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# Harmonic Maps into Principal Bundles and Generalized Magnetic Mappings


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# Outline

- Motivation
- Harmonic maps into Kaluza-Klein principal bundles
    - Kaluza–Klein metrics on principal bundles
    - Kaluza–Klein harmonic maps and generalized magnetic maps
- Examples of Kaluza-Klein harmonic maps and generalized magnetic maps

# Source

The results presented in this talk are based on the following paper:

H. Benziadi, A. López Almorox, and C. Tejero Prieto. "Harmonic maps into principal bundles and generalized magnetic mappings". In: *Journal of Geometry and Physics* (2026)

# Motivation

## Riemannian magnetic curves

Let $\gamma : I \to (M, g)$ be a smooth curve, and let $A \in \Omega^1(M)$ be a gauge potential. Its curvature is a closed 2-form

$$F = dA,$$

called the (Faraday) *magnetic field.*

The associated Lorentz endomorphism $\mathscr{F} \in T_1^1(M)$ is defined by

$$g\big(\mathscr{F}(D_1), D_2\big) = F(D_1, D_2), \qquad \forall\, D_1, D_2 \in \mathfrak{X}(M).$$

A smooth curve $\gamma$ is called a *magnetic curve* if it satisfies the Lorentz equation

$$\boxed{\nabla_{\dot\gamma}\dot\gamma = q\,\mathscr{F}(\dot\gamma),}$$

where $q \in \mathbb{R}$ is a constant representing the charge, and $\mathscr{F}(\dot\gamma)$ is the Lorentz force.

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## Kaluza-Klein formalism for Riemannian magnetic curves

Let $P = M \times U(1)$ be a $U(1)$-principal bundle over $M$ with a principal connection $\omega \in \Omega^1(P, \mathfrak{u}(1))$, where $\dim M = m$ and $\dim P = m + 1$.

We consider the commutative diagram

$$\begin{tikzcd} I \subset \mathbb{R} \arrow[r, "\widetilde{\gamma}"] \arrow[dr, "\gamma"'] & (P = M \times U(1), \widehat{g}) \arrow[d, "{\pi,\, G=U(1)}"] \\ & (M, g) \end{tikzcd}$$

The metric $\widehat{g}$ is called the *Kaluza–Klein metric* and is defined by

$$\widehat{g} = \pi^* g + \kappa \langle \omega, \omega \rangle,$$

where $\kappa > 0$ is a constant and $\langle \cdot, \cdot \rangle$ is an Ad-invariant inner product on $\mathfrak{u}(1)$.

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Using the global section $s \in \Gamma(M, P)$, the Kaluza–Klein metric is represented by

$$\widehat{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + \kappa^2 A_\mu A_\nu & \kappa A_\nu \\ \kappa A_\mu & 1 \end{pmatrix}, \quad s^*\omega = A \in \Omega^1(M, \mathfrak{g}).$$

Let $\widetilde{\gamma} : I \longrightarrow (P, \widehat{g})$ be a geodesic curve on $P$, that is:

$$\frac{d^2 x^K}{dt^2} + \Gamma^K_{BC} \frac{dx^B}{dt} \frac{dx^C}{dt} = 0, \qquad K, B, C = 1, \cdots, m+1.$$

Its projection $\gamma$ on the base manifold $M$ satisfies:

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{dt} \frac{dx^\sigma}{dt} = \kappa \left( \frac{dx^{m+1}}{dt} + \kappa A_\nu \frac{dx^\nu}{dt} \right) \mathscr{F}^\mu_\nu \frac{dx^\nu}{dt}, \quad \mu, \nu, \sigma = 1, \cdots, m \qquad (1)$$

where $\mathscr{F}^\mu_\nu = g^{\mu\sigma} F_{\sigma\nu}$ and $\mathscr{F}^\mu_\nu \frac{dx^\nu}{dt}$ represents the Lorentz force.

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Therefore,

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Hence, we obtain

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## Generalization to Arbitrary Gauge Groups $G$: Kerner's Generalization

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# Harmonic maps into Kaluza-Klein principal bundles

To extend the Kaluza-Klein formalism beyond curves, we replace the interval ($I \in \mathbb{R}$) by a Riemannian manifold $(N, h)$ with $\dim N > 1$. Since geodesics are harmonic maps when the domain is one-dimensional, harmonic maps provide the natural higher-dimensional generalization of geodesics.

