Reduction and dynamics in the curved n-body problem

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Recall the classical n-body problem in $\mathbb{R}^d$ $(d = 1, 2, 3)$

$q = (q_1, q_2, \dots q_n) \in Q = \mathbb{R}^d \setminus \text{collisions}, \qquad p = (p_1, p_2, \dots p_n) \in T^*Q$

$$\ddot{q}_i = -\sum_{j \neq i} \frac{m_j(q_j - q_i)}{|q_i - q_j|^3}$$

$$H(q, p) = \text{Kinetic} + \text{potential} = \frac{1}{2} p^T M\, p - \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|q_j - q_i|}$$

Famous solutions: three-body Lagrange triangles and Euler collinear configurations, figure-eight

A (natural?) question: If we curve the ambient space, do these solutions survive?

This talk concerns the continuation of n-body solutions from the plane to sphere. (The hyperbolic case $\rightarrow$ another time.)

The potential

– on $\mathbb{R}^2$: $d_E =$ Euclidean distance

$$\mathcal{U}(d_E) = -\frac{1}{d_E}$$

– on $\mathbb{S}^2$: $d =$ geodesic distance

$$\mathcal{U}(\mathrm{d}) = -\cot(d)$$

NOTE: The cotangent potential is the unique extension that is harmonic on the sphere. For a sphere $\mathbb{S}_R$ of radius R.

$$\frac{1}{R}\cot\left(\frac{d}{R}\right) \longrightarrow \frac{1}{d_E} \text{ as } R \longrightarrow \infty$$

History — the nineteenth century

- **1830s.** Bolyai and Lobachevsky, independently, extend gravitation to $\mathbb{H}^3$: force inversely proportional to the area of the sphere of radius r. Neither writes down an analytic expression.

- **1860.** P. J. Serret carries the force to $\mathbb{S}^2$ and solves the Kepler problem there.

- **1870.** Schering supplies the analytic form — the *cotangent potential*. (He reports that Dirichlet had discussed the problem privately; no notes.)

- **1873.** Lipschitz tries $1/\sin(r/R)$ on $\mathbb{S}^3$; solvable only in elliptic functions.

- **1885.** Killing extends Schering's law to $\mathbb{S}^3$: potential $\propto \cot r$.

- **1902–03.** Liebmann proves the Bertrand analogue — bounded orbits close — singling out cot among all candidates.

Interest faded after the advent of general relativity.

History — the modern revival

- **1992.** Kozlov & Harin, *Kepler's problem in constant curvature spaces* — the Bertrand analogue revisited.
- **1999–2006.** Chernoivan & Mamaev; Borisov, Mamaev & Kilin; Cariñena, Rañada & Santander: two-body problem on $\mathbb{S}^2$ (reduction, stochasticity, periodic orbits).
- **2012.** Diacu, Pérez-Chavela & Santoprete, *The n-body problem in spaces of constant curvature I, II*; Diacu, *Relative equilibria of the curved N-body problem*.
- **2013–14.** Martínez & Simó, stability of the Lagrangian homographic solutions on $\mathbb{S}^2$; Diacu & Popa: all Lagrangian relative equilibria have equal masses.
- **2018.** Borisov, García-Naranjo, Mamaev & Montaldi: reduction and relative equilibria for the two-body problem, in the geometric-mechanics language.
- **2020.** Bengochea, García-Azpeitia, Pérez-Chavela & Roldán: take $SO(2)$ as the symmetry group and prove that any non-degenerate planar relative equilibrium continues to small curvature of either sign.

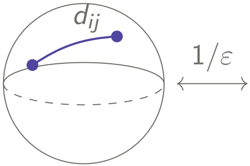
Setup on a sphere of radius $R = 1/\varepsilon$

$$\mathbb{S}^2_{1/\varepsilon} \subset \mathbb{R}^3 \text{ with } \langle q, q\rangle = 1/\varepsilon^2\,, \qquad Q^\varepsilon = \left(\mathbb{S}^2_{1/\varepsilon}\right)^n \setminus \text{collisions}\,, \qquad T^*Q^\varepsilon.$$

Geodesic distance d_{ij}; the cotangent potential

$$\mathcal{U}_\varepsilon = -\sum_{i<j} m_i m_j\, \varepsilon \cot\!\big(\varepsilon\, d_{ij}\big), \qquad \varepsilon \cot(\varepsilon d) \xrightarrow[\varepsilon\to 0]{} \frac{1}{d}.$$

Symmetry group $G_\varepsilon = SO(3)$, acting diagonally.

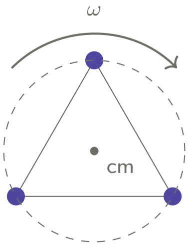


As $\varepsilon \to 0$:
the sphere flattens and $Q^\varepsilon \longrightarrow (\mathbb{R}^2)^n \setminus \Delta$

An example: the Lagrange triangle for equal masses

Planar case. Three masses at the vertices of an *equilateral* triangle, rotating rigidly about the common centre of mass. For equal masses m and side s:

$$\omega^2 = \frac{3m}{r^3} \ \ (r = \text{side}), \qquad L = m\sqrt{3m}\, r^{1/2}, \qquad \mathrm{P}_{\mathrm{tot}} = 0.$$



It is a relative equilibrium:
$z(t) = \exp(t\,\omega b_3) \cdot z(0)$

Momentum $\boldsymbol{\mu}^0 = (0,0,L) \in \mathfrak{se}(2)^*$, $L = mr^2\omega \neq 0$

On the sphere. Diacu–Pérez-Chavela–Santoprete (2012): on $\mathbb{S}^2$ a rigidly rotating equilateral triangle forces *equal masses*.

A remark of Diacu's — and this talk

Diacu observed that something structural happens at $\varepsilon = 0$:

> For $\varepsilon \neq 0$ the system has the integrals of *angular* momentum. At $\varepsilon = 0$ some of them survive as angular momentum and the rest become *linear* momentum.
>
> "So an interesting kind of bifurcation occurs in this case as we pass through $\varepsilon = 0$."

This is due to the Inönü–Wigner contraction of the symmetry algebra,

$$\mathfrak{so}(3) \;\longrightarrow\; \mathfrak{se}(2) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

This talk: the contraction w.r.t. *symplectic reduction*, and the continuation of relative equilibria (RE) and relative periodic orbits (RPOs) across it.

