Geometry of infinite-dimensional manifolds and applications to integrable systems

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Research partially financed by
FWF-NCN Project I5015-N:
"Banach Poisson-Lie Groups and Integrable Systems"
AI Austria Project PAT1179524:
"Geometric Green Learning on Groups and Quotient Spaces"

XXXIV International Fall Workshop on Geometry and Physics

Plan

1. Finite-dimensional theory of Poisson-Lie groups
2. Banach Poisson-Lie groups
3. Poisson structures on Banach unitary groups
4. Bruhat-Poisson structure on the restricted Grassmannian

If time permits...

1. Korteweg-De Vries in Lax form
2. Baker function associated to elements in Gr_{res}
3. The Grassmannian model of ΩU_n
4. From the Baker function to solutions of the KdV hierarchy

- M. Sato, Y. Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifold*, Nonlinear partial differential equations in applied science, Proc. U.S. - Jap. Semin., Tokyo 1982, North-Holland Math. Stud. 81, 259-271 (1983).
- G. Segal, G. Wilson, *Loop Groups and Equations of KdV type*, Publication mathématiques de l'IHES, n. 61, 1985, 5–65.
- A. Pressley, G. Segal, *Loop Groups*, Oxford Mathematical Monographs.
- Saša Krešić Jurić, *On the Birkhoff factorization problem for the Heisenberg magnet and nonlinear Schrödinger equations*, Journal of Mathematical Physics 47, 2006.
- A.B.Tumpach, *Banach Poisson-Lie group and Bruhat-Poisson structure of the restricted Grassmannian*, Communications in Mathematical Physics, 2020.

Motivation...

Korteweg-de Vries in Lax form

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \qquad \Leftrightarrow \qquad \frac{\partial L}{\partial t} = [(L^{\frac{3}{2}})_+, L]$$

where

- $L = D^2 - 4u \qquad \text{with} \qquad D = \frac{\partial}{\partial x}$
- For any pseudo-differential operator with Laurent series
  $R = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots,$
  $R_+ = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0.$
- $L^{\frac{1}{2}} = D + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots$ such that $(L^{\frac{1}{2}})^2 = L$

More generally, the n-th KdV hierarchy is the following hierarchy of equations indexed by $r \in \mathbb{N}$ and $n \in \mathbb{N}$

$$\frac{\partial L}{\partial t} = [(L^{\frac{r}{n}})_+, L]$$

The restricted Grassmannian

$$\mathrm{GL}_{\mathrm{res}}(H) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\mathrm{P}_{\mathrm{res}}(H) = \{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in GL(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{\mathrm{res}}(H) = \mathrm{GL}_{\mathrm{res}}(H)/\,\mathrm{P}_{\mathrm{res}}(H)$$

$$\mathrm{U}_{\mathrm{res}} = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{U}(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{\mathrm{res}} = \mathrm{U}_{\mathrm{res}}(H)/(\mathrm{U}(H_+) \times \mathrm{U}(H_-))$$

Triangular group $\mathrm{B}^+_{\mathrm{res}}(H) \subset \mathrm{GL}_{\mathrm{res}}(H)$:

An invertible operator $g \in \mathrm{GL}_{\mathrm{res}}(H)$ belongs to $\mathrm{B}^+_{\mathrm{res}}(H)$ if it is upper triangular with respect to the basis $\{\dots z^n, \dots, z^1, 1, z^{-1}, z^{-2}, \dots\}$ of H, with strictly positive coefficents on the diagonal.

Remark : $\mathrm{B}^+_{\mathrm{res}}(H)$ acts on $Gr_{\mathrm{res}}(H)$

Relation between the restricted Grassmannian and the KdV hierachy [G. Segal and G. Wilson, 1985]

$\Gamma_+ = \{g = e^f, f \text{ holomorphic in } \mathbb{D}, f(0) = 0\}$
$\Rightarrow g = e^{t_1 z + t_2 z^2 + t_3 z^3 + \cdots} \in \Gamma_+$ acts on $L^2(\mathbb{S}^1, \mathbb{C})$ by multiplication and the corresponding operator is upper triangular in $B^+_{\mathrm{res}}(H)$.

$\mathrm{Gr}^{(n)} = \{W \in \mathrm{Gr}^0_{\mathrm{res}}(H) \ : z^n W \subset W\}$.

Proposition 5.13 in [SW85] : The action of Γ_+ on $\mathrm{Gr}^{(n)}$ induces the flows of the KdV hierarchy. For $r \geq 1$, the flow $W \mapsto \exp(t_r z^r) W$ on $\mathrm{Gr}^{(n)}$ induces the r-th KdV flow.

Key Observation : $\Gamma_+ \subset \mathrm{B}^+_{\mathbf{res}}(H)$.

Goal of Today...

Poisson geometry of Gr_{res} and dressing action inducing the KdV hierarchy

Theorem [A.B.T] : The restricted unitary group $\mathrm{U}_{res}(H)$ and the restricted Grassmannian $Gr_{res}(H)$ carry natural Poisson structures such that :

1. the projection $p\ :\mathrm{U}_{\mathrm{res}}(H)\to \mathrm{Gr}_{\mathrm{res}}(H)$ is a Poisson map,
2. the natural action of $\mathrm{U}_{\mathrm{res}}(H)$ on $\mathrm{Gr}_{\mathrm{res}}(H)$ is a Poisson map,
3. the following right action of $\mathrm{B}^+_{\mathrm{res}}(H)$ on $\mathrm{Gr}_{\mathrm{res}}(H)$ is a Poisson map :

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{res}}(H)\times \mathrm{B}^+_{\mathrm{res}}(H) & \to & \mathrm{Gr}_{\mathrm{res}}(H) \\ (g\,\mathrm{P}_{\mathrm{res}}(H), b) & \mapsto & (b^{-1}g)\,\mathrm{P}_{\mathrm{res}}(H). \end{array}$$

4. the symplectic leaves of $\mathrm{Gr}_{\mathrm{res}}(H)$ are the orbits of $\mathrm{B}^+_{\mathrm{res}}(H)$.

$\Rightarrow$ the action of $\Gamma^+\subset \mathrm{B}^+_{\mathrm{res}}(H)$ is Poisson and generates the KdV hierarchy.

