Deformation of Lagrangian submanifolds and numerical analysis

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Flows, integrators and deformations

Let M be a smooth manifold and $X \in \mathcal{X}(M)$.

Definition (An integrator)

Let $I \subset \mathbb{R}$ be an interval containing 0. An integrator of order $k \in \mathbb{N}$ for X is a smooth family $\varphi = (\varphi_\epsilon)_{\epsilon \in I}$ of diffeomorphisms of M such that

- $\varphi_0 = \mathrm{Id}$,
- for any $x \in M$, $\epsilon \in I \mapsto \varphi_\epsilon(x) \in M$ is smooth,
- $\varphi_\epsilon - \phi_\epsilon^X = o(\epsilon^k)$.

Рис.: Leonhard Euler
fig.:Wikipedia

The method of approximation itself, which Geometers are accustomed to use, is bound up with a great many difficulties besides, and an unlimited multitude of small perturbations is neglected, by which fact it becomes so that this approximation by itself [only] minimally carries through the business of determining the motions, but on the contrary, for it to be completed, still more supports are desired.[a]

[a] Considerationes de motu corporum coelestium, 1766
Translated and Annotated by Sylvio R. Bistafa, arXiv: 2104.13470

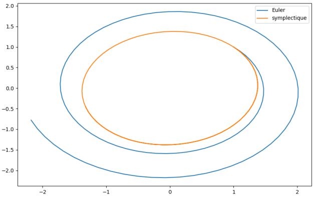
Hamiltonian flows, symplectic integrators and deformation of Lagrangian submanifolds

(M, ω) symplectic, $H \in \mathcal{C}^\infty(M)$, $X_H = \omega(dH, \cdot) \in \mathcal{X}(M)$ and Φ_ϵ^H its flow.

Definition (Symplectic integrator)

A symplectic integrator $\varphi = (\varphi_\epsilon)_{\epsilon \in I}$ of order $k \in \mathbb{N}$ for H is an integrator of order k for X_H such that for all $\epsilon \in I$, φ_ϵ is a symplectomorphism.

Example (Symplectic Euler method, DeVogelaere, 1956)



$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} q_n + \Delta t \frac{\partial H}{\partial p}(q_n, p_{n+1}) \\ p_n - \Delta t \frac{\partial H}{\partial q}(q_n, p_{n+1}) \end{pmatrix}$$

$H(q,p) = \frac{p^2+q^2}{2}$,
$\Delta t = 0.1$, $n = 100$ itérations
Quasi-periodic orbits

Theorem (Quasi-first integral, Benettin et Giorgilli, 1994)

Under some assumptions, $\exists (H_\epsilon)_\epsilon$, $C > 0$, $\epsilon^* > 0$ *s.t.*

$$\forall n \in \mathbb{N}, \forall \epsilon < \epsilon^*, \ |H_\epsilon(\varphi_\epsilon^n(q,p)) - H_\epsilon(q,p)| \leq Cn\epsilon \exp\left(-\frac{\epsilon^*}{\epsilon}\right) \qquad (1)$$

Variational principles and generating functions

Definition (The action)

$$S_\epsilon : \begin{array}{ccc} Q \times Q & \to & \mathbb{R} \\ (q_0, q_1) & \mapsto & \int_0^\epsilon L(q(s), \dot{q}(s)) \mathrm{d}s \end{array} \quad (2)$$

where $s \in [0, \epsilon] \mapsto q(s)$ the extremal path s.t. $\left\{ \begin{array}{ll} q(0) & = q_0, \\ q(t) & = q_1 \end{array} \right.$.

$$\begin{array}{ccccc} \mathrm{Graph}(dS_\epsilon) & \mapsto & & \mathrm{Graph}(\phi_\epsilon^H) & \\ \cap & & & \cap & \\ T^*(Q \times Q) & \stackrel{\sim}{\to} & T^*Q & \times & T^*Q \\ (q_0, q_1, \frac{\partial S_\epsilon}{\partial q_0}(q_0, q_1), \frac{\partial S_\epsilon}{\partial q_1}(q_0, q_1)) & \mapsto & (q_0, p_0), & & \phi_\epsilon^H(q_0, p_0) \end{array}$$

Variational integrators: discretize $\int_0^\epsilon L(q(s), \dot{q}(s)) \mathrm{d}s \approx \int_0^\epsilon L(q_0, \frac{q_1 - q_0}{\epsilon}) \mathrm{d}s$

Remark (Correspondence of Lagrangian spaces)

Variational integrators have similar stability theory than symplectic integrators: long run estimates (1) *etc...*

Algebra for the Deformation Problem of Lagrangians

Category-like structure (Weinstein...):

- *Objects* = {Symplectic manifolds}
- $Hom(M_1, M_2) = \{L \subset M_1 \times \overline{M_2}\}$
- $Id_M = \Delta_M \subset M \times \overline{M}$
- Composition:

$$L_{12} \circ L_{23} = \left\{ (m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2, \begin{array}{ccc} (m_1, m_2) & \in & L_{12} \\ (m_2, m_3) & \in & L_{23} \end{array} \right\}$$

Proposition (**Associativity**)

For all $L_{12} \in Hom(M_1, M_2)$, $L_{23} \in Hom(M_2, M_3)$, $L_{34} \in Hom(M_3, M_4)$*:*

$$(L_{12} \circ L_{23}) \circ L_{34} = L_{12} \circ (L_{23} \circ L_{34})$$

Open Questions:

- Operadic structure: Formal Lagrangian operad, Cattaneo, Dherin, Felder
- DGLA governing deformations of Lagrangian submanifolds and relation to Kontsevich formality: Ongoing with D. Calaque

Composition of Lagrangian spaces and multi-step numerical methods for non-linear dynamics

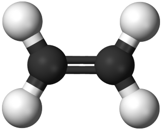
Hamiltonian flows	$\phi^{H_3} = \phi^{H_1} \circ \phi^{H_2}$
Trajectories	Concatenation of paths
Generating functions	$S_3(q_0, q_2) = \operatorname{extr}_{q_1}\big(S_1(q_0, q_1) + S_2(q_1, q_2)\big)$

Composition of Lagrangian spaces in mechanics

$\Rightarrow$ **This provides a geometric framework towards:**

Symplectic multi-step methods[a]:

- vibration of molecules
- coupled Duffing oscillators
- vibration of coupled beams



Ethylene

fig.: Wikipedia

Airplane blade

fig.: craiyon.com

[a]

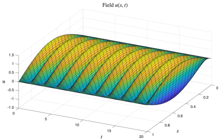
- Geometric Numerical Integration, Hairer, Lubich, Wanner
- Simulating Hamiltonian Dynamics, Leimkuhler, Reich

To be continued

Numerical study:

- Energy exchanges between vibration modes in Timoshenko model[a]

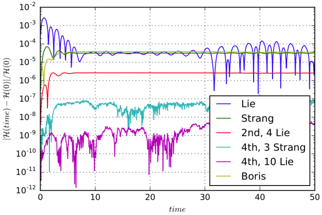
[a] Ongoing with A. Patel, V. Carlier, L. Le Marrec



Wave dynamics

fig: T. Wang

Poisson integrators in plasma physics



Landau damping: energy error[a]

[a] GEMPIC, Sonnendrucker et al., 2017

Variational integrators for field theories with S. Leyendecker, D. M. de Diego, R. Sato, T. Wang :

- ongoing: boundary Lagrangians and application to field theories

  Generating functionals and Lagrangian partial differential equations, Vankerschaver, Liao, Leok, JMP, 2013