Plan

1. **The obstruction.** $\mathfrak{so}(3) \to \mathfrak{se}(2)$ with a jump in coadjoint isotropy; reduced spaces change dimension.
2. **How to deal with this.** Reduce by a subgroup that is present for *all* ε. Pay for it with a pre-symplectic reduced space and find that for our purpose it is harmless.
3. **The charts and the Newtonian expansion.** $H_\varepsilon = H_0 + \varepsilon^2 H_2 + O(\varepsilon^4)$, explicitly. This is what makes the application computable.
4. **Relative equilibria (RE).** A continuation theorem on each isotropy stratum (and how it aligns with what is already known).
5. **Main result** The continuation of the drifting Lagrange triangle (homographic solution), an RPO that is not isolated (Implicit Function Theorem does not work directly). Method: employ Lyapunov–Schmidt reduction.
6. **Geometry of the drift on $\mathbb{S}^2$.** Seeing the answer as a picture.

The Inönü–Wigner contraction

Basis of $\mathfrak{g}_\varepsilon$ adapted to the flat limit: $\mathrm{b}_1, \mathrm{b}_2$ (translations in the limit), b_3 (rotation).

$$[\mathrm{b}_1, \mathrm{b}_2]_\varepsilon = \varepsilon^2\, \mathrm{b}_3, \qquad [\mathrm{b}_2, \mathrm{b}_3]_\varepsilon = \mathrm{b}_1, \qquad [\mathrm{b}_3, \mathrm{b}_1]_\varepsilon = \mathrm{b}_2.$$

For each $\varepsilon > 0$, the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_\varepsilon)$ is isomorphic to $\mathfrak{so}(3)$ and at $\varepsilon = 0$ the $(\mathfrak{g}, [\cdot, \cdot]_\varepsilon)$ is isomorphic to $\mathfrak{se}(2)$.

$$\boxed{\mathfrak{so}(3),\ \ \varepsilon > 0} \xrightarrow{\ \varepsilon \to 0\ } \boxed{\mathfrak{se}(2),\ \ \varepsilon = 0}$$

compact — non-compact

At $\varepsilon = 0$ the bracket $[\mathrm{b}_1, \mathrm{b}_2]$ *dies*: the two “rotations” become commuting translations.

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$$\begin{array}{ccc} \boxed{\mathfrak{so}(3),\ \ \varepsilon > 0} & \xrightarrow{\ \varepsilon \to 0\ } & \boxed{\mathfrak{se}(2),\ \ \varepsilon = 0} \\ \text{compact} & & \text{non-compact} \end{array}$$

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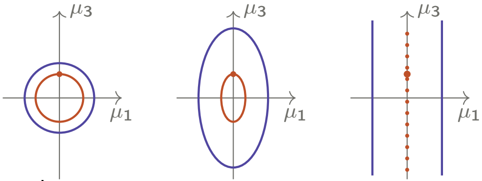
Coadjoint action: $\left\langle \mathrm{ad}^{*,\varepsilon}_{\boldsymbol{\xi}} \boldsymbol{\mu}\,, \boldsymbol{\eta} \right\rangle = \langle \boldsymbol{\mu}\,, [\boldsymbol{\xi}, \boldsymbol{\eta}]_\varepsilon \rangle$ with $\boldsymbol{\xi} = \xi_1 \mathrm{b}_1 + \xi_2 \mathrm{b}_2 + \omega \mathrm{b}_3$

$$\begin{aligned} \mathrm{ad}^{*,\varepsilon}_{\boldsymbol{\xi}} \boldsymbol{\mu} &= \big(\omega\mu_2 - \varepsilon^2 \xi_2 \mu_3, \ \ -\omega\mu_1 + \varepsilon^2 \xi_1 \mu_3, \ \ \xi_2\mu_1 - \xi_1\mu_2\big) \in \mathfrak{so}(3)^* \\ \mathrm{ad}^{*,0}_{\boldsymbol{\xi}} \boldsymbol{\mu} &= \big(\omega\mu_2, \qquad\qquad -\omega\mu_1, \qquad\qquad \xi_2\mu_1 - \xi_1\mu_2\big) \in \mathfrak{se}(2)^* \end{aligned}$$

$$\boldsymbol{\mu} = (0,0,L)\,,\ L \neq 0 \implies \mathrm{ad}^{*,\varepsilon}_{\boldsymbol{\xi}} \boldsymbol{\mu} = \big(-\varepsilon^2 \xi_2 L, \ -\varepsilon^2 \xi_1 L, \ 0\big) \neq \mathrm{ad}^{*,0}_{\boldsymbol{\xi}} \boldsymbol{\mu} = (0,0,0)$$

Coadjoint orbits degenerate

Casimir $C_\varepsilon(\boldsymbol{\mu}) = \mu_1^2 + \mu_2^2 + \varepsilon^2\mu_3^2$. Every orbit *lies in* a level set $C_\varepsilon = c$, and the level sets are ellipsoids with semi-axes $\sqrt{c}$ and $\sqrt{c}/\varepsilon$, so at a given ε they are nested and similar.



ε larger $\quad$ ε smaller $\quad$ $\varepsilon = 0$

Blue: a level set with c fixed — a single orbit for every $\varepsilon \geq 0$, stretching into the cylinder $\mu_1^2 + \mu_2^2 = c$.

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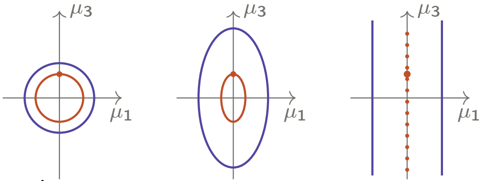


ε larger $\quad$ ε smaller $\quad$ $\varepsilon = 0$

Blue: a level set with c fixed — a single orbit for every $\varepsilon \geq 0$, stretching into the cylinder $\mu_1^2 + \mu_2^2 = c$.

Red: the orbit through $(0, 0, L)$, where $c = \varepsilon^2 L^2$: height fixed at L, width εL collapsing.

At $\varepsilon = 0$ the level set $C_0 = 0$ is the μ_3-axis and every point of this axis is fixed by the coadjoint action.
A two-dimensional orbit collapses to a point, and the isotropy jumps from 1 to 3.