Key Difficulty : $\mathrm{B}^+_{\mathrm{res}}(H)$ modeled on a non-reflexive Banach space.

Plan

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4. Bruhat-Poisson structure on the restricted Grassmannian

References :

- A.B.Tumpach, *Banach Poisson-Lie groups and Bruhat-Poisson structure of the restricted Grassmannian*, Communications in Mathematical Physics.
- A. B. Tumpach, T. Goliński, *The Banach Poisson–Lie group structure of* $\mathrm{U}(H)$, Geometric Methods in Physics XXXIX, Bialystok, Poland, Springer, 2022.
- T. Goliński, A. B. Tumpach, *Banach Poisson–Lie groups, Lax equations and the AKS theorem in infinite dimensions*, Differential Geometry and its Applications, Volume 101, 2025.
- T. Goliński, P. Rahangdale, A.B. Tumpach, *Poisson structures in the Banach setting: comparison of different approaches*, Geometric Methods in Physics XLI, Bielowieza, Poland, Springer, 2024.

Finite-dimensional Poisson–Lie groups

References for finite-dimensional Poisson-Lie groups

V.G. Drinfel'd, '83
Y. Kosmann-Schwarzbach, F. Magri, '88
J.-H. Lu, '91

From Quantum groups to Poisson-Lie groups

Kirillov, 99, Merits and Demerits of the orbit method:

The first approximation to a quantum group is a so-called **Poisson-Lie group** which is an ordinary Lie group G with an additional structure: the Poisson brackets in the space of functions on G. It arises when we consider the (non-commutative) multiplication in the function algebra on a quantum group as a deformation of the ordinary multiplication in the function algebra on G.

Namely, this deformation has the form

$$f \circ g = fg + h\{f,\ g\} + o(h),$$

where h is the deformation parameter (considered either as a real number or as a formal variable) and $\{\ ,\ \}$ is the Poisson bracket on G compatible with group multiplication.

In terms of the corresponding bivector c compatibility means

$$c(xy) = L_x^* c(y) + R_y^* c(x),$$

where L_x, R_y are the operators of left and right shifts on G. It follows that c vanishes at the unit element $e \in G$ and is completely determined by its first order jet at e. The latter gives a Lie algebra structure $[\ ,\ ]_*$ on the space $T_e^*(G) = \mathfrak{g}^*$.

Poisson–Lie groups in the finite-dimensional case

Manin triples

$$\Updownarrow$$

Lie-bialgebras

$$\Updownarrow$$

connected simply connected Poisson–Lie groups

Poisson–Lie groups

Let us start with an example of a Manin triple...

$\mathfrak{u}(n)$ = Lie-algebra of the unitary group $\mathrm{U}(n)$
= space of skew-symmetric matrices

$\mathfrak{b}(n)$ = Lie-algebra of the Borel group $\mathrm{B}(n, \mathbb{C})$
= space of upper triangular matrices with real coef. on diagonal

Then the space $M(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$ of all complex matrices decomposes :

$$M(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$$

and for the non-degenerate symmetric bilinear continuous map $\langle \cdot, \cdot \rangle$ given by

$$\langle A, B \rangle = \mathrm{Im}\,\mathrm{Tr}(AB) = \text{ imaginary part of trace}(AB)$$

the blocks $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$ are both isotropic.

Poisson–Lie groups

Definition of a Manin triple

A **Banach Manin** triple consists of a triple of Banach Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ over a field $\mathbb{K}$ and a **non-degenerate symmetric bilinear** continuous map $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that

1. the bilinear map $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is invariant with respect to the bracket $[\cdot,\cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$, i.e.

$$\langle [x,y]_{\mathfrak{g}}, z\rangle_{\mathfrak{g}} + \langle y, [x,z]_{\mathfrak{g}}\rangle_{\mathfrak{g}} = 0, \quad \forall x,y,z \in \mathfrak{g}; \tag{1}$$

2. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Banach spaces;
3. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Banach Lie subalgebras of $\mathfrak{g}$;
4. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the bilinear map $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$.

Let M be a finite-dimensional manifold.

Poisson bracket

A **Poisson bracket** on M is a bilinear map $\{\cdot,\cdot\} \ : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ with

- skew-symmetry $\{f,g\} = -\{g,f\}$
- Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
- Leibniz rule $\{f,gh\} = \{f,g\}h + g\{f,h\}$

Poisson tensor

$\{f,g\} = \pi(df,dg)$ where $\pi \in \Gamma(\Lambda^2 TM)$ is a bivector field

Example

Any symplectic manifold is a Poisson manifold

Let G be a finite-dimensional Lie group.

Poisson-Lie groups

A **Poisson-Lie group** G is a Lie group equipped with a Poisson structure compatible with the group multiplication.

Example

Any Lie group G with $\{\cdot, \cdot\} = 0$ is a Poisson Lie group
Any compact Lie group is a Poisson-Lie group in a non-trivial way.

Definition

For any Poisson-Lie group (G, π), with Lie algebra $\mathfrak{g}$, one defines $\Pi_r^G := R_g^* \pi \ : G \to \Lambda^2 \mathfrak{g}$ as

$$\Pi_r^G(g)(\alpha, \beta) := \pi(R_g^* \alpha, R_g^* \beta), \quad \alpha, \beta \in \mathfrak{g}^*$$

Let (G, π) be a finite-dimensional Poisson-Lie group.

Facts

1. the fact that the Poisson tensor π is compatible with the group multiplication implies the following cocycle condition on $\Pi_r^G := R_g^* \pi$

$$\Pi_r^G(gh) = \Pi_r^G(g) + \mathrm{Ad}_g \Pi_r^G(h)$$

2. the derivative of $\Pi_r^G \; : G \to \Lambda^2\mathfrak{g}$ at the unit of the group is a cocycle $\theta \; : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$

$$\begin{aligned} \theta\left([x,y]\right)(\alpha,\beta) = & \;\; \theta(y)(\mathrm{ad}_x^*\alpha,\beta) + \theta(y)(\alpha,\mathrm{ad}_x^*\beta) \\ & -\theta(x)(\mathrm{ad}_y^*\alpha,\beta) - \theta(x)(\alpha,\mathrm{ad}_y^*\beta) \end{aligned}$$

where $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.

