Geometry of infinite-dimensional manifolds and applications to integrable systems

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Outline

Lecture 4: Kähler and hyperkähler geometry in infinite dimensions

1. Kähler quotients
2. hyperkähler quotients
3. hyperkähler coadjoint orbits of $SL(2, \mathbb{C})$

Byproducts

1. Mostow decomposition (refinement of polar decomposition)
2. complex orbits and cotangent bundles: two facets of the same hyperkähler object
3. Ekeland alternative to Hopf-Rinow Theorem on Hilbert manifold

References:

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Definition

A **weak symplectic form** ω on a Banach manifold $\mathscr{M}$ is a closed non-degenerate 2-form on $\mathscr{M}$. In particular, for all x in $\mathscr{M}$ the map

$$\begin{array}{rccc} \varphi_x : & T_x\mathscr{M} & \rightarrow & T_x^*\mathscr{M} \\ & X & \mapsto & i_X\omega \end{array}$$

is an injection.

Definition

A **weak Kähler manifold** is a Banach manifold $\mathscr{M}$ with

- a weak symplectic form ω
- a weak Riemannian metric g
- the endomorphism I defined by $\mathrm{g}(IX, Y) = \omega(X, Y)$ satisfies $I^2 = -1$ and the Nijenhuis tensor N of I vanishes.

The Nijenhuis tensor is defined as

$$N_x(X, Y) := [\tilde{X}, \tilde{Y}](x) + I[\tilde{X}, I\tilde{Y}](x) + I[I\tilde{X}, \tilde{Y}](x) - [I\tilde{X}, I\tilde{Y}](x)$$

where $X, Y \in T_x\mathscr{M}$ and $\tilde{X}, \tilde{Y}$ are smooth extensions of X and Y.

References on Nijenhuis Operators

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- $\mathscr{M}$ be a smooth Banach manifold over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$
- G Banach Lie group (over $\mathbb{K}$) acting smoothly on $\mathscr{M}$
- $\mathfrak{g}$ Lie algebra of G

Definition

A **moment map** for a G-action on a weakly symplectic Banach manifold $\mathscr{M}$ is a map $\mu : \mathscr{M} \to \mathfrak{g}^*$, satisfying

$$d\mu_x(\mathfrak{a}) = i_{X^{\mathfrak{a}}}\,\omega,$$

for all x in $\mathscr{M}$ and for all $\mathfrak{a}$ in $\mathfrak{g}$, where $X^{\mathfrak{a}}$ denotes the vector field on $\mathscr{M}$ generated by the infinitesimal action of $\mathfrak{a} \in \mathfrak{g}$. The G-action is called *Hamiltonian* if there exists a G-equivariant moment map μ, i.e. a moment map satisfying the following condition :

$$\mu(g \cdot x) = \mathrm{Ad}^*(g)\,(\mu(x))\,.$$

Example (Finite-dimensional example)

- $M_{n,m}(\mathbb{C})$ be the space of complex $(n \times m)$ matrices with symplectic form $\omega(X, Y) = \Im \operatorname{Tr} X^* Y$.
- left action of $U(n)$ hamiltonian with $\mu_{\mathrm{U}(n)}(x)(\mathfrak{a}) = -\frac{i}{2} \operatorname{Tr} xx^* \mathfrak{a}$
- right action of $U(m)$ hamiltonian with $\mu_{U(m)}(x) = -\frac{i}{2} \operatorname{Tr} x^* x \mathfrak{a}$,

$$\begin{tikzcd} & M_{n,m}(\mathbb{C}) \arrow[dl, "\mu_{\mathrm{U}(n)}"'] \arrow[dr, "\mu_{\mathrm{U}(m)}"] & \\ \mathfrak{u}(n)^* & & \mathfrak{u}(m)^* \end{tikzcd}$$

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- **Balleier C. and Wurzbacher, T., On the geometry and quantization of symplectic Howe pairs, Math. Z., 271 (1-2), 577–591 (2012).**

H separable infinite-dimensional Hilbert space,
$H = H_+ \oplus H_-$, $\dim H_+ = \dim H_- = \infty$
On bounded operators $A \in L(H)$ acting on H, define

$$\|A\|_p = \left(\mathrm{Tr}(A^*A)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

