An introductory course on geometric field theories: the multisymplectic setting

Fourth lecture: Symmetries. Gauge symmetries. Conservation laws

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Course Programme

1. **First lecture:** Introductory remarks. Previous geometric background.
2. **Second lecture:** Multisymplectic Lagrangian field theory.
3. **Third lecture:** Multisymplectic Hamiltonian field theory.
4. **Fourth lecture:** Symmetries. Gauge symmetries. Conservation laws.

Introduction

Symmetries play a crucial role in studying dynamical systems and field theories.

Exploring symmetries often involves analyzing ordinary and partial differential equations, as they encode **conserved quantities** (i.e., first integrals of the equations) and **conservation laws** that reveal fundamental properties of physical systems.

As a cornerstone on this topic, Emmy Noether's groundbreaking work established a deep connection between certain kinds of symmetries and conservation principles.

E. Noether, "Invariant Variation Problems", *Transport Theor. Stat. Phys.* **1**(3) (1971) 186–207. (Original in *Gott. Nachr.* 1918, 235–257. Translated by Mort Tavel).

Y. Kosmann-Schwarzbach, *The Noether Theorems. Invariance and Conservation Laws in the Twentieth Century*, Sources and Studies in the History of Mathematics and Physical Sciences. Springer, NY, 2011.

Initially, symmetries in physical systems were understood as the invariance of the equations of the theory under specific transformations in phase space which, as a consequence, transform solutions to the equations into other solutions.

In modern geometric formulations of classical mechanics and field theories, symmetries are typically defined through the invariance of the fundamental geometric structures underlying the theory, ensuring the preservation of the equations of motion as a natural consequence.

The case in which symmetries are locally generated through the flux of vector fields is also relevant, since this is the usual situation in physics.

In particular, in multisymplectic field theories, symmetries and their associated conservation laws have been studied within this geometric framework, and symmetries that preserve the (pre)multisymplectic structure are known as ***Noether*** or ***Cartan symmetries***.

This intrintric formulation of Noether symmetries allows to easily prove a geometric version of ***Noether's Theorem***.

The aim of this lecture is to present the description of symmetries, gauge symmetries, and conservation laws in the multisymplectic setting of first-order classical field theories.

First, after introducing the concept of ***conserved form***, the derivation of the associated ***conservation law*** is explained.

Second, the general definition of ***symmetry*** is introduced. Then, the particular case of ***Cartan (Noether) symmetries*** is considered and the ***Noether theorem*** is stated for them, giving the way of obtaining their associated conservation laws.

Next, the concept of ***gauge symmetry*** and its geometrical meaning are discussed in this formulation.

Finally, the results are applied to study Cartan-Noether and gauge symmetries for some physical theories: Maxwell's Electromagnetisms, Wave equation and the Einstein–Palatini approach of GR.

A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, "Multivector field formulation of Hamiltonian field theories: Equations and symmetries", *J. Phys. A: Math. Gen.* **32** (1999) 8461–8484.

J. Gaset, P.D. Prieto-Martínez, N. Román-Roy, "Variational principles and symmetries on fibered multisymplectic manifolds", *Comm. Math.* **24**(2) (2016) 137–152.

J. Gaset, N. Román-Roy, "Symmetries and gauge symmetries in multisymplectic first and second-order Lagrangian field theories: electromagnetic and gravitational fields", *Rev. Acad. Cienc. Canarias (RACC)* **32**(2) 2020 (2021) 61–84.

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Symmetries, conserved forms and conservation laws

Preliminaries

Consider a Lagrangian or Hamiltonian (pre)multisymplectic field theory:

- A fiber bundle $\bar{\varrho}\colon \mathcal{F} \xrightarrow{\varrho} E \xrightarrow{\pi} M$ (multiphase space of the theory).
- An exact (pre)multisymplectic form $\Omega = -\mathrm{d}\Theta \in \mathfrak{X}^m(\mathcal{F})$ (the Poincaré–Cartan or the Hamilton–Cartan forms).
- The field equations for locally decomposable m.v.f. $\mathbf{X} \in \mathfrak{X}^m(\mathcal{F})$,

$$\iota(\mathbf{X})\Omega = 0 \quad , \quad \iota(\mathbf{X})\omega \neq 0 \ (\bar{\varrho}\text{-transverse}) \ .$$

First, consider the regular case, or the singular (almost-regular) case when these equations are compatible on $\mathcal{F}$ (final constraint manifold $\mathcal{S} = \mathcal{F}$).

This means that $(\mathcal{F}, \Omega)$ is, either $(J^1\pi, \Omega_{\mathcal{L}})$, or $(J^1\pi^*, \Omega_{\mathbf{h}})$, or $(P_\circ, \Omega^\circ_{\mathbf{h}})$.

