An introductory course on geometric field theories: the multisymplectic setting

Third lecture: Multisymplectic Hamiltonian field theory

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Course Programme

1. **First lecture:** Introductory remarks. Previous geometric background.
2. **Second lecture:** Multisymplectic Lagrangian field theory.
3. **Third lecture:** Multisymplectic Hamiltonian field theory.
4. **Fourth lecture:** Symmetries. Gauge symmetries. Conservation laws.

Introduction

The first attempt to develop a *finite-dimensional covariant Hamiltonian formalism* for field theories was made in the 1930s by *T. de Donder* and *H. Weyl*, giving rise to the so-called *de Donder–Weyl approach*.

Starting from a *Lagrangian function* $L(x^\mu, y^A, y^A_\mu)$, they defined a *Legendre transformation*, introducing the *multimomenta coordinates* in the *multimomentum phase space*,

$$p_i^\mu = \frac{\partial L}{\partial y^i_\mu} \, ,$$

in which the corresponding variational principle leads to the *(Hamilton)–de Donder–Weyl equations*,

$$\frac{\partial y^i}{\partial x^\mu} = \frac{\partial H}{\partial p_i^\mu} \quad , \quad \frac{\partial p_i^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial y^i} \, .$$

T. de Donder, *Théorie invariantive du calcul des variations.* Gauthier-Villars, Paris, 1935.

H. Weyl, "Geodesic fields in the calculus of variation for multiple integrals", *Ann. Math.* **36**(3) (1935) 607–629.

The de Donder–Weyl formalism was initially hindered due to the lack of a definition of *Poisson brackets* and, as a consequence, of a good treatment of constraints in *singular field theories* and of a method of quantization.

Instead, in theoretical physics, Hamiltonian field theories are usually studied using the *canonical Hamiltonian formalism* which arises from performing a space+time splitting in the base manifold M.

This breaks the covariance exhibited in the Lagrangian formalism and originates infinite-dimensional Lagrangian and Hamiltonian descriptions.

However, in this way, the *Dirac–Bergmann constraint analysis* can be applied for Hamiltonian singular field theories and, furthermore, this formalism gives a method for performing *canonical quantization*.

M. de León, D. Martín de Diego, A. Santamaría, "Symmetries in classical field theory", *Int. J. Geom. Meth. Mod. Phys.* **1**(5) (2004) 651–710.

M.J. Gotay, "A multisymplectic framework for classical field theory and the calculus of variations. II. Space+time decomposition", *Dif. Geom. Appl.* **1** (1991) 375–390.

M.J. Gotay, J. Isenberg, J.E. Marsden, R. Montgomery, "Momentum maps and classical relativistic fields II: Part II: Canonical analysis of field theories", arXiv:math-ph/0411032 [math-ph] (2004).

Later on, another Lorentz covariant Hamiltonian formalism was developed, first by *C. Crnković* and *E. Witten*, and soon after, generalized by *J. Lee* and *R.M. Wald*; who constructed a covariant phase space in which the canonical formulation is carried out without doing a space+time splitting.

C. Crnković, E. Witten, "Covariant description of canonical formalism in geometrical theories", in *Three Hundred Years of Gravitation*, Cambridge University Press, Princeton, 1987, 676–684.

J. Lee, R.M. Wald, "Local symmetries and constraints", *J. Math. Phys.* **31**(3) (1990) 725–743.

This covariant phase space is an infinite-dimensional *(pre)symplectic manifold*, and we can carry out the *Gotay–Nester–Hinds constraint algorithm*, which is equivalent to the *Dirac–Bergmann constraint analysis*, and perform a quantization method.

However, the development in the last decades of geometric methods for field theories (*multisymplectic, polysymplectic, k-(co)symplectic* formulations), caused the interest in the de Donder-Weyl approach, which is a finite-dimensional covariant formalism, to return forcefully.

This lecture is devoted to do a geometric presentation of the covariant Hamiltonian formalism of first-order multisymplectic classical field theories based on the de Donder–Weyl approach:

First, we introduce the multimomentum bundles and their geometric structures.

Second, the multisymplectic Hamiltonian formalism is developed and, in particular, the Hamiltonian formalism associated with regular and singular (almost-regular) Lagrangian systems.

- J.F. Cariñena, M. Crampin, L.A. Ibort, "On the multisymplectic formalism for first order field theories", *Diff. Geom. Appl.* **1**(4) (1991) 345–374.
- M. de León, J. Marín-Solano, J.C. Marrero, "A geometrical approach to Classical Field Theories: A constraint algorithm for singular theories", Proc. *New Develops. Diff. Geom.*, L. Tamassi-J. Szenthe eds., Kluwer Acad. Press, (1996) 291-312.
- A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, "Multivector field formulation of Hamiltonian field theories: Equations and symmetries", *J. Phys. A: Math. Gen.* **32** (1999) 8461–8484.
- N. Román-Roy, "Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories", *Sym. Integ. Geom. Meth. Appl.* **5** (2009) 100.