For $1 < p < 2 < q < +\infty$, one has:

$$L^1(H) \subset L^p(H) \subset L^2(H) \subset L^q(H) \subset L^\infty(H)$$

$$U(H) = \{U \in GL(H), U^*U = \mathrm{id}\}$$
$$U_1(H) = U(H) \cap \{\mathrm{id} + L^1(H)\}$$
$$U_{res}(H) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in U(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$
$$U_{1,2}(H) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in U(H), A \text{ and } D \text{ Trace-class}, B \text{ and } C \in L^2(H)\}$$

A Banach affine space

For $k \in \mathbb{R}^*$, let $\mathscr{M}_k$ be the following affine Banach space :

$$\mathscr{M}_k := \left\{ x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \in B(H_+, H) \mid \begin{array}{l} x_+ - k\mathrm{Id}_{H_+} \in L^1(H_+) \\ x_- \in L^2(H_+, H_-) \end{array} \right\}$$

Then $\mathscr{M}_k$ is modelled over the Banach space

$$L^1(H_+) \times L^2(H_+, H_-).$$

Example

$\mathscr{M}_k$ is a weak symplectic flat space modeled on $L^1(H_+) \times L^2(H_+, H_-)$ with symplectic form:

$$\omega_x(X, Y) = \Im \mathrm{Tr}\, X^* Y,$$

where $X, Y \in T_x\mathscr{M}_k$.

Example

$\mathscr{M}_k$ is a weak Kähler flat space modeled on $L^1(H_+) \times L^2(H_+, H_-)$ with symplectic form, complex structure and Riemannian metric given by:

$$\begin{array}{l} \omega_x(X, Y) = \Im \mathrm{Tr}\, X^* Y, \\ IX = iX \\ \mathrm{g}_x(X, Y) = \Re \mathrm{Tr}\, X^* Y \end{array}$$

where $X, Y \in T_x\mathscr{M}_k$.

$$\mathscr{M}_k := \left\{ x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \in B(H_+, H) \;\middle|\; \begin{array}{l} x_+ - k\mathrm{Id}_{H_+} \in L^1(H_+) \\ x_- \in L^2(H_+, H_-) \end{array} \right\}$$

Example

The group $U_1(H_+)$ acts on the right on the space $\mathscr{M}_k$ by $g \cdot x = x \circ g^{-1}$, this action preserves the symplectic form of $\mathscr{M}_k$ and is hamiltonian with momentum map:

$$\mu_{U_1}(x)(\mathfrak{a}) = \frac{i}{2}\mathrm{Tr}\, x^* x \mathfrak{a}.$$

Moreover μ_{U_1} is invariant by the left action of $U_{1,2}(H)$.

Definition

A regular value of the moment map is an element $\xi \in \mathfrak{g}^*$ such that, for every x in the level set $\mu^{-1}(\xi)$, the map $d\mu_x : T_x\mathscr{M} \to \mathfrak{g}^*$ is surjective and its kernel admits a closed complement in $T_x\mathscr{M}$.

Remark

If ξ is a regular value of μ, then $\mu^{-1}(\xi)$ is a submanifold of $\mathscr{M}$. If μ is G-equivariant and if ξ is an $\mathrm{Ad}^*(G)$-invariant element of $\mathfrak{g}^*$, then the manifold $\mu^{-1}(\xi)$ is globally G-stable, and one can consider the quotient space $\mu^{-1}(\xi)/G$.

Example

The level set $\mu_{U_1}^{-1}(\frac{i}{2}\mathrm{Tr})$ is a real submanifold of $\mathscr{M}_k$, stable by the right action of $U_1(H_+)$ and can be identified with the set:

$$\mathscr{N}_k = \{x \in \mathscr{M}_k, x^*x = k^2\mathrm{id}_{H_+}\}.$$

Proposition

The group $U_{1,2}(H)$ acts transitively on the level set $\mathscr{N}_k$.

Question Are $U_{1,2}(H)$ and $U_1(H_+)$ forming a dual pair?

Kähler quotient — finite-dimensional version

Theorem (Donaldson,Kronheimer,...)

Let $(\mathscr{M}, g, \omega)$ be a Kähler manifold endowed with an hamiltonian action of a Lie group G (with momentum map μ) preserving the Kähler structure. If the following conditions are satisfied:

- *ξ is a regular $\mathrm{Ad}^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$,*

then the Kähler structure on $\mathscr{M}$ induces a Kähler structure on $\mu^{-1}(\xi)/G$.

