Homogeneous Darboux and Frobenius theorems on graded manifolds

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The interest of gradings

There are several scenarios in geometry and physics in which a $(\mathbb{N}, \mathbb{Z}, \mathbb{Z}_2, \mathbb{R} \ldots)$ grading appears:

- the algebra of exterior forms with the exterior product $\left(\Omega^\bullet(M), \wedge\right)$,
- the spin of particles,
- intensive/extensive variables in thermodynamics,
- symplectisation/Poissonisation of contact/Jacobi manifolds,
- supermanifolds,
- higher tangent bundles.

$\mathbb{N}$-gradings in geometry and physics

Example (Vector bundles)

- Let $\pi\colon E \to M$ be a vector bundle with bundle coordinates,

$$\pi\colon (x^i, y^a) \mapsto (x^i)\,.$$

- We can declare $\deg(x^i) = 0$ and $\deg(y^a) = 1$.
- Thus, changes of bundle coordinates are changes of coordinates preserving this grading.

$\mathbb{N}$-gradings in geometry and physics

Example (Second-order tangent bundle)

- Let $\mathrm{T}^2 M = \mathrm{J}^2_0(\mathbb{R}, M)$ denote the bundle of second jets of smooth maps $(\mathbb{R}, 0) \to M$.
- Consider local coordinates (x^a) in M.
- Taylor expansions of curves on M:

$$x^a(t) = x^a(0) + \dot{x}^a(0)\, t + \ddot{x}^a(0)\, \frac{t^2}{2} + o(t^2)$$

  define coordinates $(x^a, \dot{x}^a, \ddot{x}^a)$ on $\mathrm{T}^2 M$.
- We can fix the grading $\deg(x^a) = 0$, $\deg(\dot{x}^a) = 1$ and $\deg(\ddot{x}^a) = 2$.
- In this way, changes of coordinates adapted to the jet structure are precisely those changes preserving the grading.

$\mathbb{N}$-gradings in geometry and physics

Example (Thermodynamic variables)

Relevant variables in thermodynamics are extensive or intensive, i.e. they are proportional to the size of the system or not. We can encode this information by defining

$$\deg(\text{intensive}) = 0\,, \quad \deg(\text{extensive}) = 1\,.$$

Example (Spin)

The spin of elementary particles has integer or half-integer values: $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ Of course, this defines the $\mathbb{N}$-grading

$$\deg = 2\,\text{spin}\ .$$

Gradings and homogeneity

Given a manifold M with some graded coordinates $\deg(x^i) = w_i \in \mathbb{N}$, we can encode the information about the grades onto the weight vector field

$$\nabla = w_i \cdot x^i \partial_{x^i} \,.$$

Example (Vector bundle)

In the case of a vector bundle $\pi\colon E \to M$, the weight vector field is the Euler vector field $\nabla_E \in \mathfrak{X}(E)$, i.e. the infinitesimal generator of homotheties on the fibers. In bundle coordinates (x^i, y^a),

$$\nabla_E = \sum_a y^a \partial_{y^a} \,.$$

Definition

A **homogeneity action** on a manifold M is a smooth action $h\colon \mathbb{R} \times M \to M$ of the monoid $(\mathbb{R}, \cdot)$ on M, namely,

$$h_t \circ h_s = h_{ts}\,, \quad h_1 = \mathrm{Id}_M\,, \quad \forall\, s, t \in \mathbb{R}\,.$$

Its associated **weight vector field** is given by

$$\nabla = \left.\frac{\partial}{\partial t}\right|_{t=1} h_t\,.$$

Definition

A function is called **homogeneous of degree** $w \in \mathbb{N}$ if

$$f \circ h_t = t^w \cdot f\,, \quad \forall\, t \in \mathbb{R}\,,$$

or equivalently, if $\nabla(f) = w \cdot f$.
A tensor field A on M is called **homogeneous of degree** $w \in \mathbb{N}$ if

$$\mathcal{L}_{\nabla} A = w \cdot A\,.$$

Theorem

Let $h\colon \mathbb{R}\times M \to M$ *be a homogeneity action on* M*. Then,* $h_0\colon M \to h_0(M)$ *is a fiber bundle with the typical fiber* $\mathbb{R}^k$*. For every* $x_0 \in h_0(M)$ *there exists a local trivialization* $h_0^{-1}(U) \simeq U \times \mathbb{R}^k$ *with coordinates* $(x^a; y^i)$ *such that*