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Multimomentum bundles

Extended multimomentum bundle

Let $\pi \colon E \longrightarrow M$ be a fiber bundle; $\dim M = m$, $\dim E = n + m$.

Coords. on E adapted to the bundle: (x^μ, y^A), $1 \leqslant \mu \leqslant m$, $1 \leqslant A \leqslant n$.

Definition 2.1

The ***extended multimomentum bundle*** *is the bundle of m-forms on E vanishing when contracting with two π-vertical vector fields,*

$$\mathcal{M}\pi := \Lambda_2^m \mathrm{T}^* E \ .$$

The natural projections are

$$\sigma \colon \mathcal{M}\pi \to E \quad , \quad \bar{\sigma} \colon \mathcal{M}\pi \to M \ .$$

Proposition 2.2

$\mathcal{M}\pi \equiv \Lambda_2^m \mathrm{T}^* E$ is canonically isomorphic to $\mathrm{Aff}(J^1\pi, \mathbb{R})$ (the affine maps from the fibers of $\pi^1 \colon J^1\pi \to E$ onto $\mathbb{R}$).

Natural coordinates in $\mathcal{M}\pi$ adapted to the bundle structure are (x^μ, y^A, p^μ_A, p); hence $\dim \mathcal{M}\pi = nm + n + m + 1$. They are defined as follows: Let $\tilde{y} \in \mathcal{M}\pi = \Lambda^m_2 \mathrm{T}^* E$, with $\tilde{y} \xrightarrow{\sigma} y \xrightarrow{\pi} x$; then $\tilde{y} = (y, \xi)$, where $y \in E$ and $\xi \in \Lambda^m_2 \mathrm{T}^*_y E$ whose expression in a natural chart of E is

$$\xi = \lambda^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \lambda\, \mathrm{d}^m x \ .$$

Then, if $\partial^m x \equiv \dfrac{\partial}{\partial x^1} \wedge \cdots \wedge \dfrac{\partial}{\partial x^m}$ and $\partial^{m-1}x^\mu \equiv \iota(\mathrm{d}x^\mu)\partial^m x$, we have:

$$x^\mu(\tilde{y}) = x^\mu(y) \quad , \quad y^A(\tilde{y}) = y^A(y) \ ,$$

$$p(\tilde{y}) = \xi(\partial^m x) = \lambda \quad , \quad p^\mu_A(\tilde{y}) = \xi\left(\frac{\partial}{\partial y^A} \wedge \partial^{m-1}x^\mu\right) = \lambda^\mu_A \ ;$$

and $\sigma(x^\mu, y^A, p^\mu_A, p) = (x^\mu, y^A)$ and $\bar{\sigma}(x^\mu, y^A, p^\mu_A, p) = (x^\mu)$.

Observe also that, considering the bundle $\pi^1 \colon J^1\pi \to E$, if $y \in E$, an affine map $J^1_y\pi \to \mathbb{R}$ is given by

$$y^A_\mu \mapsto \lambda^\mu_A\, y^A_\mu + \lambda \ .$$

The coordinates p^μ_A are called **multimomentum coordinates**.

Canonical structure

As $\mathcal{M}\pi$ is a bundle of forms, it is endowed with a canonical form:

Definition 2.3

The ***canonical*** *or* ***tautological form*** $\widetilde{\Theta} \in \Omega^m(\mathcal{M})$ *is defined as follows: if* $\widetilde{y} = (y, \xi) \in \Lambda_2^m \mathrm{T}^* E$*; then, for every* $\widetilde{Z}_1, \ldots, \widetilde{Z}_m \in \mathrm{T}_{\widetilde{y}}(\mathcal{M}\pi)$*,*

$$\widetilde{\Theta}_{\widetilde{y}}(\widetilde{Z}_1, \ldots, \widetilde{Z}_m)) := \xi(y; \mathrm{T}_{\widetilde{y}}(\sigma \circ \mathfrak{p})(\widetilde{Z}_1), \ldots, \mathrm{T}_{\widetilde{y}}(\sigma \circ \mathfrak{p})(\widetilde{Z}_m)) .$$

Therefore, we define

$$\widetilde{\Omega} := -\mathrm{d}\widetilde{\Theta} \in \Omega^{m+1}(\mathcal{M}\pi) .$$

They are known as the ***multimomentum Liouville*** m ***and*** $(m+1)$***-forms****,*

Proposition 2.4

$\widetilde{\Omega}$ is a multisymplectic form (and $(\mathcal{M}\pi, \widetilde{\Omega})$ is a multisymplectic manifold).