Kähler quotient — infinite-dimensional version

Theorem (A.B.T)

Let $(\mathscr{M}, g, \omega)$ be a weak Kähler Banach manifold endowed with an hamiltonian action of a Banach Lie group G (with momentum map μ) preserving the Kähler structure. If the following conditions are satisfied:

- *ξ is a regular $Ad^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$;*
- *for all x in $\mu^{-1}(\xi)$, one has*
  $T_x G.x \oplus T_x G.x^{\perp_g} = T_x \mu^{-1}(\xi)$,

then the Kähler structure of $\mathscr{M}$ induces a weak formally integrable Kähler structure on $\mu^{-1}(\xi)/G$.

Example

The quotient space

$$\mathcal{N}_k/U_1(H_+) = \{x \in \mathcal{M}_k, x^*x = k^2 \mathrm{id}_{H_+}\}/U_1(H_+)$$

is a Kähler manifold.

Definition

Let $(\mathcal{M}, \mathrm{g}, \omega)$ be a weak Kähler Banach manifold endowed with an hamiltonian action of a Banach Lie group G, and suppose that this action extend to a holomorphic action of the complexified group $G^{\mathbb{C}}$ containing G and with Lie algebra $\mathfrak{g} \oplus i\mathfrak{g}$. The stable manifold associated with the level set $\mu^{-1}(\xi)$ of the momentum map is defined as $\mathcal{M}^s = \{y \in \mathcal{M}, \exists g \in G^{\mathbb{C}} | g \cdot y \in \mu^{-1}(\xi)\}$.

Theorem

The quotient manifold $\mu^{-1}(\xi)/G$ can be identified as complex manifold with the quotient $\mathscr{M}^s/G^{\mathbb{C}}$.

Example

In the case of the right action of $U_1(H_+)$ on $\mathscr{M}_k$, the complexified group is $GL_1(H_+) = GL(H_+) \cap \{\mathrm{id}_{H_+} + L^1(H_+)\}$ and the stable manifold is

$$\mathscr{M}_k^s := \{y \in \mathscr{M}_k, y \text{ is injective}\} = \{y \in \mathscr{M}_k, \det\left(\frac{y^*y}{k^2}\right) > 0\}$$

As complex manifold one has: $\mathscr{N}_k/U_1(H_+) \simeq \mathscr{M}_k^s/GL_1(H_+)$.

Theorem

The map $p : \mathscr{M}_k^s \to Gr_{res}^0$ given by $p(y) = \mathrm{range}(y)$ is a $U_{1,2}$-equivariant and $GL_1(H_+)$-invariant holomorphic submersion.

Proof: The canonical basis of a element $P \in Gr_{res}^0$ belongs to $\mathscr{M}_1$.

Corollary

The Kähler quotient of $\mathscr{M}_k$ by the hamiltonian right action of $U_1(H_+)$ can be identified as Kähler manifold with the connected component of the restricted Grassmannian containing H_+.

Gr_{res} as Hermitian symmetric space

Denote by $\mathfrak{h} = \mathfrak{u}(H_+) \times \mathfrak{u}(H_-)$ and by
$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}, X \in L^2(H_+, H_-) \right\}$.
Then $\mathfrak{u}_{res} = \mathfrak{h} \oplus \mathfrak{m}$ and

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \text{ and } [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

Proposition

The geodesic of Gr_{res} starting at $o = H_+$ with velocity vector $V \in \mathfrak{m}$ are of the form

$$\gamma(t) = exptV \cdot o.$$

Define the following group

$$U_{1,2}(H) \;=\; \left\{ \left(\begin{array}{cc} U_{++} & U_{+-} \\ U_{-+} & U_{--} \end{array} \right) \in U(H)\;,\; \begin{array}{l} U_{++} - \mathrm{Id}_{H_+} \in L^1(H_+), \\ U_{--} - \mathrm{Id}_{H_-} \in L^1(H_-), \\ U_{+-} \in L^2(H_-, H_+), \\ U_{-+} \in L^2(H_+, H_-) \end{array} \right\},$$

Corollary

$U_{1,2}(H)$ *acts transitively on the connected component* Gr_{res}^0.

Theorem (Theorem B in Ekeland, 78)

Let M be a complete Riemannian Hilbert manifold, and take a point q on M. The set T of points $p \in M$ such that there exists a unique minimal geodesic from p to q contains a dense G_δ (i.e. a dense countable intersection of open sets).

Idea of the proof of corollary: Connect P to a geodesic ball around $H_+ \in Gr_{res}$.

