An introductory course on geometric field theories: the multisymplectic setting

Second lecture: Multisymplectic Lagrangian field theory

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Course Programme

1. **First lecture:** Introductory remarks. Previous geometric background.
2. **Second lecture:** Multisymplectic Lagrangian field theory.
3. **Third lecture:** Multisymplectic Hamiltonian field theory.
4. **Fourth lecture:** Symmetries. Gauge symmetries. Conservation laws.

Introduction

In the Lagrangian formalism of variational multisymplectic field theories, the appropriate geometrical setting for constructing finite-dimensional covariant phase spaces are jet bundles.

These are fiber bundles over spacetime whose fiber coordinates represent the fields and their spacetime derivatives; and the formalism is developed using the canonical geometric structures naturally defined on them.

In general, for a kth-order Lagrangian field theory (where the Lagrangians depend on derivatives of the fields up to order k), the theory takes place on the $(2k-1)$-order jet bundle, and the Lagrangian field equations (the Euler–Lagrange equations) are PDE's of order $2k$.

For first-order Lagrangian field theories (where the Lagrangians depend on derivatives of the fields up to first-order), the corresponding first-order jet bundle is called the multivelocity phase space, and the coordinates on the fibers representing the first derivatives of the fields, are called multivelocities. Then, the Lagrangian field equations are second-order PDE's.

In this lecture, I present the main features of the multisymplectic Lagrangian framework for first-order field theories, ranging from the geometry of jet bundles to the variational principle and field equations.

A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, "Geometry of Lagrangian first-order classical field theories". *Forts. Phys.* **44** (1996) 235-280.

A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, "Multivector fields and connections: Setting Lagrangian equations in field theories", *J. Math. Phys.* **39**(9) (1998) 4578-4603.

N. Román-Roy, "Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories", *Sym. Integ. Geom. Meth. Appl.* **5** (2009) 100.

Index

Jet bundles

First-order jet bundles

Let $\pi \colon E \longrightarrow M$ be a fiber bundle; $\dim M = m$, $\dim E = n + m$.

Coords. on E adapted to the bundle: (x^μ, y^A), $1 \leqslant \mu \leqslant m$, $1 \leqslant A \leqslant n$.

Definition 2.1

Two local sections $\phi, \psi \colon U \subset M \to E$ are ***1-equivalent at $x \in U \subset M$*** *if,*

(i) $\phi(x) = \psi(x)$. *(ii)* $\mathrm{T}_x\phi = \mathrm{T}_x\psi$*; that is,* $\left.\dfrac{\partial \phi^A}{\partial x^\mu}\right|_x = \left.\dfrac{\partial \psi^A}{\partial x^\mu}\right|_x$.

The equivalence class containing ϕ is called the ***1-jet of ϕ at x****, and is denoted $j^1_x\phi$. The* ***bundle of 1-jets*** *of sections of π, also called the* ***1-jet bundle of π*** *or the* ***first-order jet bundle of π****, is the set*

$$J^1\pi = \left\{ j^1_x\phi \mid x \in M \, , \, \phi \in \Gamma_x(\pi) \right\} \, .$$

The fibers of $J^1\pi$ at $y \in E$ are denoted $J^1_y E$ and their elements $\bar{y} \in J^1_y E$.

If $\phi \colon U \to E$ is a representative of $\bar{y} \in J^1_y E$, we write $\phi \in \bar{y}$.

$J^1\pi \longrightarrow E$ is an affine bundle with natural projections:

$$\begin{array}{rcl} \pi^1 \colon J^1\pi & \longrightarrow & E \\ \bar{y} = j^1_x\phi & \longmapsto & y = \phi(x) \end{array} \quad ; \quad \begin{array}{rcl} \bar{\pi}^1 = \pi \circ \pi^1 \colon J^1\pi & \longrightarrow & M \\ \bar{y} = j^1_x\phi & \longmapsto & x \end{array} \, .$$

Local adapted coordinates in $J^1\pi$ are (x^μ, y^A, y^A_μ), defined as follows: if $\bar{y} \in J^1\pi$, with $\pi^1(\bar{y}) = y$, $\pi(y) = x$, and $\phi \in \bar{y}$, then

$$y^A(\bar{y}) = \phi^A(x) \quad ; \quad y^A_\mu(\bar{y}) = \frac{\partial \phi^A}{\partial x^\mu}(x) \ ,$$

and $\dim J^1\pi = m + n + nm$.