Their local expressions are

$$\widetilde{\Theta} = p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{m-1}x_\mu + p\, \mathrm{d}^m x \quad , \quad \widetilde{\Omega} = -\mathrm{d}p_i^\mu \wedge \mathrm{d}y^i \wedge \mathrm{d}^{m-1}x_\mu - \mathrm{d}p \wedge \mathrm{d}^m x .$$

Restricted multimomentum bundle

Definition 2.5

Let $\Lambda^m_1 \mathrm{T}^* E$ be the bundle of πsemibasic m-forms on E (those vanishing when contracting with one π-vertical vector field). The ***restricted multimomentum bundle*** *is defined as the quotient*

$$J^1\pi^* := \mathcal{M}\pi / \Lambda^m_1 \mathrm{T}^* E \ .$$

The natural projections are depicted in the following diagram

$$\begin{tikzcd} \mathcal{M}\pi \arrow[rr, "\mathfrak{p}"] \arrow[dr, "\sigma"] \arrow[ddr, "\overline{\sigma}"'] & & J^1\pi^* \arrow[dl, "\tau"'] \arrow[ddl, "\overline{\tau}"] \\ & E \arrow[d, "\pi"] & \\ & M & \end{tikzcd}$$

In natural coordinates of E, the elements of $\Lambda^m_2 \mathrm{T}^* E$ and $\Lambda^m_1 \mathrm{T}^* E$ have the local expressions $p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + p\, \mathrm{d}^m x$ and $p\, \mathrm{d}^m x$, respectively; so, natural coordinates in $J^1\pi^*$ are (x^μ, y^A, p^μ_A).
Hence $\dim J^1\pi^* = nm + n + m = \dim J^1\pi$.

Multisymplectic field theories

The regular case: multisymplectic Hamiltonian field theories

Definition 3.1

Sections $\mathbf{h}\colon J^1\pi^* \to \mathcal{M}\pi$ of $\mathfrak{p}$ are called ***Hamiltonian sections****.*
A Hamiltonian section is locally given by

$$\mathbf{h}(x^\mu, y^i, p_i^\mu) = (x^\mu, y^i, p_i^\mu, p = -H(x^\nu, y^i, p_j^\nu)) ,$$

where H is the associated local Hamiltonian function, sometimes called ***de Donder–Weyl Hamiltonian function****.*

Hamiltonian sections allow us to define Hamiltonian field theories:

Definition 3.2

The ***Hamilton–Cartan forms*** *on $J^1\pi^*$ associated with $\mathbf{h}$ are*

$$\Theta_{\mathbf{h}} := \mathbf{h}^*\widetilde{\Theta} = p_i^\nu \mathrm{d}y^i \wedge \mathrm{d}^{m-1}x_\mu - H\,\mathrm{d}^m x \in \Omega^m(J^1\pi^*) ,$$

$$\Omega_{\mathbf{h}} := -\mathrm{d}\Theta_{\mathbf{h}} = \mathbf{h}^*\widetilde{\Omega} = -\mathrm{d}p_i^\mu \wedge \mathrm{d}y^i \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}H \wedge \mathrm{d}^m x \in \Omega^{m+1}(J^1\pi^*) .$$

$\Omega_{\mathbf{h}}$ is a multisymplectic form in $J^1\pi^*$, and $(J^1\pi^*, \Omega_{\mathbf{h}})$ is said to be a ***regular*** *or* ***multisymplectic Hamiltonian system****.*

Variational principle and Hamiltonian field equations

Definition 3.3 (Hamilton–Jacobi Variational Principle)

Let $(J^1\pi^*, \Omega_{\mathbf{h}})$ be a Lagrangian system and $\Gamma(\overline{\tau})$ the set of sections of $\overline{\tau}$. Consider the following functional:

$$\begin{aligned} \mathbf{H} \colon \Gamma(\overline{\tau}) &\longrightarrow \mathbb{R} \\ \psi &\longmapsto \int_{D\subset M} \psi^*\Theta_{\mathbf{h}} \ . \end{aligned}$$

Then, the ***Hamilton–Jacobi variational problem*** *for the Hamiltonian system $(J^1\pi^*, \Omega_{\mathbf{h}})$ is the search for the critical (or stationary) sections of the functional* ***H*** *with respect to the variations of ψ given by $\psi_t = \rho_t \circ \psi$, where $\{\rho_t\}$ is a local 1-parameter group of any compact-supported $\overline{\tau}$-vertical vector field $Z \in \mathfrak{X}^{\mathrm{V}(\overline{\tau})}(J^1\pi^*)$; that is,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_D \psi_t^*\Theta_{\mathbf{h}} = 0 \ .$$

The corresponding Hamiltonian field equations derived from this variational principle can be stated in several alternative ways:

Theorem 3.4

The following assertions on a section $\psi \in \Gamma(\overline{\tau})$ are equivalent:

1. *The section ψ is critical for the Hamilton–Jacobi variational problem.*
2. *ψ is a solution to the **Hamiltonian field equations for sections**:*
$$\psi^* \iota(X)\Omega_{\mathbf{h}} = 0 \ ; \ \text{for every } X \in \mathfrak{X}(J^1\pi^*) \ . \tag{1}$$
3. *ψ is an integral section of a class of $\overline{\tau}$-transverse locally decomp. multivector fields, $\{\mathbf{X_h}\} \subset \mathfrak{X}^m(J^1\pi^*)$, whose elements are solutions to the **Hamiltonian field equations for multivector fields**:*
$$\iota(\mathbf{X_h})\Omega_{\mathbf{h}} = 0 \ . \tag{2}$$
4. *ψ is an integral section of an Ehresmann connection $\nabla_{\mathbf{h}}$ in $J^1\pi^*$ which is a solution to the **Hamiltonian field eqs. for connections**:*
$$\iota(\nabla_{\mathbf{h}})\Omega_{\mathbf{h}} = (m-1)\Omega_{\mathbf{h}} \ . \tag{3}$$
5. *ψ is a solution to the **Hamilton-de Donder-Weyl equations**:*
$$\frac{\partial(y^i\circ\psi)}{\partial x^\mu} = \frac{\partial H}{\partial p_i^\mu}\circ\psi \quad , \quad \frac{\partial(p_i^\mu\circ\psi)}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}\circ\psi \, . \tag{4}$$

In natural coordinates of $J^1\pi^*$, if $\psi(x) = \left(x^\mu, y^A(x), p_A^\mu(x)\right)$, then:

- Using the local expressions in Def. 3.2, equation (1) reads as (4) (the Hamilton-de Donder-Weyl equations).
- The local expressions of a $\overline{\tau}$-transverse loc. decomp. multivector field and its associated Ehreshmann connection are

$$\mathbf{X_h} = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + F_\mu^A \frac{\partial}{\partial y^A} + G_{A\mu}^\rho \frac{\partial}{\partial p_A^\rho} \right) \ ,$$
$$\nabla_{\mathbf{h}} = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + F_\mu^A \frac{\partial}{\partial y^A} + G_{A\mu}^\rho \frac{\partial}{\partial p_A^\rho} \right) \ ;$$

  Then, from (2) and (3) we get

$$F_\mu^A = \frac{\partial H}{\partial p_A^\mu} \quad , \quad G_{A\mu}^\mu = -\frac{\partial H}{\partial y^A} .$$

- If $\mathbf{X_h}$ and $\nabla_{\mathbf{h}}$ are integrable, then the equations for their integral sections

$$\frac{\partial (y^A \circ \psi)}{\partial x^\mu} = F_\mu^A \circ \psi \quad , \quad \frac{\partial (p_A^\rho \circ \psi)}{\partial x^\mu} = G_{A\mu}^\rho \circ \psi .$$

  lead to the Hamilton-de Donder-Weyl equations (4).

Hamiltonian formalism for regular Lagrangian field theories

Definition 3.5

Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a Lagrangian system.

- *The* ***extended Legendre map*** $\widetilde{\mathcal{FL}}\colon J^1\pi \to \mathcal{M}\pi$, *associated with* $\mathcal{L}$ *is defined as follows: if* $\bar{y} \in J^1\pi$, *then*

$$\mathcal{FL}(\bar{y}) = \tilde{y} \equiv (\pi^1(\bar{y}), \xi) \in \mathcal{M}\pi \ ,$$

such that, for every $Z_1, \ldots, Z_m \in \mathrm{T}_{\bar{y}} J^1\pi$, *then*

$$\xi\big(\mathrm{T}_{\bar{y}}\pi^1(Z_1), \ldots, \mathrm{T}_{\bar{y}}\pi^1(Z_m)\big) := (\Theta_{\mathcal{L}})_{\bar{y}}(Z_1, \ldots, Z_m),$$

- *The* ***restricted Legendre map*** $\mathcal{FL}\colon J^1\pi \to J^1\pi^*$ *associated with* $\mathcal{L}$

$$\mathcal{FL} := \mathfrak{p} \circ \widetilde{\mathcal{FL}} \ .$$

Their local expressions are:

$$\widetilde{\mathcal{FL}}^* x^\mu = x^\mu \ , \quad \widetilde{\mathcal{FL}}^* y^i = y^i \ , \quad \widetilde{\mathcal{FL}}^* p_i^\mu = \frac{\partial L}{\partial y_\mu^i} \ , \quad \widetilde{\mathcal{FL}}^* p = L - y_\mu^i \frac{\partial L}{\partial y_\mu^i} \ ,$$

$$\mathcal{FL}^* x^\mu = x^\mu \ , \quad \mathcal{FL}^* y^i = y^i \ , \quad \mathcal{FL}^* p_i^\mu = \frac{\partial L}{\partial y_\mu^i} \ .$$

Proposition 3.6

The Lagrangian $\mathcal{L}$ is regular if, and only if, $\mathcal{F}\mathcal{L}$ is a local diffeomorphism.

