

Conformal symmetries in Galilean spacetimes

XXXIV Fall Workshop on Geometry and Physics

Tarragona

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Introduction



SUR
LES VARIÉTÉS À CONNEXION AFFINE
ET
LA THÉORIE DE LA RELATIVITÉ GÉNÉRALISÉE
(PREMIÈRE PARTIE)
PAR E. CARTAN

La plupart des Chapitres de ce Mémoire, qui paraîtra en deux parties, sont consacrés à la théorie purement géométrique de ce que j'appelle les *variétés à connexion affine*, et qui comprennent, comme cas particulier, les variétés à connexion *métrique* et les variétés à connexion *euclidienne*. Le terme « connexion affine » est emprunté à M. H. Weyl⁽¹⁾, mais il prend dans ce Mémoire une signification élargie. Comme je l'ai indiqué à grands traits dans des Notes aux *Comptes rendus de l'Académie des Sciences*⁽²⁾, une variété à connexion affine est une variété qui, au voisinage *immédiat* de chaque point, a tous les caractères d'un espace affine, et pour laquelle on a une loi de repérage des domaines entourant deux points *infinitement voisins* : cela veut dire que si, en chaque point, on se donne un système de coordonnées cartésiennes ayant ce point pour origine, on connaît les formules de transformations (de même nature que dans l'espace affine) qui permettent de passer d'un système de référence à tout autre système de référence d'origine infiniment voisine. Dans la théorie de

(¹) Dans son beau Livre : *Temps, espace, matière*, traduit sur la quatrième édition allemande par Gustave Juvet et Robert Leroy; Paris, A. Blanchard, 1922.

(²) *C. R.*, t. 174, p. 437, 593, 734, 857, 1104.

★ ¹ ² The affine connection is not metric in general.

¹E. Cartan (1923). Les variétés à connexion affine. *Ann. Ec. Norm. Sup.* **40**, pp. 1–25.

²E. Cartan (1924). Les variétés à connexion affine (suite). *Ann. Ec. Norm. Sup.* **41**, pp. 325–412.

★ ^{1 2} The affine connection is not metric in general.

★ ³ Clocks and rules + compatible connection $\Rightarrow$ **Galilean geometry**.

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³A. N. Bernal, M. López and M. Sanchez (2002), Fundamental units of length and time. *Found. Phys.* **32**, 77–108

- Covariance (Trautman 1966, Kunzle 1972).
- Time dilation (Hansen et al. 2020).
- Gauge theory (Duval-Kunzle 1984).
- Curvature and matter content are dynamic: Poisson equation (Bernal-Sánchez 2003).

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-
- $1/c$ expansion of General Relativity (Friedrichs 1928, Dautcourt 1964).
 - Null reduction of Lorentzian spacetimes (Duval et al. 1985, Minguzzi 2006).

Galilean spacetimes

Let M^{n+1} a connected, Hausdorff and paracompact smooth manifold.

$\Omega \in \Lambda^1(M)$ non-vanishing one form, called **absolute clock**.

If $An\Omega := \{v \in TM : \Omega(v) = 0\}$, take a positive definite metric on the n-distribution $An\Omega$,

$$g: \Gamma(An\Omega) \times \Gamma(An\Omega) \rightarrow C^\infty(M).$$

g is called the **absolute metric** and $(An\Omega, g)$ the **absolute space**.

⁴A. N. Bernal, M. Sánchez (2003). Leibnizian, Galilean and Newtonian structures of space-time. *Journal of Mathematical Physics* **32**, 77-108.

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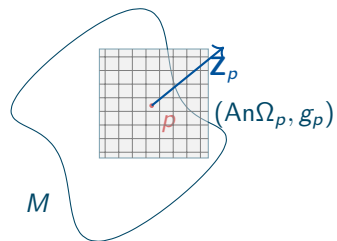
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The triad (M, Ω, g) is a **Leibnizian spacetime** .

⁴A. N. Bernal, M. Sánchez (2003). Leibnizian, Galilean and Newtonian structures of space-time. *Journal of Mathematical Physics* **32**, 77-108.



Given an **event** $p \in M$,

- $v \in T_p M$ is **timelike** (**spacelike**) if $\Omega_p(v) \neq 0$ ($= 0$),
- A timelike vector $v \in T_p M$ is **future-pointing** (**past-pointing**) if $\Omega_p(v) > 0$ (< 0).

$\gamma: I \subseteq \mathbb{R} \rightarrow M$ is an **observer** if $\Omega(\gamma') = 1$.