The isotropy jumps

Solving $\mathrm{ad}^{*,\varepsilon}_{\boldsymbol{\xi}}\boldsymbol{\mu} = 0$ we find the coadjoint isotropy subalgebra:

$$\mathfrak{g}^{\varepsilon}_{\boldsymbol{\mu}} = \mathbb{R}\cdot\big(\mu_1,\ \mu_2,\ \varepsilon^2\mu_3\big)$$

stratum	$\dim \mathfrak{g}^{\varepsilon}_{\boldsymbol{\mu}^0}$, $\varepsilon > 0$	$\dim \mathfrak{g}^{0}_{\boldsymbol{\mu}^0}$	
generic $(\mu_1, \mu_2) \neq (0,0)$	1	1	no jump
axial $(0,0,L)$, $L \neq 0$	1	3	jump
zero $(0,0,0)$	3	3	no jump

On the axial stratum $(0,0,L)$, $L \neq 0$

- for $\varepsilon > 0$ the isotropy is $\mathbb{R}b_3$ — just rotations.
- for $\varepsilon = 0$ it is all of $\mathfrak{se}(2)$, because $(0,0,L)$ is a fixed point of the coadjoint action.

This breaks a "ε-continuous" reduction

Marsden–Weinstein–Meyer reduction at $\boldsymbol{\mu}^0$ quotients $\mathcal{J}_\varepsilon^{-1}(\boldsymbol{\mu}^0)$ by $G^\varepsilon_{\boldsymbol{\mu}^0}$:

$$\dim P^\varepsilon_{\mathrm{MWM}} = \dim T^*Q^\varepsilon - \dim G_\varepsilon - \dim G^\varepsilon_{\boldsymbol{\mu}^0}.$$

On the axial stratum $(0,0,L)$, $L \neq 0$

$$\dim P^\varepsilon_{\mathrm{MWM}} = 4n-4 \;\; (\varepsilon > 0), \qquad \dim P^0_{\mathrm{MWM}} = 4n-6 \;\; (\varepsilon = 0).$$

The reduced spaces are not a smooth family.
No direct application of the Implicit Function Theorem.

The planar problem has *more* symmetry at this momentum value and this is the reason the curved problem is not a perturbation of the flat one in the ordinary sense.

The device: reduce by less

Choose a subgroup contained in the isotropy for *every* $\varepsilon \geq 0$:

$$K := SO(2) = \exp(\mathbb{R}\,\mathrm{b}_3) \;\subseteq\; G^{\varepsilon}_{\boldsymbol{\mu}^0} \quad \text{for all } \varepsilon \geq 0.$$

Define

$$P^{\varepsilon} := \mathcal{J}_{\varepsilon}^{-1}(0,0,L)/K, \qquad \dim P^{\varepsilon} = 4n-4 \quad \text{for } \textit{every } \varepsilon \geq 0.$$

$$\boxed{\begin{array}{c}\varepsilon > 0\\ K = G^{\varepsilon}_{\boldsymbol{\mu}^0}\text{: MWM reduction}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\varepsilon = 0\\ K \subsetneq G^{0}_{\boldsymbol{\mu}^0}\text{: under-reduced}\end{array}}$$

P^0 carries a *presymplectic* form,
with a 2-dimensional kernel

The price: at $\varepsilon = 0$ we have quotiented the rotations but not the two translations.

Before proceeding, we need charts

To compare $\varepsilon > 0$ with $\varepsilon = 0$ we must put everything on one fixed space.

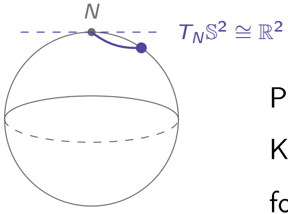
The idea: use a single chart centred at the North pole $N = (0, 0, 1/\varepsilon)$.

$$\mathrm{Exp}_N^{\varepsilon} : \mathbb{R}^2 \longrightarrow Q^{\varepsilon}$$

$$(u, v) \longrightarrow \left(u\, \frac{\sin(\varepsilon\rho)}{\varepsilon\rho},\ v\, \frac{\sin(\varepsilon\rho)}{\varepsilon\rho},\ \frac{\cos(\varepsilon\rho)}{\varepsilon} \right)$$

where $\rho = \|(u, v)\|$

$$\Phi^{\varepsilon} :\ T^*U^n \longrightarrow T^*Q^{\varepsilon}, \qquad U \subset \mathbb{R}^2, \quad \varepsilon\|(u, v)\| < \pi/2.$$



Pull back ω_ε, H_ε, $\mathcal{J}_\varepsilon$.

Key point: the pulled-back symplectic form is ε-independent $= \omega^0$.

So all the ε-dependence is pushed into the pulled-back H_ε and $\mathcal{J}_\varepsilon$, on a *fixed* symplectic manifold. Everything is real-analytic in ε^2.

Flat limit: as $\varepsilon \to 0$,

$$\mathrm{Exp}_N^\varepsilon(u, v) \to (u, v, +\infty).$$

(u, v) become standard Cartesian coordinates.

The n-body problem in the exponential chart

Expansion of the kinetic energy T_ε

$$T_\varepsilon = \sum_{i=1\ldots n} \frac{\|p_i\|^2}{2m_i} \;+\; \varepsilon^2 \sum_{i=1\ldots n} \frac{1}{6m_i}\Big(\|q_i\|^2\|p_i\|^2 - (q_i \cdot p_i)^2\Big) \;+\; O(\varepsilon^4).$$

or

$$T_\varepsilon = \sum_{i=1\ldots n} \frac{\|p_i\|^2}{2m_i} \;+\; \varepsilon^2 \sum_{i=1\ldots n} \frac{1}{6m_i}\|q_i \times p_i\|^2 \;+\; O(\varepsilon^4).$$

The correction is $\frac{1}{6m_i}\|q_i \times p_i\|^2$ — it penalises *angular* motion far from the pole. Radial motion is untouched at this order.

Note: a particle moving radially through the pole feels no $O(\varepsilon^2)$ kinetic correction, as it should — a geodesic through N is a great circle and the chart is isometric along it.