Definition 3.7

The Lagrangian $\mathcal{L}$ is ***hyperregular*** *if $\mathcal{F}\mathcal{L}$ is a global diffeomorphism.*

Then, $\mathbf{h} := \widetilde{\mathcal{F}\mathcal{L}} \circ \mathcal{F}\mathcal{L}^{-1}$ is a (local) *Hamiltonian section*

$$\begin{tikzcd}
 & & \mathcal{M}\pi \arrow[d, shift right, "\mathfrak{p}"'] \\
J^1\pi \arrow[urr, "\widetilde{\mathcal{F}\mathcal{L}}"] \arrow[rr, "\mathcal{F}\mathcal{L}"] & & J^1\pi^* \arrow[u, shift right, "\mathbf{h}"']
\end{tikzcd}$$

and we can define the corresponding Hamilton-Cartan forms $\Theta_{\mathbf{h}}$ and $\Omega_{\mathbf{h}}$.
We have:

$$\widetilde{\mathcal{F}\mathcal{L}}^*\widetilde{\Theta} = \mathcal{F}\mathcal{L}^*\Theta_{\mathbf{h}} = \Theta_{\mathcal{L}} \ , \ \widetilde{\mathcal{F}\mathcal{L}}^*\widetilde{\Omega} = \mathcal{F}\mathcal{L}^*\Omega_{\mathbf{h}} = \Omega_{\mathcal{L}} \ , \ E_{\mathcal{L}} = \mathcal{F}\mathcal{L}^* H \ .$$

Definition 3.8

The pair $(J^1\pi^*, \Omega_{\mathbf{h}})$ is the ***Hamiltonian system associated with the regular Lagrangian system*** *$(J^1\pi, \Omega_{\mathcal{L}})$.*

Theorem 3.9 (Equivalence theorem)

Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a (hyper)regular Lagrangian system and $(J^1\pi^*, \Omega_{\mathbf{h}})$ the associated Hamiltonian system.

- *If $\phi \in \Gamma(\pi)$ is a solution to the Hamilton variational problem, then $\psi = \mathcal{FL} \circ j^1\phi \in \Gamma(\bar{\tau})$ is a solution to the Hamilton–Jacobi problem. Conversely, if ψ is a solution to the Hamilton–Jacobi problem, then $\phi = \tau \circ \psi$ is a solution to the Hamilton problem.*

$$
\begin{tikzcd}
J^1\pi \arrow[rr, "\mathcal{FL}"] \arrow[dr, "\pi^1"] & & J^1\pi^* \arrow[dl, "\tau"'] \\
& E \arrow[d, shift left, "\pi"] & \\
& M \arrow[u, shift left, "\phi"] \arrow[uul, "j^1\phi"] \arrow[uur, "\psi"'] &
\end{tikzcd}
$$

- *The holonomic multivector fields and connections solutions to the Lagrangian field equations are $\mathcal{FL}$-related with the multivector fields and connections solutions to the Hamiltonian field equations (i.e.; their associated distributions in $J^1\pi$ and $J^1\pi^*$ are $\mathcal{FL}$-related).*

The singular case: premultisymplectic Hamiltonian field theories

Singular Hamiltonian systems in field theory are defined on submanifolds $\jmath_\circ \colon P_\circ \hookrightarrow J^1\pi^*$, fibered over E and M.

$$
\begin{tikzcd}
 & & \mathcal{M}\pi \arrow[d, shift right=1, "\mathfrak{p}"'] \\
P_\circ \arrow[urr, "\mathrm{h}_\circ"] \arrow[rr, "\jmath_\circ"] \arrow[dr, "\tau_\circ"] \arrow[ddr, "\overline{\tau}_\circ"'] & & J^1\pi^* \arrow[u, shift right=1, "\mathrm{h}"'] \arrow[dl, "\tau"'] \arrow[ddl, "\overline{\tau}"] \\
 & E \arrow[d, "\pi"] & \\
 & M &
\end{tikzcd}
$$

Definition 3.10

The ***Hamilton–Cartan forms*** *on $P_\circ$ are obtained using Hamiltonian sections to define* ***Hamiltonian maps*** $\mathbf{h}_\circ = \mathbf{h} \circ \jmath_\circ \colon P_\circ \to \mathcal{M}\pi$,

$$\Theta^\circ_{\mathbf{h}} = \mathbf{h}_\circ^* \widetilde{\Theta} \in \mathit{\Omega}^m(P_\circ) \quad , \quad \Omega^\circ_{\mathbf{h}} = -\mathrm{d}\Theta^\circ_{\mathbf{h}} = \mathbf{h}_\circ^* \widetilde{\Omega} \in \mathit{\Omega}^{m+1}(P) \ .$$

$(P_\circ, \Omega^\circ_{\mathbf{h}})$ *is a* ***singular*** *or* ***premultisymplectic Hamiltonian system****.*

The *Hamilton–Jacobi variational principle* for $(P_\circ, \Omega^\circ_{\mathbf{h}})$ is stated as in the regular case, but for sections $\psi_\circ \colon M \to P_\circ$. Then, the corresponding *Hamiltonian field equations* read:

1. $\psi_\circ^* \iota(X_\circ)\Omega^\circ_{\mathbf{h}} = 0$, for every $X_\circ \in \mathfrak{X}(P_\circ)$.
2. $\iota(\mathbf{X}^\circ_{\mathbf{h}})\Omega^\circ_{\mathbf{h}} = 0$, for $\mathbf{X}^\circ_{\mathbf{h}} \in \mathfrak{X}^m(P_\circ)$.
3. $\iota(\nabla^\circ_{\mathbf{h}})\Omega^\circ_{\mathbf{h}} = (m-1)\Omega^\circ_{\mathbf{h}}$, for Ehresmann connections $\nabla^\circ_{\mathbf{h}}$ on $P_\circ$.