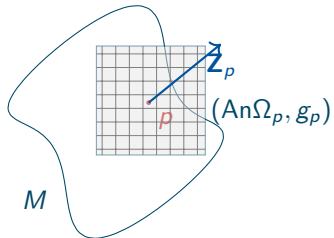
$Z \in \Gamma(TM)$ is a **field of observers** if $\Omega(Z) = 1$.

Given an event $p \in M$,

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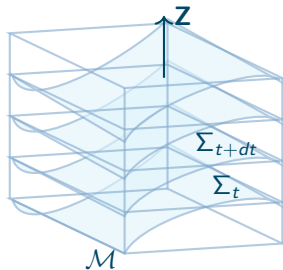
- **Locally synchronizability:** $\Omega \wedge d\Omega = 0$.

Locally, $\Omega = \lambda dT$ for smooth functions $\lambda > 0$ and T .

- **Proper time locally synchronizability:** $d\Omega = 0$.

Locally, $\Omega = dT$, for certain smooth function T .

$\Rightarrow$ Foliation in terms of **(absolute) restspaces** Σ .



Inertia principle must be included via an affine connection ∇ compatible with the Leibnizian structure,

- $\nabla\Omega = 0$, i.e., $X(\Omega(Y)) = \Omega(\nabla_X Y)$,
- $\nabla g = 0$, i.e., $X(g(V, W)) = g(\nabla_X V, W) + g(V, \nabla_X W)$,

for all $X, Y \in \Gamma(TM)$ and $V, W \in \Gamma(\text{An}\Omega)$.

A **Galilean spacetime** is a tuple (M, Ω, g, ∇) .

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A **Galilean spacetime** is a tuple (M, Ω, g, ∇) .

Absolute clock Ω and ∇ -torsion are related:

$$\Omega \circ \text{Tor} = d\Omega$$

Hence, ∇ torsion-free implies $d\Omega = 0$.

Let $Z \in \Gamma(TM)$ be a field of observers and consider the natural spacelike projection $P^Z X = X - \Omega(X)Z \in \Gamma(\text{An}\Omega)$, for every $X \in \Gamma(TM)$. Define

- $\mathcal{G}_Z = \nabla_Z Z \in \Gamma(\text{An}\Omega)$ **gravitational field**,
- $\omega_Z = \frac{1}{2} \text{Rot}(Z) \in \Gamma(\Lambda^2 \text{An}\Omega)$ **vorticity** or **Coriolis field**.

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For a Leibnizian structure (M, Ω, g) , torsionfree $\nabla \longmapsto (\mathcal{G}_Z, \omega_Z)$ is one-to-one and onto.

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For a Leibnizian structure (M, Ω, g) , torsionfree $\nabla \mapsto (\mathcal{G}_Z, \omega_Z)$ is one-to-one and onto.

In fact, for any Z , there exists a unique torsionfree Galilean connection ∇^Z such that

$$\mathcal{G}_Z^{\nabla^Z} = 0 \quad \text{and} \quad \omega_Z^{\nabla^Z} = 0.$$

Moreover, if $\hat{\omega}_Z$ is the $(1,1)$ -tensor g -associated to ω_Z , any torsionfree Galilean connection ∇ is given by

$$\nabla = \nabla^Z + \mathcal{G}_Z \otimes \Omega \otimes \Omega + \Omega \otimes (\hat{\omega}_Z \circ P^Z) + (\hat{\omega}_Z \circ P^Z) \otimes \Omega.$$

Let (M, Ω, g) be a Leibnizian spacetime and let $Z \in \Gamma(TM)$ be a field of observers. The unique torsion-free Galilean connection ∇ with given $\mathcal{G}_Z$ and ω_Z is given by

$$\nabla_X Y = P^Z (\nabla_X Y) + X(\Omega(Y))Z, \quad \forall X, Y \in \Gamma(TM),$$

via a **formula “à la Koszul”**:

$$\begin{aligned} 2g(P^Z(\nabla_X Y), V) = & X(g(P^Z Y, V)) + Y(g(P^Z X, V)) - V(g(P^Z X, P^Z Y)) \\ & + 2\Omega(X)\Omega(Y)g(\mathcal{G}_Z, V) + 2\Omega(X)\omega_Z(P^Z Y, V) + 2\Omega(Y)\omega_Z(P^Z X, V) \\ & + \Omega(X)(g([Z, P^Z Y], V) - g([Z, V], P^Z Y)) \\ & - \Omega(Y)(g([Z, P^Z X], V) + g([Z, V], P^Z X)) \\ & + g([P^Z X, P^Z Y], V) - g([P^Z Y, V], P^Z X) - g([P^Z X, V], P^Z Y). \end{aligned}$$

for all $V \in \Gamma(\text{An}(\Omega))$.

