Singularity theorems from a mathematical perspective

(based on works with Calisti-Hafemann-Kunzinger-Steinbauer and with Kontou-Ohanyan-Schinnerl)

Melanie Graf

University of Hamburg

XXXIV Fall Workshop in Geometry and Physics, Tarragona, September 2, 2026

Outline

Singularity Theorems – A brief Pop Science Intro 1/2

- Global geometric results with physical relevance in General Relativity

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- General Relativity: Gravity = curvature of spacetime
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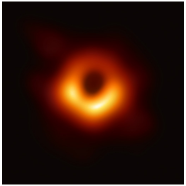
Singularity Theorems: "Typical" spacetimes are "singular" (i.e., contain incomplete causal geodesics)



Picture of Black Hole M87 ©EHT Collaboration

Singularity Theorems – A brief Pop Science Intro 2/2

History:

- The first singularity theorem was published by R. Penrose in 1965; further results by Hawking, Hawking-Penrose, ...
- This was after knowledge of exact singular solutions (like Schwarzschild, Kerr (barely!), "Big Bang" models, . . .) but before widespread evidence of/belief in singularities
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'Pop-Differential Geometry' Intro

Lorentzian analogues of well-known Riemannian results like Bonnet-Myers

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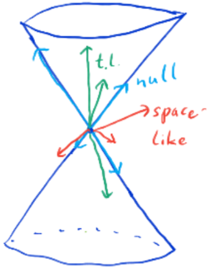
Figure: $\mathbb{R}^{2+1}$, $g = \mathrm{diag}(-1,1,1)$



A non-zero vector X is called

- spacelike $\iff$ $g(X,X) > 0$
- timelike $\iff$ $g(X,X) < 0$
- null (or lightlike) $\iff$ $g(X,X) = 0$
- causal $\iff$ X is timelike or null

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  - E.g., for $M = \mathbb{R}^n$, smooth matrix valued map $g: \mathbb{R}^n \to \mathbb{R}^{n\times n}$;
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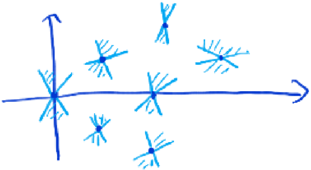
- Spacetime: Lorentzian manifold with a time orientation (smoothly) distinguishing one connected component of the cones

E.g. $M = \mathbb{R}^n$ with a Lorentzian metric with $g_{00} < 0$: Can choose the "upper" cone as "future". Locally, any spacetime looks like this!



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then (M, g) contains at least one incomplete timelike or null geodesic.

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- existence of causal geodesics γ from p to q with $L(\gamma) = \max$

Energy conditions/curvature bounds

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SEC (U timelike)	$T_{\mu\nu} U^\mu U^\nu - \frac{T\, g_{\mu\nu}}{n-2} \geq 0$	$\mathrm{Ric}(U,U) \geq 0$	$\rho + P \geq 0$ and $(n-3)\rho + (n-1)P \geq 0$

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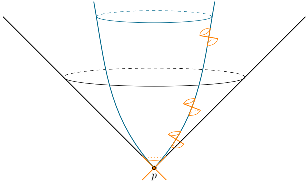
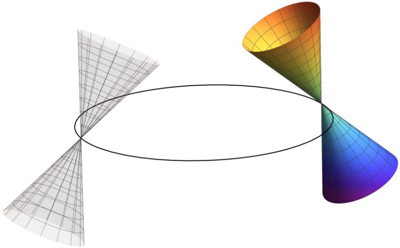
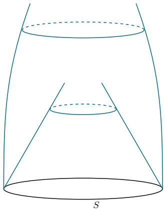
- Ensure that certain geodesic congruences don't expand faster than in Minkowski



Initial conditions

- Trapped surface (Interpretation: Black hole setting, surface of strong gravity)





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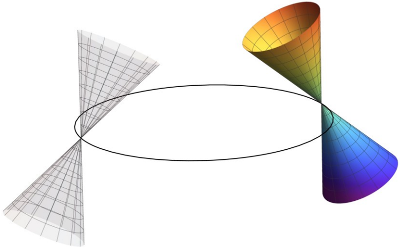

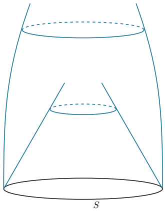

- Time-slice with initial contraction to the future or past (Interpretation: Cosmological setting, strict sign of the Hubble parameter)