Furthermore, in the case that M is an orientable manifold with volume form $\omega_{_M} \in \Omega^m(M)$, the coordinates (x^μ) in M are taken such that

$$\omega_{_M} = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x \, .$$

Finally, we denote:

$$\mathrm{d}^{m-1}x_\mu \equiv \iota\Big(\frac{\partial}{\partial x^\mu}\Big)\mathrm{d}^m x \ .$$

We also denote $\omega = \bar{\pi}^{1*}\omega_{_M}$.

Definition 2.2

Let $\phi \colon U \subset M \to E$ be a local section of π. For every $x \in U$, ϕ defines an element $\bar{y} = j^1_x\phi \in J^1_x\pi$. Therefore, the local section of $\bar{\pi}^1$,

$$\begin{array}{rcccc} j^1\phi & : & U \subseteq M & \longrightarrow & J^1\pi \\ & & x & \longmapsto & \bar{y} = j^1_x\phi \end{array} ,$$

is called the ***canonical lift*** *of ϕ to $J^1\pi$.*
Sections of $\bar{\pi}^1$ which are canonical lifts of sections of π are called ***holonomic sections****.*

If $\phi(x) = \left(x^\mu, \phi^A(x)\right)$, then $j^1\phi(x) = \left(x^\mu, \phi^A(x), \dfrac{\partial \phi^A}{\partial x^\mu}(x)\right)$.

Canonical structures

Canonical structures in $J^1\pi$: The ***canonical endomorphism*** in $J^1\pi$:

$$\mathcal{J} = \left(\mathrm{d}y^A - y^A_\mu \mathrm{d}x^\mu\right) \otimes \frac{\partial}{\partial y^A_\nu} \otimes \frac{\partial}{\partial x^\nu} \ .$$

The ***contact module*** or ***Cartan codistribution***:

$$\mathcal{C} = \left\langle \mathrm{d}y^A - y^A_\mu \mathrm{d}x^\mu \right\rangle \ .$$

Definition 2.3

*If M is an orientable manifold with volume form $\omega_{_M} \in \Omega^m(M)$, and $\omega = \bar{\pi}^{1*}\omega_{_M} \in \Omega^m(J^1\pi)$, the **canonical endomorphism** associated with ω is*

$$\mathcal{J}_\omega := \iota(\mathcal{J})\omega$$

$$\mathcal{J}_\omega = \left(\mathrm{d}y^A - y^A_\mu \mathrm{d}x^\mu\right) \otimes \mathrm{d}^{m-1}x_\nu \otimes \frac{\partial}{\partial y^A_\nu} \ .$$

Multivector fields and connections on jet bundles

Let $\{x^\mu, y^A, y^A_\mu\}$ be adapted coordinates in the bundle $J^1\pi \to E \to M$. An Ehresmann connection ∇ in $J^1\pi \to M$, its associated horizontal bundle $\mathrm{H}(\nabla)$, and a representative of the class of locally decomposable $\bar{\pi}^1$-transverse multivector fields associated with ∇ read as:

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + f^A_\mu \frac{\partial}{\partial y^A} + F^A_{\mu\nu} \frac{\partial}{\partial y^A_\nu}\right), \tag{1}$$

$$\mathrm{H}(\nabla) = \left\langle \frac{\partial}{\partial x^\mu} + f^A_\mu \frac{\partial}{\partial y^A} + F^A_{\mu\nu} \frac{\partial}{\partial y^A_\nu} \right\rangle,$$

$$\mathbf{X} = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + f^A_\mu \frac{\partial}{\partial y^A} + F^A_{\mu\nu} \frac{\partial}{\partial y^A_\nu}\right), \tag{2}$$

Holonomic multivector fields and connections (SOPDE's)

Definition 2.4

A locally decomposable m.v.f. $\mathbf{X} \in \mathfrak{X}^m(J^1\pi)$ is ***holonomic*** *or a* ***Second Order Partial Differential Equation (SOPDE)*** *if*

1. *It is $\bar{\pi}^1$-transverse.*
2. *It is integrable.*
3. *Its integral sections $\psi\colon M \to J^1\pi$ are holonomic.*

An Ehresmann connection ∇ in $J^1\pi$ is ***holonomic*** *or a* ***Second Order Partial Differential Equation*** *if*