Note that for $X \in \mathfrak{m}$, $\exp tX$ belongs to $U_{1,2} \subset U_2$. From the properties of the Riemannian exponential map, there exists a neighborhood $\mathscr{V}$ of H_+ in $\mathrm{Gr}^0_{\mathrm{res}}$ such that every element in $\mathscr{V}$ can be joined to H_+ be a (minimal) geodesic. Hence an arbitrary element $W \in \mathrm{Gr}^0_{\mathrm{res}}$ can be joined to an element $W' \in \mathscr{V}$ by a geodesic

$$\beta_1(t) = (\exp tX_1) \cdot W', \quad X_1 \in \mathfrak{m}_{W'}, \quad t \in [0\,, 1],$$

and W' can be joined to H_+ by a geodesic

$$\beta_2(t) = (\exp tX_2) \cdot H_+, \quad X_2 \in \mathfrak{m}, \quad t \in [0\,, 1].$$

Consequently $W = \beta_1(1) = (\exp X_1) \cdot W' = (\exp X_1)(\exp X_2) \cdot H_+$. But X_1 belongs to $\mathfrak{m}_{W'} = \exp(X_2)\mathfrak{m}\exp(-X_2)$, hence

$$W = (\exp X_2 \exp X_3) \cdot H_+$$

where $X_3 = \mathrm{Ad}\left(\exp(-X_2)\right)(X_1)$ belongs to $\mathfrak{m}$. Since $\exp X_3$ and $\exp X_2$ are elements of the unitary group $U_{1,2}$, it follows that their product belongs to $U_{1,2}$.

Definition

A hyperkähler manifold can be defined as a smooth manifold $\mathscr{M}$ endowed with

- one Riemannian metric g,
- three (formally integrable) complex structures I_1, I_2, I_3 satisfying the same identities as the quaternions, in particular $I_1 I_2 I_3 = -1$,
- three symplectic forms ω_1, ω_2, ω_3 linked to the Riemannian metric g and the complex structures I_i by $\omega_i(X, Y) = g(X, I_i Y)$.

- $\mathscr{M}$ endowed with I_1 and $\omega_{\mathbb{C}} := \omega_2 + i\omega_3$ is a complex symplectic manifold.
- for any $(a, b, c) \in \mathbb{S}^2$, $I_{(a,b,c)} = aI_1 + bI_2 + cI_3$ is a complex structure on $\mathscr{M}$.
- $\Rightarrow$ any hyperkähler manifold admits a family of complex symplectic structures indexed by the two-sphere of complex structures defined by I_1, I_2, I_3.

hyperkähler quotients

Theorem (HKLR87- Hyperkähler quotient — finite-dimensional version)

Let $(\mathscr{M}, g, \omega_1, \omega_2, \omega_3)$ be a hyperkähler manifold endowed with a tri-hamiltonian action of a Lie group G (with momentum map $\mu = (\mu_1, \mu_2, \mu_3)$) preserving the kähler structure. If the following conditions are satisfied:

- *$\xi = (\xi_1, \xi_2, \xi_3)$ is a regular $Ad^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$;*

then the hyperkähler structure of $\mathscr{M}$ induces a hyperkähler structure on $\mu^{-1}(\xi)/G$.

hyperkähler quotients

Theorem (T07- Hyperkähler quotient — infinite-dimensional version)

Let $(\mathscr{M}, g, \omega_1, \omega_2, \omega_3)$ be a weak hyperkähler Banach manifold endowed with an hamiltonian action of a Banach Lie group G (with momentum map $\mu = (\mu_1, \mu_2, \mu_3)$) preserving the hyperkähler structure. If the following conditions are satisfied:

- *ξ is a regular $Ad^*(G)$-invariant value of the momentum map ;*
- *G acts properly and freely on $\mu^{-1}(\xi)$;*
- *for all x in $\mu^{-1}(\xi)$, there exists a closed linear subspace H_x of $T_x\mu^{-1}(\xi)$ such that*

$$T_x G.x \oplus I_1 T_x G.x \oplus I_2 T_x G.x \oplus I_3 T_x G.x \oplus H_x = T_x\mathscr{M},$$

then the hyperkähler structure of $\mathscr{M}$ induces a weak formally integrable hyperkähler structure on $\mu^{-1}(\xi)/G$.