Notation:

$$\ker^m \Omega := \{\mathbf{X} \in \mathfrak{X}^m(\mathcal{F}) \,|\, \iota(\mathbf{X})\Omega = 0\} \ ,$$

$$\ker^m_{ld} \Omega := \ker^m \Omega \cap \mathfrak{X}^m_{ld}(\mathcal{F}) \ ,$$

$$\ker^m_{ld(\omega)} \Omega := \{\mathbf{X} \in \mathfrak{X}^m_{ld}(\mathcal{F}) \ | \ \iota(\mathbf{X})\Omega = 0, \, \iota(\mathbf{X})\omega \neq 0\} \ ,$$

$$\ker^m_{\mathrm{I}(\omega)} \Omega := \{\mathbf{X} \in \ker^m_{ld(\omega)} \Omega \ | \ \mathbf{X} \text{ is integrable}\} \ .$$

Conserved forms and conservation laws

Definition 2.1

*A **conserved form** is a differential form $\xi \in \Omega^{m-1}(\mathcal{F})$ such that*

$$\mathrm{L}_{\boldsymbol{X}}\xi = (-1)^{m+1}\iota(\boldsymbol{X})\mathrm{d}\xi = 0 \ , \ \text{for every } \mathbf{X} \in \ker^m_{Id(\omega)}\Omega \ .$$

Proposition 2.2

If $\xi \in \Omega^{m-1}(\mathcal{F})$ is a conserved form and $\boldsymbol{X} \in \ker^m_{\mathrm{I}(\omega)}\Omega$, then ξ is closed on the integral submanifolds of $\boldsymbol{X}$; that is, if $j_S \colon S \hookrightarrow \mathcal{F}$ is an integral submanifold of $\boldsymbol{X}$, then $\mathrm{d}j_S^*\xi = 0$.

Lemma 2.3

For $\xi \in \Omega^{m-1}(\mathcal{F})$ and $\boldsymbol{X} \in \ker^m_{\mathrm{I}(\omega)}\Omega$, if $\psi \colon M \to \mathcal{F}$ is an integral section of $\boldsymbol{X}$, then $\psi^*\xi \in \Omega^{m-1}(M)$, and $\exists! X_{\psi^*\xi} \in \mathfrak{X}(M)$ such that

$$\iota(X_{\psi^*\xi})\omega_M = \psi^*\xi \, .$$

Furthermore, $\mathrm{div}X_{\psi^*\boldsymbol{\xi}} \in \mathrm{C}^\infty(M)$ (the divergence of $X_{\psi^*\boldsymbol{\xi}}$), is defined by

$$\mathrm{L}_{X_{\psi^*\boldsymbol{\xi}}}\omega_{_M} := (\mathrm{div}X_{\psi^*\boldsymbol{\xi}})\,\omega_{_M}\,.$$

Then, from the above Lemma 2.3 we have that

$$(\mathrm{div}X_{\psi^*\boldsymbol{\xi}})\,\omega_{_M} = \mathrm{d}(\psi^*\boldsymbol{\xi})\,,$$

and, as a consequence of Proposition 2.2, we conclude that:

Corollary 2.4

If $\boldsymbol{\xi} \in \Omega^{m-1}(\mathcal{P})$ is a conserved form, for every integral section $\psi\colon M \to \mathcal{F}$ of any $\boldsymbol{X} \in \ker^m_{\mathrm{I}(\omega)}\Omega$, we have that $\mathrm{d}(\psi^*\boldsymbol{\xi}) = 0$ and, as a consequence,

$$\mathrm{div}X_{\psi^*\boldsymbol{\xi}} = 0\,. \tag{1}$$

Definition 2.5

Equation (1) is the ***conservation law*** *associated with the conserved form $\boldsymbol{\xi}$, and $\psi^*\boldsymbol{\xi}$ is called the* ***conserved current*** *associated with ψ and $\boldsymbol{\xi}$.*

In coords., if $\psi^*\boldsymbol{\xi} = f^\mu\,\mathrm{d}x^{m-1}x_\mu$, then $X_{\psi^*\boldsymbol{\xi}} = f^\mu\dfrac{\partial}{\partial x^\mu}$, and (1) is written

$$\mathrm{div}X_{\psi^*\boldsymbol{\xi}} = \frac{\partial f^\mu}{\partial x^\mu} = 0\,.$$

General symmetries

Definition 2.6

1. *A **general symmetry** is a diffeomorphism $\Phi\colon \mathcal{F} \to \mathcal{F}$ such that*
$$\Phi_*(\ker^m_{ld(\omega)} \Omega) \subset \ker^m_{ld(\omega)} \Omega \ .$$
2. *An **infinitesimal general symmetry** is a vector field $Y \in \mathfrak{X}(\mathcal{F})$ whose local diffeomorphisms generated by the flux of Y are local general symmetries; that is,*
$$[Y, \ker^m_{ld(\omega)} \Omega] \subset \ker^m_{ld(\omega)} \Omega \ .$$

Proposition 2.7

If $\Phi_1, \Phi_2 \in \mathrm{Diff}(\mathcal{F})$ are general symmetries, then so is $\Phi_2 \circ \Phi_1$.
If $Y_1, Y_2 \in \mathfrak{X}(\mathcal{F})$ are infinitesimal general symmetries, then so is $[Y_1, Y_2]$.

Proposition 2.8

Let $\xi \in \Omega^{m-1}(\mathcal{F})$ be a conservative form. Therefore:

1. *If $\Phi \in \mathrm{Diff}(\mathcal{F})$ is a general symmetry, then so is $\Phi^*\xi$.*
2. *If $Y \in \mathfrak{X}(\mathcal{F})$ is an infinitesimal general symmetry, then so is $\mathrm{L}_Y\, \xi$.*

General symmetries transform m.v.f. solutions into m.v.f. solutions. However, the states of the field are not m.v.f., but their integral sections. To translate this requirement to sections, symmetries must be $\bar{\varrho}$-fiber preserving transformations in the bundle $\mathcal{F}$; that is, diffeomorphisms and vector fields on $\mathcal{F}$ which restrict to diffeoms. and vector fields on M.