Expansion of the gravitational potential U_ε

Write $\psi_{ij} = \varepsilon d_{ij}$ for the angle subtended, $r_{ij} = \|q_i - q_j\|$ the *chart* distance. Then, from the ambient inner product,

$$\cos\psi_{ij} = 1 - \frac{\varepsilon^2 r_{ij}^2}{2} + \frac{\varepsilon^4}{24}\left(r_{ij}^4 + 4\|q_i \times q_j\|^2\right) + O(\varepsilon^6) \implies \psi_{ij} = \varepsilon r_{ij} - \frac{\varepsilon^3\|q_i \times q_j\|^2}{6 r_{ij}} + O(\varepsilon^5),$$

and therefore

$$\varepsilon \cot\psi_{ij} = \frac{1}{r_{ij}} + \varepsilon^2\left(\frac{\|q_i \times q_j\|^2}{6 r_{ij}^3} - \frac{r_{ij}}{3}\right) + O(\varepsilon^4).$$

$$U_\varepsilon = -\sum_{i<j} \frac{m_i m_j}{r_{ij}} \;+\; \varepsilon^2 \sum_{i<j} m_i m_j \left(\frac{r_{ij}}{3} - \frac{\|q_i \times q_j\|^2}{6 r_{ij}^3}\right) + O(\varepsilon^4)$$

Note: we have the two competing corrections: an attracting term linear in r_{ij}, and a repelling term proportional to the area $A_{ij} = \|q_i \times p_i\|^2$ spanned by the two position vectors.

The curvature Hamiltonian H_2

Collecting,

$$H_\varepsilon = H_0 + \varepsilon^2 H_2 + \mathcal{O}(\varepsilon^4),$$

$$\boxed{H_2 = \sum_{i=1\ldots n} \frac{1}{6m_i}\|q_i \times p_i\|^2 + \sum_{1\le i<j\le n} m_i m_j \left(\frac{r_{ij}}{3} - \frac{\|q_i \times q_j\|^2}{6\, r_{ij}^3}\right)}$$

Likewise the momentum map expands,

$$\mathcal{J}_\varepsilon = \mathcal{J}_0 + \varepsilon^2 \mathcal{J}_2 + \mathcal{O}(\varepsilon^4)$$

,

$$\mathcal{J}_2 = \left(\tfrac{1}{3} \sum_{i=1\ldots n} \left(q_i\,(q_i \cdot p_i) - \|q_i\|^2 p_i\,, 0\right)\right),$$

Co-moving Hamiltonian:

$$H_\varepsilon^{\boldsymbol{\xi}} := H_\varepsilon - \langle \mathcal{J}_\varepsilon, \boldsymbol{\xi}\rangle = \left(H_0 - \langle \mathcal{J}_0, \boldsymbol{\xi}\rangle\right) + \varepsilon^2\left(H_2 - \langle \mathcal{J}_2, \boldsymbol{\xi}\rangle\right) + \mathcal{O}(\varepsilon^4) = H_0^{\boldsymbol{\xi}} + \varepsilon^2 H_2^{\boldsymbol{\xi}} + \mathcal{O}(\varepsilon^4)\,.$$

Back to $\boldsymbol{\mu} = (0, 0, L)$, $L \neq 0$.

At $\varepsilon = 0$, linear momentum$= 0$, so translating the whole configuration does not change the momentum

Let $\mathrm{z}^0(t) = \exp(t\boldsymbol{\xi})\mathrm{z}^0 = R_{\omega t}\mathrm{z}^0$ be a a planar RE with $\boldsymbol{\xi} = \omega \mathrm{b}_3$, $\boldsymbol{\mu} = (0, 0, L)$ and centre of mass at the origin. Then

$$\mathrm{z}^0_{\mathrm{c}}(t) := \mathrm{z}^0(t) + \mathrm{c} = R_{\omega t}\mathrm{z}^0(0) + \mathrm{c} \;\text{ is an RE with }\; \boldsymbol{\mu} = (0, 0, L) \text{ for any } \mathrm{c} \in \mathbb{R}^2$$

In the reduced space $P^\varepsilon := \mathcal{J}_\varepsilon^{-1}(0, 0, L)/K = \mathcal{J}_\varepsilon^{-1}(0, 0, L)/SO(2)$ the class of $\left[\mathrm{z}^0_{\mathrm{c}}\right]$ is

$$\left[\mathrm{z}^0_{\mathrm{c}}\right] = \left\{R_{\omega t}\mathrm{z}^0(0) + \mathrm{c} \,|\, t \in \mathbb{R}\right\}$$

At the same momentum value $\boldsymbol{\mu} = (0, 0, L)$ we have triangles that:

1) spin about the origin and

2) spin about c, for any $c \in \mathbb{R}^2$ fixed.

Meaning for the dynamics

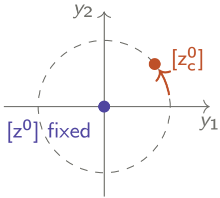
Let $c \in \mathbb{R}^2$ be fixed. Since

$$R_{\omega t} \cdot z^0 + c = R_{\omega t}\left(z^0 + R_{-\omega t} c\right)$$

quotienting by $K = SO(2)$ removes the leading $R_{\omega t}$; what survives is a translation by $R_{-\omega t}(c)$. Hence we may represent the reduced trajectory through $[z_c^0]$ as

$$\boxed{y(t) = R_{-\omega t}(c)}$$

and the set of translates $\left\{[z_c^0] \mid c \in \mathbb{R}^2\right\}$ is invariant under the reduced flow.



The translate is *not* a fixed point: it circles the original at rate ω.

Linearising: $\dot{y} = -\omega \Omega\, y$, eigenvalues $\pm i\omega$.

$L \neq 0$ makes the triangle spin and this spinning is what makes the unwanted directions nondegenerate.

Local product structure of $P_0 = \mathcal{J}_0^{-1}(0,0,L)/SO(2)$

Informally: *Take* $\mathrm{z}^0 \in \mathcal{J}_0^{-1}(0,0,L)$ *and look at its class modulo rotations. Any nearby class is parametrised by a pair* $(\mathrm{y}, w) \in \mathbb{R}^2 \times P^0_{\mathrm{MWM}}$.