1. *It is integrable.*
2. *Its integral sections $\psi\colon M \to J^1\pi$ are holonomic.*

(We also say that they verify the ***Second-Order Condition****).*

A holonomic section $j^1\phi(x) = \left(x^\mu, \phi^A(x), \dfrac{\partial \phi^A}{\partial x^\mu}(x)\right)$ is an integral section of ∇ and its associated $\mathbf{X}$ if, and only if, in (1) and (2),

$$f^A_\mu(j^1\phi(x)) = \frac{\partial \phi^A}{\partial x^\mu} \quad , \quad F^A_{\mu\nu}(j^1\phi(x)) = \frac{\partial^2 \phi^A}{\partial x^\mu \partial x^\nu} \; . \tag{3}$$

This means that

$$\mathbf{X} = \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + y_\mu^A \frac{\partial}{\partial y^A} + F_{\mu\nu}^A \frac{\partial}{\partial y_\nu^A} \right) , \tag{4}$$

$$\nabla = \mathrm{d}x^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + y_\mu^A \frac{\partial}{\partial y^A} + F_{\mu\nu}^A \frac{\partial}{\partial y_\nu^A} \right) . \tag{5}$$

Definition 2.5

*A locally decomposable $\bar{\pi}^1$-transverse multivector field $\mathbf{X} \in \mathfrak{X}^m(J^1\pi)$ and its associated connection ∇ are **semiholonomic** if (4) and (5) hold or, equivalently,*

$$\iota(\mathcal{J})\mathbf{X} = \iota(\mathcal{J}_\omega)\mathbf{X} = 0 \quad ; \quad \iota(\mathcal{J})\nabla = \iota(\mathcal{J}_\omega)\nabla = 0 \, .$$

Proposition 2.6

A locally decomposable $\bar{\pi}^1$-transverse multivector field $\mathbf{X} \in \mathfrak{X}^m(J^1\pi)$ and its associated connection ∇ are holonomic if, and only if, they are semiholonomic and integrable.

Lagrangian field theories

Lagrangian systems

Geometric framework for first-order Lagrangian field theories:

- An m-dimensional orientable manifold M, with volume form $\omega_{_M} \in \Omega^m(M)$; (typically *spacetime*).
- A fiber bundle $\pi \colon E \longrightarrow M$, with $\dim E = m + n$, which is the ***covariant configuration bundle*** of the theory.
  The ***fields*** are the sections of the projection π.
- The first-order jet bundle $\bar{\pi}^1 \colon J^1\pi \xrightarrow{\pi^1} E \xrightarrow{\pi} M$, which is called the ***Lagrangian (multi)phase space*** of the theory.
  The ***states of the field*** are holonomic sections of the projection $\bar{\pi}^1$.
  In the charts of coordinates adapted to the bundle structure, (x^μ, y^A, y^A_μ), the coordinates (y^A_μ) (in the fibers of π^1) are called ***multivelocities***, and $J^1\pi$ is also known as the ***multivelocity phase space***.

The physical (or mathematical) information of the theory:

Definition 3.1

*A **Lagrangian density** is a $\bar{\pi}^1$-semibasic m-form $\mathcal{L} \in \Omega^m(J^1\pi)$.
As M is an orientable manifold, then $\mathcal{L} = L\,\omega$, where $L \in \mathrm{C}^\infty(J^1\pi)$ is the **Lagrangian function** associated with $\mathcal{L}$ and ω.*

Definition 3.2

*Given a Lagrangian density $\mathcal{L} \in \Omega^m(J^1\pi)$:
The **Poincaré–Cartan m-form** associated with $\mathcal{L}$ is defined as:*

$$\Theta_{\mathcal{L}} := i(\mathcal{J})\mathrm{d}\mathcal{L} + \mathcal{L} = i(\mathcal{J}_\omega)\mathrm{d}L + \mathcal{L} \in \Omega^m(J^1\pi) \ .$$

*The form $\vartheta_{\mathcal{L}} = i(\mathcal{J})\mathrm{d}\mathcal{L}$ is called the **Lagrangian form** associated with $\mathcal{L}$.
The **Poincaré–Cartan $(m+1)$-form** associated with $\mathcal{L}$ is*

$$\Omega_{\mathcal{L}} := -\mathrm{d}\Theta_{\mathcal{L}} \in \Omega^{m+1}(J^1\pi) \ .$$