Definition 2.9

1. *A **restricted** or **fiber-preserving general symmetry** is a general symmetry $\Phi\colon \mathcal{F} \to \mathcal{F}$ such that there exists $\varphi \in \mathrm{Diff}(M)$, and*
$$\bar{\varrho} \circ \Phi = \varphi \circ \bar{\varrho} \ .$$
2. *An **infinitesimal restricted** or **fiber-preserving general symmetry** is a vector field $Y \in \mathfrak{X}(\mathcal{F})$ whose flux are local restricted symmetries; that is, Y is $\bar{\varrho}$-projectable: $\bar{\varrho}_* Y = Z \in \mathfrak{X}(M)$.*

Theorem 2.10

Let $\Phi \in \mathrm{Diff}(\mathcal{F})$ be a restricted general symmetry. Then, for every $\boldsymbol{X} \in \ker^m_{\mathrm{I}(\omega)} \Omega$, we have that $\Phi_* \boldsymbol{X} \in \ker^m_{\mathrm{I}(\omega)} \Omega$, and Φ transforms integral sections of $\boldsymbol{X}$ into integral sections of $\Phi_* \boldsymbol{X}$.
The same result holds for infinitesimal restricted general symmetries.

In natural coordinates $(x^\mu, y^A, z^{A\mu})$ in $\mathcal{F}$, they are:

$$\Phi(x^\mu, y^A, z^{A\mu}) = (\Phi^\nu(x^\mu), \Phi^B(x^\mu, y^A, z^{A\mu}), \Phi^{B\nu}(x^\mu, y^A, z^{A\mu}))\,,$$

$$Y = f^\nu(x^\mu)\frac{\partial}{\partial x^\nu} + F^B(x^\mu, y^A, z^{A\mu})\frac{\partial}{\partial y^B} + G^{B\nu}(x^\mu, y^A, z^{A\mu})\frac{\partial}{\partial z^{B\nu}}\,.$$

Analogously, other particularly relevant symmetries in physics are those corresponding to transformations in the phase space induced by transformations in the configuration space. In this respect, for the Lagrangian and Hamiltonian formalisms of field theories, diffeomorphisms $\Phi_E \in \mathrm{Diff}(E)$ which restrict to diffeomorphisms on M, and π-projectable vector fields $Y_E \in \mathfrak{X}(E)$ can be canonically lifted to $J^1\pi$, $J^1\pi^*$, and $P_\circ$.

In particular, in natural coordinates of $J^1\pi$ and $J^1\pi^*$, if

$$Y_E = f^\mu(x^\nu)\frac{\partial}{\partial x^\mu} + F^A(x^\nu, y^B)\frac{\partial}{\partial y^A} \in \mathfrak{X}(E)\,,$$

these canonical liftings are:

$$j^1 Y_E = f^\mu\frac{\partial}{\partial x^\mu} + F^A\frac{\partial}{\partial y^A} + \left(\frac{\partial F^A}{\partial x^\mu} - y^A_\nu\frac{\partial f^\nu}{\partial x^\mu} + y^B_\mu\frac{\partial F^A}{\partial y^B}\right)\frac{\partial}{\partial y^A_\mu} \in \mathfrak{X}(J^1\pi)\,,$$

$$j^{1*} Y_E = f^\mu\frac{\partial}{\partial x^\mu} + F^A\frac{\partial}{\partial y^A} + \left(\frac{\partial f^\mu}{\partial x^\nu}p^\nu_A - \frac{\partial f^\nu}{\partial x^\nu}p^\mu_A - \frac{\partial F^B}{\partial y^A}p^\mu_B\right)\frac{\partial}{\partial p^\mu_A} \in \mathfrak{X}(J^1\pi^*)\,.$$

DEFINITION 2.11

1. *A **natural general symmetry** is a restricted general symmetry $\Phi \colon \mathcal{F} \to \mathcal{F}$ which is the canonical lift of a diffeomorphism $\Phi_E \in \mathrm{Diff}\,(E)$ which restricts to a diffeomorphism on M.*
2. *An **infinitesimal natural general symmetry** is an infinitesimal restricted general symmetry $Y \in \mathfrak{X}(\mathcal{F})$ which is the canonical lift of a π-projectable vector field $Y_E \in \mathfrak{X}(E)$.*

Remark: Asking for symmetries to be restricted is also justified because, actually, the most usual situation are classical field theories which exhibit symmetries that are induced by transformations on the base manifold M. In order to lift these base space transformations to the multiphase space $\mathcal{F}$, they must be lifted, first from M to E, and then from E to $\mathcal{F}$ as natural symmetries.

In general, these lifts from M to E can be done in a canonical way when E is a bundle whose fibres have some additional natural structure (for instance, a principal bundle, a bundle of frames, a tensor bundle, etc.), as it happens in most field theories.