Conformally Leibnizian vector fields

Conformally Leibnizian vector fields

Infinitesimal symmetries

A vector field $K \in \Gamma(TM)$ is **conformally Leibnizian** if its flow conformally preserves the Leibnizian structure, i.e.,

$$L_K \Omega = \theta \Omega \quad \text{and} \quad L_K g = \rho g ,$$

for a **conformal scalar** $\rho \in C^\infty(M)$ and some $\theta \in C^\infty(M)$, where L_K is the Lie derivative with respect to K .

If $\theta = \rho = 0$, then K is **Leibnizian**.

⁵D. de la Fuente, R. M. Rubio, and J. Torrente-Teruel (2025). “Geometric structure of Galilean spacetimes admitting a conformally Leibnizian vector field”. *Phys. Scr.* **100**:9, p. 095001.

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If $\theta = \rho = 0$, then K is Leibnizian.

If also $L_K \nabla = 0$, K is **Galilean**.

⁵D. de la Fuente, J.A.S. Pelegrín and R. M. Rubio (2021). “On the geometry of stationary Galilean spacetimes”. *Gen. Relativ. Gravit.* **53**:8, pp. 1–15

⁶D. de la Fuente, R. M. Rubio, and J. Torrente-Teruel (2025). “Geometric structure of Galilean spacetimes admitting a conformally Leibnizian vector field”. *Phys. Scr.* **100**:9, p. 095001.

A vector field $K \in \Gamma(TM)$ is **torse-forming** if it satisfies

$$\nabla_X K = \rho X + \alpha(X)K, \quad \forall X \in \Gamma(TM),$$

for a so-called **conformal scalar** $\rho \in C^\infty(M)$ and **generating form** $\alpha \in \Lambda^1(M)$.

If $V(\rho) = 0$ for all $V \in \Gamma(\text{An}\Omega)$, K is called **spatially invariant**.

- If $\alpha(K) = 0$, K is called a torqued vector field.
- If $\alpha = 0$ everywhere, K is concircular.
- If $\alpha = 0$ everywhere and ρ is a non-zero constant, K is a concurrent.

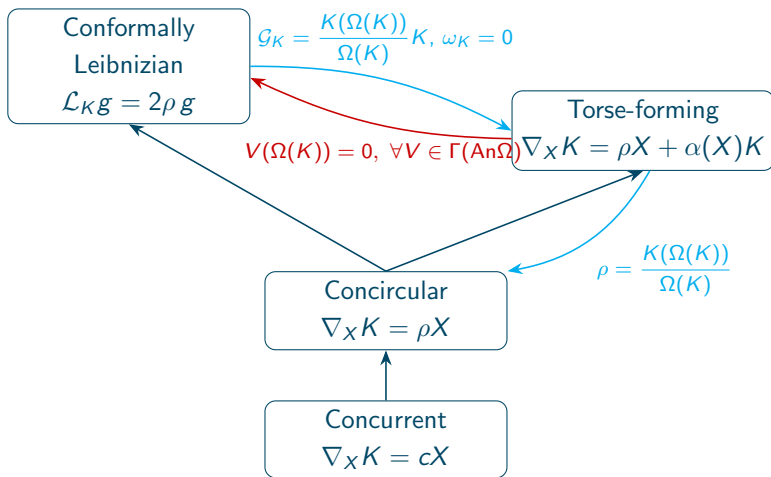
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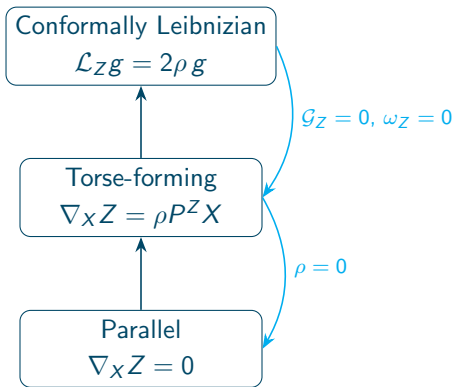
If $V(\rho) = 0$ for all $V \in \Gamma(\text{An}\Omega)$, K is called spatially invariant.

- If $\alpha(K) = 0$, K is called a **torqued** vector field.
- If $\alpha = 0$ everywhere, K is **concircular**.
- If $\alpha = 0$ everywhere and ρ is a non-zero constant, K is a **concurrent**.



$K \in \Gamma(TM)$ **timelike vector field.**

$X \in \Gamma(TM)$, $c \in \mathbb{R}$, $\alpha \in \Lambda^1(M)$ and $\rho \in C^\infty(M)$.



