dt \int d^3x(\pi_\phi \partial_t \phi - N\mathcal{H}^M - N^j \mathcal{H}_j^M),$$

for π_ϕ the field-momentum, the superhamiltonian $\mathcal{H}^M$ and the supermomenta $\mathcal{H}_j^M$ of the matter theory:

$$\mathcal{H}^M = \frac{\pi_\phi^2}{2\sqrt{|h|}} + \frac{\sqrt{|h|}}{2}\left(h^{jk}\nabla_j\phi\nabla_k\phi + m^2\phi^2\right),$$

$$\mathcal{H}_j^M = \pi_\phi \nabla_j \phi.$$

Classical geometrodynamics III

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$$\mathcal{H}^M = \frac{\pi_\phi^2}{2\sqrt{|h|}} + \frac{\sqrt{|h|}}{2} \left(h^{jk} \nabla_j \phi \nabla_k \phi + m^2 \phi^2 \right),$$

$$\mathcal{H}_j^M = \pi_\phi \nabla_j \phi.$$

Combining both actions:

$$S_T = S_G + S_M = \int dt \int d^3x (\pi^{jk} \partial_t h_{jk} + \pi_\phi \partial_t \phi - N\mathcal{H}^T - N^j \mathcal{H}_j^T),$$

where $\mathcal{H}^T = \mathcal{H}^G + \mathcal{H}^M$ and $\mathcal{H}_j^T = \mathcal{H}_j^G + \mathcal{H}_j^M$.

Classical geometrodynamics IV

Dynamics is then obtained from the action as extremal points of the variations with respect to the different variables:

Classical geometrodynamics IV

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Dynamical equations

- Variations with respect to the fields lead to equations which can be combined in the form

$$\delta F(h,\pi,\phi,\pi^\phi) = \int d\mu(x) \left(\{F\left(h,\pi,\phi,\pi^\phi\right), \mathcal{H}^T\}\delta N(x) + \{F\left(h,\pi,\phi,\pi^\phi\right), \mathcal{H}_j^T(x)\}\delta N^j(x) \right),$$

  where $\{\cdot,\cdot\} = \{\cdot,\cdot\}_G + \{\cdot,\cdot\}_M$ is the Poisson bracket defined as the sum of the (compatible) canonical Poisson brackets defined on the phase-spaces of gravitational and matter fields, respectively.
- Variations with respect to N and N^j lead to the constraint equations:

$$\mathcal{H}^T = 0, \qquad \mathcal{H}_j^T = 0, \quad \forall j.$$

Classical geometrodynamics V

The superhamiltonians and supermomenta act therefore as Hamiltonian generators of the transformation which the space-time foliation induces on the sequence of fields phase-spaces.

Classical geometrodynamics V

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Dirac algebra of constraints

Constraints must be preserved by the dynamical equations. This is achieved by asking the Hamiltonian generators to close a Lie algebra structure (that we will call **Dirac algebra** in the following) which reads:

$$\{\mathcal{H}^T_x, \mathcal{H}^T_{x'}\} = {\mathcal{H}^j}^T_x \partial_j \delta(x, x') - {\mathcal{H}^j}^T_{x'} \partial_j \delta(x', x)$$

$$\{\mathcal{H}^T_x, \mathcal{H}^T_{jx'}\} = \mathcal{H}^T_x \partial_j \delta(x, x')$$

$$\{\mathcal{H}^T_{jx}, \mathcal{H}^T_{kx'}\} = \mathcal{H}^T_{jx'} \partial_k \delta(x, x') - \mathcal{H}^T_{kx} \partial_j \delta(x', x)$$

These are the same commutation relations of the hypersurface deformations in the space-time foliation.

Classical geometrodynamics VI

Path independence principle

The preservation of the (first class) constraints ensures the fulfillment of the *path independence principle* for the deformation of the hypersurfaces in the foliation: the evolution of a hypersurface Σ_{τ_1} to a hypersurface Σ_{τ_2} becomes independent of the choice of N, N^j.

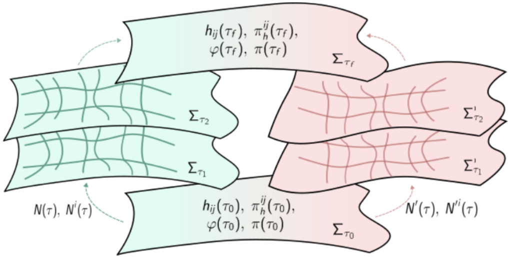

Figure: Path independence principle for two sets of hsf. deformations.