Namely, a configuration w *with centre of mass at the origin, translated out by* y *with* w *read modulo rotations, after a choice of local section.*

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Let $\mathrm{z}^0 \in \mathcal{J}_0^{-1}(0,0,L)$ have the centre of mass at the origin and $[\mathrm{z}^0] \in P^0 = \widehat{\mathcal{J}}_0^{-1}(0,0,L)/SO(2)$. Further, let $q_{\mathrm{cm}} : Q^0 \to \mathbb{R}^2$ the *centre-of-mass map* (so we choose a slice transverse to the translations) and

$$\widehat{N}^0_{\mathrm{cm}} := \mathcal{J}_0^{-1}(0,0,L) \cap \{q_{\mathrm{cm}} = 0\}\,.$$

Then, near $[\mathrm{z}^0] \in P^0$ we have

(1) $\mathrm{y} \mapsto [T_{\mathrm{y}} \cdot \mathrm{z}^0]$ and $\widehat{N}^0_{\mathrm{cm}} \to \widehat{N}^0_{\mathrm{cm}}/SO(2) =: P^0_{\mathrm{MWM}}$ provide local coordinates

$$(\mathrm{y}, \mathrm{w}) \in \mathbb{R}^2 \times P^0_{\mathrm{MWM}} \text{ on } P^0 \;\text{ near }\; [\mathrm{z}^0]$$

(2) P^0_{MWM} is locally symplectomorphic to the $SE(2)$ Marsden–Weinstein-Meyer (MWM) reduced space at $(0,0,L)$, via reduction by stages for $\mathbb{R}^2 \lhd SE(2) \twoheadrightarrow SO(2)$;

(3) if $z^0(t) = \exp(t\boldsymbol{\xi})z^0$ is a RE with $\boldsymbol{\xi} = \omega \mathrm{b}_3$, $\omega \neq 0$, then for every $\mathrm{c} \in \mathbb{R}^2$ fixed, z^0_c of has coordinates $(\mathrm{y}, \mathrm{w}) = \big(\mathrm{c}, [\mathrm{z}^0]_{\mathrm{MWM}}\big)$, the submanifold $\{\mathrm{y} = 0\}$ is invariant under the reduced flow $X_{[H]_0}$, and

$$D_{[\mathrm{z}]} X_{[H]_0}\Big|_{[\mathrm{z}^0]} = \begin{pmatrix} -\omega\,\Omega & 0 \\ * & D_\mathrm{w} X_{[H]_0}|_{\mathrm{y}=0} \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

The upper-left block has the eigenvalues $\pm i\omega$ and is invertible, and $D_w X_{[H]_0}|_{\mathrm{y}=0}$ is the linearisation at the point corresponding to $[\mathrm{z}^0]$ of the MWM-reduced Hamiltonian vector field on P^0_{MWM}.

NOTE: so the spectrum at an RE splits:

- upper-left: $\pm i\omega$ — purely kinematic, independent of n, of the masses and the shape;
- lower-right: the Marsden–Weinstein-Meyer linearisation.

Continuation of relative equilibria

Definition A RE is non-degenerate if it is non-degenerate in the reduced space. In our context, a RE z^ε is non-degenerate iff $[\mathrm{z}^\varepsilon]$ is a non-degenerate critical point of the reduced Hamiltonian $[H]_\varepsilon$.

Let K_ε be the subgroup we reduce by: $G^\varepsilon_{\boldsymbol{\mu}^0}$ on the generic stratum, $SO(2)$ on the axial one, and G_ε on the zero stratum.

Theorem (Continuation of RE)

Let z^0 be a non-degenerate RE of the flat problem with momentum $\boldsymbol{\mu}^0$. Then for ε small there is a smooth branch z^ε of relative equilibria of H_ε with

$$\mathcal{J}_\varepsilon(\mathrm{z}^\varepsilon) = \boldsymbol{\mu}^0, \qquad \boldsymbol{\xi}^\varepsilon = \boldsymbol{\xi}^0 + O(\varepsilon^2), \qquad \mathrm{z}^\varepsilon \to \mathrm{z}^0.$$

It is unique near z^0 *up to the coadjoint isotropy group* $G^\varepsilon_{\boldsymbol{\mu}^0}$.

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Theorem (Continuation of RE)

Let z^0 be a non-degenerate RE of the flat problem with momentum $\boldsymbol{\mu}^0$. Then for ε small there is a smooth branch z^ε of relative equilibria of H_ε with

$$\mathcal{J}_\varepsilon(z^\varepsilon) = \boldsymbol{\mu}^0, \qquad \boldsymbol{\xi}^\varepsilon = \boldsymbol{\xi}^0 + O(\varepsilon^2), \qquad z^\varepsilon \to z^0.$$

It is unique near z^0 *up to the coadjoint isotropy group* $G^\varepsilon_{\boldsymbol{\mu}^0}$.

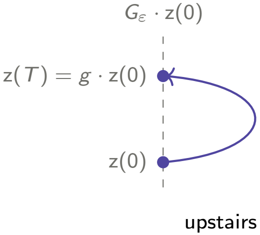
Credit where it is due. For attraction interaction only, the only RE are on the axial stratum $\boldsymbol{\mu} = (0, 0, L)$, $L \neq 0$. When considering the $SO(2)$ symmetry only, Bengochea et al. (2020) solved it via Lagrange multipliers.

Relative periodic orbits (RPOs)

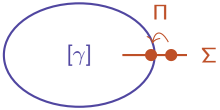
Definition. A solution $\mathrm{z}(t)$ of X_{H_ε} is a relative periodic orbit if there are $T > 0$ and $g \in G_\varepsilon$ with
$$\mathrm{z}(t+T) = g \cdot \mathrm{z}(t) \qquad \text{for all } t.$$

The element g is the *drift* of $\mathrm{z}(t)$, and $\boldsymbol{\xi} \in \mathfrak{g}_\varepsilon$ with $g = \exp(T\boldsymbol{\xi})$ is a *drift generator*.

We have the drift $g \in G_{\boldsymbol{\mu}}$, since $\mathcal{J}_\varepsilon$ is conserved and equivariant; and z projects to a genuine *closed* orbit $[\gamma]$ of period T in the reduced space P^ε.



Definition Let $\gamma(t)$ be a RPO and $[\gamma](t)$ the corresponding periodic orbit in the reduced space. Then γ is *non-degenerate* when the monodromy of $[\gamma]$ (the linearised reduced flow over one period) has 1 as a Floquet multiplier with algebraic multiplicity two (one from the direction of the flow and one from the energy).