We consider the following commutative diagram:

$$\begin{tikzcd}
(N,h) \arrow[r, "\widetilde{\Phi}"] \arrow[dr, "\Phi"'] & (P,\widehat{g}) \arrow[d, "\pi{,}G"] \\
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$$\dim M = m, \qquad \dim G = d, \qquad \dim P = m + d.$$

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# Harmonic maps

- Let $(N, h)$ and $(M, g)$ be two Riemannian manifolds, with $N$ compact and oriented.
- Let $\Phi : N \longrightarrow M$ be a smooth map.
- We define the Dirichlet energy functional

$$E(\Phi) = \frac{1}{2} \int_N \|d\Phi\|^2 \, \mathrm{Vol}_h \, .$$

- A smooth map $\Phi : N \longrightarrow M$ is said to be harmonic if it is a critical point of $E(\Phi)$, i.e., it satisfies the Euler-Lagrange equation:

$$\tau(\Phi) = \mathrm{Tr}_h(\nabla d\Phi) = 0,$$

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- In terms of local coordinates $\{y^s\}$ on $N$ and $\{x^\mu\}$ on $M$, we have

$$g^{st}\left[\frac{\partial^2\Phi^\lambda}{\partial y^s \partial y^t} - \Gamma^l_{st}\frac{\partial\Phi^\lambda}{\partial y^l} + \frac{\partial\Phi^\mu}{\partial y^t}\frac{\partial\Phi^\nu}{\partial y^s}(\overline{\Gamma}^\lambda_{\mu\nu} \circ \Phi)\right] = 0, \quad \lambda = 1, \cdots, m.$$

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## Special case: Geodesics ($\dim N = 1$)

For $N = I \subset \mathbb{R}$, a harmonic map $\gamma : I \to (M, g)$ is precisely a geodesic satisfying

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{dt} \frac{dx^\sigma}{dt} = 0.$$

# Kaluza-Klein metrics on principal $G$-bundles

**Definition 1**

A Kaluza–Klein metric $\widehat{g}$ on a principal $G$-bundle $\pi : P \longrightarrow M$ is a Riemannian metric on $P$ that is $G$-invariant.

**Theorem 1**

The space $\mathrm{Met}_{KK}(P)$ of Kaluza-Klein metrics on $P$ is equipped with an $\mathrm{Aut}(P)$-equivariant bijection

$$\begin{aligned} \Psi : \mathrm{Met}_{KK}(P) &\xrightarrow{\sim} \mathcal{A}(P) \times \mathrm{Met}(\mathrm{ad}P) \times \mathrm{Met}(M) \\ \widehat{g} &\longmapsto (\omega, \quad g_{\mathrm{ad}}, \quad g) \end{aligned}$$

$$TP = \mathcal{V}P \oplus (\mathcal{V}P)^{\perp}, \qquad \pi : (P, \widehat{g}) \to (M, g) \text{ is a Riemannian submersion.}$$

The inverse bijection is given by

$$\widehat{g} := \Psi^{-1}(\omega, g_{\mathrm{ad}}, g) = (\omega^* g_{\mathrm{ad}})^{\mathcal{V}} + \pi^* g.$$

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# Kaluza-Klein harmonic maps and generalized magnetic maps

Let $\pi : (P, \widehat{g}) \longrightarrow (M, g)$ be a Kaluza-Klein principal $G$-bundle

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We establish the relationship between the tension fields of $\widetilde{\Phi}$ and $\Phi$.

## Proposition 1

The second fundamental form $\mathrm{II}_\Phi \in S^2(N, \Phi^*TM)$ of the composition $\Phi = \pi \circ \widetilde{\Phi} : (N, h) \longrightarrow (M, g)$ satisfies

$$\mathrm{II}_\Phi = \pi_* \circ \mathrm{II}_{\widetilde{\Phi}} + \widetilde{\Phi}^*(\mathrm{II}_\pi)$$

and its $h$-Riemannian trace satisfies the identity

$$\tau(\Phi) = \pi_*(\tau(\widetilde{\Phi})) + \mathrm{Tr}_h\left[\widetilde{\Phi}^*(\mathrm{II}_\pi)\right].$$