3. the Jacobi identity verified by the Poisson structure implies that $\theta^* : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$

Finite-dimensional Lie bialgebras

Lie bialgebra

Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. One says that $(\mathfrak{g}, \mathfrak{g}^*)$ form a **Lie bialgebra** if there is a Lie bracket $\Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ whose dual map $\mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$

$$\begin{aligned}\theta\left([x, y]\right)(\alpha, \beta) = \; & \theta(y)(\mathrm{ad}^*_x\alpha, \beta) + \theta(y)(\alpha, \mathrm{ad}^*_x\beta) \\ & -\theta(x)(\mathrm{ad}^*_y\alpha, \beta) - \theta(x)(\alpha, \mathrm{ad}^*_y\beta)\end{aligned}$$

Poisson–Lie groups in the finite-dimensional case

Manin triples

$$\Updownarrow$$

Lie-bialgebras

$$\Updownarrow$$

connected simply connected Poisson–Lie groups

Example

- $M(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$ with $\langle A, B\rangle = \mathrm{Im}\,\mathrm{Tr} AB$ is a Manin triple.
- U(n) and B($n,\mathbb{C}$) are dual Poisson-Lie groups with

$$\Pi_r^G(g)(x_1, x_2) = \langle p_{\mathfrak{u}}(g^{-1}x_1 g), p_{\mathfrak{b}}(g^{-1}x_2 g)\rangle.$$

- Moreover $GL(n,\mathbb{C}) = U(n) \times B(n)$ because of Iwasawa dec.
- This gives a dressing action

$$\varphi : B(n) \times U(n) \to U(n)$$

by $\varphi(b)(k) = k'$ where k' is the unique element of $U(n)$ such that $bk = k'b'$ with $b' \in B(n)$.

Reference :

J.-H. Lu, A. Weinstein, *Poisson Lie groups, Dressing Transformations, and Bruhat Decompositions*, Journal of Differential Geometry, 1990.

Plan

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2. **Banach Poisson-Lie groups**
3. Poisson structure on the unitary group of an Hilbert space
4. Bruhat-Poisson structure on the restricted Grassmannian

Poisson manifold modeled on a non-separable Banach space

Problems :

(1) no bump functions available (norm not even $\mathscr{C}^1$ away from the origin)

(2) there exist derivations of order greater then 1 (called queer) [Kriegl, Michor, '97]

(3) there exist Poisson bracket without Poisson tensor (called queer) (Leibniz rule does not imply existence of Poisson tensor) [Beltita, Golinski, T., 2018]

(4) existence of Hamiltonian vector field is not automatic

References for Poisson geometry on Banach manifods

A. A. Odzijewicz, T. Ratiu, 2003
P. Cabau, F. Pelletier, 2011
K.H. Neeb, H. Sahlmann, T. Thiemann, 2013
D. Beltita, T. Golinski, A.B.Tumpach, 2018
T. Goliński, P. Rahangdale, A.B. Tumpach, 2024.

Let us recall the definition of a duality pairing.

Definition

Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ be two normed vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$$

be a continuous bilinear map. One says that the map $\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-}$ is a **duality pairing** between $\mathfrak{g}_+$ and $\mathfrak{g}_-$ if and only if it is **non-degenerate**, i.e. if the following two conditions hold :

$$\begin{aligned} &\left(\langle x, y \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} = 0, \quad \forall x \in \mathfrak{g}_+\right) \Rightarrow y = 0 \quad \text{and} \\ &\left(\langle x, y \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} = 0, \quad \forall y \in \mathfrak{g}_-\right) \Rightarrow x = 0. \end{aligned}$$

Definition (Poisson tensor)

M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM. π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha,\beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) = 0.$$

Definition (Poisson Manifold)

A Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Banach symplectic manifold

Any Banach symplectic manifold (M, ω) is naturally a generalized Banach Poisson manifold $(M, \mathbb{F}, \pi)$ with

1. $\mathbb{F} = \omega^\sharp(TM)$;
2. $\pi\ : \omega^\sharp(TM) \times \omega^\sharp(TM) \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \omega(X_\alpha, X_\beta)$ where X_α and X_β are uniquely defined by $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$.

Definition

Consider a duality pairing $\langle \cdot, \cdot \rangle_{\mathfrak{g},\mathfrak{b}}\ : \mathfrak{g} \times \mathfrak{b} \to \mathbb{K}$ between 2 Banach spaces. $\mathfrak{g}$ is a **Banach Lie–Poisson space with respect to** $\mathfrak{b}$ if

- $\mathfrak{b}$ is a Banach Lie algebra $(\mathfrak{b}, [\cdot, \cdot]_{\mathfrak{b}})$
- $\mathfrak{b}$ acts continuously on $\mathfrak{g} \hookrightarrow \mathfrak{b}^*$ by coadjoint action, i.e.

$$\mathrm{ad}^*_\alpha x \in \mathfrak{g},$$

for all $x \in \mathfrak{g}$ and $\alpha \in \mathfrak{b}$, and $\mathrm{ad}^*\ : \mathfrak{b} \times \mathfrak{g} \to \mathfrak{g}$ is continuous.

Banach Poisson-Lie groups

A **Banach Poisson-Lie group** G is a Banach Lie group endowed with a Banach Poisson manifold structure compatible with the multiplication.

Proposition

Let G be a Banach Lie group with Lie algebra $\mathfrak{g}$, and $(G, \mathbb{F}, \pi)$ a Banach Poisson structure on G. Then G is a Banach Poisson-Lie group if and only if

1. $\mathbb{F}$ is invariant under left and right multiplications by elements in G,
2. the fiber $\mathbb{F}_e \subset \mathfrak{g}^*$, where e is the unit element of G, is invariant under the coadjoint action of G on $\mathfrak{g}^*$ and the map

$$\begin{array}{rccl} \Pi_r^G & : & G & \to & \Lambda^2\mathbb{F}_e^* \\ & & g & \mapsto & R_{g^{-1}}^{**}\pi_g, \end{array}$$

is a 1-cocycle on G with respect to the coadjoint representation of G in $\Lambda^2\mathbb{F}_e^*$.