A suitable Lie groupoid

The construction above is not unique since different embeddings of the hypersurface in the space-time M are possible. In this 3+1 formalism, we can recover the general covariance of GR by considering an adapted version of the diffeomorphism group. In this new context, the system must be invariant under the hypersurface deformations, parametrized by the lapse and the shift vector fields:

$$D(N, \vec{N}) = \int_{\Sigma} d^3x \left(N(x) \overbrace{n^{\mu} \frac{\delta}{\delta X^{\mu}(x)}}^{D_n} + N^j(x) \overbrace{\partial_{x^j} X^{\mu}(x) \frac{\delta}{\delta X^{\mu}(x)}}^{D_{t_j}} \right).$$

A suitable Lie groupoid

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The bracket relations among the constraints of general relativity are identical to the bracket relations in the Lie algebroid of a groupoid consisting of diffeomorphisms between space-like hypersurfaces in spacetimes.

A suitable Lie groupoid II

Deformation groupoid(Blohmann, Fernandes, Weinstein, 2013)

The set of hypersurface deformations $D(N, \vec{N})$ defines (the set of morphisms) of a Lie groupoid, whose objects are the isometric embeddings of hypersurfaces Σ in a Lorentzian manifold M.

A suitable Lie groupoid II

Deformation groupoid(Blohmann, Fernandes, Weinstein, 2013)

The set of hypersurface deformations $D(N, \vec{N})$ defines (the set of morphisms) of a Lie groupoid, whose objects are the isometric embeddings of hypersurfaces Σ in a Lorentzian manifold M.

Generators of the deformation define thus a Lie algebroid with the bracket:

$$[D(N, \vec{N}), D(M, \vec{M})] = D\left(\vec{N}(M) - \vec{M}(N) + \mathrm{div}_h(\vec{N})M - \mathrm{div}_h(\vec{M})N,\right.$$
$$\left.[\vec{N}, \vec{M}] + N\mathrm{grad}_h M - M\mathrm{grad}_h N\right)$$

where $(\mathrm{grad}_h N)^j = h^{jk}\partial_k N$ and $\mathrm{div}_h(\vec{N}) = \frac{1}{\sqrt{|h|}}\partial_j\left(\sqrt{h}N^j\right)$.

A suitable Lie groupoid III

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GROUPOID SYMMETRY AND CONSTRAINTS IN GENERAL RELATIVITY

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Dedicated to Darryl Holm, for his 65th birthday

A suitable Lie groupoid IV

PHYSICAL REVIEW D **94,** 104032 (2016)

Hypersurface-deformation algebroids and effective spacetime models

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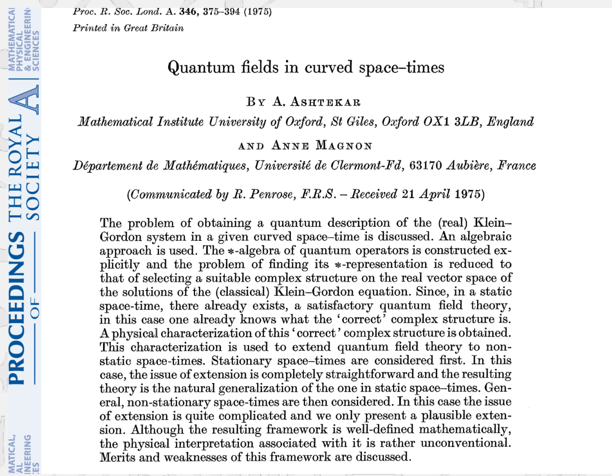
(Received 23 September 2016; published 11 November 2016)

In canonical gravity, covariance is implemented by brackets of hypersurface-deformation generators forming a Lie algebroid. Lie-algebroid morphisms, therefore, allow one to relate different versions of the brackets that correspond to the same spacetime structure. An application to examples of modified brackets found mainly in models of loop quantum gravity can, in some cases, map the spacetime structure back to the classical Riemannian form after a field redefinition. For one type of quantum corrections (holonomies), signature change appears to be a generic feature of effective spacetime, and it is shown here to be a new quantum spacetime phenomenon which cannot be mapped to an equivalent classical structure. In low-curvature regimes, our constructions not only prove the existence of classical spacetime structures assumed elsewhere in models of loop quantum cosmology, they also show the existence of additional quantum corrections that have not always been included.

