Dynamical Similarity: A Bridge Between Symplectic and Contact Geometry from Physics

Callum Bell
Lancaster University, UK
In collaboration with David Sloan

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Universitat Rovira i Virgili

UNIVERSITAT ROVIRA i VIRGILI

Introduction and Motivation

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- Motivation from physics: degeneration
- Interesting from a mathematical perspective

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- The goal is to eliminate the redundant 'scaling' degree of freedom

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- The reduced theory reproduces the dynamics of the original system without reference to the scaling variable

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$$\text{Symplectic} \longrightarrow \text{Multisymplectic}$$
$$\text{Contact} \longrightarrow \text{Multicontact}$$

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- Physically, we interpret action dependence as non-conservative or frictional behaviour
- $\mathcal{L}^H$ has no notion of scale, but must reproduce the dynamics of the full system. The frictional degrees of freedom therefore compensate the eliminated scale

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- Carrying out the variational calculation, we obtain the dynamical equations

$$D\Sigma_{IJ} = 0, \qquad G_I := \frac{1}{2}\varepsilon_{IJKL} e^J \wedge R^{KL} = 0$$

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- There is some degree of choice in the decomposition of ω

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- Substituting the decomposed fields into $\mathcal{L}$, we carry out the reduction procedure, obtaining

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  2. Non-vanishing torsion

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- In fact, we find an interesting mathematical structure with a nice physical interpretation...

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- Despite different algebraic structures, Lagrangians derived from points in the moduli space all reproduce Palatini dynamics (at least as a subset)

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    2. 1^{st} order Hamiltonian formalism (constraint structure)
    3. Is it possible to construct a well-defined path integral for action-dependent field theories?
- If it is possible to construct a path integral, does our multicontact Lagrangian evade any of the problems typically associated with the construction of a (Euclidean) gravitational path integral, such as the conformal instability?

References

A few relevant arXiv references for further details:

- 1803.04472 Dynamical similarity
- 2206.09911 Contact reduction (mathematical formulation + examples)
- 2509.16099 Generalisation to multisymplectic field theories
- 2409.08340 Action-dependent field theories
- 2608.24140 Reduction of 1^{st} order Palatini gravity

Thank You for your Attention!