## Corollary 1

If $\widetilde{\Phi}$ is harmonic, then

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## Proposition 2

Let $\pi : (P, \widehat{g}) \longrightarrow (M, g)$ be a Kaluza-Klein principal $G$-bundle. For every $\pi$-projectable vector field $D \in \mathfrak{X}(P)$, we have

$$\boxed{\mathrm{II}_\pi(D, D) := (\nabla d\pi)(D, D) = [-\pi_* \circ (2\mathbf{A} + \mathbf{T})](D, D),}$$

where $\nabla$ is the connection on $T^*P \otimes \pi^*TM$ induced by the Levi-Civita connections of $(P, \widehat{g})$ and $(M, g)$, $\mathbf{A}$ and $\mathbf{T}$ are the O'Neill's tensors defined by

$$\boldsymbol{A}(D_1, D_2) = \left[\nabla^P_{D_1^{\mathcal{H}}} D_2^{\mathcal{V}}\right]^{\mathcal{H}} + \left[\nabla^P_{D_1^{\mathcal{H}}} D_2^{\mathcal{H}}\right]^{\mathcal{V}},$$

$$\boldsymbol{T}(D_1, D_2) = \left[\nabla^P_{D_1^{\mathcal{V}}} D_2^{\mathcal{H}}\right]^{\mathcal{V}} + \left[\nabla^P_{D_1^{\mathcal{V}}} D_2^{\mathcal{V}}\right]^{\mathcal{H}}.$$

## Theorem 2

Let $\pi : (P, \widehat{g}) \to (M, g)$ be a Kaluza–Klein principal $G$-bundle, $\widetilde{\Phi} : (N, h) \to (P, \widehat{g})$ a smooth map, and let $\Phi = \pi \circ \widetilde{\Phi} : (N, h) \to (M, g)$. Then the tension field of $\Phi$ is given by

$$\tau(\Phi) = \pi_*(\tau(\widetilde{\Phi})) - \pi_* \left( \mathrm{Tr}_h \left[ \widetilde{\Phi}^*(2\mathbf{A} + \mathbf{T}) \right] \right).$$

In particular, $\widetilde{\Phi}$ is (horizontally) harmonic if and only if

$$\boxed{\tau(\Phi) = -\pi_* \left( \mathrm{Tr}_h \left[ \widetilde{\Phi}^*(2\mathbf{A}) \right] \right) - \pi_* \left( \mathrm{Tr}_h \left[ \widetilde{\Phi}^* \mathbf{T} \right] \right).}$$

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Let $\pi : (P, \widehat{g}) \longrightarrow (M, g)$ be a Kaluza-Klein principal $G$-bundle.

## Proposition 3

The vertical component of the tension field of a smooth map $\widetilde{\Phi} : (N, h) \longrightarrow (P, \widehat{g})$ is given by

$$\boxed{[\tau(\widetilde{\Phi})]^{\mathcal{V}} = -\delta^{(\nabla^N, \widetilde{\Phi}^* \nabla^{P,\mathcal{V}})}(\widetilde{\Phi}^* \omega) + \omega(\mathrm{Tr}_h(\widetilde{\Phi}^* \mathbf{T})).}$$

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Thus, if $\pi$ has totally geodesic fibers, then $\widetilde{\Phi}^* \omega$ satisfies the Hodge gauge equation

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## Lorentz Endomorphism

Let $\pi : (P, \widehat{g}) \longrightarrow (M, g)$ be a Kaluza-Klein principal bundle with a principal connection $\omega \in \Omega^1(P, \mathcal{V}P)$. Its curvature $\Omega^\omega \in \Omega^2(P, \mathcal{V}P)$ induces the **Lorentz endomorphism** $\mathscr{F}^\omega \in T_1^1(P) \otimes_{C^\infty(P)} \mathfrak{X}^{\mathcal{V}}(P)$, defined by

$$\langle V, \Omega^\omega \rangle_{\mathcal{V}P}(D_1, D_2) = \widehat{g}(\langle V, \mathscr{F}^\omega \rangle (D_1), D_2)\,,$$

for every vertical vector field $V$ and all $D_1, D_2 \in \mathfrak{X}(P)$

## Definition 2

**The curvature-modified metric** $\widehat{g}_{\mathscr{F}^\omega}$ is defined by

$$\widehat{g}_{\mathscr{F}^\omega}(D_1, D_2) = \frac{1}{2}\left\{\langle D_1^{\mathcal{V}}, \mathscr{F}^\omega \rangle_{\mathcal{V}P}(D_2^{\mathcal{H}}) + \langle D_2^{\mathcal{V}}, \mathscr{F}^\omega \rangle_{\mathcal{V}P}(D_1^{\mathcal{H}})\right\}.$$

