GenDis: Transformations between Lagrangians of different order

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Outline

GenDis: a compass through Modified Gravity?

- **Modified Gravity**: extensions of General Relativity (GR) put forward to overcoming its observational and theoretical limitations. Several proposals are **higher order**.

- Disformal transformations and generalizations (henceforth **GenDis**) are emerging as a powerful tool to organize the space of beyond-GR proposals and to build new ones.

$$\tilde{g}_{\mu\nu} = \underbrace{C(\varphi, X)\, g_{\mu\nu}}_{\text{conformal rescaling}} + \underbrace{D(\varphi, X)\, \partial_\mu \varphi\, \partial_\nu \varphi}_{\text{disformal term}} \qquad X := -\frac{1}{2}\, g^{\mu\nu} \partial_\mu \varphi\, \partial_\nu \varphi.$$

[Bekenstein, 9211017]

- The formal challenges underlying GenDis motivate their study within **mechanics**.

A **generalized disformal transformation (GenDis)** is a triple (Q, L, ϕ) of a manifold Q, a first order Lagrangian $L : TQ \to \mathbb{R}$ and a smooth map $\phi : TQ \to Q$.

Formalization of GenDis in mechanics

A **generalized disformal transformation (GenDis)** is a triple (Q, L, ϕ) of a manifold Q, a first order Lagrangian $L : TQ \to \mathbb{R}$ and a smooth map $\phi : TQ \to Q$.

Example. Let $Q = \mathbb{R}$, $L = \frac{1}{2}\dot{q}^2$ and

$$\begin{aligned} \phi : T\mathbb{R} &\longrightarrow \mathbb{R} \\ (q, \dot{q}) &\longmapsto q + \dot{q} \end{aligned} \qquad \begin{aligned} \phi^{(1)} : T^{(2)}\mathbb{R} &\longrightarrow T\mathbb{R} \\ (q, \dot{q}, \ddot{q}) &\longmapsto (q + \dot{q}, \dot{q} + \ddot{q}) \end{aligned}$$

Applying ϕ to L, we get $L' = {\phi^{(1)}}^* L = \frac{1}{2}\left(\dot{q} + \ddot{q}\right)^2$.

Relating Lagrangians systems

- Seed Lagrangian $L : TQ \to \mathbb{R}$; its dynamics is encoded in (TQ, ω_L, E_L).
- Transformed Lagrangian $L' : T^{(2)}Q \to \mathbb{R}$; its dynamics is encoded in $\left(T^{(3)}Q, \omega_{L'}, E_{L'}\right)$.

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Let (M, ω, H), (M', ω', H') be (pre)symplectic dynamical systems. $f : M \to M'$ smooth s.t. $\omega = f^*\omega'$ and $H = f^*H'$ is a **morphism of (pre)symplectic dynamical systems**.
If f is also a diffeomorphism, then we call it an **isomorphism**.

Isomorphism of symplectic dynamical systems

$$\Downarrow$$

same trajectories, symmetries, stability, brackets.

GenDis in a commutative diagram

Standard framework

$$\begin{tikzcd} TQ \arrow[r, "Tf"'] \arrow[d, "\tau_Q"'] \arrow[rr, bend left, "L' = (Tf)^* L"] & TQ \arrow[r, "L"'] \arrow[d, "\tau_Q"] & \mathbb{R} \\ Q \arrow[r, "f"] & Q & \end{tikzcd}$$

GenDis in a commutative diagram

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\end{tikzcd}
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GenDis framework

$$
\begin{tikzcd}
T^{(3)}Q \arrow[d, "\rho^3_2"'] \arrow[dr, "\phi^{(2)}"] & & \\
T^{(2)}Q \arrow[d, "\rho^2_1"'] \arrow[dr, "\phi^{(1)}"] & T^{(2)}Q \arrow[d, "\rho^2_1"] & \\
TQ \arrow[d, "\tau_Q"'] \arrow[dr, "\phi"] & TQ \arrow[d, "\tau_Q"] \arrow[r, "L"'] & \mathbb{R} \\
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\end{tikzcd}
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TQ \arrow[d, "\tau_Q"'] \arrow[dr, "\phi"] & TQ \arrow[r, "L"'] \arrow[d, "\tau_Q"] & \mathbb{R} \\
Q & Q &
\end{tikzcd}
$$

$\Phi := \rho^2_1 \circ \phi^{(2)}$ is the **relating map**.