*A **Lagrangian system** is a pair $(J^1\pi, \Omega_{\mathcal{L}})$.*

Remark: $\Omega_{\mathcal{L}}$ is variational, since, for every $X, Y, Z \in \mathfrak{X}^{\mathrm{V}(\bar{\pi}^1)}(J^1\pi)$,

$$\iota(X)\iota(Y)\Theta_{\mathcal{L}} = 0 \implies \iota(X)\iota(Y)\iota(Z)\Omega_{\mathcal{L}} = 0 \ .$$

$$\Theta_{\mathcal{L}} = \frac{\partial L}{\partial y^A_\mu} \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu - \left(\frac{\partial L}{\partial y^A_\mu} y^A_\mu - L \right) \mathrm{d}^m x \ ,$$

$$\Omega_{\mathcal{L}} = -\mathrm{d}\left(\frac{\partial L}{\partial y^A_\mu} \right) \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\mu + \mathrm{d}\left(\frac{\partial L}{\partial y^A_\mu} y^A_\mu - L \right) \wedge \mathrm{d}^m x \ . \qquad (6)$$

The local function

$$E_{\mathcal{L}} := \frac{\partial L}{\partial y^A_\mu} y^A_\mu - L$$

is the ***Lagrangian energy*** associated with L. It cannot be intrinsically defined as a function on $J^1\pi$, unless a connection in $\pi\colon E \to M$ is used.

Definition 3.3

A Lagrangian density and the corresponding Lagrangian system are ***regular*** *if $\Omega_{\mathcal{L}}$ is a multisymplectic form (i.e., 1-nondegenerate). Elsewhere, they are* ***singular*** *or* ***non-regular*** *(and $\Omega_{\mathcal{L}}$ is a premultisymplectic form).*

Remark: In coordinates, this regularity condition is locally equivalent to demanding that the partial Hessian matrix $\left(\frac{\partial^2 L}{\partial y^A_\mu \partial y^B_\nu} \right)$ is regular.

Variational principle

In general, in the standard formulation of variational principles, a **variational problem** consists of the following elements:

- A set of functions which are the variables of the problem.
  In geometric field theories this is the set of sections which represent the states of the fields.
- A set of admissible transformations of the functions.
  In geometric field theories these transformations are locally generated by vector fields in the multiphase bundle, which are vertical for the projection onto the base manifold M.
- An action functional defined in the above set of functions from a Lagrangian density. It is given by an integral (on compact-supported domains) of the Lagrangian density evaluated on those functions.
  In geometric field theories, this integral can also be stated using the geometric structure constructed from the Lagrangian density.

Then, the variational problem consists in searching for the critical or stationary sections of the functional with respect to admissible variations.

Definition 3.4 (Hamilton Variational Principle)

*Let $(J^1\pi, \Omega_\mathcal{L})$ be a Lagrangian system and $\Gamma(\pi)$ the set of sections of π. Consider the following functional (the **action** defined by $\mathcal{L}$):*

$$\begin{aligned} \mathbf{L}\colon \Gamma(\pi) &\longrightarrow \mathbb{R} \\ \phi &\longmapsto \int_{D\subset M} (j^1\phi)^*\mathcal{L} = \int_{D\subset M} (j^1\phi)^*\Theta_\mathcal{L} \ . \end{aligned}$$

*Then, the **Hamilton variational problem** for the Lagrangian system $(J^1\pi, \Omega_\mathcal{L})$ is the search for the critical (or stationary) sections of the functional $\mathbf{L}$ with respect to the variations of ϕ given by $\phi_t = \sigma_t \circ \phi$, where $\{\sigma_t\}$ is a local 1-parameter group of any compact-supported π-vertical vector field $Z \in \mathfrak{X}^{\mathrm{V}(\pi)}(E)$; that is,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_D \left(j^1\phi_t\right)^*\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_D \left(j^1\phi_t\right)^*\Theta_\mathcal{L} = 0 \ .$$

The corresponding Lagrangian field equations derived from this variational principle can be stated in several alternative ways:

Lagrangian field equations

Theorem 3.5

The following assertions on a section $\phi \in \Gamma(\pi)$ are equivalent:

1. *The section ϕ is critical for the Hamilton variational problem.*
2. *$j^1\phi$ is a solution to the* ***Lagrangian field equations for sections****:*
$$(j^1\phi)^*\iota(X)\Omega_{\mathcal{L}} = 0 \; ; \; \text{for every } X \in \mathfrak{X}(J^1\pi) \; . \tag{7}$$
3. *$j^1\phi$ is an integral section of a class of holonomic multivector fields, $\{\mathbf{X}_{\mathcal{L}}\} \subset \mathfrak{X}^m(J^1\pi)$, whose elements are solutions to the* ***Lagrangian field equations for multivector fields****:*
$$\iota(\mathbf{X}_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \; . \tag{8}$$
4. *$j^1\phi$ is an integral section of a holonomic connection $\nabla_{\mathcal{L}}$ in $J^1\pi$ which is a solution to the* ***Lagrangian field eqs. for connections****:*
$$\iota(\nabla_{\mathcal{L}})\Omega_{\mathcal{L}} = (m-1)\Omega_{\mathcal{L}} \; . \tag{9}$$
5. *$j^1\phi$ is a solution to the* ***Euler–Lagrange equations****:*
$$\frac{\partial L}{\partial y^A} \circ j^1\phi - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial y^A_\mu} \circ j^1\phi\right) = 0 \; . \tag{10}$$

References

Remarks:

- There are Lagrangian densities or, equivalently, Lagrangian functions, that, being different, give rise to the same Poincaré–Cartan form $\Theta_{\mathcal{L}}$ and, hence, to the same Euler–Lagrange equations.
  They are generically called ***equivalent Lagrangians***.
- Even more, different Lagrangians can originate different Poincaré–Cartan forms, but which lead to the same variational problem and, hence, to the same Euler–Lagrange equations.
  These are called ***Lepage equivalent forms*** and their corresponding Lagrangians are ***Lepage equivalent Lagrangians***.

D. Krupka, *Introduction to Global Variational Geometry*, Atlantis Studies in Variational Geometry, Atlantis Press, Elsevier, 2013.

In natural coordinates of $J^1\pi$, if $\phi(x) = \left(x^\mu, y^A_\mu(x)\right)$, then $(j^1\phi)(x) = \left(x^\mu, y^A_\mu(x), \dfrac{\partial y^A}{\partial x^\mu}(x)\right)$. Therefore:

- Using (6), equation (7) reads as (10) (the Euler-Lagrange eqs).
- Using (1), (2), and (6); from (8) and (9) we get

$$\left(f^B_\mu - y^B_\mu\right)\frac{\partial^2 L}{\partial y^A_\nu \partial y^B_\mu} = 0\,, \tag{11}$$

$$\frac{\partial L}{\partial y^A} - \frac{\partial^2 L}{\partial x^\mu \partial y^A_\mu} - \frac{\partial^2 L}{\partial y^B \partial y^A_\mu} f^B_\mu - \frac{\partial^2 L}{\partial y^B_\nu \partial y^A_\mu} F^B_{\nu\mu} + \frac{\partial^2 L}{\partial y^A \partial y^B_\mu}\left(f^B_\mu - y^B_\mu\right) = 0 \tag{12}$$

- If $\mathbf{X}_\mathcal{L}$ and $\nabla_\mathcal{L}$ are holonomic, they are semiholonomic; then $f^B_\mu = y^B_\mu$. Therefore, eqs. (11) hold identically and eqs. (12) are

$$0 = \frac{\partial L}{\partial y^A} - \frac{\partial^2 L}{\partial x^\mu \partial y^A_\mu} - \frac{\partial^2 L}{\partial y^B \partial y^A_\mu} y^B_\mu - \frac{\partial^2 L}{\partial y^B_\nu \partial y^A_\mu} F^B_{\nu\mu}\ ; \tag{13}$$

  and, bearing in mind (3), eqs. (13) give the Euler-Lagrange equations (10).
- When $\mathcal{L}$ is regular, semiholonomy comes from eqs. (11), and eqs. (13) are compatible for $F^B_{\nu\mu}$.