Cartan (Noether) symmetries. Noether's Theorem

Definition 2.12

1. *A **Cartan** or **Noether symmetry** is a diffeomorphism $\Phi\colon \mathcal{F} \to \mathcal{F}$ such that, $\Phi^*\Omega = \Omega$.*
   *A Noether symmetry $\Phi \in \mathrm{Diff}(\mathcal{F})$ is said to be **restricted** or **fiber-preserving** if it restricts to a diffeomorphism on M.*
   *A restricted Noether symmetry is **natural** if it is the canonical lift to $\mathcal{F}$ of a diffeomorphism $\Phi_{\mathcal{E}} \in \mathrm{Diff}\,(E)$ which restricts to M.*
2. *An **infinitesimal Cartan** or **Noether symmetry** is a vector field $Y \in \mathfrak{X}(\mathcal{F})$ such that $\mathrm{L}_Y\,\Omega = 0$.*
   *An infinitesimal Noether symmetry $Y \in \mathfrak{X}(\mathcal{F})$ is said to be **restricted** or **fiber-preserving** if Y is a $\bar{\varrho}$-projectable vector field.*
   *An infinitesimal restricted Noether symmetry $Y \in \mathfrak{X}(\mathcal{F})$ is **natural** if it is the canonical lift of a π-projectable vector field $Y_E \in \mathfrak{X}(\mathcal{E})$.*

*As a particular case, if $\Omega = -\mathrm{d}\Theta$ and $\Phi^*\Theta = \Theta$ or $\mathrm{L}_Y\,\Theta = 0$, then the (infinitesimal) Cartan (Noether) symmetry is said to be **exact**.*

Proposition 2.13

Every (infinitesimal) Cartan (Noether) symmetry is a (infinitesimal) general symmetry.

Proposition 2.14

- *If* $\Phi_1, \Phi_2 \in \mathrm{Diff}(\mathcal{F})$ *are Cartan (Noether) symmetries, then so is* $\Phi_2 \circ \Phi_1$.
- *If* $Y_1, Y_2 \in \mathfrak{X}(\mathcal{F})$ *are infinitesimal Cartan (Noether) symmetries, then so is* $[Y_1, Y_2]$.

Remark: Notice that, for infinitesimal Cartan (Noether) symmetries,

$$0 = \mathrm{L}_Y \Omega = \mathrm{d}\iota(Y)\Omega \ .$$

Then, for $p \in \mathcal{F}$, there exists $U_p \ni p$, and $\xi_Y \in \mathit{\Omega}^{m-1}(U_p)$, such that

$$\iota(Y)\Omega \underset{U_p}{=} \mathrm{d}\xi_Y \ .$$

Thus, an infinitesimal Cartan symmetry is a locally Hamiltonian vector field for the multisymplectic form Ω, and ξ_Y is a local Hamiltonian form.

Then, the classical Noether's theorem can be stated as follows:

Theorem 2.15 (Noether)

If $Y \in \mathfrak{X}(\mathcal{F})$ is an infinitesimal Cartan (Noether) symmetry, with $\iota(Y)\Omega = \mathrm{d}\xi_Y$. Then, for every $\mathbf{X} \in \ker^m_{Id(\omega)}\Omega$ we have that

$$\mathrm{L}_{\mathbf{X}}\xi_Y = 0 \; ;$$

that is, every Hamiltonian $(m-1)$-form ξ_Y associated with Y is a conserved form

If $\mathbf{X} \in \ker^m_{\mathrm{I}(\omega)}\Omega$ and ψ is an integral section of $\mathbf{X}$, then the conserved current form $\psi^*\xi_Y$ is called a ***Noether current***.

Corollary 2.16

If $Y \in \mathfrak{X}(\mathcal{F})$ is an exact infinitesimal Cartan (Noether) symmetry, then $\xi_Y = \iota(Y)\Theta$ is a conserved form.

The general singular (premultisymplectic) case

Now, consider the general case of singular (almost-regular) classical field theories where the final constraint submanifold is a submanifold $\mathcal{S} \hookrightarrow \mathcal{F}$.

In this case, the definitions and results above conserved forms and conservation laws hold, but at support on S.

Furthermore, all types of symmetries introduced earlier are defined in the same way (with some technical assumptions to define *natural* symmetries), with the additional condition that the diffeomorphisms $\Phi \in \mathrm{Diff}(\mathcal{F})$ must leave $\mathcal{S}$ invariant, and the vector fields $Y \in \mathfrak{X}(\mathcal{F})$ must be tangent to $\mathcal{S}$.

Gauge symmetries

General ideas

The term gauge is used in physics to refer to different situations in relation to certain kinds of symmetries which do not change physically the system. Related to this is the notion of gauge freedom.

However, one of the standard uses of the concept *gauge* is for describing symmetries related to the non-regularity of the theory. This leads to the existence of states that are physically equivalent and degrees of freedom which are physically irrelevant which are called gauge degrees of freedom. So, they are a special type of symmetries which generate transformations that have no physical transcendence, and are called gauge symmetries.

Next, we give a geometric interpretation of these topics in the multisymplectic framework of field theories.

M.J. Bergvelt, E.A. de Kerf, "The Hamiltonian structure of Yang-Mills theories and instantons" (Part I), *Physica* **139A** (1986) 101–124.

M.J. Gotay, J.M. Nester, "Presymplectic Hamilton and Lagrange systems, gauge transformations and the Dirac theory of constraints", *Lect. Notes in Phys.* **94**, 272-279; Springer, Berlin, 1979.

Geometric gauge symmetries

Let $(\mathcal{F}, \Omega)$ be a premultisymplectic system and $\mathcal{S} \subseteq \mathcal{F}$ the final constraint submanifold, where sections representing the physical states of the fields take their image.

Local generators of gauge symmetries will be called ***gauge vector fields***.