Disformal defect

Let $\delta L_j := \frac{\partial L}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^j}\right)$ be the Euler-Lagrange equations of the seed system. Then:

- Lagrangian Energy: $E_{L'} = \Phi^* E_L + \left(\boldsymbol{\phi^{(2)^*}\delta L_j}\right) \boldsymbol{\frac{\partial \phi^j}{\partial \dot{q}^i} \dot{q}^i}$;
- Lagrangian 2-form: $\omega_{L'} = \Phi^* \omega_L \boldsymbol{- d}\left[\left(\boldsymbol{\phi^{(2)^*}\delta L_j}\right) \boldsymbol{\frac{\partial \phi^j}{\partial \dot{q}^i} dq^i}\right]$.

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Let $\delta L = \delta L_j dq^j \in \Omega^1\left(T^{(2)}Q\right)$ be the Euler-Lagrange 1-form and

$$J^* = dq^i \otimes \frac{\partial}{\partial \dot q^i} + d\dot q^i \otimes \frac{\partial}{\partial \ddot q^i} + d\ddot q^i \otimes \frac{\partial}{\partial \dddot q^i} \in \operatorname{End}\left(T^*(T^{(3)}Q)\right).$$

The **disformal defect** is the semibasic form $\kappa = J^*\left({\phi^{(2)}}^*\delta L\right) \in \Omega^1\left(T^{(3)}Q\right)$.

$$E_{L'} = \Phi^* E_L + \boldsymbol{\widehat{\kappa}}, \qquad \omega_{L'} = \Phi^*\omega_L - \boldsymbol{d\kappa}.$$

Relating dynamics: the emergence of extra solutions

- Equation for dynamical vector field:

$$0 = \iota_X \omega_{L'} - dE_{L'} = \Phi^* \left(\iota_{\Phi_* X} \omega_L - dE_L\right) - \boldsymbol{\mathcal{L}_X \kappa}.$$

- Euler Lagrange equations:

$$0 = \frac{d}{dt}\left[\left({\phi^{(2)}}^* \delta L_j\right) \frac{\partial \phi^j}{\partial \dot{q}^i}\right] - \left({\phi^{(2)}}^* \delta L_j\right) \frac{\partial \phi^j}{\partial q^i}$$

Dynamics is not preserved: solutions of the seed system pullback to solutions, but in general there are more solutions.

A no-equivalence theorem for regular GenDis

Recall: $E_{L'} = \Phi^* E_L + \widehat{\boldsymbol{\kappa}}, \qquad \omega_{L'} = \Phi^* \omega_L - \boldsymbol{d\kappa}.$

Theorem. Let (Q, L, ϕ) be a GenDis such that $L' = {\phi^{(1)}}^* L$ is regular. Then, the relating map Φ is **NOT** a morphism of the presymplectic dynamical systems $(T^{(3)}Q, \omega_{L'}, E_{L'})$ and (TQ, ω_L, E_L). Moreover, there always exist solutions of the transformed system that do not project to solutions of the seed.

- For a notion of equivalence to exist, we need a **singular transformed system**.
- Let $i_{Q_f} : Q_f \hookrightarrow T^{(3)}Q$ be the inclusion of the final constraint submanifold into $T^{(3)}Q$. For equivalence, we need $i_{Q_f}^* \widehat{\kappa} = 0$ and $i_{Q_f}^* d\kappa = 0$.

Recovering dynamics and geometry

Intuitive definition. A GenDis is a **dynamical relation** if:

- every seed solution has a transformed counterpart which is a solution;
- every solution of the transformed system projects to a solution of the seed.

If the map between solutions is one-to-one, the GenDis is a **dynamical equivalence**.

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- every solution of the transformed system projects to a solution of the seed.

If the map between solutions is one-to-one, the GenDis is a **dynamical equivalence**.

Generally, a correspondence between solutions does not lead to equivalence. However:

> ***Theorem***. If (Q, L, ϕ) is a dynamical relation, then $i_{Q_f}^* \kappa = 0$. Moreover, $\Phi \circ i_{Q_f}$ is a morphism of the presymplectic dynamical systems $(Q_f, i_{Q_f}^* \omega_{L'}, i_{Q_f}^* E_{L'})$ and (TQ, ω_L, E_L). If (Q, L, ϕ) is a dynamical equivalence and L is regular, then $\Phi \circ i_{Q_f}$ is an isomorphism of symplectic dynamical system.