Theorem [T20] :

Let $(G, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group. Then $\mathfrak{g}$ is a Banach Lie bialgebra with respect to $\mathbb{F}_e$. The Lie bracket in $\mathbb{F}_e$ is given by

$$[\alpha_1, \beta_1]_{\mathbb{F}_e} := T_e\Pi_r(\cdot)(\alpha_1, \beta_1) \in \mathbb{F}_e \subset \mathfrak{g}^*, \quad \alpha_1, \beta_1 \in \mathbb{F}_e \subset \mathfrak{g}^*, \tag{2}$$

where $\Pi_r := R^{**}_{g^{-1}}\pi : G \to \Lambda^2\mathbb{F}_e^*$, and $T_e\Pi_r : \mathfrak{g} \to \Lambda^2\mathbb{F}_e^*$ denotes the differential of Π_r at the unit element $e \in G$.

Theorem [T20] :

Let $(G, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group. If the map $\pi^\sharp : \mathbb{F} \to \mathbb{F}^*$ defined by $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in $TG \subset \mathbb{F}^*$, then $\mathfrak{g}$ is a Banach Lie–Poisson space with respect to $\mathbb{F}_e$.

Poisson–Lie groups in the infinite-dimensional case

Manin triple

$$\Updownarrow$$

Banach Lie-bialgebra + Banach Lie-Poisson space

$$\Uparrow$$

Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

A.B.Tumpach, *Bruhat-Poisson structure of the restricted Grassmannian and the KdV hierarchy*, Communications of Mathematical Physics.

Poisson–Lie groups in the infinite-dimensional case

Manin triple

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Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

A.B.Tumpach, *Bruhat-Poisson structure of the restricted Grassmannian and the KdV hierarchy*, Communications of Mathematical Physics.
P. Rahangdale, *Drinfeld Correspondence in Infinite Dimensions*, ArXiv.

Plan

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H separable infinite-dimensional Hilbert space,
$H = H_+ \oplus H_-$, $\dim H_+ = \dim H_- = \infty$
On bounded operators $A \in L(H)$ acting on H, define

$$\|A\|_p = \left(\mathrm{Tr}(A^*A)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

For $1 < p < 2 < q < +\infty$, one has:

$$L^1(H) \subset L^p(H) \subset L^2(H) \subset L^q(H) \subset L^\infty(H)$$

$U(H) = \{U \in GL(H), U^*U = \mathrm{id}\}$
$U_1(H) = U(H) \cap \{\mathrm{id} + L^1(H)\}$
$U_{res}(H) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in U(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$
$U_{1,2}(H) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in U(H), A \text{ and } D \text{ Trace-class}, B \text{ and } C \in L^2(H)\}$
$L_{1,2}(H) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A \text{ and } C \text{ Trace class}, B \text{ and } C \text{ Hilbert-Schmidt}\}$
$\mathfrak{u}_2 = \text{skew-Hermitian Hilbert-Schmidt}$
$\mathfrak{b}_2 = \text{upper triangular Hilbert-Schmidt with real diagonal}$

Example of bounded operator with unbounded triangular truncation [Davidson, Nest Algebras]

$$\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \frac{1}{n} & \ddots \\
\ddots & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \ddots \\
\ddots & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \ddots \\
\ddots & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ddots \\
\ddots & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \ddots \\
\ddots & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \ddots \\
\ddots & -\frac{1}{n} & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

- the triangular truncation is unbounded on the Banach space of trace class operators
- Explicit example of a trace class operator whose triangular truncation is not trace class?

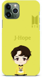
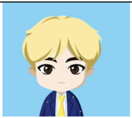
Theorem

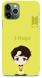
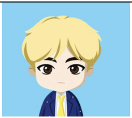
Consider the Banach Lie group $\mathrm{U}(H)$ *and define*

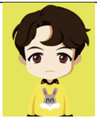
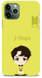
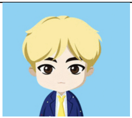
- *the precotangent bundle* $\mathbb{F} = T_* \mathrm{U}(H) \subset T^* \mathrm{U}(H)$ *by right translations*
$$\mathbb{F}_g = R^*_{g^{-1}} \left(\mathrm{L}_1(H)/\mathfrak{u}_1(H) \right)$$
- *the map* $\Pi_r : \mathrm{U}(H) \to \Lambda^2 \mathbb{F}^*_e$ *by*
$$\Pi_r(g)([x_1]_{\mathfrak{u}_1}, [x_2]_{\mathfrak{u}_1}) = \Im\ Tr\ \left(g^{-1}\, p_{\mathfrak{b}_2}(x_1)\, g\ p_{\mathfrak{u}_2}(g^{-1}\, p_{\mathfrak{b}_2}(x_2)\, g)\right), \tag{3}$$
- *the tensor* $\pi \in \Lambda^2 \mathbb{F}^*$ *by* $\pi(g) = R^{**}_g \Pi_r(g)$.

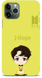
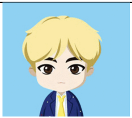
Then $(\mathrm{U}(H), \mathbb{F}, \pi)$ *is a Banach Poisson–Lie group. On the space of smooth functions with differentials in* $\mathbb{F}$, *the Poisson bracket reads:*

$$\{f, h\}(g) = \Pi_r(g)(R^*_g df_g, R^*_g dh_g) = \Pi_r(g)(df_g \circ R_g, dh_g \circ R_g).$$

				
Double Lie group	Group	Lie algebra	Fiber $\mathbb{F}_e \subset \mathfrak{g}^*$	Group in duality
$GL_n(\mathbb{C})$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$	$\mathfrak{b}(n)$	$\mathrm{B}(n)$
$\mathrm{GL}_2(H)$	$\mathrm{U}_2(H)$	$\mathfrak{u}_2(H)$	$\mathfrak{b}_2(H)$	$\mathrm{B}_2(H)$
$\mathrm{GL}_p(H), 1 < p \leq 2$	$\mathrm{U}_p(H)$	$\mathfrak{u}_p(H)$	$\mathfrak{b}_p(H)$	$\mathrm{B}_p(H)$