$Z \in \Gamma(TM)$ **field of observers.**

$X \in \Gamma(TM)$ and $\rho \in C^\infty(M)$.

Conformally Leibnizian vector fields

Galilean covering maps

A timelike vector field $K \in \Gamma(TM)$ is a *uniform global generator* if there exists an open interval $I \subset \mathbb{R}$ and a restspace Σ such that the restricted flow

$$\psi: I \times \Sigma \longrightarrow M$$

is well-defined and onto.

★ It holds **locally**!

A timelike vector field $K \in \Gamma(TM)$ is a *uniform global generator* if there exists an open interval $I \subset \mathbb{R}$ and a restspace Σ such that the restricted flow

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is well-defined and onto.

Theorem dFRT

Let (M, Ω, g, ∇) be a Galilean spacetime and let $Z \in \Gamma(TM)$ be a field of observers. If Z is a **uniform global generator** on some $I \times \Sigma$ with $I \subset \mathbb{R}$ an open interval and Σ a restspace, the restricted flow

$$\psi: (I \times \Sigma, dt, \psi^*g, \psi^*\nabla) \longrightarrow (M, \Omega, g, \nabla),$$

is a Galilean covering map. If Z is complete, then ψ is normal.

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is well-defined and onto.

Theorem dFRT

Let (M, dt, g, ∇) be a Galilean spacetime with **absolute time function** $T \in C^\infty(M)$ and let Z be a field of observers. If Z is a **uniform global generator** on some $I \times \Sigma$, $I \subset \mathbb{R}$ an open interval and Σ a restspace, then the restricted flow $\psi|_{I \times \Sigma}$ is a Galilean diffeomorphism.

Corollary

Let (M, Ω, g, ∇) be a Galilean spacetime and $Z \in \Gamma(TM)$ a **conformally Leibnizian** field of observers. If Z is a uniform global generator on some $I \times \Sigma$, with $I \subset \mathbb{R}$ an open interval and Σ a restspace, then the restricted flow

$$\Psi: (I \times \Sigma, dt, f^2 \pi_\Sigma^* g|_\Sigma, \tilde{\nabla}) \longrightarrow (M, \Omega, g, \nabla),$$

where the twisting function $f \in C^\infty(I \times \Sigma)$ is

$$f(t, q) := \exp \left(\int_0^t \bar{\rho}(\Psi(s, q)) ds \right), \quad (t, q) \in I \times \Sigma,$$

and the pullback connection $\tilde{\nabla} = \Psi^* \nabla$ is

$$\tilde{\nabla} = \nabla^{\partial_t} + \mathcal{G}_{\partial_t} \otimes dt \otimes dt + dt \otimes (\hat{\omega}_{\partial_t} \circ P^{\partial_t}) + (\hat{\omega}_{\partial_t} \circ P^{\partial_t}) \otimes dt,$$

respectively, is a Galilean covering map.

Corollary

Let (M, Ω, g, ∇) be a Galilean spacetime and $Z \in \Gamma(TM)$ a **torse-forming** field of observers. If Z is a uniform global generator on some $I \times \Sigma$, with $I \subset \mathbb{R}$ an open interval and Σ a restspace, then the restricted flow

$$\Psi: (I \times \Sigma, dt, f^2 \pi_{\Sigma}^* g|_{\Sigma}, \nabla^{\partial_t}) \longrightarrow (M, \Omega, g, \nabla),$$

is a Galilean covering map on a **Galilean Twisted (GT)** spacetime.

D. de la Fuente, R. M. Rubio and J. Torrente-Teruel (2025). On Twisted spacetimes: a new class of Galilean cosmological models. *Adv. Nonlinear Stud.*

Corollary

Let (M, Ω, g, ∇) be a Galilean spacetime and $Z \in \Gamma(TM)$ a **torse-forming** field of observers. If Z is a uniform global generator on some $I \times \Sigma$, with $I \subset \mathbb{R}$ an open interval and Σ a restspace, then the restricted flow

$$\Psi: (I \times \Sigma, dt, f^2 \pi_{\Sigma}^* g|_{\Sigma}, \nabla^{\partial_t}) \longrightarrow (M, \Omega, g, \nabla),$$

is a Galilean covering map on a **Galilean Twisted (GT)** spacetime.

If Z is **spatially invariant**, then Ψ can be defined on a **Galilean Generalized Robertson-Walker (GGRW)** spacetime.

If Z is complete, then Ψ is normal.