Proposition 3.6

If $(J^1\pi, \Omega_{\mathcal{L}})$ is a regular Lagrangian system, then:

1. *The Lagrangian field equations for multivector fields (8) and connections (9) have solutions on $J^1\pi$. These solutions are not unique if $m > 1$.*
2. *The $\bar{\pi}^1$-transverse multivector fields $\mathbf{X}_{\mathcal{L}} \in \mathfrak{X}^m(J^1\pi)$ and their associated Ehresmann connections $\nabla_{\mathcal{L}}$ which are solutions to equations (8) and (9) are semiholonomic.*

Remark: Equations (8) and (9) are equations for locally decomposable m.v.f. and connections (not for sections), and, if they are compatible, their solutions can be integrable or not. So, once solved, the integrability condition must be additionally imposed; which could originate constraints defining a submanifold of $J^1\pi$ where these integrable solutions exist.

If $\mathbf{X}_{\mathcal{L}} = \bigwedge_{\mu=1}^{m} X_\mu$ and $\nabla_{\mathcal{L}} = \mathrm{d}x^\mu \otimes X_\mu$, the necessary and sufficient condition for the existence of integral sections (solutions to eqs. (3)) is:

$$[X_\mu, X_\nu] = 0 \ , \ \ (\text{for every } \mu, \nu) \ .$$

The singular case: constraint algorithm

When the system is not regular, in general, the Lagrangian field equations have no solutions everywhere on $J^1\pi$. In the most favourable cases, they are compatible only on a submanifold of $J^1\pi$, which is obtained by applying a constraint algorithm.

Furthermore, solutions to the Lagrangian field equations for multivector fields (8) and connections (9) are not necessarily semiholonomic and, as a consequence, if they are integrable, their integral sections are not necessarily holonomic; so this requirement must be imposed as an additional condition.

Thus, the objective is to find the maximal submanifold of $J^1\pi$ where these SOPDE solutions to the Lagrangian field equations (8) and (9) exist and are tangent to the submanifold.

- M. de León, J. Marín-Solano, J.C. Marrero, M.C. Muñoz-Lecanda, N. Román-Roy, "Premultisymplectic constraint algorithm for field theories". *Int. J. Geom. Meth. Mod. Phys.* **2** (2005) 839–871.

Operational implementation of the **constraint algorithm** for eqs. (8,9):

1. Find the maximal compatibility submanifold $S_1 \hookrightarrow J^1\pi$ where semiholonomic multivector fields and connection solution to eqs. (8) and (9) exist. It is (locally) defined by compatibility constraints $\zeta_j^{(1)}$.
2. Impose the tangency conditions to find the points of S_1 where there are semiholonomic solutions $\mathbf{X} = \bigwedge_{\mu=1}^{m} X_\mu$ and $\mathrm{H}(\nabla) = \langle X_\mu \rangle$, tangent to S_1. These conditions read locally
$$\mathrm{L}_{X_\mu} \zeta_j^{(1)} \underset{S_1}{=} 0 \ .$$
3. If these conditions originate new stability constraints $\zeta_{j'}^{(2)}$ that define a submanifold $S_2 \hookrightarrow S_1$, the tangency conditions apply again.
4. The procedure continues until, in the best situations, the Lagrangian field equations have semiholonomic solutions tangent to a maximal final constraint submanifold S_f, that projects on M,
$$S_f \hookrightarrow \cdots\cdots \hookrightarrow S_1 \hookrightarrow J^1\pi \ .$$
5. Finally, we have to find holonomic solutions. Then, the integrability conditions can originate new constraints; i.e., new submanifols, where the tangency conditions have to be imposed again.

Example

Maxwell's electromagnetism

Maxwell equations (in void):

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_o}\quad,\qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad;$$

$$\nabla\cdot\mathbf{B}=0\quad,\qquad \nabla\times\mathbf{B}=\mu_o\mathbf{j}+\mu_o\epsilon_o\frac{\partial\mathbf{E}}{\partial t}\quad.$$

$\mathbf{E}$: *electric field*. $\mathbf{B}$: *magnetic field*. ρ: *density of electric charge*.

$\mathbf{j}$: *density of electric current*. ϵ_o: *electric constant*. μ_o: *magnetic constant*

$\nabla\cdot\mathbf{B}=0\Longrightarrow\mathbf{B}=\nabla\times\mathbf{A}$; $\mathbf{A}=(A^1,A^2,A^3)$: ***vector potential***.

$0=\nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right)\Longrightarrow\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}=-\nabla\Phi$; Φ: ***scalar potential***.

$\mathcal{A}=\left(-\frac{\Phi}{c},\mathbf{A}\right)=(A_\mu)$: ***electromagnetic four-potential form***.