For premultisymplectic Hamiltonian system, in general, these Hamiltonian field equations are not compatible on $P_\circ$. In the best situations, they have consistent solutions on a final constraint submanifold $P_f \hookrightarrow P_\circ$, which is obtained by using a suitable constraint algorithm, like in the Lagrangian case:

$$P_f \hookrightarrow \cdots\cdots \hookrightarrow P_1 \hookrightarrow P_\circ \hookrightarrow J^1\pi^* \ .$$

Then, sections solution to the field equations have their images on P_f and the distributions associated with multivector fields and connections solution to the corresponding field equations are tangent to P_f.

Hamiltonian formalism for singular Lagrangian field theories

Definition 3.11

Let $(J^1\pi, \Omega_{\mathcal{L}})$ be a Lagrangian system. The Lagrangian $\mathcal{L}$ (and the Lagrangian system) is said to be ***almost-regular*** *if:*

1. $\widetilde{P}_\circ := \widetilde{\mathcal{FL}}(J^1\pi)$ *is a submanifold of* $\mathcal{M}\pi$*, with natural embedding* $\widetilde{\jmath_0} \colon \widetilde{P}_\circ \hookrightarrow \mathcal{M}\pi$.
2. $\widetilde{\mathcal{FL}}$ *is a submersion onto its image.*
3. *For every* $\bar{y} \in J^1\pi$*, the fibers* $\widetilde{\mathcal{FL}}^{-1}(\widetilde{\mathcal{FL}}(\bar{y})) \subset J^1\pi$ *are connected.*

Proposition 3.12

If $\mathcal{L}$ is an almost-regular Lagrangian then:

1. $P_\circ := \mathcal{FL}(J^1\pi)$ *is a submanifold of* $J^1\pi^*$, $\jmath_0 \colon P_\circ \hookrightarrow J^1\pi^*$ *(**primary constraint submanifold** in Dirac's terminology).*
2. $\mathcal{FL}$ *is a submersion onto its image.*
3. *For every* $\bar{y} \in J^1\pi$*, the fibers* $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y})) \subset J^1\pi$ *are connected.*
4. *The restriction of* $\mathfrak{p} \colon \mathcal{M}\pi \to J^1\pi^*$ *to* $\widetilde{P}_\circ$*,* $\mathfrak{p}_\circ \colon \widetilde{P}_\circ \to P_\circ$*, is a diffeomorphism.*

Let $\widetilde{\mathcal{FL}}_\circ$ be the restriction map of $\widetilde{\mathcal{FL}}$ onto $\widetilde{P}_\circ$ given by $\widetilde{\mathcal{FL}} = \widetilde{\jmath}_\circ \circ \widetilde{\mathcal{FL}}_\circ$.

Let $\mathcal{FL}_\circ$ be the restriction map of $\mathcal{FL}$ onto $P_\circ$ given by $\mathcal{FL} = \jmath_\circ \circ \mathcal{FL}_\circ$.

Taking $\widetilde{\mathbf{h}} := \widetilde{\mathfrak{p}}_\circ^{-1}$, we have the Hamiltonian map $\mathbf{h}_\circ := \widetilde{\jmath}_\circ \circ \widetilde{\mathbf{h}}$.

$$
\begin{tikzcd}
 & \widetilde{P}_\circ \arrow[rr, "\widetilde{\jmath}_\circ"] \arrow[d, shift left, "\widetilde{\mathfrak{p}}_\circ"] & & \mathcal{M}\pi \arrow[d, shift right, "\mathfrak{p}"'] \\
J^1\pi \arrow[ur, "\widetilde{\mathcal{FL}}_\circ"] \arrow[r, "\mathcal{FL}_\circ"'] & P_\circ \arrow[u, shift left, "\widetilde{\mathrm{h}}"] \arrow[urr, "\mathrm{h}_\circ"'] \arrow[rr, "\jmath_\circ"'] \arrow[dr, "\tau_\circ"] \arrow[ddr, "\overline{\tau}_\circ"'] & & J^1\pi^* \arrow[u, shift right, "\mathrm{h}"'] \arrow[dl, "\tau"'] \arrow[ddl, "\overline{\tau}"] \\
 & & E \arrow[d, "\pi"] & \\
 & & M &
\end{tikzcd}
$$