QFT on a curved background

QFT on a curved background



Proc. R. Soc. Lond. A. **346**, 375–394 (1975)
Printed in Great Britain

Quantum fields in curved space–times

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(*Communicated by R. Penrose, F.R.S. – Received* 21 *April* 1975)

The problem of obtaining a quantum description of the (real) Klein–Gordon system in a given curved space–time is discussed. An algebraic approach is used. The $*$-algebra of quantum operators is constructed explicitly and the problem of finding its $*$-representation is reduced to that of selecting a suitable complex structure on the real vector space of the solutions of the (classical) Klein–Gordon equation. Since, in a static space-time, there already exists, a satisfactory quantum field theory, in this case one already knows what the 'correct' complex structure is. A physical characterization of this 'correct' complex structure is obtained. This characterization is used to extend quantum field theory to non-static space-times. Stationary space–times are considered first. In this case, the issue of extension is completely straightforward and the resulting theory is the natural generalization of the one in static space–times. General, non-stationary space-times are then considered. In this case the issue of extension is quite complicated and we only present a plausible extension. Although the resulting framework is well-defined mathematically, the physical interpretation associated with it is rather unconventional. Merits and weaknesses of this framework are discussed.

QFT on a curved background

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Nuclear Physics B 530 (1998) 247–278

NUCLEAR PHYSICS B

The Schrödinger wave functional and vacuum states in curved spacetime

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Received 6 May 1996; revised 23 April 1998; accepted 13 May 1998

Abstract

New developments in the Schrödinger picture description of vacuum states in curved spacetime are presented. General solutions for the vacuum wave functional are given for both static and dynamic (Bianchi type I) spacetimes and for conformally static spacetimes of Robertson–Walker type. The formalism is illustrated for a simple 'cosmological' model with a time-dependent metric and the phenomenon of particle creation is related to a special form of the kernel in the vacuum wave functional.

QFT on a curved background

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Annals of Physics 313 (2004) 446–478

ANNALS of PHYSICS

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Schrödinger and Fock representation for a field theory on curved spacetime

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Available online 1 June 2004

Abstract

Linear free field theories are one of the few quantum field theories that are exactly soluble. There are, however, (at least) two very different languages to describe them, Fock space methods and the Schrödinger functional description. In this paper, the precise sense in which the two representations are related is explored. Several properties of these representations are studied, among them the well-known fact that the Schrödinger counterpart of the usual Fock representation is described by a Gaussian measure. A real scalar field theory is considered, both on Minkowski spacetime for arbitrary, non-inertial embeddings of the Cauchy surface, and for arbitrary (globally hyperbolic) curved spacetimes. We present both the Gaussian representation with a non-trivial measure and the homogeneous representation where the non-triviality lies in the vacuum state. As concrete examples, the Schrödinger representations on stationary and homogeneous cosmological spacetimes are constructed.

Modelling classical fields

For each hypersurface Σ_τ, consider the set of matter fields $\mathcal{M}_M^\tau$ on the hypersurface Σ_τ. If we consider the corresponding cotangent bundles $T^*\mathcal{M}_M^\tau$, we have a time-dependent sequence of classical field theories.

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Mathematically, we will model the fields using the space of test functions of compact support on Σ, $\mathcal{N} = C_c(\Sigma)$ to define the differentiable structure on the manifold and the corresponding dual space $\mathcal{N}'$ to model our fields and their momenta. We also consider a rigged Hilbert space structure

$$\mathcal{N} = C_c(\Sigma) \subset \overbrace{\mathcal{L}^2(\Sigma, d\mathrm{Vol}_h)}^{\mathcal{H}_\Delta} \subset \mathcal{N}'.$$

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$$\mathcal{N} = C_c(\Sigma) \subset \overbrace{\mathcal{L}^2(\Sigma, d\mathrm{Vol}_h)}^{\mathcal{H}_\Delta} \subset \mathcal{N}'.$$

We will use the notation ϕ_x to represent a test function $\phi \in \mathcal{N}$ and ξ^x to represent a distribution $\xi \in \mathcal{N}'$. Thus, $\xi^x \phi_x = \int d\mathrm{Vol}_h(x)\, \xi(x)\phi(x)$.