The classical theorems

The Penrose singularity theorem (Penrose ('65))

A (smooth) spacetime is future null geodesically incomplete if

1. $\mathrm{Ric}(\ell, \ell) \geq 0$ for every null vector ℓ
2. There exists a non-compact Cauchy hypersurface Σ in M
3. There exists a trapped surface S (precise definition: compact smooth achronal spacelike codim. 2 submf. with past-pt. timelike mean curvature vector field $\sum_i (\nabla_{E_i} E_i)^\perp$)

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The Hawking singularity theorem (Hawking ('67))

A (smooth) spacetime is future timelike geodesically incomplete if

1. $\mathrm{Ric}(U, U) \geq 0$ for every timelike vector U
2. There exists a (achronal) compact smooth spacelike hypersurface Σ in M (Easier version: $+$ Σ is Cauchy, $-$ Σ compact)
3. The mean curvature vector of Σ is uniformly past pointing timelike, i.e. $\exists \beta > 0$ with $g(\vec{H}, \vec{n}) \geq \beta > 0$

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$$0 \geq \frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) = -\frac{n-1}{b} + \underbrace{g(\vec{H}, \vec{n})}_{\geq \beta} + \int_0^b (1 - \frac{t}{b})^2 \underbrace{\mathrm{Ric}(\dot\gamma, \dot\gamma)}_{\geq 0} dt \geq -\frac{n-1}{b} + \beta$$

as long as γ is maximizing (i.e. $L(\gamma) = \tau(\gamma(0), \gamma(b))$)

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$\rightsquigarrow$ contradiction to completeness (more or less directly depending on global assumptions, usually using "causality theory" to get a "non-compact $\cong$ compact" contradiction)

Some modern developments

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- Singularity Theorems in Lorentzian Length Spaces (Cavalletti, Mondino; Alexander, G., Kunzinger, Sämann; Ketterer; McCann ...)
- Related topics: (In-)extendibility results (Ling, Sbierski, ...), study of spacetime boundaries, looking at more concrete models, ...

Geodesics and Curvature for low regularity metrics

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Definition (Distributional null energy condition; G. '20)

A spacetime with $\mathcal{C}^1$-metric satisfies the distributional null energy condition if for any compact $K \subseteq M$ and any small $\delta > 0$ there exists $\varepsilon(\delta, K)$ s.t.

$$(\mathrm{Ric}(\mathcal{X},\mathcal{X}))|_U > -\delta \qquad \text{(as distributions on U)}$$

for any (local) smooth vector field $\mathcal{X} \in \mathfrak{X}(U)$ $(U \subseteq K)$ with $||\mathcal{X}||_h = 1$ (where h is some fixed Riemannian background metric) and $|g(\mathcal{X},\mathcal{X})| < \varepsilon(\delta, K)$ on $U \subseteq K$.

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G. '20: With these definitions the Penrose and the Hawking theorem hold for $g \in C^1$.

For Hawking's theorem we can do better!

Theorem ($C^{0,1}$-Hawking singularity theorem, I (Calisti, G., Hafemann, Kunzinger, Steinbauer '25))

Let (M, g) be a spacetime with a locally Lipschitz metric g. Assume

(i) $\mathrm{Ric}_g(X, X) \geq 0$ *in the distributional sense for all timelike* $X \in \mathfrak{X}(M)$.

(ii) *There is a smooth spacelike Cauchy hypersurface* Σ *with slab mean curvature bounded above by some* $\beta < 0$.

Then

$$\tau_\Sigma(q) := \sup\{\tau(p, q) : \ p \in \Sigma\} \leq \frac{n-1}{|\beta|}$$

for all $q \in M$.

Conclusion avoids geodesics & $\exists$ version for metric measure spacetimes with non-negative timelike Ricci curvature in an OT sense ("timelike curvature dimension condition") (Cavalletti-Mondino '22)

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- Equivalently: For any t.l. VF X there exists small enough $U \supset \Sigma$ s.t. $\operatorname{ess\,sup} \mathcal{H}^X[g] < \beta$ on U

Reminder: Statement globally hyperbolic version

Theorem ($C^{0,1}$-Hawking singularity theorem, I (Calisti, G., Hafemann, Kunzinger, Steinbauer '25))

Let (M, g) be a spacetime with a locally Lipschitz metric g. Then the following cannot hold simultaneously:

(i) $\mathrm{Ric}_g(X, X) \geq 0$ *in the distributional sense for all timelike* $X \in \mathfrak{X}(M)$.