Hyperkähler structure on the cotangent bundle of a complex Lie group

Theorem (Kähler structure on T^*G, Kronheimer '88)

*For G a **compact Lie group** with Lie algebra $\mathfrak{g}$, the **cotangent bundle***

$$T^*G \simeq G \times \mathfrak{g} \simeq G^{\mathbb{C}}$$

*is a **Kähler manifold** where the symplectic form is the canonical symplectic form of the cotangent space, and the complex structure is the complex structure of the complexification $G^{\mathbb{C}}$ of G with the identification $T^*G \ni (x, \xi) \mapsto x \exp(i\xi) \in G^{\mathbb{C}}$.*

Ex: $G = S^1$, $T^*S^1 = \mathbb{C}^*$
$G = SU(2)$, $T^*SU(2) = SL(2, \mathbb{C})$.

Hyperkähler structure on the cotangent bundle of a complex Lie group

Theorem (hyperkähler structure on $T^*G^{\mathbb{C}}$ (Kronheimer '88))

*For G a compact Lie group with Lie algebra $\mathfrak{g}$, the **cotangent bundle** $T^*G^{\mathbb{C}}$ of the corresponding **complex group** carries a **hyperkähler structure** which is invariant under left- and right-translations by elements of G.*

$$T^*G^{\mathbb{C}} \simeq G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \simeq G^{\mathbb{C}} \times \mathfrak{g} \times \mathfrak{g}$$

is a hyperkähler manifold where two real symplectic forms are given by the real and imaginary parts of the canonical complex symplectic form of the complex cotangent space, and one complex structure is induced by the natural complex structure of the complexification $G^{\mathbb{C}}$ of G.

Ex: $G = S^1$, $T^*G^{\mathbb{C}} = S^1 \times \mathbb{R}^3$

Nahm's equations

Definition

Nahm's equations are the following system of equations on four Lie-algebra-valued functions of one variable, $A_i : \mathbb{R} \mapsto \mathfrak{g}$, i $= 0, 1, 2, 3$:

$$\frac{dA_1}{ds} + [A_0, A_1] + [A_2, A_3] = 0 \tag{1}$$

$$\frac{dA_2}{ds} + [A_0, A_3] + [A_3, A_1] = 0 \tag{2}$$

$$\frac{dA_3}{ds} + [A_0, A_4] + [A_1, A_2] = 0. \tag{3}$$

The set of $\mathscr{C}^1$-solutions will be denoted by $\mathscr{N}$.

Nahm's equations

Remark

The gauge group $\mathscr{G}$ of $\mathscr{C}^2$ maps $g : [0,1] \to G$ acts on the space of solutions $\mathscr{N}$ by

$$A_0 \mapsto gA_0g^{-1} + g\frac{dg^{-1}}{ds} \tag{4}$$

$$A_i \mapsto gA_ig^{-1}, i = 1, 2, 3. \tag{5}$$

Remark

Nahm's equation correspond to $SU(2)$ invariant selfdual connexions on a trivial principal G-bundle over $\mathbb{R}^4 \setminus \{0\}$

Nahm's equations

Theorem (Kronheimer '88)

Consider the normal subgroup $\mathscr{G}_0$ of $\mathscr{G}$ consisting of paths g with $g(0) = g(1) = 1$. Then

- $\mathscr{N}/\mathscr{G}_0$ *is a hyperkähler manifold.*
- *For one of the complex structures, the underlying holomorphic symplectic manifold is $T^*G^{\mathbb{C}}$.*

Remark

Replacing the closed interval $[0, 1]$ by $[0, \infty)$ and adding some boundary conditions (and a lot of analysis) allows to construct hyperkähler structures on complex coadjoint orbits.

Theorem (T.)

The cotangent space $T^*\mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_+)$ *of the restricted Grassmannian is a hyperkähler manifold isomorphic to the complexification* $\mathrm{Gr}_{\mathrm{res}}^{\mathbb{C}}(\mathscr{H}_+)$ *of the Grassmannian defined as*

$$\mathrm{Gr}_{\mathrm{res}}^{\mathbb{C}}(\mathscr{H}_+) = \{(P, Q) \in \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_+) \times \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_-), P \oplus Q = \mathscr{H}\}$$