Criteria for dynamical relation and equivalence

Theorem. Let (Q, L, ϕ) be a GenDis such that:

- $\phi^{(1)}$ and $\phi^{(2)}$ are surjective.
- Around every $y \in TQ$ there exist coordinates in which the matrices $A = \dfrac{\partial \phi^i}{\partial q^j}$ and $N = A^{-1}B = \left(\dfrac{\partial \phi^i}{\partial q^k}\right)^{-1} \dfrac{\partial \phi^k}{\partial \dot{q}^j}$ are respectively regular and strictly lower triangular.

Then, (Q, L, ϕ) is a dynamical relation.

If moreover A is lower triangular,

$$L = \frac{1}{2}\dot{x}^i \eta_{ij} \dot{x}^j - V(x), \qquad V(x) = \sum_{k=1}^{n} V^{(k)}(x^k), \qquad \eta_{ij} = \delta_{ij} m_{(i)},$$

and $\Phi \circ i_{Q_f}$ is a bijective submersion, then (Q, L, ϕ) is a dynamical equivalence.

Conclusions & outlook

Summary

- We have put forward a framework to study **GenDis**-related Lagrangians in mechanics.
- GenDis distort the geometry of the seed theories and enlarge their solution space. The **disformal defect** causes these features.
- Notions of **equivalence** can be retained through the constraint structure of the transformed system.

Conclusions & outlook

Summary

- We have put forward a framework to study **GenDis**-related Lagrangians in mechanics.
- GenDis distort the geometry of the seed theories and enlarge their solution space. The **disformal defect** causes these features.
- Notions of **equivalence** can be retained through the constraint structure of the transformed system.

Future work

- Characterization of additional dynamics arising in the general case.
- Extend the GenDis framework beyond second order Lagrangians:

$$\phi : T^{(k)}Q \to T^{(r)}Q, \qquad 0 \leq r \leq k.$$

- Generalizing the formalism to classical field theory.

Thank you very much for your attention!

Example 1

Seed Lagrangian: $L = \frac{1}{2}\dot q_1^2 - \frac{1}{2}q_1^2 + \frac{1}{2}\dot q_2^2 \in C^\infty\left(T\mathbb{R}^2\right)$

$$\mathrm{EoM}_L : \begin{cases} \ddot q_1 + q_1 = 0 \\ \ddot q_2 = 0 \end{cases} \qquad E_L = \frac{1}{2}\dot q_1^2 + \frac{1}{2}q_1^2 + \frac{1}{2}\dot q_2^2 \qquad \omega_L = dq_1 \wedge d\dot q_1 + dq_2 \wedge d\dot q_2$$

$\phi\left(q_1, q_2, \dot q_1, \dot q_2\right) = \left(\dot q_1, \dot q_2\right)$ yields $L' := \phi^{(1)*}L = \frac{1}{2}\ddot q_1^2 + \frac{1}{2}\ddot q_2^2 - \frac{1}{2}\dot q_1^2 \in C^\infty\left(T^{(2)}\mathbb{R}^2\right)$

$$\phi^*\mathrm{EoM}_L : \begin{cases} \dddot q_1 + \dot q_1 = 0 \\ \dddot q_2 = 0 \end{cases} \qquad \mathrm{EoM}_{L'} : \begin{cases} q_1^{(4)} + \ddot q_1 = 0 \\ q_2^{(4)} = 0 \end{cases}$$

Solutions pullback to solutions but **new solutions appear**

$$\Phi^* E_L = \frac{1}{2}\ddot q_1^2 + \frac{1}{2}\dot q_1^2 + \frac{1}{2}\ddot q_2^2 \qquad E_{L'} = \Phi^* E_L - \dot q_1\left(\dddot q_1 + \dot q_1\right) - \dot q_2 \dddot q_2$$

$$\Phi^*\omega_L = d\dot q_1 \wedge d\ddot q_1 + d\dot q_2 \wedge d\ddot q_2 \qquad \omega_{L'} = \Phi^*\omega_L - dq_1 \wedge d\left(\dddot q_1 + \dot q_1\right) - dq_2 \wedge d\dddot q_2$$