In order to do an accurate geometric definition, we should note that gauge vector fields must have the following properties:

- As the flux of gauge vector fields connect physically equivalent states, these vector fields must be tangent to $\mathcal{S}$.
- As the existence of gauge symmetries is a characteristic of the non-regularity of the theory, gauge vector fields are related with the premultisymplectic character of Ω. Hence, the elements of $\ker \Omega$ are candidates to be gauge vector fields.
- The base manifold M does not contain gauge degrees of freedom; so they must be just in the fibres of $\bar{\varrho}$. Then, gauge vector fields must be $\bar{\varrho}$-vertical, and so, gauge transfs. restrict to the identity on M.
- It is also usual to demand that physical symmetries are natural. This means that they are canonical lifts to $\mathcal{F}$ of vector fields in E.

Definition 3.1

Let $\underline{\mathfrak{X}(\mathcal{S})} \subset \mathfrak{X}(\mathcal{F})$ be the set of vector fields in $\mathcal{F}$ tangent to $\mathcal{S}$, and let

$$\mathcal{G} = \ker \Omega \cap \mathfrak{X}^{\mathrm{V}(\bar{\varrho})}(\mathcal{F}) \cap \underline{\mathfrak{X}(\mathcal{S})} \ .$$

*The elements $Z \in \mathcal{G}$ are called **(geometric) gauge vector fields** or **infinitesimal (geometric) gauge symmetries**.*
*In particular, if Z is the canonical lift of a vector field in E,then it is called a **natural (geometric) gauge vector field**.*
*The local diffeomorphisms generated by the flux of (geometric) gauge vector fields are called **(geometric) gauge transformations**.*

Remarks:

- As $\mathcal{G} \subseteq \ker \Omega$, for every $Z \in \mathcal{G}$ we have $\iota(Z)\Omega = 0 \Longrightarrow \mathrm{L}_Z \Omega = 0$. Then, gauge vector fields are infinitesimal Cartan symmetries. (The non $\bar{\varrho}$-vertical vector fields of $\ker \Omega \cap \underline{\mathfrak{X}(\mathcal{S})}$ are not gauge vector fields, but just infinitesimal Noether symmetries).
- If $Z_1, Z_2 \in \mathcal{G}$, then $[Z_1, Z_2] \in \mathcal{G}$; so $\mathcal{G}$ generates an involutive distribution. The quotient $\widetilde{\mathcal{S}} = \mathcal{S}/\mathcal{G}$ is assumed to be a differentiable manifold and that the projection $\widetilde{\varrho}_{\mathcal{S}} \colon \mathcal{S} \to \widetilde{\mathcal{S}}$ is a submersion.

Definition 3.2

*For $\tilde{p} \in \widetilde{\mathcal{S}}$, the points $\widetilde{\varrho}_{\mathcal{S}}^{-1}(\tilde{p}) \in \mathcal{S}$ are called **gauge equivalent points**. Two sections $\psi_1, \psi_2 \colon M \to \mathcal{S}$ are **gauge equivalent states** if, for every $x \in M$, the points $\psi_1(x)$ and $\psi_2(x)$ are gauge equivalent.*

Obviously, the integral curves of gauge vector vector fields are made of gauge equivalent points.

Therefore, the following statement is assumed:

Statement 3.3 (Gauge principle)

Gauge equivalent states are physically equivalent or, what means the same thing, they represent the same physical state of the field.

Note: The discussion on whether $\mathcal{G}$ contains or not all gauge vector fields of the theory (those generating all the transformations that relate the states representing the same physical state) is open. This means whether there are secondary gauge vector fields (like in mechanics), or not.

Conjecture: The set of all gauge vector fields: $\mathscr{G} = \underline{\ker(\jmath_{\mathcal{S}}^*\Omega)} \cap \mathfrak{X}^{\mathrm{V}(\bar{\varrho})}(\mathcal{F})$

Gauge reduction

When a physical system has gauge symmetries, a relevant problem consists in removing the unphysical redundant information introduced by the existence of gauge equivalent states.

If $\mathscr{G}$ is the set of all gauge vector fields, this procedure can be done taking the quotient set $\widetilde{\mathcal{S}} = \mathcal{S}/\mathscr{G}$ which, assumed to be a manifold, is made of the true physical degrees of freedom, by definition.

In addition, gauge vector fields are $\bar{\varrho}$-vertical vector fields; so all the gauge degrees of freedom are in the fibers of $\bar{\varrho}\colon \mathcal{F} \to M$ and, hence, the base manifold M does not contain gauge equivalent points. In this way, after doing the quotient $\mathcal{S}/\mathscr{G}$, the base manifold M remains unchanged and then $\widetilde{\mathcal{S}}$ fibers over M.

$$
\begin{tikzcd}
\mathcal{S} \arrow[r, "\jmath_{\mathcal{S}}"] \arrow[d, "\widetilde{\varrho}_{\mathcal{S}}"'] \arrow[dr, "\pi_{\mathcal{S}}"] & \mathcal{F} \arrow[d, "\bar{\varrho}"] \\
\widetilde{\mathcal{S}} \arrow[r, "\widetilde{\pi}_{\mathcal{S}}"] & M
\end{tikzcd} .
$$

Sections of the projection $\widetilde{\pi}_{\mathcal{S}} \colon \widetilde{\mathcal{S}} \to M$ represent the "real" physical states. They are equivalence classes of sections made of gauge equivalent states.

Definition 3.4

This procedure is known as the ***gauge reduction procedure*** *for removing the (unphysical) gauge degrees of freedom of the theory.*

Remark: An alternative way to remove the gauge redundance consists in taking a (local) section of the projection $\tilde{\varrho}_{\mathcal{S}} \colon \mathcal{S} \to \widetilde{\mathcal{S}}$; that is, a representative in each gauge equivalence class of sections.