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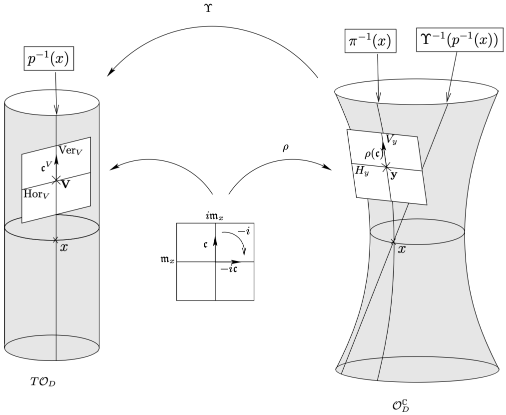
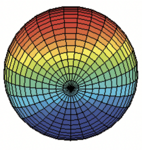
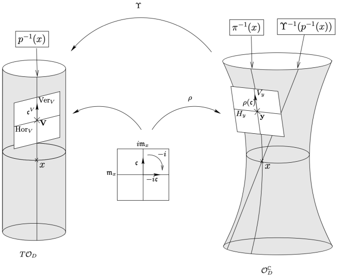
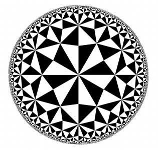


[2] A. B. Tumpach, *hyperkähler structures and infinite-dimensional Grassmannians,* Journal of Functional Analysis, https://doi.org/10.1016/j.jfa.2006.05.019

The 2-sphere and the hyperbolic disc are living in the same coadjoint orbit of $SL(2,\mathbb{C})$







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The 2-sphere and the hyperbolic disc are living in the same coadjoint orbit of $SL(2, \mathbb{C})$

- we will identify the 2-sphere with the complex projective line $\mathbb{CP}(1)$ endowed with its Kähler metric known as the Fubini-Study metric.
- The corresponding Riemannian metric is $\frac{1}{4}$ times the Riemannian metric of the round sphere of radius 1,
- the complex structure is the one of the Riemannian sphere $\mathbb{S}^2 \simeq \mathbb{C} \cup \{\infty\}$.
- The hyperbolic Poincaré disc will be denoted by $\mathbb{H}^2 = \{z \in \mathbb{C}, |z| < 1\}$.

$$G = SU(2) := \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1, a, b \in \mathbb{C} \right\},$$

$$G^{\mathbb{C}} = SL(2, \mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbb{C} \right\},$$

$$x := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2).$$

Let G and $G^{\mathbb{C}}$ act by adjoint action on their Lie algebras, and denote by $\mathscr{O}_x$ the compact adjoint orbit of x under G, and $\mathscr{O}_x^{\mathbb{C}}$ the complex adjoint orbit of x under $G^{\mathbb{C}}$. One has:

$$\begin{array}{ccc} \mathscr{O}_x & \simeq & \mathbb{CP}(1) \\ \left(\begin{smallmatrix} a & -\bar{c} \\ c & \bar{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} a & -\bar{c} \\ c & \bar{a} \end{smallmatrix}\right)^{-1} & \mapsto & \mathrm{Eig}(i) = \mathbb{C}\begin{pmatrix} a \\ c \end{pmatrix}, \end{array}$$

where $\mathrm{Eig}(i)$ is the eigenspace for the eigenvalue i, and similarly

$$\begin{array}{ccc} \mathscr{O}_x^{\mathbb{C}} & \simeq & \{(\ell_1, \ell_2) \in \mathbb{CP}(1) \times \mathbb{CP}(1), \ell_1 \oplus \ell_2 = \mathbb{C}^2\} \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} & \mapsto & \left(\mathrm{Eig}(i) = \mathbb{C}\begin{pmatrix} a \\ c \end{pmatrix}, \mathrm{Eig}(-i) = \mathbb{C}\begin{pmatrix} b \\ d \end{pmatrix}\right). \end{array}$$

The stabilizers of x for the adjoint actions of G and $G^{\mathbb{C}}$ are respectively:

$$\mathfrak{t} := \mathrm{Stab}_{SU(2)}(x) = U(1) = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \theta \in \mathbb{R}\right\},$$
$$\mathfrak{t}^{\mathbb{C}} := \mathrm{Stab}_{SL(2,\mathbb{C})}(x) = \mathbb{C}^* = \left\{\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \in \mathbb{C}^*\right\}.$$

- the orthogonal $\mathfrak{m}$ of $\mathfrak{t}$ in $\mathfrak{su}(2)$ with respect to the Killing form $\kappa(A, B) := \mathrm{Tr}\,(\mathrm{ad}(A)\mathrm{ad}(B)) = 4\mathrm{Tr}\,AB$:

$$\mathfrak{m} := \left\{ \begin{pmatrix} 0 & -\bar{c} \\ c & 0 \end{pmatrix}, c \in \mathbb{C} \right\}.$$

- $\mathfrak{g} = \mathfrak{su}(2) = \mathfrak{t} \oplus \mathfrak{m}$
- $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{t} \oplus \mathfrak{m} \oplus i\mathfrak{t} \oplus i\mathfrak{m}$, with the commutation relations $[\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{t}$.