(ii) *There is a smooth spacelike Cauchy hypersurface* Σ *with slab mean curvature bounded above by some* $\beta < 0$.

(iii) *There is a* $q \in M$ *with* $\tau_\Sigma(q) > \frac{n-1}{|\beta|}$

The approximation scheme

- Regularize $g \in C^{0,1}$ by smoothing via convolution $\rightsquigarrow$ obtain g_ε (with w.l.o.g. well behaved lightcones)

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  - becomes arbitrarily close to zero in L^p_{loc} for all $1 \leq p < \infty$ (CGHKS '25) but not for $p = \infty$

Side Remark

Smoothing techniques & techniques for getting improved estimates from distributional inequalities are very general (any signature, other quantities constructed from a low-regularity metric tensor, interact well with e.g. weighted Sobolev spaces in asymptotically flat manifolds, etc.)

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Example (Low-regularity Positive Mass Theorems)

Recent proofs (without spin assumptions!) for

- $g \in C^0 \cap W^{1,p}_{loc}$ with $R \geq 0$ distributionally (smooth outside a compact set with usual decay incl. $R|_{M\setminus K} \in L^1$; $p \geq n$) (Hafemann '26)
- $g \in C^0 \cap W^{1,p}_{-\tau}$ with $R \geq 0$ a finite measure (with usual decay $\tau > \frac{n-2}{2}$; $p > n$) (G.-Hafemann 'soon)

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These do NOT give a contradiction for fixed ε, but they do lead to a contradiction in the limit. Why?

Volume of nice subsets of Σ can be quantitatively controlled by spacetime-L^1 Ricci bounds

Theorem (G.-Kontou-Ohanyan-Schinnerl '24)

Let (M, g) ($g \in \mathcal{C}^\infty$) be a globally hyperbolic spacetime with Cauchy surface Σ and timelike Ricci curvature bounded below by κ and mean curvature of Σ bounded above by β. We assume $\kappa < 0$ and $0 > \beta \geq -(n-1)\sqrt{|\kappa|}$. Let $T, \eta > 0$.

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Then for any $B \subseteq \mathrm{Reg}^+_\eta(T) \subseteq \Sigma$ with $0 < \mathrm{vol}_\Sigma(B) < \infty$ and some $T, \eta > 0$

$$\mathrm{vol}_\Sigma(B)\, c(n,\kappa,\eta,T)\, (|\beta| - \frac{n-1}{T}) \leq \int_{\Omega^+_T(B)} |\,\mathrm{Ric}(U,U)_-|\, dvol_g,$$

where $c(n,\kappa,\eta,T)$ is a (positive) explicit constant only depending on the parameters n, κ, η, T.

A worldvolume singularity theorem

Control achieved by the previous theorem can also be leveraged to prove a singularity theorem for smooth spacetimes satisfying the following energy conditions instead of the classical SEC (cf. G.-Kontou-Ohanyan-Schinnerl):

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(i)(a) There is a $\kappa < 0$ such that $\mathrm{Ric}(U, U) \geq n\kappa$ (for $U = \partial_t$)

(i)(b) There exist (small) $Q_1, Q_2 > 0$ such that for all $F \in C_c^\infty(\mathcal{M}^+)$ we have the following integral bound on the Ricci tensor:

$$\int_M \mathrm{Ric}(U_p, U_p) F(p)^2 \, dvol_g(p) \geq -Q_1 \|F\|^2_{L^2(M)} - Q_2 \|\dot{F}\|^2_{L^2(M)}.$$

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Weaker conditions than the classical SEC!

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- Comparison of Lipschitz-Hawking with the LLS version of Cavalletti-Mondino: Under investigation (cf. Braun, Salamo-Candal), but still unclear at the moment
- Can one leverage L^1-control of the null Ricci curvature in a similar way? To...
  - ...obtain a $C^{0,1}$ Penrose theorem? Many open questions! (But there is a version based on "null Optimal Transport" by Cavalletti, Manini, Mondino '25)
  - ...obtain a worldvolume Penrose singularity theorem? (Ideally already with improvements to make it semi-classically relevant)

Thank you for your attention!