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Let $\pi : (P, \widehat{g}) \longrightarrow (M, g)$ be a Kaluza-Klein principal $G$-bundle. A smooth map $\widetilde{\Phi} : (N, h) \longrightarrow (P, \widehat{g})$ is (horizontally) harmonic if and only if $\Phi = \pi \circ \widetilde{\Phi} : (N, h) \longrightarrow (M, g)$ satisfies

$$\boxed{\tau(\Phi) = -\pi_* \left( \mathrm{Tr}_h \left[ \widetilde{\Phi}^* \widehat{g}_{\mathscr{F}^\omega} \right] \right) - \pi_* \left( \mathrm{Tr}_h \left[ \widetilde{\Phi}^* \mathbf{T} \right] \right).}$$

## Definition 3

The equation above is called the **intrinsic generalized Lorentz equation** satisfied by the **generalized magnetic map** $\Phi$.

## Definition 4

A generalized magnetic map $\Phi$ is called **uncharged** if

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## Theorem 5

Let $\widetilde{\Phi} : (N, h) \longrightarrow (P, \widehat{g})$ be a smooth map and $\Phi = \pi \circ \widetilde{\Phi}$. Then $\widetilde{\Phi}$ is harmonic if and only if

$$\tau(\Phi) = -\pi_*[\mathrm{Tr}_h(\widetilde{\Phi}^* \widehat{g}_{\mathscr{F}\omega})] - \pi_*[\mathrm{Tr}_h(\widetilde{\Phi}^* \mathbf{T})] \tag{2}$$

and

$$\delta^{(\nabla^N, \widetilde{\Phi}^* \nabla^{P,\mathcal{V}})}(\widetilde{\Phi}^* \omega) = \omega[\mathrm{Tr}_h(\widetilde{\Phi}^* \mathbf{T})]. \tag{3}$$

These equations are called **generalized Wong's equations**

## Definition 5

A **Kaluza-Klein harmonic map** is a harmonic map whose target is a Kaluza-Klein principal bundle.

## Theorem 5

Let $\widetilde{\Phi} : (N, h) \longrightarrow (P, \widehat{g})$ be a smooth map and $\Phi = \pi \circ \widetilde{\Phi}$. Then $\widetilde{\Phi}$ is harmonic if and only if

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## Lorentz strength endomorphism

Let $B = \{\xi_1, \dots, \xi_d\}$ be a basis of the Lie algebra $\mathfrak{g}$.
Since the curvature form $\Omega^\omega$ is $G$-invariant, it descends to

$$\Omega_\omega = \sum_{a=1}^{d} \Omega_\omega^a \otimes \xi_a^* \in \Omega^2(M) \otimes_{C^\infty(M)} \mathfrak{X}^{\mathcal{V}}(P)^G,$$

which induces the Lorentz strength endomorphism

$$\mathscr{F}_\omega = \sum_{a=1}^{d} \mathscr{F}_\omega^a \otimes \widetilde{\boldsymbol{\xi}}_a \in T_1^1(M) \otimes_{C^\infty(M)} \mathfrak{X}^{\mathcal{V}}(P)^G.$$

## Theorem 6

The map $\widetilde{\Phi} : (N, h) \longrightarrow (P, \widehat{g})$ is (horizontally) harmonic if and only if $\Phi = \pi \circ \widetilde{\Phi}$ satisfies

$$\boxed{\tau(\Phi) = -\sum_{a=1}^{k} \left[\Phi^*(\mathscr{F}^a_\omega)\right] \Big(p_h^{-1}\big(\widetilde{\Phi}^*[p_{\widehat{g}}(\widetilde{\boldsymbol{\xi}}_a)]\big)\Big) - \pi_*\Big(\mathrm{Tr}_h \left[\widetilde{\Phi}^*\mathbf{T}\right]\Big)}$$

## Corollary 2

If, for each $a$, the vector field $\widetilde{\boldsymbol{\xi}}_a$ is a Killing vector field, then

$$\boldsymbol{K}_a := p_h^{-1}\big(\widetilde{\Phi}^*[p_{\widehat{g}}(\widetilde{\boldsymbol{\xi}}_a)]\big) \in \mathfrak{X}(N)$$

represents the **conserved current** along $\widetilde{\Phi}$.