$$\mathfrak{g}^{n.c.} := \mathfrak{su}(1, 1) = \mathfrak{t} \oplus i\mathfrak{m} = \left\{ \begin{pmatrix} i\theta & \bar{\beta} \\ \beta & -i\theta \end{pmatrix}, \theta \in \mathbb{R}, \beta \in \mathbb{C} \right\}$$

$$SU(1, 1) := \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

Proposition

The non-compact orbit $\mathscr{O}_x^{n.c.}$ is diffeomorphic to the quotient space $SU(1,1)/U(1)$ hence to $\mathbb{H}^2$,

$$\mathscr{O}_x^{n.c.} \simeq SU(1,1)/U(1) \simeq \mathbb{H}^2$$

and can be parameterized by $\beta \in \mathbb{C}$ as follows

$$\begin{array}{ccccc} \mathscr{O}_x^{n.c.} & \longrightarrow & SU(1,1)/U(1) & \longrightarrow & \mathbb{H}^2 \\ Ad\begin{pmatrix} \cosh|\beta| & \frac{\beta}{|\beta|}\sinh|\beta| \\ \frac{\bar{\beta}}{|\beta|}\sinh|\beta| & \cosh|\beta| \end{pmatrix}(x) & \longmapsto & \left[\begin{pmatrix} \cosh|\beta| & \frac{\beta}{|\beta|}\sinh|\beta| \\ \frac{\bar{\beta}}{|\beta|}\sinh|\beta| & \cosh|\beta| \end{pmatrix}\right]_{U(1)} & \longmapsto & \frac{\beta}{|\beta|}\tanh|\beta|. \end{array}$$

Proposition

Mostow decomposition $SL(2,\mathbb{C}) = SU(2) \times \exp i\mathfrak{m} \times \exp i\mathfrak{t}$, *maps a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ *to the product* $k \cdot f \cdot e$, *where*

$$e := \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \in \exp i\mathfrak{t}, \;\; \textit{with } \alpha := \sqrt{\tfrac{|a|^2+|c|^2}{|b|^2+|d|^2}},$$

$$f := \begin{pmatrix} \cosh|\beta| & \frac{\beta}{|\beta|}\sinh|\beta| \\ \frac{\bar{\beta}}{|\beta|}\sinh|\beta| & \cosh|\beta| \end{pmatrix} \in \exp i\mathfrak{m},$$

with

$$\beta := \frac{\bar{a}b+\bar{c}d}{|\bar{a}b+\bar{c}d|}\,\frac{\ln\left(\sqrt{(|a|^2+|c|^2)(|b|^2+|d|^2)}+|\bar{a}b+\bar{c}d|\right)}{2}$$

$$k := \begin{pmatrix} a & b \\ c & d \end{pmatrix} e^{-1} f^{-1}.$$

Theorem

There is a $SU(2)$-equivariant fibration $\pi : \mathscr{O}_x^{\mathbb{C}} \to \mathscr{O}_x$ of the complex (co-)adjoint orbit

$$\mathscr{O}_x^{\mathbb{C}} \simeq \left\{(\ell_1, \ell_2) \in \mathbb{CP}(1) \times \mathbb{CP}(1), \ell_1 \oplus \ell_2 = \mathbb{C}^2\right\}$$

over the compact (co-)adjoint orbit $\mathscr{O}_x \simeq \mathbb{CP}(1) \simeq \mathbb{S}^2$ given by

$$\begin{array}{rccc} \pi : & \mathscr{O}_x^{\mathbb{C}} & \longrightarrow & \mathscr{O}_x \\ & Ad(k \cdot f \cdot e)(x) & \longmapsto & Ad(k)(x), \end{array}$$

where $(k, f, e) \in SU(2) \times \exp i\mathfrak{m} \times \exp i\mathfrak{t}$. In particular, the fiber $\pi^{-1}(x)$ over $x \in \mathscr{O}_x^{\mathbb{C}}$ is $Ad(e^{i\mathfrak{m}})(x) = \mathscr{O}_x^{n.c.} \simeq \mathbb{H}^2$.