$$\left(x^1, \dots x^n; y^1, \dots, y^k\right) \circ h_t = \left(x^1, \dots x^n; t^{w_1} \cdot y^1, \dots, t^{w_k} \cdot y^k\right) , \quad \forall\, t \in \mathbb{R} ,$$

for some positive integers w_i*. In these coordinates,*

$$\nabla = \sum_{i=1}^{k} w_i \cdot y^i \partial_{y^i} .$$

Definition

A distribution $D \subseteq \mathrm{T}M$ is **homogeneous** if

$$\mathrm{T}h_t(D) \subseteq D\,, \quad \forall\, t \in \mathbb{R}\,.$$

Equivalently, if

$$[\nabla, \Gamma(D)] \subseteq \Gamma(D)\,.$$

Theorem

A distribution $D \subseteq \mathrm{T}M$ of rank r is homogeneous if and only if it is locally generated by homogeneous vector fields $X_1, \ldots, X_r$ on a local trivialization $h_(0)^{-1}(U)$ of each point.

Corollary

A distribution $D \subseteq \mathrm{T}M$ of rank r is homogeneous and involutive if and only if it is locally generated by homogeneous vector fields $X_1, \ldots, X_k, Y_1, \ldots, Y_{r-k}$ with

$$\mathcal{L}_{\nabla} X_i = [\nabla, X_i] = 0\,, \quad \mathcal{L}_{\nabla} Y_a = [\nabla, Y_a] = w_a \cdot Y_a\,, \quad w_a \in \mathbb{Z}\,,$$

which pairwise commute:

$$[X_i, X_j] = [X_i, Y_a] = [Y_a, Y_b] = 0\,, \quad 1 \leq i, j \leq k\,, \quad 1 \leq a, b \leq r-k\,.$$

Proposition

Let $X_1, \ldots, X_k, Y_1, \ldots, Y_{r-k}$ be linearly independent, homogeneous and pairwise commuting. Then, around each point, there exists a system of homogeneous bundle coordinates (x^μ, y^ρ) straightening them:

$$X_i = \partial_{x^i}\,, \quad Y_a = \partial_{y^a}\,, \quad 1 \leq i,j \leq k\,, \quad 1 \leq a,b \leq r-k\,.$$

The proof is essentially the same one as the standard straightening theorem, bearing in mind the homogeneity of the coordinates and the vector fields considered.

Homogeneous Frobenius theorem

Combining the corollary and the proposition above, we obtain our main result:

Theorem (Grabowski and L. G., 2026)

Let $h \colon \mathbb{R} \times M \to M$ be a homogeneity action on M. A distribution $D \subseteq \mathrm{T}M$ of rank r is homogeneous and involutive if and only if, around each point, there exists a system of homogeneous bundle coordinates (x^μ, y^ρ),

$$x^\mu \circ h_t = x^\mu \,, \quad y^\rho \circ h_t = t^{w_\rho} \cdot y^\rho \,, \quad \forall\, t \in \mathbb{R} \,,$$

such that

$$D = \left\langle \partial_{x^1}, \ldots, \partial_{x^k}, \partial_{y^1}, \ldots, \partial_{y^{r-k}} \right\rangle \,.$$

Example

Given a homogeneous involutive distribution $D \subseteq \mathrm{T}E$ on a vector bundle $\pi\colon E \to M$, one can find vector bundle coordinates (x^a, y^i) such that

$$D = \left\langle \partial_{x^1}, \ldots, \partial_{x^k}, \partial_{y^1}, \ldots, \partial_{y^{r-k}} \right\rangle .$$

Remark

- The definitions and results above generalize naturally, *mutatis mutandis*, to supermanifolds.
- In particular, the Frobenius theorem for N-manifolds proven by Bursztyn, Cueca and Mehta[a] can be recovered as a corollary of our homogeneous Frobenius theorem.
- Our proof is more simple and direct.

[a] H. Bursztyn, M. Cueca and R. A. Mehta, "A geometric characterization of N-graded manifolds and the Frobenius theorem". J. Noncommut. Geom. (2026)

Moltes gràcies per la vostra atenció!

✉ Feel free to contact me at alopez-gordon@impan.pl

🌐 These slides are available at www.alopezgordon.xyz