				
Double Lie group	Group	Lie algebra	Lie algebra	Group in duality
$GL_n(\mathbb{C})$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$	$\mathfrak{b}(n)$	$\mathrm{B}(n)$
$\mathrm{GL}_2(H)$	$\mathrm{U}_2(H)$	$\mathfrak{u}_2(H)$	$\mathfrak{b}_2(H)$	$\mathrm{B}_2(H)$
$\mathrm{GL}_p(H), 1 < p \leq 2$	$\mathrm{U}_p(H)$	$\mathfrak{u}_p(H)$	$\mathfrak{b}_p(H)$	$\mathrm{B}_p(H)$

				
Double Lie group	Group	Lie algebra	Fiber $\mathbb{F}_e \subset \mathfrak{g}^*$	Group in duality
?	$\mathrm{U}(H)$	$\mathfrak{u}(H)$	$L^1(H)/\mathfrak{u}_1$	?
?	$\mathrm{U}_{res}(H)$	$\mathfrak{u}_{res}(H)$	$L^{1,2}(H)/\mathfrak{u}_{1,2}$	?
?	$\mathrm{U}_1(H)$	$\mathfrak{u}_1(H)$	$L^1(H)/\mathfrak{u}_1$	?

				
Double Lie group	Group	Lie algebra	Lie algebra	Group in duality
?	$\mathrm{U}(H)$	$\mathfrak{u}(H)$	$L^1(H)/\mathfrak{u}_1$	?
?	$\mathrm{U}_{res}(H)$	$\mathfrak{u}_{res}(H)$	$L^{1,2}(H)/\mathfrak{u}_{1,2}$	?
?	$\mathrm{U}_1(H)$	$\mathfrak{u}_1(H)$	$L^1(H)/\mathfrak{u}_1$	?

Theorem (T20)

Consider the Banach Lie group $\mathrm{U}_{\mathrm{res}}(H)$, *and*

1. $\mathfrak{g}_+ := \mathrm{L}_{1,2}(H)/\mathfrak{u}_{1,2}(H) \subset \mathfrak{u}^*_{\mathrm{res}}(H)$,
2. $\mathbb{U} \subset T^*\,\mathrm{U}_{\mathrm{res}}(H)$, $\mathbb{U}_g = R^*_{g^{-1}}\mathfrak{g}_+$,
3. $\tilde{\pi}_r \; : \mathrm{U}_{\mathrm{res}}(H) \to \Lambda^2\mathfrak{g}^*_+$ *defined by*

$$\tilde{\pi}_r(g)([x_1]_{\mathfrak{u}_{1,2}}, [x_2]_{\mathfrak{u}_{1,2}}) = \operatorname{Im}\operatorname{Tr}(g^{-1}\,p_{\mathfrak{b}_2^+}(x_1)\,g)\left[p_{\mathfrak{u}_2}(g^{-1}\,p_{\mathfrak{b}_2^+}(x_2)\,g)\right],$$

4. $\tilde{\pi}(g) = R_g^{**}\tilde{\pi}_r(g)$.

Then $(\mathrm{U}_{\mathrm{res}}(H), \mathbb{U}, \tilde{\pi})$ ***is a Banach Poisson-Lie group.***

Plan

1. Finite-dimensional theory of Poisson-Lie groups
2. Banach Poisson-Lie groups
3. Poisson structures on Banach unitary groups
4. **Bruhat-Poisson structure on the restricted Grassmannian**

Theorem (T20)

Consider the Banach Lie group $\mathrm{B}^+_{\mathrm{res}}(H)$, *and*

1. $\mathfrak{g}_- := \mathrm{L}_{1,2}(H)/\mathfrak{b}^+_{1,2}(H) \subset \mathfrak{b}^+_{\mathrm{res}}(H)^*$,
2. $\mathbb{B} \subset T^* \mathrm{B}^+_{\mathrm{res}}(H)$, $\mathbb{B}_b := R^*_{b^{-1}}\mathfrak{g}_-$,
3. $\pi_r \ : \mathrm{B}^+_{\mathrm{res}}(H) \to \Lambda^2 \mathfrak{g}^*_-$ *defined by*

$$\pi_r(b)([x_1]_{\mathfrak{b}^+_{1,2}}, [x_2]_{\mathfrak{b}^+_{1,2}}) = \operatorname{Im}\operatorname{Tr}(b^{-1}\, p_{\mathfrak{u}_2}(x_1)\, b) \left[p_{\mathfrak{b}^+_2}(b^{-1}\, p_{\mathfrak{u}_2}(x_2)\, b)\right],$$

4. $\pi(b) = R_b^{**}\pi_r(b)$.

Then $\left(\mathrm{B}^+_{\mathrm{res}}(H), \mathbb{B}, \pi\right)$ ***is a Banach Poisson-Lie group.***

Theorem (T20)

The restricted unitary group $\mathrm{U}_{\mathrm{res}}(H)$ *and the restricted Grassmannian* $Gr_{res}(H)$ *carry natural Poisson structures such that :*

1. *the projection* $p \;:\mathrm{U}_{\mathrm{res}}(H) \to \mathrm{Gr}_{\mathrm{res}}(H)$ *is a Poisson map,*
2. *the natural action of* $\mathrm{U}_{\mathrm{res}}(H)$ *on* $\mathrm{Gr}_{\mathrm{res}}(H)$ *is a Poisson map,*
3. *the following right action of* $\mathrm{B}^+_{\mathrm{res}}(H)$ *on* $\mathrm{Gr}_{\mathrm{res}}(H)$ *is a Poisson map :*

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{res}}(H) \times \mathrm{B}^+_{\mathrm{res}}(H) & \to & \mathrm{Gr}_{\mathrm{res}}(H) \\ (g\,\mathrm{P}_{\mathrm{res}}(H), b) & \mapsto & (b^{-1}g)\,\mathrm{P}_{\mathrm{res}}(H). \end{array}$$

4. *the symplectic leaves of* $\mathrm{Gr}_{\mathrm{res}}(H)$ *are the orbits of* $\mathrm{B}^+_{\mathrm{res}}(H)$.

A.B.Tumpach, *Banach Poisson-Lie groups and Bruhat-Poisson structure of the restricted Grassmannian*, Communications in Mathematical Physics, 2020.