The corresponding Hamilton–Cartan forms are

$$
\Theta_{\mathbf{h}}^\circ := \mathbf{h}_\circ^* \widetilde{\Theta} \in \Omega^m(P_\circ) \quad , \quad \Omega_{\mathbf{h}}^\circ := -\mathrm{d}\Theta_{\mathbf{h}}^\circ = \mathbf{h}_\circ^* \widetilde{\Omega} \in \Omega^{m+1}(P_\circ)
$$

They verify that

$$\mathcal{FL}_\circ^*\Theta_{\mathbf{h}}^\circ = \Theta_{\mathcal{L}} \quad , \quad \mathcal{FL}_\circ^*\Omega_{\mathbf{h}}^\circ = \Omega_{\mathcal{L}} \ ,$$

and there exists a ***de Donder–Weyl Hamiltonian function*** $H_\circ \in C^\infty(P_\circ)$ such that $E_{\mathcal{L}} = \mathcal{FL}_\circ^* H_\circ$.

DEFINITION 3.13

$(P_\circ, \Omega_h^\circ)$ *is the* ***premultisymplectic Hamiltonian system associated with the premultisymplectic Lagrangian system*** $(J^1\pi, \Omega_{\mathcal{L}})$.

Equivalence: For the Lagrangian and Hamiltonian constraint algorithms:

$$\begin{tikzcd}
S_f \arrow[r] \arrow[d, "\mathcal{FL}_f"'] & \cdots\cdots \arrow[r] & S_1 \arrow[r] \arrow[d, "\mathcal{FL}_1"'] & J^1\pi \arrow[d, "\mathcal{FL}_\circ"'] \arrow[dr, "\mathcal{FL}"] & \\
P_f \arrow[r] & \cdots\cdots \arrow[r] & P_1 \arrow[r] & P_\circ \arrow[r] & J^1\pi^*
\end{tikzcd}$$

The Equivalence theorem is stated like in the regular case (Theor. 3.9); but now sections solution to the Lagrangian and Hamiltonian field eqs. have their images on S_f and P_f, respectively; and the distributions associated with m.v.f. and connections solution are tangent to S_f and P_f.

Example

Maxwell's electromagnetism

Multisymplectic Hamiltonian setting:

Spacetime Minkowskian manifold (M,η): coordinates (x^μ), $\mu = 0,1,2,3$.

Configuration bundle $\pi \colon E \to M$, $E = M \times \Lambda^1(M)$
($\Lambda^1(M)$ one-form potentials on M): coordinates (x^μ, A_μ).

Multivelocity phase bundle $J^1\pi$: coordinates $(x^\mu, A_\mu, A_{\mu\nu})$.

Extended multimomentum bundle $J^1\pi$: coordinates $(x^\mu, A_\mu, p^{\mu\nu}, p)$.

Restricted multimomentum bundle $J^1\pi^*$: coordinates $(x^\mu, A_\mu, p^{\mu\nu})$.

Maxwell's Lagrangian function: $L(x^\mu, A_\nu, A_{\mu\nu}) = -\dfrac{1}{4\mu_\circ} F_{\mu\nu}F^{\mu\nu} - A_\alpha J^\alpha$;
where $F_{\mu\nu} = A_{\mu\nu} - A_{\nu\mu}$. It is singular (almost-regular) because

$$\frac{\partial^2 L}{\partial A_{\alpha\beta}\partial A_{\mu\nu}} = \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\mu}\eta^{\beta\nu} \,.$$

Restricted Legendre map $\mathcal{F}\mathcal{L}\colon J^1\pi \to J^1\pi^*$,

$$\mathcal{F}\mathcal{L}^* p^{\mu\nu} = \frac{\partial L}{\partial A_{\mu\nu}} = -\frac{1}{\mu_\circ} F^{\mu\nu} \ . \tag{5}$$

where coordinates $p^{\mu\nu}$ have the same properties as $F^{\mu\nu}$ (skew-symmetry).

Primary constraint submanifold $P_\circ = \mathcal{F}\mathcal{L}(J^1\pi)$, given by the constraints

$$\zeta^{\mu\mu} \equiv p^{\mu\mu} = 0 \quad , \quad \zeta^{\mu\nu} \equiv p^{\mu\nu} + p^{\nu\mu} = 0 \ .$$

Coordinates $(x^\mu, A_\mu, p^{\mu\nu})$, $\mu > \nu$.