Modelling classical fields

We define a complex structure on $T^*\mathcal{M}^\tau_M$ in order to quantize the theory. This complex structure J_τ and the quantization mapping $\mathcal{Q}_\tau$ depend on the geometry and the gravitational degrees of freedom. Ashtekar & Magnon proved in 1975 that, for static space-times, the complex structure can be determined using the Hamiltonian of the classical field theory. In general, the definition of a suitable complex structure is an open problem.

Modelling classical fields

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Hence, we model the classical field phase space using the set of distributions $\mathcal{N}'$, and defining a suitable complex function on the space $\mathcal{N}' \times \mathcal{N}'$. Within this framework, we will be able to define a suitable Hilbert space to host the states of the quantum fields.

Defining quantum states

Minlos theorem

There exists a unique Borel probability measure $d\mu$ on $\mathcal{N}'_{\mathbb{C}}$ satisfying that

$$C(\bar{\rho}, \rho) = \int_{\mathcal{N}'_{\mathbb{C}}} d\mu(\bar{\phi}, \phi) e^{i(\phi^x \rho_x + \bar{\phi}^x \bar{\rho}_x)} = e^{-\bar{\rho}_x \Delta^{xy} \rho_y}$$

where Δ is a Hermitian covariance form on $C_c(\Sigma)$ and

$$\bar{\rho}_x \Delta^{xy} \rho_y = \int_{\Sigma_\tau} d\mathrm{Vol}_h(x) d\mathrm{Vol}_h(y) \bar{\rho}(x) \Delta(x, y) \rho(y).$$

Defining quantum states

Minlos theorem

There exists a unique Borel probability measure $d\mu$ on $\mathcal{N}'_{\mathbb{C}}$ satisfying that

$$C(\bar{\rho},\rho)=\int_{\mathcal{N}'_{\mathbb{C}}} d\mu(\bar{\phi},\phi)e^{i(\phi^x\rho_x+\bar{\phi}^x\bar{\rho}_x)}=e^{-\bar{\rho}_x\Delta^{xy}\rho_y}$$

where Δ is a Hermitian covariance form on $C_c(\Sigma)$ and

$$\bar{\rho}_x\Delta^{xy}\rho_y=\int_{\Sigma_\tau} d\mathrm{Vol}_h(x)d\mathrm{Vol}_h(y)\bar{\rho}(x)\Delta(x,y)\rho(y).$$

This result allows us to consider a well-defined Gaussian measure on the space of classical fields on each hypersurface Σ_τ and thus define the set of pure quantum states of the matter field in the form

$$\mathcal{H}_\tau=\mathcal{L}^2(\mathcal{N}'_{\mathbb{C}},d\mu_\tau).$$

Defining quantum states II

Associated with the Hilbert space $\mathcal{H}_\tau$ we can build the **Segal isomorphism**

$$\mathcal{I} : \mathcal{H}_\tau \to \Gamma\mathcal{H}_\Delta \otimes \Gamma\mathcal{H}_\Delta^*$$

which relates the Hilbert space with the (bosonic) Fock space constructed on **the space of 1-particle states** $\mathcal{H}_\Delta$ by a unitary isomorphism of Hilbert spaces. The isomorphism is built using the **Wiener-Ito chaos decomposition** of the system, which, roughly speaking, generalizes Hermite polynomials to the infinite dimensional realm.

Defining quantum states II

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$$\mathcal{I} : \mathcal{H}_\tau \to \Gamma\mathcal{H}_\Delta \otimes \Gamma\mathcal{H}^*_\Delta$$

which relates the Hilbert space with the (bosonic) Fock space constructed on **the space of 1-particle states** $\mathcal{H}_\Delta$ by a unitary isomorphism of Hilbert spaces. The isomorphism is built using the **Wiener-Ito chaos decomposition** of the system, which, roughly speaking, generalizes Hermite polynomials to the infinite dimensional realm.