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We now introduce the natural generalization of the classical example of geodesic curves.

## Example 1: Generalized magnetic curves

Let $\widetilde{\gamma} : (I, h) \longrightarrow (P, \widehat{g})$ be a geodesic curve in $P$, that is, $\nabla^P_{\dot{\widetilde{\gamma}}} \dot{\widetilde{\gamma}} = 0$, and let $\gamma = \pi \circ \widetilde{\gamma}$ the induced curve in $M$, then $\widetilde{\gamma}$ satisfies

$$\boxed{\begin{aligned} \nabla^M_{\dot{\gamma}} \dot{\gamma} &= -\pi_*[\mathrm{Tr}_h(\widetilde{\gamma}^* \widehat{g}_{\mathscr{F}^\omega})] - \pi_*[\mathrm{Tr}_h(\widetilde{\gamma}^* \mathbf{T})]\,, \\ \delta^{(\nabla^I, \widetilde{\gamma}^* \nabla^{P,\mathcal{V}})}(\widetilde{\gamma}^* \omega) &= \omega[\mathrm{Tr}_h(\widetilde{\gamma}^* \mathbf{T})]\,. \end{aligned}}$$

The first equation is equivalent to

$$\nabla^M_{\dot\gamma}\dot\gamma = -\sum_{a=1}^{k} \widetilde{\gamma}^{\#}\widehat{g}(\widetilde{\gamma}^{\#}\widetilde{\boldsymbol{\xi}}_a, \dot{\widetilde{\gamma}})\mathscr{F}^a_\omega(\dot\gamma) - \pi_*[\mathbf{T}(\dot{\widetilde{\gamma}}, \dot{\widetilde{\gamma}})] \tag{4}$$

Suppose that, for each $a$, the vector field $\widetilde{\boldsymbol{\xi}}_a$ is a Killing vector field with respect to $\widehat{g}$ on $P$. It follows that

$$\boldsymbol{\kappa}_a = -\widetilde{\gamma}^{\#}\widehat{g}(\widetilde{\gamma}^{\#}\widetilde{\boldsymbol{\xi}}_a, \dot{\widetilde{\gamma}})$$

is constant. Therefore,

$$\boxed{\begin{aligned} \nabla^M_{\dot\gamma}\dot\gamma &= \sum_{a=1}^{d} \boldsymbol{\kappa}_a \cdot \mathscr{F}^a_\omega(\dot\gamma) - \pi_*\left[\mathbf{T}(\dot{\widetilde{\gamma}}, \dot{\widetilde{\gamma}})\right], \\ \delta^{(\nabla^I, \widetilde{\gamma}^*\nabla^{P,\mathcal{V}})}(\widetilde{\gamma}^*\omega) &= \omega\left[\mathbf{T}(\dot{\widetilde{\gamma}}, \dot{\widetilde{\gamma}})\right]. \end{aligned}}$$

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## Example 2

If $N = P$, $g = \widehat{g}$, then $\mathrm{Id}_P : (P, \widehat{g}) \longrightarrow (P, \widehat{g})$ is harmonic.

$$\begin{tikzcd}
(P, \widehat{g}) \arrow[rr, "\mathrm{Id}_P"] \arrow[drr, "\pi"'] & & (P, \widehat{g}) \arrow[d, "\pi"] \\
& & (M, g)
\end{tikzcd}$$

The Wong's equations satisfied by $\mathrm{Id}_P$ are

$$\boxed{\tau(\pi) = -d \cdot \pi_*(\mathbf{H}^{\mathcal{V}}), \quad \delta^{(\nabla^N, \nabla^\omega)}\omega = \mathbf{V},}$$

with $\mathbf{H}^{\mathcal{V}} := \frac{1}{d} \operatorname{Tr}_{\widehat{g}} \mathbf{T}^{\mathcal{V},\mathcal{V}}$ the vertical mean curvature vector field of the fibers of $\pi$ and $\mathbf{V} = \left[\sum_{a=1}^{d} ad^*_{\xi_a}(\xi_a)\right]^*$, where $\{\xi_a^*\}_{a=1}^{d}$ is a frame $\widehat{g}$-ortonormal of $\mathcal{V}P$.