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Plan part II

1. Korteweg-De Vries in Lax form
2. Baker function associated to elements in Gr_{res}
3. The Grassmannian model of ΩU_n
4. From the Baker function to solutions of the KdV hierarchy

Introduction

Korteweg-De Vries Equation :

$$\dot{u} = u_{xxx} - 6uu_x$$

- equation incoding a pseudo-spherical surface with foliations by curves [Sasaki'79, Chern-Tenenblat'81]
- bihamiltonian structure, hence by Magri-Lenard scheme [Magri '78], infinitely many conserved quantities
- admits a recursion operator [Gardner-Greene-Kruskal-Miura'74, Olver'77]
- Lax formulation [Kosmann-Schwarzbach–Magri '96]
- The Korteweg-De Vries equation is the Euler equation describing the geodesic flow on the Virasoro group Vir with respect to the right-invariant L^2-metric. [Khesin-Misiolek '02]
- related to Poisson-Lie group of pseudodifferential operators [Khesin-Zakharevich '95]
- related to the restricted Grassmannian [Segal-Wilson'85]

Plan

1. **Korteweg-De Vries in Lax form**
2. Baker function associated to elements in Gr_{res}
3. The Grassmannian model of ΩU_n
4. From the Baker function to solutions of the KdV hierarchy

Korteweg-de Vries in Lax form

Pseudo-Differential Operators

Consider Pseudo-Differential Operators as Laurent series in $D = \frac{\partial}{\partial x}$

$$R = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots,$$

where

$$\begin{cases} R_+ = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0 = \text{Differential part of } R \\ R_- = a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots = \text{Integral part of } R \end{cases}$$

and the coefficients are in some space of functions.

Action of Pseudo-Differential Operators on functions

For a function f, $D(f) = \frac{\partial f}{\partial x}$ and $a_0(f) = a_0 \cdot f$ is the multiplication by the function a_0. Since

$$Du(f) = D(uf) = D(u) \cdot f + u \cdot D(f) = \left(\frac{\partial u}{\partial x} + uD\right)(f),$$

one has $Du = u_x + uD$. To get an associative ring, set

$$D^{-1}u = \sum_{j=0}^{\infty} \frac{\partial^j u}{\partial x^j} D^{-1-j}$$

to ensure $D(D^{-1}u) = u$ and $D^{-1}(Du) = u$.

Exercice

Let $L = D^2 - 4u$ and $P = D^3 - \frac{3}{16}(uD + Du)$. Compute $[P, L] = PL - LP$.

Korteweg-de Vries in Lax form

$$\dot{u} = u_{xxx} - 6uu_x$$

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x}$$

$$\Leftrightarrow \frac{\partial L}{\partial t} = [P, L]$$

where

- $L = D^2 - 4u$
- $P = D^3 - \frac{3}{16}(uD + Du)$
- $D = \frac{\partial}{\partial x}$

In fact

$$P = (L^{\frac{3}{2}})_+$$

Korteweg-de Vries in Lax form

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} \qquad \Leftrightarrow \qquad \frac{\partial L}{\partial t} = [(L^{\frac{3}{2}})_+, L]$$

where

- $L = D^2 - 4u$ and $D = \frac{\partial}{\partial x}$
- For any pseudo-differential operator with Laurent series $R = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0 + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots,$ $R_+ = a_n D^n + \cdots + a_2 D^2 + a_1 D + a_0.$
- $L^{\frac{1}{2}} = D + a_{-1} D^{-1} + a_{-2} D^{-2} + \ldots$ such that $(L^{\frac{1}{2}})^2 = L$

Korteweg-de Vires hierarchy of equations

The n-th KdV hierarchy is the following hierarchy of equations indexed by $r \in \mathbb{N}$

$$\frac{\partial L}{\partial t} = [(L^{\frac{r}{n}})_+, L]$$

(initial KdV equation for $n = 2$ and $r = 3$)

Plan

1. Korteweg-De Vries in Lax form
2. **Baker function associated to elements in** Gr_{res}
3. The Grassmannian model of ΩU_n
4. From the Baker function to solutions of the KdV hierarchy

Relation between KdV and the restricted Grassmannian

The restricted Grassmannian

$H = L^2(\mathbb{S}^1, \mathbb{C}) = H_+ \oplus H_-$
$H_+ = \{f \in H, f(z) = a_0 + a_1 z + a_2 z^2 + \dots\}$ where $z = e^{i\theta}$
$H_- = \{f \in H, f(z) = a_{-1} z^{-1} + a_{-2} z^{-2} + a_{-3} z^{-3} + \dots\}$

$B \in GL(H_\pm, H_\pm)$ is Hilbert-Schmidt iff $\mathrm{Tr} B^* B < +\infty$

The restricted Grassmannian Gr_{res} **:** A closed subspace W of H belongs to the restricted Grassmannian Gr_{res} iff

1. $p_- \; : W \to H_-$ is Hilbert-Schmidt,
2. $p_+ \; : W \to H_+$ is Fredholm

The restricted Grassmannian

$$GL_{res} = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL(\mathscr{H}), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$P_{res} = \{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in GL(\mathscr{H}), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = GL_{res}/P_{res}$$

$$U_{res} = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in U(\mathscr{H}), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = U_{res}/(U(\mathscr{H}_+) \times U(\mathscr{H}_-))$$

Triangular group $B^+_{res} \subset GL_{res}$:

An invertible operator $g \in GL_{res}$ belongs to B^+_{res} if it is upper triangular with respect to the basis $\{\ldots, z^n, \ldots, z, 1, z^{-1}, z^{-2}, \ldots, z^{-n}, \ldots\}$ of $\mathscr{H}$, with strictly positive coefficents on the diagonal.

Remark : B^+_{res} acts on Gr_{res}

Relation between the restricted Grassmannian and the KdV hierachy [G. Segal and G. Wilson, 1985]

$\Gamma_+ = \{g = e^f, f \text{ holomorphic in } \mathbb{D}, f(0) = 0\}$
$\Rightarrow g = e^{t_1 z + t_2 z^2 + t_3 z^3 + \cdots} \in \Gamma_+$ acts on $L^2(\mathbb{S}^1, \mathbb{C})$ by multiplication and the corresponding operator is upper triangular in B^+_{res}.
$\Gamma^+_W = \{g \in \Gamma_+ \; : g^{-1} W \cap \mathscr{H}_- = \{0\}\}$.