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Defining quantum states II

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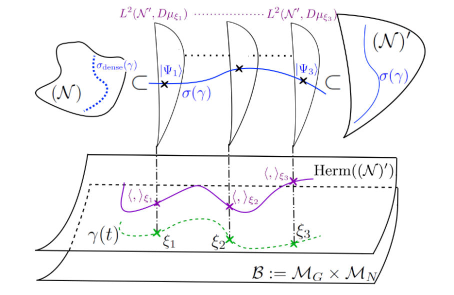
Therefore, the chosen framework, even if not being the most frequent in Physics, offers a consistent set of mathematical tools with a straightforward physical meaning.

Bundle of quantum states

Notice that the construction above depends on the complex structure chosen on Σ_τ, and on the geometrical properties of its metric tensor and therefore it defines different Hilbert spaces for each τ.



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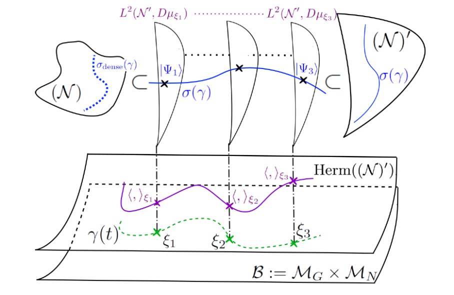

To build a differentiable structure on $\mathcal{H}_\tau$ a suitable space of test functions is chosen. It corresponds to the set of Hida test functions $(\mathcal{N})$. Using another rigged Hilbert space $(\mathcal{N}) \subset \mathcal{H} \subset (\mathcal{N})'$, states become a subset of the space of Hida distributions $(\mathcal{N})'$.

Bundle of quantum states II

- Physical states become thus sections of a bundle $\mathcal{F} \to \mathcal{B}$, whose base manifold $\mathcal{B}$ corresponds to the foliation and the gravitational degrees of freedom. Hence the model is τ–dependent.

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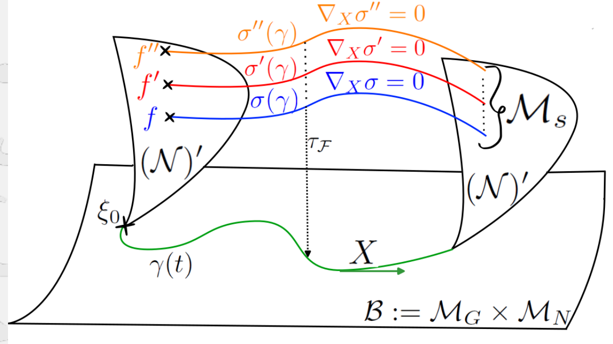
- The quantum state changes along the foliation because the gravitational background does. The equations of motion for our system must be written in terms of covariant derivatives, adding the purely quantum evolution. The norm of the quantum state will be thus preserved along the evolution.
- Analogously to the covariant differentiation of the states, we can covariantly quantize classical functionals and define quantum operators. In particular, we can quantize the superhamiltonian and supermomenta of the classical matter theory.

Bundle of quantum states III



A covariant section $\sigma : \mathcal{B} \to \mathcal{F}$ defines, at each fiber, a normalized quantum state for each Hilbert space $\mathcal{H}_\tau$. Connection compensates the changes in the Hilbert space structure along the evolution

$$\nabla_X \sigma = 0.$$

Bundle of quantum states IV

Analogously, other bundles can be considered to encode the different tensorial objects of the theory (the complex structure J_τ, classical magnitudes quantized to be defined on the Hilbert space $\mathcal{H}_\tau$, etc). Connection affects each tensor according to its nature:

- The complex structure is a $(1,1)$ tensor field

$$\nabla_X J_\tau = 0;$$

- A classical magnitude $f \in C^\infty(T^*\mathcal{M}_M)$ defines, by quantization, a linear operator on $\mathcal{H}_\tau$:

$$\nabla_X \mathcal{Q}_\tau(f) = 0$$

- ...

A Poisson structure

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Considering, for the sake of simplicity, holomorphic quantization with coordinates $(\bar{\Psi}, \Psi)$ and using again a rigged Hilbert space

$$(\mathcal{N}) \subset \mathcal{H} \subset (\mathcal{N})',$$

quantum operators can be represented by their expectation value functions on $\mathcal{H}$:

$$f_A(\bar{\Psi}, \Psi) = \int_{\mathcal{H}} d\mu(\Psi, \bar{\Psi}) \bar{\Psi} \hat{A} \Psi,$$

where $\hat{A}$ is a quantum linear operator on the Hilbert space $\mathcal{H}$ with dense domain $\mathcal{O}_{\mathcal{D}}(\mathcal{H})$.