## Example 3

Let

$$\widetilde{\Phi}_\alpha : S^1 \times S^1 \longrightarrow S^3 \subset \mathbb{R}^4$$

be the map defined by

$$\widetilde{\Phi}_\alpha(z_1, z_2) = \cos\alpha\, z_1 + \sin\alpha\, z_2 \cdot j.$$

The following diagram is commutative:

$$\begin{tikzcd} S^1 \times S^1 \arrow[r, "\widetilde{\Phi}_\alpha"] \arrow[dr, "\Phi_\alpha"'] & S^3 \subset \mathbb{R}^4 \arrow[d, "\pi"] \\ & S^2 \subset \mathbb{R}^3 \end{tikzcd}$$

where $\Phi_\alpha : S^1 \times S^1 \longrightarrow S^2 \subset \mathbb{R}^3$ is the composition $\Phi_\alpha = \pi \circ \widetilde{\Phi}_\alpha$.

The vertical and horizontal subbundles of the Hopf $U(1)$-bundle $(S^3, R)$ are given by $V(S^3) = \langle i^* \rangle, \quad H(S^3) = \langle j^*, k^* \rangle$.

The principal connection is given by $\omega = \omega^1 \otimes i^*$.

The curvature of the connection of $\omega$ is

$$\Omega^\omega = d\omega^1 \otimes i^* = -2\omega^2 \wedge \omega^3 \otimes i^*.$$

The Lorentz strength endomorphism of $(S^3, \widehat{g})$ is

$$\mathscr{F}^\omega = -2(\omega^2 \otimes D_3 - \omega^3 \otimes D_2) \otimes i^*.$$

The curvature-modified metric of $\widehat{g}$ is

$$\begin{aligned}\widehat{g}_{\mathscr{F}^\omega} &= -[\omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1] \otimes D_3 + [\omega^1 \otimes \omega^3 + \omega^3 \otimes \omega^1] \otimes D_2 \\ &= \omega^1 \cdot \omega^3 \otimes D_2 - \omega^1 \cdot \omega^2 \otimes D_3\end{aligned}$$

The Wong's equations satisfied by $\widetilde{\Phi}$ are

$$\boxed{\tau(\Phi_\alpha) = -\pi_*\left(\mathrm{Tr}_g\left[\widetilde{\Phi}_\alpha^*\left(-\omega^1 \cdot \omega^2 \otimes D_3 + \omega^1 \cdot \omega^3 \otimes D_2\right)\right]\right)}$$

$$\boxed{\delta^{(\nabla^{S^1 \times S^1}, \widetilde{\Phi}_\alpha^* \nabla^\omega)}(\widetilde{\Phi}_\alpha^*(\omega^1 \otimes i^*)) = 0}$$

If

$$\alpha \neq \frac{\pi}{2}\mathbb{Z},$$

then $\widetilde{\Phi}_\alpha$ is a spherical harmonic immersion, called the $\alpha$-twisted Clifford torus; that is,

$$\tau(\widetilde{\Phi}_\alpha) = 0$$

(in particular, for $\alpha = \frac{\pi}{4}$). Hence, $\Phi_\alpha$ is a generalized magnetic map.

A straightforward computation shows that $\Phi_\alpha$ is uncharged (i.e., $\tau(\Phi_\alpha) = 0$) if and only if

$$\alpha \in \frac{\pi}{4}\mathbb{Z},$$

that is, if and only if $\widetilde{\Phi}_\alpha$ is the standard Clifford torus.

## Example 4

Let

$$\widetilde{\Phi}_\alpha : S^3 \times S^3 \longrightarrow S^7 \subset \mathbb{R}^8$$

be a map given by

$$\widetilde{\Phi}_\alpha(q_1, q_2) = cos\alpha q_1 + sin\alpha q_2 \cdot l.$$

We have the following commutative diagram:

$$\begin{tikzcd}
S^3 \times S^3 \arrow[r, "\widetilde{\Phi}_\alpha"] \arrow[dr, "\Phi_\alpha"'] & S^7 \subset \mathbb{R}^8 \arrow[d, "\pi"] \\
 & S^4 \subset \mathbb{R}^5
\end{tikzcd}$$

where $\Phi_\alpha : S^3 \times S^3 \longrightarrow S^4 \subset \mathbb{R}^5$ is the composition $\Phi_\alpha = \pi \circ \widetilde{\Phi}_\alpha$.