Proposition 5.1 in [SW85] :
$\forall W \in \mathrm{Gr}^0_{\mathrm{res}}(\mathscr{H})$, $\exists ! \Phi_W(g, z)$ called the Baker function of W, defined for $g \in \Gamma^+_W$ and $z \in \mathbb{S}^1$, such that

(i) $\Phi_W(g, \cdot) \in W$ for each fixed $g \in \Gamma^+_W$
(ii) $\Phi_W(g, z) = g(z)(1 + \sum_1^\infty a_i(g) z^{-i})$, a_i are analytic functions on Γ^+_W and extend to meromorphic functions on the whole of Γ^+.

Proof: $\Phi_W(g, z) = g(z)(p_{+|g^{-1}W})^{-1}(\mathbf{1})$, where $\mathbf{1}$ denotes the constant function equal to 1 on $\mathbb{S}^1$. In particular $z \mapsto \Phi_W(g, z) \in W$.

Grassmannians $Gr^{(n)}$

$$\mathrm{Gr}^{(n)} = \{W \in \mathrm{Gr}^0_{\mathrm{res}}(\mathscr{H}) \ : z^n W \subset W\}.$$

Proposition 5.5 in [SW85] :
Set $D = \frac{\partial}{\partial x}$. For each integer $r \geq 2$, there is a unique differential operator P_r of the form $P_r = D^r + p_{r2} D^{r-2} + \cdots + p_{r,r-1} D + p_{rr}$ such that $\frac{\partial \Phi_W}{\partial t_r} = P_r \Phi_W$.

Denote by $\mathscr{C}^{(n)}$ the space of all operators P_n associated to subspaces W in $\mathrm{Gr}^{(n)}$ evaluated at $t_2 = t_3 = \cdots = 0$.

Proposition 5.13 in [SW85] : The action of Γ_+ on $\mathrm{Gr}^{(n)}$ induces the flows of the KdV hierarchy. For $r \geq 1$, the flow $W \mapsto \exp(t_r z^r) W$ on $\mathrm{Gr}^{(n)}$ induces the r-th KdV flow.
In particular, since Γ_+ is commutative, these flows commutes.

Remark

The previous Proposition says that the flow of the KdV equation

$$\dot{u} = u_{xxx} - 6uu_x$$

is generated by the action by multiplication by the function

$$t \mapsto \exp tz^3$$

on the Grassmannian

$$\mathrm{Gr}^{(2)} = \{W \in \mathrm{Gr}^0_{res}(\mathscr{H}) \ : z^2 W \subset W\}$$

Plan

1. Korteweg-De Vries in Lax form
2. Baker function associated to elements in Gr_{res}
3. **The Grassmannian model of ΩU_n**
4. From the Baker function to solutions of the KdV hierarchy

The Grassmannian model of ΩU_n

Instead of specifying $H = L^2(\mathbb{S}^1, \mathbb{C})$, let us choose
$H^{(n)} = L^2(\mathbb{S}^1, \mathbb{C}^n) = H_+^{(n)} \oplus H_-^{(n)}$
$H_+^{(n)} = \{f \in H, f(z) = a_0 + a_1 z + a_2 z^2 + \dots\}$ where $z = e^{i\theta}$ and $a_i \in \mathbb{C}^n$
$H_-^{(n)} = \{f \in H, f(z) = a_{-1} z^{-1} + a_{-2} z^{-2} + a_{-3} z^{-3} + \dots\}$

Denote by ε_i the standard basis of $\mathbb{C}^n$. Then

$$\begin{array}{ccc} H^{(n)} & \leftrightarrow & H \\ \varepsilon_i z^k & \leftrightarrow & z^{nk+i-1} \end{array}$$

In particular, multiplication by z on $H^{(n)}$ corresponds to multiplication by z^n on H and

$$\begin{aligned} \mathrm{Gr}^{(n)}(H) &= \{W \in \mathrm{Gr}^0_{\mathrm{res}}(H) \ : z^n W \subset W\} \\ &= \{W \in \mathrm{Gr}^0_{\mathrm{res}}(H^{(n)})) \ : zW \subset W\} \\ &= \mathrm{Gr}^{(1)}(H^{(n)}) \end{aligned}$$

Proposition (Proposition 6.3.1 in Pressley-Segal)

If $\gamma : \mathbb{S}^1 \to \mathrm{GL}_n(\mathbb{C})$ is continuously differentiable loop in $\mathrm{GL}_n(\mathbb{C})$, then the multiplication operator by γ belongs to $\mathrm{GL}_{res}(H^{(n)})$.

Norm induced on Loops by the norm in $\mathrm{GL}_{res}(H^{(n)})$

The norm induced on a loop $\gamma(z) = \sum_{k\in\mathbb{Z}} \gamma_k z^k$ by the norm

$$\|A\|_{res} = \|A\| + \|[A, p_+ - p_-]\|$$

from $\mathrm{GL}_{res}(H^{(n)})$ is

$$|||\gamma||| = \|\gamma\|_\infty + \sum |k|\|\gamma_k\|^2.$$

Notation

Denote by $L_{\frac{1}{2}}\,\mathrm{GL}_n(\mathbb{C})$ the group of invertible in the Banach algebra of matrix valued measurable loops γ with $|||\gamma||| < +\infty$. Denote by $L_{\frac{1}{2}}\,\mathrm{U}_n$ the subgroup of U_n valued loops.

Proposition (Proposition 8.3.2 in Pressley- Segal)

The group $L_{\frac{1}{2}}\,\mathrm{U}_n$ acts transitively on $Gr^{(n)}$ and the isotropy group of H_+ is the group U_n of contant loops. Consequently

$$Gr^{(n)} = L_{\frac{1}{2}}\,\mathrm{U}_n\,/U_n = \Omega_{\frac{1}{2}}\,\mathrm{U}_n$$

where $\Omega_{\frac{1}{2}}\,\mathrm{U}_n$ is the subgroup of based loops.