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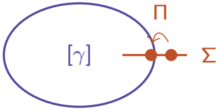


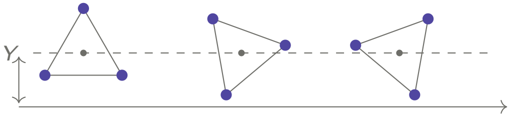
reduced

If γ is non-degenerate then $\det\left(\mathrm{d}\Pi - \mathbb{I}\right) \neq 0$, the implicit function theorem applies to $\Pi(y) = y$, and γ *continues*: for ε small there is a relative periodic orbit γ^{ε} near γ, unique modulo time-phase and K_{ε}, with

$$T^{\varepsilon} \to T^{0}\,, \qquad g^{\varepsilon} \to g^{0}\,.$$

The drifting Lagrange triangle

Consider the elliptic Lagrange homographic motions with a nonzero centre-of-mass *translational drift*.
The centre of mass slides along a straight line at constant speed.



rigid triangle (internal orbit = a circle)

Each body follows a Kepler ellipse around the centre of mass. If internal orbit - ellipse, then the triangle breathes periodically.

This is an RPO on the generic stratum: $(\mu_1, \mu_2) \neq 0$, no isotropy jump. Great!

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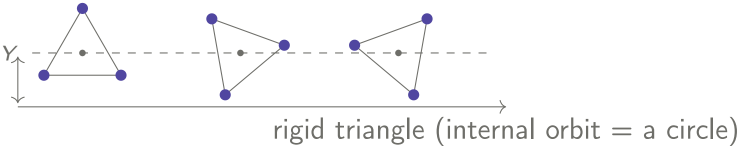

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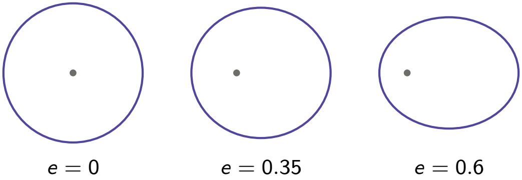
But we have a degeneracy of the classical Kepler problem - bad luck ?!

A two-parameter degeneracy

Informally: keep the semi-major axis fixed (i.e. the energy).

Two knobs: eccentricity, and the angle from a fixed direction to the periapsis points. A two-parameter family, parametrised by the eccentricity vector :

$$\boldsymbol{\eta} = (\eta_1, \eta_2)\,, \qquad |\boldsymbol{\eta}| = e\,.$$



All these orbits have the same energy E, the same momentum $\boldsymbol{\mu}$, the same period T^0.

All these orbits have the same energy E, the same momentum $\boldsymbol{\mu}$, the same period T^0 ($\kappa =$ gravitational parameter)

- the period is the same because Kepler's third law involves only the semi-major axis and not the eccentricity,

$$T^0 = 2\pi\sqrt{\frac{a^3}{\kappa}}\,,$$

so however we deform the ellipse, the triangle still comes back after exactly T^0;

- the energy is the same for the same reason,

$$E^0 = -\frac{\kappa}{2a}\,;$$

- momentum is arranged by sliding the line sideways a little as the eccentricity grows: the transverse offset trades against the internal angular momentum, keeping the total fixed.

So the planar problem has a two-parameter family $\gamma^0_{\boldsymbol{\eta}}$ of RPOs, all with the same energy E^0, period T^0 and momentum $\boldsymbol{\mu}^0$ $\Rightarrow$ the linearised return map has a 2-dimensional unit eigenspace.

Lyapunov–Schmidt reduction: the basics in $\mathbb{R}^N$

Let $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$. Problem: solve $F(y, \delta) = 0$ near $F(y_0, 0) = 0$

$$L := D_y F(y_0, 0) \text{ singular i.e. } \dim\ker L = \dim \mathcal{W} := (\dim \operatorname{coker} L) = k \geq 1$$

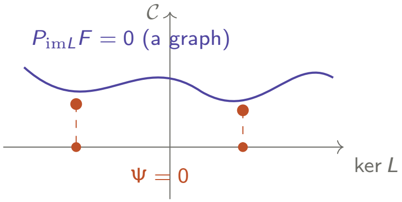
$$\mathbb{R}^N = (\ker L) \oplus \mathcal{C} \qquad \text{and} \qquad \mathbb{R}^N = (\operatorname{im} L) \oplus \mathcal{W}$$

$$y = y_K + y_C$$

(a) We have $L|_{\mathcal{C}} : \mathcal{C} \to \operatorname{im} L$ isomorphism.

$$P_{\operatorname{im} L} F(y_K + y_C, \delta) = 0$$

Implicit Function Th. $\Rightarrow y_C = y_C(y_K, \delta)$



(b) Bifurcation equation

$$\Psi(y_K, \delta) := P_{\mathcal{W}} F\big(y_K + y_C(y_K, \delta), \delta\big) = 0, \quad (k \text{ equations with } k \text{ unknows})$$

In functional setting: $F : X \times \Delta \to Y$ with X, Y Banach spaces and $\delta \in \Delta$ parameter, with dim(ker $F_x(x, \delta)$) and dim(range$F_x(x, \delta)$) finite, and range$F_x(x, \delta)$ a closed subspace of Y.

In our context

- $\boldsymbol{\eta} =$ eccentricity vector

  $$\boldsymbol{\eta}^0 \longrightarrow \gamma^0_{\boldsymbol{\eta}}(t)\,\text{circle} \ \longrightarrow \gamma^0_{\boldsymbol{\eta}}(0) =: y(\boldsymbol{\eta}) =: y \in \mathcal{N}_\varepsilon = H_\varepsilon^{-1}(E^0) \cap \mathcal{J}_\varepsilon^{-1}(\boldsymbol{\mu}^0) \subset \mathbb{R}^{4n-4}$$

- Parameters: $(T, c) \in \mathbb{R}^2 =$ period and drift rate
  (c is such that $\boldsymbol{\xi} = c\,\boldsymbol{\zeta}_\varepsilon$, with $\boldsymbol{\xi} \in \mathfrak{g}^\varepsilon_{\boldsymbol{\mu}} = \operatorname{span}\{\mu_1 \mathrm{b}_1 + \mu_2 \mathrm{b}_2 + \varepsilon^2 \mu_3 \mathrm{b}_3\}$)

- Question: for $\varepsilon > 0$ does $\gamma^\varepsilon_{\boldsymbol{\eta}}(t)$ survive? and if yes, do we have more than one? and how do they look?