$\mathcal{F}=(F_{\mu\nu})$; $F_{\mu\nu}=\frac{\partial A_\nu}{\partial x^\mu}-\frac{\partial A_\mu}{\partial x^\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$: ***electromagnetic tensor***.

$J=(\rho c,\mathbf{j})=(J^\mu)$: ***electromagnetic four-current***.

Multisymplectic Lagrangian setting:

Spacetime Minkowskian manifold (M,η): coordinates (x^μ), $\mu = 0,1,2,3$.

Configuration bundle $\pi\colon E \to M$, $E = M \times \Lambda^1(M)$
($\Lambda^1(M)$: four-potential one-forms on M): coordinates (x^μ, A_μ).

Multivelocity phase bundle $J^1\pi$: coordinates $(x^\mu, A_\mu, A_{\mu\nu})$.

Maxwell's Lagrangian function: $L(x^\mu, A_\nu, A_{\mu\nu}) = -\dfrac{1}{4\mu_\circ} F_{\mu\nu}F^{\mu\nu} - A_\alpha J^\alpha$;
where $F_{\mu\nu} = A_{\mu\nu} - A_{\nu\mu}$ and $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$. It is singular because the corresponding partial Hessian matrix is singular:

$$\frac{\partial^2 L}{\partial A_{\alpha\beta}\partial A_{\mu\nu}} = \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\mu}\eta^{\beta\nu} \,.$$

Poincaré–Cartan forms:

$$\begin{aligned}
\Theta_{\mathcal{L}} &= \frac{1}{\mu_\circ} F^{\alpha\mu}\mathrm{d}A_\alpha \wedge \mathrm{d}^3x_\mu + \Big(-\frac{1}{4\mu_\circ} F_{\mu\nu}F^{\mu\nu} + A_\alpha J^\alpha \Big)\mathrm{d}^4x \,, \\
\Omega_{\mathcal{L}} &= -\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta}\right)\mathrm{d}A_{\alpha\beta} \wedge \mathrm{d}A_\nu \wedge \mathrm{d}^3x + F^{\alpha\beta}\mathrm{d}A_{\alpha\beta} \wedge \mathrm{d}^4x \\
&\quad - J^\alpha \mathrm{d}A_\alpha \wedge \mathrm{d}^4x \,.
\end{aligned}$$

Field equations (for m.v.f.): For loc. decomposable $\bar{\pi}^1$-transverse multivector fields $\mathbf{X}_{\mathcal{L}} = X_0 \wedge \cdots \wedge X_3 \in \mathfrak{X}^4(J^1\pi)$, where

$$X_\mu = \frac{\partial}{\partial x^\mu} + D_{\mu\nu}\frac{\partial}{\partial A_\nu} + H_{\mu\alpha\beta}\frac{\partial}{\partial A_{\alpha\beta}} \ ,$$

equation (8) gives

$$\begin{cases} 0 = & F^{\mu\nu} + (\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\beta\mu}\eta^{\alpha\nu})D_{\alpha\beta} = F^{\mu\nu} + D^{\mu\nu} - D^{\nu\mu} \ , \\ 0 = & (\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta})H_{\mu\alpha\beta} + \mu_\circ J^\nu = H^{\mu\nu}_\mu - H^{\nu\mu}_\mu + \mu_\circ J^\nu \ . \end{cases} \tag{17}$$

As $\mathcal{F}$ is skewsymmetric, the first equations (17) give the semiholonomy condition for $\mathbf{X}_{\mathcal{L}}$, $D_{\mu\nu} = A_{\mu\nu}$; and for their holonomic integral sections,

$$D_{\mu\nu} = A_{\mu\nu} = \partial_\mu A_\nu \implies H_{\mu\alpha\beta} = \partial_\mu A_{\alpha\beta} = \partial_\mu\partial_\alpha A_\beta \ ,$$

and equations (17) lead to the Euler-Lagrange equations

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad , \quad \partial_\mu F^{\mu\nu} = \mu_\circ J^\nu \ . \tag{18}$$

The first eqs. (18) can be written as $\mathcal{F} = \mathrm{d}\mathcal{A}$; which means $\mathrm{d}\mathcal{F} = 0$.
They are the first pair of Maxwell equations (field constitutive equations).

The second eqs. (18) are the (second pair of) Maxwell equations.

Eqs. (17) are compatible on $J^1\pi$; so no constraint algorithm is needed.

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