In particular, Grassmannian

$$\mathrm{Gr}^{(2)} = \{W \in \mathrm{Gr}^0_{\mathrm{res}}(H)\ : z^2 W \subset W\}$$

can be identified with the space of (measurable) based loops

$$\Omega_{\frac{1}{2}}\,\mathrm{U}(2) \quad = \mathrm{L}_{\frac{1}{2}}\,\mathrm{U}(2)/\,\mathrm{U}(2)$$

Plan

1. Korteweg-De Vries in Lax form
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3. The Grassmannian model of ΩU_n
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From the Baker function to solutions of the KdV hierarchy

Group of Pseudo-Differential Operators

Pseudo-Differential Operators of the form $K = 1 + \sum_{i=1}^{+\infty} a_i D^{-i}$ form a group

$$G = \left\{ K = 1 + \sum_{i=1}^{+\infty} a_i D^{-i} \right\}.$$

Adjoint orbit of D^n

Any differential operator of the form

$$L = D^n + u_{\mathbf{n-2}} D^{\mathbf{n-2}} + u_{n-3} D^{n-3} + \cdots + u_1 D + u_0, \tag{4}$$

belongs to the orbit of D^n under the action by conjugation by G. In other words, there exists $K = 1 + \sum_{i=1}^{+\infty} a_i D^{-i} \in G$ such that

$$L = K D^n K^{-1}$$

Consequence

For operators of the form

$$L = D^n + u_{\mathbf{n-2}} D^{\mathbf{n-2}} + u_{n-3} D^{n-3} + \cdots + u_1 D + u_0,$$

there exists an Pseudo-Differential Operator

$$Q = D + \sum_{\mathbf{i=1}}^{+\infty} q_i D^{-i}, \tag{5}$$

verifying $Q^n = L$. Moreover

$$K^{-1} Q K = D$$

Module generated by exponentials

Consider the pseudo-differential module

$$\mathscr{M} = \left\{ f = e^{(xz + t_2 z^2 + t_3 z^3 + \dots)} \sum_{-\infty}^{-1} f_i z^i \right\},$$

generated by $e^{(xz+t_2z^2+t_3z^3+\dots)}$. There is a bijection between the module $\mathscr{M}$ and the group of pseudo-differential operator G

$$\begin{array}{ccc} \mathscr{M} & \longrightarrow & G \\ f = e^{(xz+t_2z^2+t_3z^3+\dots)} \sum_{-\infty}^{-1} f_i z^i & \longmapsto & K = 1 + \sum_{-\infty}^{-1} f_i D^i \end{array}$$

Baker function

To each element W of the restricted Grassmannian one can associated a unique function $\Psi_W \in \mathscr{M}$ (the Backer function of W)

$$\Psi_W = e^{(xz+t_2z^2+t_3z^3+\dots)} \sum_{-\infty}^{-1} f_i z^i.$$

This function determines for each $r \in \mathbb{N}$ a unique differential operator P_r of order r of the form

$$P_r = D^r + u_{r-2}D^{r-2} + u_{r-3}D^{r-3} + \dots + u_1 D + u_0,$$

that satisfies :

$$\frac{\partial \Psi_W}{\partial t_r} = P_r \Psi_W, r \in \mathbb{N}.$$

By the bijection $\mathscr{M} \leftrightarrow G$, one associates to the Baker function $\Psi_W \in \mathscr{M}$ an element $K \in G$ in the group of pseudo-differential operators.

Proposition

The operator $Q = D + \sum_{i=1}^{+\infty} q_i D^{-i}$, given by $Q = KDK^{-1}$ is a solution of the following Kadomtsev-Petviashvili (KP) hierarchy :

$$\frac{\partial Q}{\partial t_r} = [(Q^r)_+, Q], r \in \mathbb{N}. \tag{6}$$

Proof: differentiating $Q = KDK^{-1}$ with respect to the variable t_r one gets :

$$\begin{aligned} \frac{\partial Q}{\partial t_r} &= \frac{\partial K}{\partial t_r} DK^{-1} - KDK^{-1} \frac{\partial K}{\partial t_r} K^{-1} \\ &= \frac{\partial K}{\partial t_r} K^{-1} KDK^{-1} - KDK^{-1} \frac{\partial K}{\partial t_r} K^{-1} \\ &= \left[\frac{\partial K}{\partial t_r} K^{-1}, Q \right]. \end{aligned} \tag{7}$$

But equation $\frac{\partial \Psi_W}{\partial t_r} = P_r \Psi_W$ can be written as :

$$\frac{\partial K}{\partial t_r} + KD^r = P_r K. \tag{8}$$

In particular, $P_r = (KD^r K^{-1})_+$ and $\frac{\partial K}{\partial t_r} K^{-1} = P_r - KD^r K^{-1} = (Q^r)_+ - Q^r$, so the conclusion follows since Q and Q^r commute.

When W belongs to the sub-grassmannian $Gr_{res}^{(n)}$ consisting of elements $W \in Gr_{res}$ stable under the multiplication by z^n, the above defined pseudo-differential operator Q has the property that Q^n is a differential operator of the form

$$L = D^n + u_{\mathbf{n-2}} D^{\mathbf{n-2}} + u_{n-3} D^{n-3} + \cdots + u_1 D + u_0$$

which gives a solution of the nth KdV hierarchy.

Conclusion

$H = L^2(\mathbb{S}^1, \mathbb{C}) = H_+ \oplus H_-$
$H_+ = \{f \in \mathscr{H}, f(z) = a_0 + a_1 z + a_2 z^2 + \ldots\}$ where $z = e^{i\theta}$
$H_- = \{f \in \mathscr{H}, f(z) = a_{-1} z^{-1} + a_{-2} z^{-2} + a_{-3} z^{-3} + \ldots\}$

The flow of the KdV equation is generated by the multiplication by $\exp tz^3$ on the Grassmannian

$$\mathrm{Gr}^{(2)} = \{W \in \mathrm{Gr}^0_{\mathrm{res}}(H) \ : z^2 W \subset W\}$$

which can be identified with the space of (measurable) based loops

$$\Omega_{\frac{1}{2}} \mathrm{U}(2) \quad = \mathrm{L}_{\frac{1}{2}} \mathrm{U}(2)/\, \mathrm{U}(2)$$

consisting of loops $\gamma(z) = \sum_{k \in \mathbb{Z}} \gamma_k z^k$ in $\mathrm{U}(2)$ such that $\gamma(1) = Id$ and $\|\gamma\|_\infty + \sum |k| \|\gamma_k\|^2 < +\infty$.

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