- The function $F : \mathbb{R}^{4n-2} \times \mathbb{R} \longrightarrow \mathbb{R}^{4n-2}$

  $$((y, T, c), \varepsilon) \to F\,((y, T, c), \varepsilon) = \big(\ \varphi^{\varepsilon,c}_T(y) - y\ ,\ \ \ell_1(y - y_0)\ ,\ \ \ell_2(y - y_0)\ \big) \in \mathbb{R}^{4n-2}$$

  where $\varphi^{\varepsilon,c}_T =$ flow of $H^{\boldsymbol{\xi}}_\varepsilon$, and (ℓ_1, ℓ_2) coordinates on $\ker \omega^0$

$$\mathbb{R}^{4n-2} \times \mathbb{R} \longrightarrow \mathbb{R}^{4n-2}$$

$$((y,T,c),\varepsilon) \longrightarrow F(y,T,c) = \left(\ \varphi_T^{\varepsilon,c}(y) - y\ ,\ \ \ell_1(y-y_0)\ ,\ \ \ell_2(y-y_0)\ \right) \in \mathbb{R}^{4n-2}$$

where $\varphi_T^{\varepsilon,c}$ = flow of $H_\varepsilon^{\boldsymbol{\xi}}$, and (ℓ_1, ℓ_2) coordinates on $\ker \omega^0$.

– Linearisation

$$(\dot{y}, \dot{T}, \dot{c}) \to L := D_y F(y_0, T^0, c^0) = \Big((\mathrm{d}\varphi_{T^0}^{0,c^0} - \mathbb{I})\dot{y} + Z\dot{T} + W\dot{c},\ \ell_1(\dot{y}),\ \ell_2(\dot{y}) \Big)$$

where $\{Z, W\} :=$ basis of $\ker \omega^0$. We have

$$\begin{aligned} \ker L &= \mathrm{span}\{ \big(\partial_{\eta_1}\gamma_{\boldsymbol{\eta}}^0(0),0,0\big)\,,(\partial_{\eta_2}\gamma_{\boldsymbol{\eta}}^0(0),0,0)\} = \mathrm{span}\{v_1, v_2\} \\ &= \text{ tangent to the family } \gamma_{\boldsymbol{\eta}}^0 \end{aligned}$$

$$\operatorname{coker} L = \text{the } \omega^0\text{-dual of } \ker L = \mathrm{span}\{\omega^0(\cdot, v_1)\,, \omega^0(\cdot, v_2)\}$$

The bifurcation equation - in fact two equations -

$$L : (\ker L) \oplus \mathcal{C} \to (\mathrm{im} L) \oplus \mathcal{W}$$

Solve first $(y, T, c) = (y, T, c)(\boldsymbol{\eta}, \varepsilon)$ where $\boldsymbol{\eta} = \eta_1 v_1 + \eta_2 v_2 =: (\eta_1, \eta_2) \in \ker$ L.

Moving $\boldsymbol{\eta}$ within ker L means moving from the circular solutions toward eccentric ones. Thus *when we choose a member of the planar family then the period T, the drift rate c, and all the remaining phase-space directions y are determined.*

The next step chooses a specific member of the planar family. Pick the complement $\mathcal{W}$ to $(\mathrm{im}L)$:

$$\mathcal{W} \simeq (\mathrm{im}L)^{\mathrm{ann}} \simeq \mathrm{coker}L = \mathrm{span}\{\omega^0(\cdot, v_1)\,,\ \omega^0(\cdot, v_2)\} =: \mathrm{span}\{\lambda_1\,,\ \lambda_2\}$$

and we have two equations

$$\omega^0(r(\eta_1, \eta_2, \varepsilon), v_1) = 0 \ \text{ and } \ \omega^0(r(\eta_1, \eta_2, \varepsilon), v_2) = 0$$

We must solve $\quad \omega^0\left(r(\eta_1,\eta_2,\varepsilon),\ v_1\right) = 0$ and $\omega^0\left(r(\eta_1,\eta_2,\varepsilon),\ v_2\right) = 0.$

What are these equations? Consider the vector valued function $t \to v_j(t) = \partial_{\eta_j}\gamma^0_{\boldsymbol{\eta}}(t),$

NOTE:

1. On the fixed (T^*U^n, ω^0) the symplectic form is independent of ε

$$\omega^0\left(X_H,\ v_j(t)\right) = dH\left(v_j(t)\right) = dH\left(\partial_{\eta_j}\gamma^0_{\boldsymbol{\eta}}(t)\right) = \partial_{\eta_j}\left[\, H\left(\gamma^0_{\boldsymbol{\eta}}(t)\right)\right].$$

2. Every member of the family has the *same* period T^0, so the limits do not depend on $\boldsymbol{\eta}$ and ∂_{η_j} commutes with $\int_0^{T^0}$.

Now:

1) take H to be the co-moving $H^{\boldsymbol{\xi}^0} = H_0^{\boldsymbol{\xi}^0} + \varepsilon^2 H_2^{\boldsymbol{\xi}^0} + \ldots$

2) apply variation of parameters along the unperturbed orbit $\gamma^0_{\boldsymbol{\eta}}(t)$ and calculate the change in the return map induced by curvature

The bifurcation equation

$$\text{“}\omega^0\left(r(\eta_1,\eta_2,\varepsilon),\ v_j\right)\text{''} = \varepsilon^2 \int_0^{T^0} dH_2^{\xi^0}\left(v_j(t)\right)\, dt = \varepsilon^2 \partial_{\eta_j}\mathcal{B}(\boldsymbol{\eta}),$$

Reduced equation:

$$\mathrm{d}\mathcal{B}(\boldsymbol{\eta}) + O(\varepsilon^2) = 0.$$

with

$$\mathcal{B}(\boldsymbol{\eta}) = \int_0^{T^0} H_2^{\xi^0}\left(\gamma_{\boldsymbol{\eta}}^0(t)\right)\, dt.$$

Thus the bifurcation function is the time-average of the curvature Hamiltonian along each unperturbed member (a Melnikov integral).

Interesting: the curvature “selects" from the two-parameter family the members that survive.