Example 2

Seed $L = \frac{1}{2}\dot{q}_1^2 - \frac{1}{2}q_1^2 + \frac{1}{2}\dot{q}_2^2 \in C^\infty\left(T\mathbb{R}^2\right)$ and GenDis $\phi\left(q_1, q_2, \dot{q}_1, \dot{q}_2\right) = \left(q_1, q_2 + \dot{q}_1\right)$ yield

$$L' := \phi^{(1)^*}L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\left(\dot{q}_2 + \ddot{q}_1\right)^2 - \frac{1}{2}q_1^2 \in C^\infty\left(T^{(2)}\mathbb{R}^2\right) \qquad \text{Singular Lagrangian}$$

$$\phi^*\mathrm{EoM}_L : \begin{cases} \ddot{q}_1 + q_1 = 0 \\ \ddot{q}_2 + \dddot{q}_1 = 0 \end{cases}$$

$\phi^*\mathrm{EoM}_L$ are constraints, **no new solutions** appear

$$E_{L'} = \Phi^* E_L - \dot{q}_1\left(\ddot{q}_2 + \dddot{q}_1\right), \qquad \omega_{L'} = \Phi^*\omega_L - dq_1 \wedge d\left(\ddot{q}_2 + \dddot{q}_1\right).$$

Let $i_{Q_f} : Q_f \to T^{(3)}\mathbb{R}^2$ be the inclusion of final constraint submanifold into $T^{(3)}\mathbb{R}^2$.

$\Phi \circ i_{Q_f}$ is a Hamiltonian morphism between $\left(Q_f, i_{Q_f}^*\omega_{L'}, i_{Q_f}^* E_{L'}\right)$ and (TQ, ω_L, E_L).

Two faces of the GenDis framework

Seed $L = \frac{1}{2}\dot{q}_1^2 - \frac{1}{2}q_1^2 + \frac{1}{2}\dot{q}_2^2 \in C^\infty\left(T\mathbb{R}^2\right)$

	Example 1	**Example 2**
GenDis	$(q_1, q_2, \dot{q}_1, \dot{q}_2) \mapsto (\dot{q}_1, \dot{q}_2)$	$(q_1, q_2, \dot{q}_1, \dot{q}_2) \mapsto (q_1, q_2 + \dot{q}_1)$
Regularity	Yes	No
Dynamics	Pullback solutions are solutions, but new dynamics appears	Pullback solutions are the only solutions, no new dynamics appears
Lagrangian Energy & Lagrangian 2-form	NOT pullback of seed counterparts	Pullback of seed ones on final constraint submanifold
Diffeomorphism seed - transformed systems	No	Yes, at the level of final constraint submanifolds

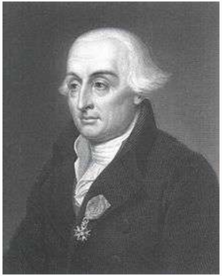
Presymplectic formulation of first and second order Lagrangian systems

- **First order Lagrangian**: $L = L(q, \dot{q}) : TQ \to \mathbb{R}$.

  Associated presymplectic dynamical system: (TQ, ω_L, E_L), where

$$E_L = \dot{q}^i \widehat{p}_i - L, \qquad \omega_L = dq^i \wedge d\widehat{p}_i.$$

  Legendre transformation: $\widehat{p}_i = \dfrac{\partial L}{\partial \dot{q}^i}$.



Giuseppe Luigi Lagrangia

(1736-1813)

- **Second order Lagrangian**: $L = L(q, \dot{q}, \ddot{q}) : T^{(2)}Q \to \mathbb{R}$.

  Associated presymplectic dynamical system: $(T^{(3)}Q, \omega_L, E_L)$, where

$$E_L = \dot{q}^i \widehat{p}_i + \ddot{q}^i \widehat{\Pi}_i - L, \qquad \omega_L = dq^i \wedge d\widehat{p}_i + d\dot{q}^i \wedge d\widehat{\Pi}_i.$$

  Legendre transformation: $\widehat{p}_i = \dfrac{\partial L}{\partial \dot{q}^i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \ddot{q}^i}, \qquad \widehat{\Pi}_i = \dfrac{\partial L}{\partial \ddot{q}^i}$.