A Poisson structure II

Then, given two operators $A, B \in \mathcal{O}_{\mathcal{D}}(\mathcal{H})$, we can define a canonical Poisson bracket structure on the space of sesquilinear functions as:

$$\{f_A, f_B\}_Q = \frac{\delta f_A}{\delta \bar{\Psi}} \frac{\delta f_B}{\delta \Psi} - \frac{\delta f_B}{\delta \bar{\Psi}} \frac{\delta f_A}{\delta \Psi} = f_{[A,B]},$$

where $[A, B] := -i(AB - BA)$.

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Quantum Kähler structure

On the space of quantum states there exists a Kähler structure having $\{\cdot, \cdot\}_Q$ as Poisson tensor.

Dynamics

How do we define dynamics of the quantum state? The evolution is defined as a dynamical equation on the space quantum states resulting of the evaluation of covariant sections of $\sigma : \mathcal{B} \to \mathcal{F}$, where $\sigma(\tau) = (\xi_\tau, \Psi_\tau)$:

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Generalized Schrödinger equation

Dynamics is defined by a generalized Schrödinger equation:

$$\nabla_\tau \Psi_\tau = \partial_\tau \Psi_\tau - \Gamma(\partial_\tau)\Psi_\tau = -\frac{i}{\hbar}\hat{H}_\tau \Psi_\tau = -\frac{i}{\hbar} X_H \Psi_\tau .$$

where Γ represents the connection one-form and encodes the dependence of the section σ on the geometrical degrees of freedom.

Application: KG on a FLRW background

If we consider a quantum scalar field evolving on a (fixed) FLRW metric, the Schrödinger equation becomes, for a the scale factor, M the mass scaled by a and Vol_k the volume:

$$
\begin{aligned}
i\left[\partial_t + \frac{\dot a}{a}\phi^y\left(2 - \frac{M^2}{2(M^2-\nabla^2)}\right)_y^{\;x}\partial_{\phi^x}\right]\Psi = \\
\dot a a^3\left[\mathrm{Vol}_k\left(\frac{1}{2}+\frac{M^2}{4(M^2-\nabla^2)}\right)\frac{1}{\sqrt{M^2-\nabla^2}}\right]^{xy}\partial_{\phi^x}\partial_{\phi^y}\Psi \\
-\frac{\dot a}{a^3}\left[\frac{1}{\mathrm{Vol}_k}\left(\frac{1}{2}+\frac{M^2}{4(M^2-\nabla^2)}\right)\sqrt{M^2-\nabla^2}\right]_{xy}\phi^x\phi^y\Psi \\
+\frac{1}{a}\phi^x\left(\sqrt{M^2-\nabla^2}\right)_x^{\;y}\partial_{\phi^y}\Psi.
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Hybrid geometrodynamics

New framework

Our goal is to adapt the ideas of classical geometrodynamics to a framework where classical matter fields are replaced by quantum ones. In order to do that we need:

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- Build the brackets of the generators and verify that they close the same Dirac algebra as their classical counterparts

Quantizing matter super-magnitudes

Notice that classical supermagnitudes are at most quadratic in field momenta:

$$\mathcal{H}^M = \frac{\pi_\phi^2}{2\sqrt{|h|}} + \frac{\sqrt{|h|}}{2}\left(h^{jk}\nabla_j\phi\nabla_k\phi + m^2\phi^2\right),$$

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$$\mathcal{H}_j^M = \pi_\phi\nabla_j\phi.$$

Furthermore, the quantization mapping is assumed to satisfy

$$\nabla_{\vec{\xi}}\mathcal{Q}_\tau(f) = \mathcal{Q}_\tau(\xi^j\partial_j f),$$

where $\vec{\xi}$ represents a vector field on the base manifold $\mathcal{B}$, which may change the leaf and the gravitational degrees of freedom. This implies a compatibility condition of the connection and the quantization mapping.