Hamiltonian section $\mathbf{h}\colon P_\circ \to \mathcal{M}\pi$: It is given by the de Donder–Weyl–Hamiltonian function (for the freee coordinates $p^{\mu\nu}$ in $P_\circ$):

$$H_\circ = -\frac{\mu_\circ}{4} p_{\mu\nu} p^{\mu\nu} + A_\alpha J^\alpha \in \mathrm{C}^\infty(P_\circ) \ .$$

Hamilton–Cartan forms:

$$\Theta^0_{\mathbf{h}} = p^{\mu\nu} \mathrm{d}A_\nu \wedge \mathrm{d}^3 x_\mu + \left(\frac{\mu_\circ}{4} p_{\mu\nu} p^{\mu\nu} - A_\alpha J^\alpha\right) \mathrm{d}^4 x \ ,$$

$$\Omega^0_{\mathbf{h}} = -\mathrm{d}p^{\mu\nu} \wedge \mathrm{d}A_\nu \wedge \mathrm{d}^3 x_\mu - \frac{\mu_\circ}{2} p_{\mu\nu} \mathrm{d}p^{\mu\nu} \wedge \mathrm{d}^4 x + J^\alpha \mathrm{d}A_\alpha \wedge \mathrm{d}^4 x \ .$$

Field equations (for m.v.f.): For locally decomposable $\bar{\tau}^1$-transverse multivector fields $\mathbf{X_h} = X_0 \wedge \cdots \wedge X_3 \in \mathfrak{X}^4(P_\circ)$, where

$$X_\mu = \frac{\partial}{\partial x^\mu} + D_{\mu\nu}\frac{\partial}{\partial A_\nu} + H_\mu^{\alpha\beta}\frac{\partial}{\partial p\alpha\beta} \ ,$$

equation (2) gives

$$\iota(\mathbf{X_h})\Omega_{\mathbf{h}}^0 = (H_\mu^{\mu\nu} + J^\nu)\mathrm{d}A_\nu - \left(D_{\mu\nu} + \frac{\mu_\circ}{2}p_{\mu\nu}\right)\mathrm{d}p^{\mu\nu} \ ;$$

which, using the skew-symmetry of $p^{\mu\nu}$, leads to

$$(D_{\mu\nu} - D_{\nu\mu}) + \mu_\circ p_{\mu\nu} = 0 \quad , \quad H_\mu^{\mu\nu} + J^\nu = 0 \ . \tag{7}$$

For the integral sections of $\mathbf{X_h}$ we have

$$D_{\mu\nu} = \partial_\mu A_\nu \quad , \quad H_\mu^{\alpha\beta} = \partial_\mu p^{\alpha\beta} \ ,$$

then, equations (7) lead to the Hamilton–de Donder–Weyl equations which, using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the Legendre map (5), read as:

$$F_{\mu\nu} = -\mu_\circ p_{\mu\nu} \quad , \quad -\partial_\mu p^{\mu\nu} = J^\nu \iff \partial_\mu F^{\mu\nu} = \mu_\circ J^\nu \ . \tag{8}$$

The first eqs. (8) are the first pair of Maxwell eqs. (field constitutive eqs.), and the second eqs. (8) are the (second pair of) Maxwell equations.

Eqs. (7) are compatible on $P_\circ$; so no constraint algorithm is needed.

Other references

- M. de León, J. Marín-Solano, J.C. Marrero, M.C. Muñoz-Lecanda, N. Román-Roy, "Premultisymplectic constraint algorithm for field theories". *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 839–871.
- A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, "Geometry of multisymplectic Hamiltonian first-order field theories", *J. Math. Phys.* **41** (2000) 7402–44.
- M. Forger, C. Paufler, H. Römer, "Hamiltonian multivector fields and Poisson forms in multisymplectic field theory", *J. Math. Phys.* **46**(11) (2005) 112901.
- F. Gay-Balmaz, J.C. Marrero, N. Martínez-Alba, "A new canonical affine bracket formulation of Hamiltonian classical field theories of first order", *Rev. R. Acad. Cienc. Exact. Fis. Nat. Ser. A-Mat.* (2024) **118** (2024) 103.
- M.J. Gotay, "A multisymplectic framework for classical field theory and the calculus of variations. I. Covariant Hamiltonian formalism", *Anal. Geom.: 200 years after Lagrange*. North-Holland. Amsterdam (1991) 203–235.
- M.J. Gotay, J. Isenberg, J.E. Marsden, R. Montgomery, "Momentum maps and classical relativistic fields I: Covariant theory", arXiv:physics/9801019 [math-ph] (2004).

- F. Hélein, J. Kouneiher, "Finite dimensional Hamiltonian formalism for gauge and quantum field theories", *J. Math. Phys.* **43** (2002) 2306–2347.
- F. Hélein, J. Kouneiher, "Covariant Hamiltonian formalism for the calculus of variations with several variables: Lepage–Dedecker versus de Donder–Weyl", *Adv. Theor. Math. Phys.* **8** (2004) 565–601.
- J. Kijowski, W. Szczyrba, "A canonical structure for classical field theories", *Comm. Math. Phys.* **46** (1976) 183–206.
- O. Krupkova, "Hamiltonian field theory", *J. Geom. Phys.* **43**(2–3) (2002) 93–132.
- J.E. Marsden, S. Shkoller, "Multisymplectic geometry, covariant Hamiltonians and nonlinear PDEs", *Comm. Math. Phys.* **199**(2) (1998) 351–395.
- C. Paufler, H. Römer, "Geometry of Hamiltonian n-vector fields in multisymplectic field theory", *J. Geom. Phys.* **44** (2002) 52–69.