Definition 3.5

This procedure is called a ***gauge fixing****, and the freedom in choosing this section is then referred as the* ***gauge freedom*** *of the theory.*

Examples

Maxwell's electromagnetism (Lagrangian setting)

Coords. in $J^1\pi$: $(x^\mu, A_\mu, A_{\mu\nu})$, where $(A_\mu) = \left(-\frac{\Phi}{c}, \mathbf{A}\right)$; $(\mu = 0,1,2,3)$.

$\Omega_{\mathcal{L}} = -(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta})\, \mathrm{d}A_{\alpha\beta} \wedge \mathrm{d}A_\nu \wedge \mathrm{d}^3x + F^{\alpha\beta}\mathrm{d}A_{\alpha\beta} \wedge \mathrm{d}^4x - J^\alpha \mathrm{d}A_\alpha \wedge \mathrm{d}^4x\,.$

As it is well-known, the gauge freedom is the arbitrariness in the choice of electromagnetic potentials: for any function $f(x^\mu) \equiv f(t, x^i)$,

$$\mathbf{A'} = \mathbf{A} + \nabla f \quad , \quad \phi' = \phi + \frac{\partial f}{\partial t}\ .$$

So, gauge transformations for sections $\psi = (x^\mu, A_\alpha(x^\mu), A_{\alpha\beta}(x^\mu))$, are

$$A'_\alpha(x^\mu) = A_\alpha(x^\mu) + \frac{\partial f(x^\mu)}{\partial x^\alpha}\ ,$$

which are locally generated by the vector fields

$$Y_E = \frac{\partial f}{\partial x^\alpha}\frac{\partial}{\partial A_\alpha} \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)\ .$$

Then, the gauge vector fields $Y \in \mathcal{G}$ are

$$Y \equiv j^1 Y_E = \frac{\partial f}{\partial x^\alpha}\frac{\partial}{\partial A_\alpha} + \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}\frac{\partial}{\partial A_{\alpha\beta}} \in \ker \Omega_{\mathcal{L}} \cap \mathfrak{X}^{\mathrm{V}(\bar{\pi}^1)}(J^1\pi)\ .$$

One-dimensional wave equation (Vibrating string)

Multisymplectic Lagrangian setting:

Spacetime manifold (M,ω_M): coordinates (t,x), with $\omega_M = \mathrm{d}t \wedge \mathrm{d}x$.

Configuration bundle $\pi\colon E \to M$, $E = \mathbb{R}^2 \times \mathbb{R}$: coordinates (t,x,y).

Multivelocity phase bundle $J^1\pi = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$: coords. (t,x,y,y_t,y_x).

Lagrangian function: $L(t,x,y,y_t,y_x) = \frac{1}{2}(\rho y_t^2 - \tau y_x^2)$ (regular); $\rho,\tau \in \mathbb{R}$.

Poincaré–Cartan forms:

$$\Theta_{\mathcal{L}} = -\rho y_t \mathrm{d}y \wedge \mathrm{d}x - \tau y_x \mathrm{d}y \wedge \mathrm{d}t + \frac{1}{2}\left(\rho y_t^2 - \tau y_x^2\right)\mathrm{d}t \wedge \mathrm{d}x\,,$$

$$\Omega_{\mathcal{L}} = \rho \mathrm{d}y_t \wedge \mathrm{d}y \wedge \mathrm{d}x + \tau \mathrm{d}y_x \wedge \mathrm{d}y \wedge \mathrm{d}t - \rho y_t \mathrm{d}y_t \wedge \mathrm{d}t \wedge \mathrm{d}x + \tau y_x \mathrm{d}y_x \wedge \mathrm{d}t \wedge \mathrm{d}x$$

Field equation: for $\bar{\pi}^1$-transverse m.v.f. $\mathbf{X}_{\mathcal{L}} = X_1 \wedge X_2 \in \mathfrak{X}^2(J^1\pi)$, where

$$X_1 = A_1\frac{\partial}{\partial t} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial y_t} + A_4\frac{\partial}{\partial y_x}\,,$$

$$X_2 = B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial y_t} + B_4\frac{\partial}{\partial y_x}\,.$$

The field equation $\iota(\mathbf{X}_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$ yields

$$A_2 = y_t \quad , \quad B_2 = y_x \quad , \quad A_3 = \frac{\tau}{\rho} B_4 \ . \tag{2}$$

For holonomic integral sections $\psi(t,x) = \left(t, x, y(t,x), \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}\right)$,

$$A_2 = \frac{\partial y}{\partial t} \quad , \quad B_2 = \frac{\partial y}{\partial x} \quad , \quad A_3 = \frac{\partial y_t}{\partial t} \quad , \quad B_4 = \frac{\partial y_x}{\partial t} \ ;$$

and (2) lead to the *Euler–Lagrange equation*

$$\frac{\partial^2 y}{\partial t^2} - \frac{\tau}{\rho} \frac{\partial^2 y}{\partial x^2} = 0 \, . \tag{3}$$

$Y = \dfrac{\partial}{\partial y}$ is an infinitesimal exact Cartan (Noether) symmetry. Then,

$$\boldsymbol{\xi}_Y = \iota(Y)\Theta_{\mathcal{L}} = -\rho y_t \mathrm{d}x - \tau y_x \mathrm{d}t$$

is a conserved form. To obtain the corresponding conservation law, take a holonomic section $\psi(t,x)$ solution to (3); then,

$$\psi^* \boldsymbol{\xi}_Y = -\rho \frac{\partial y}{\partial t} \mathrm{d}x - \tau \frac{\partial y}{\partial x} \mathrm{d}t \quad , \quad X_{\psi^* \boldsymbol{\xi}_Y} = -\rho \frac{\partial y}{\partial t} \frac{\partial}{\partial t} + \tau \frac{\partial y}{\partial x} \frac{\partial}{\partial x} \, ,$$

and the corresponding conservation law is the field equation (3) itself.