Hybrid Dirac algebra

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- In the space of sections of the bundle $\mathcal{F}$, we consider as possible quantum states just the values of covariant sections $\sigma : \mathcal{B} \to \mathcal{F}$ with respect to the quantum connection

$$\Psi_\xi = \sigma(\xi); \qquad \xi \in \mathcal{B}.$$

Hybrid Dirac algebra II

In this situation, consider the quadratic functions

$$\mathcal{H}^M_Q = \langle \Psi_\xi, \mathcal{Q}_\xi(\mathcal{H}^M)\Psi_\xi \rangle; \qquad \mathcal{H}^M_{jQ} = \langle \Psi_\xi, \mathcal{Q}(\mathcal{H}^M_j)\Psi_\xi \rangle,$$

and with them, the corresponding hybrid functions:

$$\mathcal{H}^H_Q = \mathcal{H}^G + \mathcal{H}^M_Q; \qquad \mathcal{H}^H_{jQ} = \mathcal{H}^G_j + \mathcal{H}^M_{jQ}.$$

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Hybrid Dirac algebra

Generators $\mathcal{H}_Q^H$ and $\mathcal{H}_{jQ}^H$ satisfy the Dirac algebra with Poisson bracket

$$\{\cdot,\cdot\}_H = \{\cdot,\cdot\}_G + \{\cdot,\cdot\}_Q.$$

This implies that fixing constraint equations on the sections of $\mathcal{F}$

$$\mathcal{H}_Q^H = 0 = \mathcal{H}_{jQ}^M$$

defines a set of dynamical equations which reproduce General Relativity with quantum matter sources.

Hybrid evolution equation

Hybrid dynamical equations

$$\frac{d}{ds}\Psi = \{\Psi, f_H\}_Q + \{\Psi, f_H\}_G + \dot{N}^x\partial_{N^x}\Psi + \dot{N}^{ix}\partial_{N^{ix}} =$$

$$\left(-\frac{i}{\hbar}\hat{H}(h, N(s), \vec{N}(s)) - \{h_{jk;x}, f_H\}_G\hat{\Gamma}_{h_{jk;x}} - \dot{N}^x\hat{\Gamma}N^x - \dot{\vec{N}}^x\hat{\Gamma}_{\vec{N}^x}\right)\Psi;$$

$$\frac{d}{ds}h_{jk}(x) = \{h_{jk}, f_H\}_G = 2N(x)G_{jklm}\pi^{lm}(x) + 2(D_jN^l)h_{lk};$$

$$\frac{d}{ds}\pi^{jk}(x) = \{\pi^{jk}, f_H\}_G = -\partial_{h_{jk}(x)}\mathcal{H}^G - \left\langle\Psi\middle|\mathcal{Q}\left(\partial_{h_{jk}(x)}\mathcal{H}^M\right)\Psi\right\rangle.$$

Conclusions and Outlook

- We have designed a formalism for a mathematically consistent description of a quantum scalar field on an arbitrary background defined on a globally hyperbolic space-time where
  - States of the system are modelled in a 3+1 formalism Schrödinger formalism with a well-defined Gaussian measure to define the scalar product, which may depend on the spatial hypersurface where the quantum functionals are defined. This defines a bundle of Hilbert spaces.
  - This framework requires the introduction of a connection on the bundle of quantum (hybrid) states, which defines covariant geometric objects to represent states, observables and geometric structures.
  - Evolution of quantum states is implemented by a generalized Schrödinger equation using covariant derivatives. Normalization of the quantum state is preserved along the evolution by construction.
  - This formalism can be written in geometrical terms considering a canonical Kähler structure.

Conclusions and Outlook II

- This formalism can be coupled with a dynamical gravitational field
  - The construction generalizes classical geometrodynamics replacing the classical matter sources by a quantum version based on the geometrical description of quantum states.
  - Hybrid dynamics become thus a non-linear theory due to the classical terms.
  - Dynamics of the gravitational field suffers the back-reaction of the quantum matter via the expectation value of the gradient of the quantized superhamiltonian.
  - Dynamics of the quantum fields are defined by a covariant Schrödinger equation defined on the sections of the state bundle.
  - With this, we constructed a consistent QFT theory which evolves coupled to a classical gravitational field which has the quantum field as matter source.
- We are currently working in the extension of this formalism to fermionic and gauge fields.

Thanks for your attention!