D. de la Fuente, R. M. Rubio and J. Torrente-Teruel (2025). On Twisted spacetimes: a new class of Galilean cosmological models. *Adv. Nonlinear Stud.*

D. de la Fuente and R.M. Rubio (2018). Galilean Generalized Robertson-Walker spacetimes: a new family of Galilean geometrical models, *J. Math. Phys.*, **59**, 022903 (1–10)

Proposition

Let (M, Ω, g, ∇) be a **spatially compact** Galilean spacetime and let $Z \in \Gamma(TM)$ be a field of observers. Then, Z is a uniform global generator.

Theorem dFRT

Let (M, Ω, g, ∇) be a spatially compact Galilean spacetime and let $Z \in \Gamma(TM)$ be a field of observers. Then, there exists an open interval $I \subset \mathbb{R}$ and a restspace Σ such that the restricted flow $\Psi|_{I \times \Sigma}$ is a Galilean covering map.

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Conformally Leibnizian vector fields

Uniqueness of splitting

Theorem dFRT

Let (M, Ω, g, ∇) be a Galilean spacetime admitting an **irrotational conformally Leibnizian (ICL)** field of observers $Z \in \Gamma(TM)$. Denote the conformal scalar by $\bar{\rho} \in C^\infty(M)$. If the set $\{p \in M : \bar{\rho}_p \text{ spatially invariant}\}$ is of measure zero in M , then the unique ICL field of observers with gravitational field $\mathcal{G}_Z$ is Z .

Theorem dFRT

Let (M, Ω, g, ∇) be a Galilean spacetime admitting an irrotational conformally Leibnizian (ICL) field of observers $Z \in \Gamma(TM)$. Denote the conformal scalar by $\bar{\rho} \in C^\infty(M)$. If the set $\{p \in M : \bar{\rho}_p \text{ spatially invariant}\}$ is of measure zero in M , then the unique ICL field of observers with gravitational field $\mathcal{G}_Z$ is Z .

Corollary

Let (M, Ω, g, ∇) be a Galilean spacetime admitting a **torse-forming** field of observers $Z \in \Gamma(TM)$. If the set $\{p \in M : \bar{\rho}_p \text{ spatially invariant}\}$ is of measure zero in M , then Z **unique torse-forming** field of observers.

Theorem dFRT

Let (M, Ω, g, ∇) be a Galilean spacetime admitting an irrotational conformally Leibnizian (ICL) field of observers $Z \in \Gamma(TM)$. Denote the conformal scalar by $\bar{\rho} \in C^\infty(M)$. If the set $\{p \in M : \bar{\rho}_p \text{ spatially invariant}\}$ is of measure zero in M , then the unique ICL field of observers with gravitational field $\mathcal{G}_Z$ is Z .

Corollary

Let (M, Ω, g, ∇) be a Galilean spacetime admitting a torse-forming field of observers $Z \in \Gamma(TM)$. If the set $\{p \in M : \bar{\rho}_p \text{ spatially invariant}\}$ is of measure zero in M , then Z is the unique torse-forming field of observers.

Consider the Twisted spacetime $(I \times \Sigma, dt, f^2 \pi_\Sigma^* g_\Sigma, \nabla^{\partial_t})$.

If $f \in C^\infty(I \times \Sigma)$ satisfies $df(V) \neq 0$ on a spacelike vector field V , then the coordinate field of observers ∂_t is the unique torse-forming.

(M, Ω, g, ∇) is **spatially complete** if every restspace is complete.

Theorem dFRT

Let $(M^{n+1}, \Omega, g, \nabla)$ be a spatially complete Galilean spacetime. Assume that there exist spatially invariant ICL fields of observers $Z_1, Z_2 \in \Gamma(TM)$ with $\mathcal{G}_{Z_2} = \mathcal{G}_{Z_1}$. If every restspace Σ in M is non-Euclidean and does not admit non-trivial parallel vector fields, then $Z_2 = Z_1$.

(M, Ω, g, ∇) is spatially complete if every restspace is complete.

Theorem dFRT

Let $(M^{n+1}, \Omega, g, \nabla)$ be a **spatially complete** Galilean spacetime. Assume that there exist **spatially invariant ICL** fields of observers $Z_1, Z_2 \in \Gamma(TM)$ with $\mathcal{G}_{Z_2} = \mathcal{G}_{Z_1}$. If every restspace Σ in M is non-Euclidean and does not admit non-trivial parallel vector fields, then $Z_2 = Z_1$.

Consider the GGRW spacetime $(\mathbb{R} \times \mathbb{S}^n, dt, (f \circ \pi_{\mathbb{R}})^2 \pi_{\mathbb{S}^n}^* g_{\mathbb{S}^n}, \nabla)$.

Then, the coordinate vector field ∂_t is the unique torse-forming field of observers.

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