The vertical and horizontal subbundles of the Hopf $SU(2)$-bundle $(S^7, R)$ are

$$V(S^7) = \langle D_1 = \boldsymbol{i}^*,\, D_2 = \boldsymbol{j}^*,\, D_3 = \boldsymbol{k}^* \rangle,$$

$$H(S^7) = \langle D_4 = \boldsymbol{l}^*,\, D_5 = \boldsymbol{m}^*,\, D_6 = \boldsymbol{n}^*,\, D_7 = \boldsymbol{o}^* \rangle.$$

The principal connection is

$$\omega = \omega^1 \otimes \boldsymbol{i}^* + \omega^2 \otimes \boldsymbol{j}^* + \omega^3 \otimes \boldsymbol{k}^*.$$

The Curvature of the connection $\omega$ is

$$\begin{aligned}\Omega^\omega = 2[&(-\omega^4 \wedge \omega^5 + \omega^6 \wedge \omega^7) \otimes \boldsymbol{i}^* - (\omega^4 \wedge \omega^6 + \omega^5 \wedge \omega^7) \otimes \boldsymbol{j}^* + \\ &+ (-\omega^4 \wedge \omega^7 + \omega^5 \wedge \omega^6) \otimes \boldsymbol{k}^*].\end{aligned}$$

The Lorentz strength endomorphism of $(S^7, \widehat{g})$ is

$$\mathscr{F}^\omega = 2[(-\omega^4 \otimes D_5 + \omega^5 \otimes D_4 + \omega^6 \otimes D_7 - \omega^7 \otimes D_6) \otimes \boldsymbol{i}^* + \\ - (\omega^4 \otimes D_6 - \omega^6 \otimes D_4 + \omega^5 \otimes D_7 - \omega^7 \otimes D_5) \otimes \boldsymbol{j}^* + \\ + (-\omega^4 \otimes D_7 + \omega^7 \otimes D_4 + \omega^5 \otimes D_6 - \omega^6 \otimes D_5) \otimes \boldsymbol{k}^*].$$

The curvature-modified metric of $\widehat{g}$ is

$$\widehat{g}_{\mathscr{F}^\omega} = -\omega^1 \cdot \omega^4 \otimes D_5 + \omega^1 \cdot \omega^5 \otimes D_4 + \omega^1 \cdot \omega^6 \otimes D_7 + \omega^1 \cdot \omega^7 \otimes D_6 - \\ - \omega^2 \cdot \omega^4 \otimes D_6 - \omega^2 \cdot \omega^5 \otimes D_7 + \omega^2 \cdot \omega^6 \otimes D_4 + \omega^2 \cdot \omega^7 \otimes D_5 - \\ - \omega^3 \cdot \omega^4 \otimes D_7 + \omega^3 \cdot \omega^5 \otimes D_6 - \omega^3 \cdot \omega^6 \otimes D_5 + \omega^3 \cdot \omega^7 \otimes D_4.$$

The Wong's equations satisfied by $\widetilde{\Psi}_\alpha$ are

$$\begin{aligned}\tau(\Psi_\alpha) = -\pi_*\Big(\mathrm{Tr}_g[\widetilde{\Psi}_\alpha((\omega^1\cdot\omega^5+\omega^2\cdot\omega^6+\omega^3\cdot\omega^7)\otimes D_4-\\ -(\omega^1\cdot\omega^4-\omega^2\cdot\omega^7+\omega^3\cdot\omega^6)\otimes D_5+\\ +(\omega^1\cdot\omega^7-\omega^2\cdot\omega^4+\omega^3\cdot\omega^5)\otimes D_6+\\ -(\omega^1\cdot\omega^6-\omega^2\cdot\omega^5-\omega^3\cdot\omega^4)\otimes D_7)]\Big)\end{aligned}$$

and

$$\delta^{(\nabla^{S^3\times S^3},\widetilde{\Psi}^*_\alpha\nabla^\omega)}(\widetilde{\Psi}^*_\alpha(\omega^1\otimes \boldsymbol{i}^*+\omega^2\otimes\boldsymbol{j}^*+\omega^3\otimes\boldsymbol{k}^*))=0.$$

If

$$\alpha \neq \frac{\pi}{2}\mathbb{Z},$$

then $\widetilde{\Phi}_\alpha$ is a spherical harmonic immersion, called the $\alpha$-twisted spherical immersion; that is,

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# Thank you for your attention!

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