Theorem

Moreover the complex (co-)adjoint orbit is isomorphic to the (co-)tangent bundle of $\mathbb{CP}(1)$ via the following $SU(2)$-equivariant diffeomorphism

$$\begin{array}{rccc} \Phi : & T\mathbb{CP}(1) = T\mathcal{O}_x & \longrightarrow & \mathcal{O}_x^{\mathbb{C}} \\ & (z = g\ x\ g^{-1}, v = g\ \mathfrak{a}\ g^{-1}) & \longmapsto & g\ e^{i\mathfrak{a}}\ x\ e^{-i\mathfrak{a}}\ g^{-1} \end{array}$$

with $g \in SU(2)$ and $\mathfrak{a} \in \mathfrak{m}$.

In other words, $\mathcal{O}_x^{\mathbb{C}}$ is the complexification of the compact orbit $\mathcal{O}_x \simeq \mathbb{CP}(1)$ and is isomorphic to the tangent bundle of $\mathbb{CP}(1)$. The hyperbolic space $\mathbb{H}^2$ sits inside $\mathcal{O}_x^{\mathbb{C}}$ as the fiber over $x \in \mathcal{O}_x \simeq \mathbb{CP}(1)$ and is the orbit of x under the adjoint action of $SU(1,1) \subset SL(2,\mathbb{C})$.

Theorem (Biquard-Gauduchon)

1. *The complex adjoint orbit $\mathcal{O}_x^{\mathbb{C}}$ admits a G-invariant hyperkähler structure compatible with the complex symplectic form $\omega_{\mathbb{C}}$ of Kirillov-Kostant-Souriau and extending the natural Kähler structure of the Hermitian-symmetric orbit of compact type $\mathcal{O}_x$.*
2. *The Kähler form ω_1 associated with the natural complex structure $I_1 := i$ of the complex orbit $\mathcal{O}_x^{\mathbb{C}}$ is given by $\omega_1 = dd^c K$, where the potential K has the following expression*

$$K(y) = \alpha\,\mathrm{Re}\langle y, \pi(y)\rangle, \textit{where } y \in \mathcal{O}_x^{\mathbb{C}}, \tag{6}$$

where the constant α is defined by $ad_x^2 = -\alpha^2$ on $\mathfrak{m}_x$.

Theorem (Biquard-Gauduchon)

3. *The explicit expression of the symplectic form ω_1 is the following*

$$\omega_1(X^{\mathfrak{c}+i\mathfrak{c}'}, X^{\mathfrak{d}+i\mathfrak{d}'}) = \alpha \operatorname{Im}\left(\langle X^{i\mathfrak{c}'}, \pi_*(X^{\mathfrak{d}})\rangle - \langle X^{i\mathfrak{d}'}, \pi_*(X^{\mathfrak{c}})\rangle\right), \tag{7}$$

where $\mathfrak{c}, \mathfrak{c}', \mathfrak{d}$ and $\mathfrak{d}'$ belong to $\mathfrak{m}_x$.

4. *The Riemannian metric is given by*

$$g(X^{\mathfrak{c}}, X^{\mathfrak{d}}) = g(X^{i\mathfrak{c}}, X^{i\mathfrak{d}}) = \alpha \operatorname{Re}\langle X^{\mathfrak{c}}, \pi_*(X^{\mathfrak{d}})\rangle, \qquad g(X^{\mathfrak{c}}, X^{i\mathfrak{d}}) = 0, \tag{8}$$

where $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{d}$ and $\mathfrak{d}'$ belong to $\mathfrak{m}_x$.

5. *The complex structure I_2, extending the complex structure I of the Hermitian-symmetric space of compact type $\mathscr{O}_x$, is given at $y \in \pi^{-1}(x)$ by*

$$I_2 X^{\mathfrak{d}} = X^{\frac{1}{\alpha} ad_x \mathfrak{d}}, \qquad I_2 X^{i\mathfrak{d}} = -X^{\frac{1}{\alpha} ad_x i\mathfrak{d}}, \tag{9}$$

where $\mathfrak{c}$ and $\mathfrak{d}$ belong to $\mathfrak{m}_x$.

Theorem (T.)

The cotangent space $T^*\mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_+)$ of the restricted Grassmannian is a hyperkähler manifold isomorphic to the complexification $\mathrm{Gr}^{\mathbb{C}}_{\mathrm{res}}(\mathscr{H}_+)$ of the Grassmannian defined as

$$\mathrm{Gr}^{\mathbb{C}}_{\mathrm{res}}(\mathscr{H}_+) = \{(P, Q) \in \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_+) \times \mathrm{Gr}_{\mathrm{res}}(\mathscr{H}_-), P \oplus Q = \mathscr{H}\}$$

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