Higher order tangent bundles

Definition. Let Q be a manifold, its **k-th order tangent bundle** $T^{(k)}Q$ is the space of equivalence classes of curves $\gamma : \mathbb{R} \to Q$ up to order k, i.e. for any chart (U, φ) on Q

$$\gamma_1 \sim \gamma_2 \iff \partial_t^r(\varphi \circ \gamma_1)(0) = \partial_t^r(\varphi \circ \gamma_2)(0), \quad r = 0, \ldots, k.$$

- Coordinates $\left(q^i\right)$ on $U \subset Q$ induce coordinates on $T^{(k)}U \subset T^{(k)}Q$:

$$\left(q^i, \dot{q}^i, \ddot{q}^i, \ldots, q^{(k)i}\right), \quad q^{(r)i}([\gamma]) := \frac{d^r}{dt^r}\left(q^i \circ \gamma\right)(0), \quad r = 0, \ldots, k.$$

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- There are natural projections onto tangent bundles of lower order

$$\begin{aligned} \rho^k_r : T^{(k)}U &\longrightarrow T^{(r)}U \qquad (0 \leq r \leq k) \\ \left(q^i, \dots, q^{(k)i}\right) &\longmapsto \left(q^i, \dots, q^{(r)i}\right) \end{aligned}$$

For instance: $\rho^3_2\left(q^i, \dot{q}^i, \ddot{q}^i, \dddot{q}^i\right) = \left(q^i, \dot{q}^i, \ddot{q}^i\right)$.

$$\begin{tikzcd} T^{(3)}Q \arrow[d, "\rho^3_2"'] \\ T^{(2)}Q \arrow[d, "\rho^2_1"'] \\ TQ \arrow[d, "\rho^1_0 = \tau_Q"'] \\ Q \end{tikzcd}$$

Lifts of maps

> **Definition**. Let $f : Q \to P$ be a smooth map, $k \in \mathbb{N}$, its **k-th tangent lift** is the map:
>
> $$T^{(k)}f : T^{(k)}Q \longrightarrow T^{(k)}P, \quad [\gamma] \longmapsto [f \circ \gamma]\,.$$

Choosing coordinates (q^i) and (Q^a) on Q and P, we get:

$$T^{(k)}f\left(q^i, \dot{q}^i, \ddot{q}^i, \ldots, q^{(k)i}\right) = \left(Q^i, \dot{Q}^i, \ddot{Q}^i, \ldots Q^{(k)i}\right),$$

$$Q^{(r)i} = \frac{d^r}{dt^r}\bigg|_{t=0} \left(f^i(q(t))\right), \quad r = 0, ..., k.$$

E.g. $\quad Q^i = f^i(q), \quad \dot{Q}^i = \dfrac{\partial f^i}{\partial q^j}\dot{q}^j, \quad \ddot{Q}^i = \dfrac{\partial f^i}{\partial q^j}\ddot{q}^j + \dfrac{\partial^2 f^i}{\partial q^j \partial q^k}\dot{q}^j\dot{q}^k, \ \ldots$

$$\begin{tikzcd}
T^{(3)}Q \arrow[r, "T^{(3)}f"] \arrow[d, "\rho^3_2"'] & T^{(3)}P \arrow[d, "\rho^3_2"] \\
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TQ \arrow[r, "T^{(1)}f=Tf"] \arrow[d, "\tau_Q"'] & TP \arrow[d, "\tau_P"] \\
Q \arrow[r, "f"] & P
\end{tikzcd}$$

Proposal of definition and holonomic lifts

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$$j_{k+1}\left(q^i, \dots, q^{(k+1)i}\right) = \left(\left(q^i, \dot{q}^i\right), \dots, \left(q^{(k)i}, q^{(k+1)i}\right)\right)$$

$$\phi^{(k)}\left(q^i, \dot{q}^i, \ddot{q}^i, \dots, q^{(k+1)i}\right) = \left(Q^i, \dot{Q}^i, \ddot{Q}^i, \dots Q^{(k)i}\right),$$

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E.g. $\quad Q^i = \phi^i\left(q, \dot{q}\right), \quad \dot{Q}^i = \dfrac{\partial \phi^i}{\partial q^j}\dot{q}^j + \dfrac{\partial \phi^i}{\partial \dot{q}^j}\ddot{q}^j, \ \dots$

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