The bifurcation function

$$\mathcal{B}(\boldsymbol{\eta}) = \int_0^{T^0} H_2^{\boldsymbol{\xi}^0}\big(\gamma^0_{\boldsymbol{\eta}}(t)\big)\,dt, \qquad H_2^{\boldsymbol{\xi}^0} = H_2 - \langle \mathcal{J}_2, \boldsymbol{\xi}^0\rangle.$$

Reduced equation: $\quad \mathrm{d}\mathcal{B}(\boldsymbol{\eta}) + O(\varepsilon^2) = 0.$

Now compute. Evaluating H_2 and $\mathcal{J}_2$ on the drifting family, we end up with a polynomial of degree one in $e^2 = |\boldsymbol{\eta}|^2$:

$$\boxed{\frac{1}{T^0}\mathcal{B}(\boldsymbol{\eta}) = \mathcal{B}_0 + \mathcal{B}_2\,|\boldsymbol{\eta}|^2, \quad \text{with} \;\; \mathcal{B}_2 > 0.}$$

Because $\mathcal{B}_2 > 0$, the Hessian $D^2\mathcal{B}(0) = (2T^0)\,\mathcal{B}_2\,\mathbb{I}$ is nonsingular; the degeneracy at the equal-masses Lagrange triangular solution is *resolved* by curvature

Theorem(Continuation of the equal masses Lagrangian homographic RPO)

For each small $\varepsilon > 0$ *there is a relative periodic solution of the curved problem on* $\mathbb{S}^2_{1/\varepsilon}$*, with momentum* $\boldsymbol{\mu}^0$ *and energy* E^0*, close to the drifting Lagrange family, with*

$$T^\varepsilon = T^0 + O(\varepsilon^2), \qquad \boldsymbol{\xi}^\varepsilon = \boldsymbol{\xi}^0 + O(\varepsilon^2), \qquad \boldsymbol{\eta}^\varepsilon = O(\varepsilon^2),$$

unique modulo time-phase and K_ε*.*

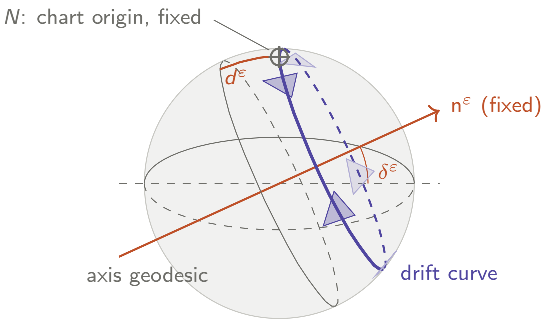
What the continued object is:

1) not a rigid triangle. It is $O(\varepsilon^2)$-close to the member with eccentricity $O(\varepsilon^2)$: the triangle *pulsates* with period T^ε and amplitude $O(\varepsilon^2)$, while the whole configuration drifts.

2) not necessarily equilateral in the geodesic distances of $\mathbb{S}^2_{1/\varepsilon}$ — only that it is close to a planar member of the family.

Where does it drift to?

On the sphere there are no translations. The drift generator must lie in $\mathfrak{g}^{\varepsilon}_{\boldsymbol{\mu}^{\mathbf{o}}} = \mathbb{R} \cdot (\mu_1, 0, \varepsilon^2 L)$, so it is a *rotation* — about an axis that is nearly horizontal, but tilted: $\mathsf{n}^{\varepsilon} \propto (0, \mu_1, \varepsilon L)$, $\sin \delta^{\varepsilon} = \varepsilon L / \sqrt{\mu_1^2 + \varepsilon^2 L^2}$.



The sphere, the axis and N stay fixed; the *configuration* is carried round by $\exp(t\boldsymbol{\xi}^{\varepsilon})$.

The drift curve

The centre of mass sits at signed geodesic distance d^ε from the great circle polar to n^ε, and

$$\boxed{\tan\left(\varepsilon d^\varepsilon\right) = \frac{\varepsilon L}{\mu_1}} \qquad \text{exactly} \qquad \Longrightarrow \qquad d^\varepsilon = \frac{L}{\mu_1} - \frac{\varepsilon^2 L^3}{3\mu_1^3} + O(\varepsilon^4).$$

The drift curve is the circle at that distance, with geodesic curvature

$$k_g = \varepsilon \tan(\varepsilon d^\varepsilon) = \frac{\varepsilon^2 L}{\mu_1}.$$

The flat limit is now readable. As $\varepsilon \to 0$:

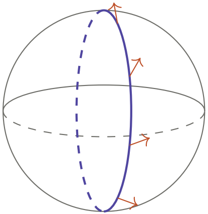
$$d^\varepsilon \longrightarrow \frac{L}{\|\mathrm{P}\|}, \qquad k_g \longrightarrow 0.$$

$L/\|\mathrm{P}\|$ is the classical *impact parameter*: the distance from the origin to the line about whose points the total angular momentum vanishes.

Holonomy, and two time scales

Parallel transport around a circle of angular radius $\pi/2 - \delta^\varepsilon$ produces a rotation by the enclosed solid angle. Relative to the transported frame, the configuration turns by

$$\Delta\varphi = 2\pi \sin\delta^\varepsilon = \frac{2\pi\varepsilon L}{\mu_1} + O(\varepsilon^3) \qquad \text{per circuit.}$$



Two time scales:
fast — internal period $T^\varepsilon \approx T^0$
slow — one circuit of the drift curve
after $\approx 2\pi M/(\varepsilon\mu_1 T^0)$ periods

So the solution closes up only after $O(1/\varepsilon)$ periods — and only for those ε where the two frequencies are commensurate. For all other ε the orbit is quasi-periodic and fills a two-torus.

Summary[1]

1. The symmetry algebra contracts, $\mathfrak{so}(3) \to \mathfrak{se}(2)$, and coadjoint isotropy *jumps* on the axial stratum: the curved problem is no naive perturbation of the flat one.
2. Reducing by $SO(2)$ restores a constant-dimension family, presymplectic at $\varepsilon = 0$. The leftover directions are translations, contributing $\pm i\omega$ — invertible because $L \neq 0$ forces $\omega \neq 0$.
3. For RE this recovers what Bengochea et al. obtain by pinning the centre of rotation.
4. For RPOs we looked at the drifting homographic Lagrange triangle. As it displays a two-parameter degenerate family, the implicit function theorem does not apply.
5. We applied Lyapunov–Schmidt and proved the continuation of the Lagrangian homographic family. The curvature "resolves" the degeneracy.
6. On $\mathbb{S}^2_{1/\varepsilon}$ the drift is a small circle and holonomy $2\pi \sin \delta^{\varepsilon}$ per circuit, closing only after $O(1/\varepsilon)$ periods.

[1] To appear in Nonlinearity. The arXiv version is not correct.

Thank you for your attention!