Einstein–Palatini (metric-affine) approach in General Relativity

J. Gaset, N. Román-Roy, "New multisymplectic approach to the metric-affine (Einstein-Palatini) action for gravity", *J. Geom. Mech.* **11**(3) (2019) 361–396.

The Einstein-Palatini Lagrangian (without energy-matter sources):

Configuration bundle $\pi\colon E \to M$.

M is 4-dimensional spacetime.

$E = \mathcal{E} \times_M C(LM)$, where $\mathcal{E}$ is the manifold of Lorentzian metrics on M and $C(LM)$ is the manifold of *linear connections* in $\mathrm{T}M$.

Adapted coords. in E are $(x^\mu, g_{\alpha\beta}, \Gamma^\lambda_{\mu\nu})$, $(0 \leq \mu, \nu \leq 3;\ 0 \leq \alpha \leq \beta \leq 3)$, and the induced coordinates in $J^1\pi$ are $(x^\mu, g_{\alpha\beta},\ \Gamma^\lambda_{\mu\nu},\ g_{\alpha\beta,\mu},\ \Gamma^\lambda_{\mu\nu,\rho})$. Thus dim $E = 78$ and dim $J^1\pi = 374$.

The *Einstein-Palatini Lagrangian* is a singular 1*st*-order Lagrangian depending on the metric g and on an arbitrary connection Γ,

$$L_{EP} = \mathrm{g}\, g^{\alpha\beta} R_{\alpha\beta} = \mathrm{g}\, g^{\alpha\beta} \left(\Gamma^\gamma_{\beta\alpha,\gamma} - \Gamma^\gamma_{\gamma\alpha,\beta} + \Gamma^\gamma_{\beta\alpha}\Gamma^\sigma_{\sigma\gamma} - \Gamma^\gamma_{\beta\sigma}\Gamma^\sigma_{\gamma\alpha} \right) \in \mathrm{C}^\infty(J^1\pi)\,.$$

where $\mathrm{g} \equiv \sqrt{|det(g_{\alpha\beta})|}$.

Multisymplectic Lagrangian setting:

The premultisymplectic *Poincaré-Cartan forms* for the Lagrangian density $\mathcal{L} = L_{EP}\,\mathrm{d}^4x \in \mathit{\Omega}^4(J^1\pi)$ are

$$\Theta_{\mathcal{L}} = \left(L_{\rm EP} - \frac{\partial L_{\rm EP}}{\partial \Gamma^{\alpha}_{\beta\gamma,\mu}}\Gamma^{\alpha}_{\beta\gamma,\mu}\right)\mathrm{d}^4x + \frac{\partial L_{\rm EP}}{\partial \Gamma^{\alpha}_{\beta\gamma,\mu}}\mathrm{d}\Gamma^{\alpha}_{\beta\gamma} \wedge \mathrm{d}^3x_{\mu} \in \mathit{\Omega}^4(J^1\pi)\,,$$

$$\Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}} \in \mathit{\Omega}^5(J^1\pi)\ ;$$

We look for the holonomic multivector fields $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^4(J^1\pi)$ solutions to the field equation for m.v.f. $\iota(\mathbf{X}_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$.

Introducing the functions:

$$L^{\alpha\beta,\mu} = \frac{\partial L}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3}\frac{1}{n(\mu\nu)}D_{\nu}\left(\frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}}\right) = \frac{\partial L_0}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3} D_{\nu}L^{\alpha\beta,\mu\nu}\ ,$$

$$H = \sum_{\alpha\leq\beta;\mu\leq\nu} L^{\alpha\beta,\mu\nu}g_{\alpha\beta,\mu\nu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu}g_{\alpha\beta,\mu} - L\ .$$

the premultisymplectic constraint algorithm leads to the constraints:

$$0 = \frac{\partial H}{\partial g_{\mu\nu}} - \frac{\partial L_\alpha^{\beta\gamma,\sigma}}{\partial g_{\mu\nu}} \Gamma^\alpha_{\beta\gamma,\sigma} \ , \tag{4}$$

$$0 = g_{\rho\sigma,\mu} - g_{\sigma\lambda}\Gamma^\lambda_{\mu\rho} - g_{\rho\lambda}\Gamma^\lambda_{\mu\sigma} - \frac{2}{3} g_{\rho\sigma} T^\lambda_{\lambda\mu} \ , \tag{5}$$

$$0 = T^\alpha_{\beta\gamma} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta} \ , \tag{6}$$

$$0 = T^\alpha_{\beta\gamma,\nu} - \frac{1}{3}\delta^\alpha_\beta T^\mu_{\mu\gamma,\nu} + \frac{1}{3}\delta^\alpha_\gamma T^\mu_{\mu\beta,\nu} \ , \tag{7}$$

$$0 = g_{\rho\gamma}\Gamma^\gamma_{[\nu\lambda}\Gamma^\lambda_{\mu]\sigma} + g_{\sigma\gamma}\Gamma^\gamma_{[\nu\lambda}\Gamma^\lambda_{\mu]\rho} + g_{\rho\lambda}\Gamma^\lambda_{[\mu\sigma,\nu]} + g_{\sigma\lambda}\Gamma^\lambda_{[\mu\rho,\nu]} + \frac{2}{3} g_{\rho\sigma} T^\lambda_{\lambda[\mu,\nu]} \tag{8}$$

where $T^\alpha_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta}$, and $T^\gamma_{\mu\nu,\lambda} = \dfrac{\partial T^\gamma_{\mu\nu}}{\partial x^\lambda}$.

They define the submanifold $\mathcal{S}_f \hookrightarrow J^1\pi$ where there are holonomic multivector fields solution to the field equations tangent to $\mathcal{S}_f$, which are

$$\mathbf{X} = \bigwedge_{\nu=0}^{3} \left(\frac{\partial}{\partial x^\nu} + \sum_{\rho\leq\sigma} \left(g_{\rho\sigma,\nu}\frac{\partial}{\partial g_{\rho\sigma}} + f_{\rho\sigma\mu,\nu}\frac{\partial}{\partial g_{\rho\sigma,\mu}} \right) + \Gamma^\alpha_{\beta\gamma,\nu}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}} + f^\alpha_{\beta\gamma\mu,\nu}\frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \right).$$

$$\mathcal{G} = \left\langle \delta^\alpha_\gamma \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma}}, \frac{\partial}{\partial g_{\alpha\beta,\mu}}, \frac{\partial}{\partial \Gamma^\alpha_{\beta\gamma,\mu}} \right\rangle \ .$$

- A consequence of the singularity of $\mathcal{L}$ is that $\Omega_{\mathcal{L}}$ is π^1-projectable onto a form in E and then, the Euler-Lagrange equations (Einstein's equations) are 1st-order PDE's, instead of 2nd-order. So they are defined as a submanifold of $J^1\pi$, and appear also as constraints (4).
- The so-called pre-metricity constraints (5) are related to the metricity condition for the *Levi-Civita connection*.
- There are the torsion constraints that impose conditions on the torsion of the connection (6), and on their derivatives (7).
- Finally, a family of additional integrability constraints (8) appear as a consequence of demanding the integrability of the multivector fields which are solutions to the field equations (**??**).
- The Einstein–Palatini model is a gauge theory (as $\mathcal{L}$ is singular). The conditions of the connection to be torsionless and metric (which allows us to recover the Einstein–Hilbert model from the Einstein–Palatini model) are partially recovered as a consequence of the constraints and an additional partial fixing of the gauge degrees of freedom of the theory.

Other references

- M. de León, D. Martín de Diego, A. Santamaría, "Symmetries in classical field theory", *Int. J. Geom. Meth. Mod. Phys.* **1**(5) (2004) 651–710.
- M. Forger, B.L. Soares, "Local symmetries in gauge theories in a finite-dimensional setting", *J. Geom. Phys.* **62**(9) (2012) 1925–1938.
- A. Guerra IV, N. Román-Roy, "More insights into symmetries in multisymplectic field theories", *Symmetry* **2023**, 15, 390 (2023).
- A. Guerra, N. Román-Roy, "Canonical lifts in multisymplectic de Donder-Weyl Ham. field theor.", *J. Phys. A Math. Theor.* **57**(33) (2024) 335203.
- J. Herman, "Noether's theorem in multisymplectic geometry", *Diff. Geom. Appl.* **56**(2) (2018) 260–294.
- L. Ryvkin, T. Wurzbacher, M. Zambon, "Conserved quantities on multisymplectic manifolds", *J. Austral Math. Soc.* **108**(1) (2020) 120–144.

Further remarks

Some other interesting topics in multisymplectic classical field theories:

- The multisymplectic formulation has been used to study many other relevant theories in theoretical physics; such as Generalized theories of gravitation (Lovelok, Honderski, Einstein-Cartan, ...), string and brane theories, Carrollian models, etc.
- The (pre)multisymplectic Hamiltonian formalism can also be stated in the extended multimomentum bundle, yielding the so-called **multisymplectic extended Hamiltonian formalism**.
- The problem of the construction of Poisson brackets in the multisymplectic Hamiltonian formalism has been studied, giving rise to several different proposals.
- There exists the **unified Lagrangian–Hamiltonian formalism** (which is an extension to multisymplectic field theories of the Skinner–Rusk formalism in mechanics). This is a mixed formalism which is specially interesting when dealing with singular theories.
- Geometric noncovariant space-time formulations of classical field theories have been developed from the multisymplectic framework.

- The multisymplectic framework for higher-order classical field theories (mainly the second-order case) has also been developed and applied to describe physical theories; such as, the *Einstein–Hilbert approach* in General Relativity or the *Kortewg–de Vries equation*.
- The reduction of field theories with symmetries has been addressed in the multisymplectic framework; but it has been only solved in particular situations. However, the problem of developing a procedure analogous to the Marsden-Weinstein-Meyer reduction for (pre)multisymplectic field theories remains still open.
- The study of other kinds of symmetries and their associated conservation laws is also a field of active research.
- In general, the geometry of multisymplectic manifolds is a fruitful topic with many interesting problems to study.
- A new geometric structure called ***multicontact***, inspired by contact and multisymplectic geometries, has been recently developed to describe non-conservative or action-dependent Lagrangian and Hamiltonian classical field theories. Some of the above mentioned topics are starting to be investigated in this context.

Thanks for your attention !