Geometry of infinite-dimensional manifolds and applications to integrable systems

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Research partially financed by
FWF-NCN Project I5015-N:
"Banach Poisson-Lie Groups and Integrable Systems"
AI Austria Project PAT1179524:
"Geometric Green Learning on Groups and Quotient Spaces"

XXXIV International Fall Workshop on Geometry and Physics

End of Lecture 1: Fundamental building blocks

3. Tools from Analysis
4. (some) Geometric structures

Lecture 2: Subtilities of infinite-dim Geometry

1. Darboux's Theorem
2. Poisson structures
3. Poisson–Lie groups (finite-dim version)

What are the Tools from Functional Analysis?

Banach-Picard fixed point Theorem or Contraction Theorem

(E, d) complete metric space
$f \ : E \to E$ contraction of E : $d(f(x), f(y)) \leq kd(x, y)$ where $k \in (0, 1)$

$$\Rightarrow \begin{cases} \exists \,!\; x \in E, f(x) = x \\ \forall x_0 \in E, \; \textit{the sequence } x_{n+1} = f(x_n) \textit{ converges to } x \end{cases}$$

What are the Tools from Functional Analysis?

Hahn-Banach Theorem

E locally convex space
$A \subset E$ a convex
$x \in E$, $x \notin \overline{A}$

$$\Rightarrow \exists \textit{ continuous functional } \ell \ : E \to \mathbb{R} \textit{ with } \ell(x) \notin \overline{\ell(A)}$$

What are the Tools from Functional Analysis?

Open mapping Theorem

$$\left\{ \begin{array}{l} F \;\; \textit{Fréchet} \\ G \;\; \textit{Fréchet} \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} F \;\; \textit{webbed locally convex} \\ G \;\; \textit{inductive limit of Baire locally convex spaces} \end{array} \right.$$

$L \; : F \to G$ *continuous, linear, and surjective*

$\Rightarrow L$ *is open*

What are the Tools from Functional Analysis?

Cauchy-Lipschitz Theorem in the Banach case

- $\mathcal{I} \subset \mathbb{R}$ be an interval containing 0
- $\mathcal{U}$ open set of a Banach space $\mathcal{B}$
- $P \; : \mathcal{I} \times \mathcal{U} \to \mathcal{B}$

such that

- $\|P(t,f)\| \leq C \quad \forall (t,f) \in \mathcal{I} \times \mathcal{U}$
- $\|P(t,f_1) - P(t,f_0)\| \leq C' \|f_1 - f_0\| \quad \forall t \in \mathcal{I}, \forall f_0, f_1 \in \mathcal{U}$

For any $f_0 \in \mathcal{U}$ we can find a neighboorhood $\tilde{\mathcal{U}}$ of f_0 and an $\varepsilon > 0$ such that for any $f \in \tilde{\mathcal{U}}$ the Cauchy problem

$$\frac{d}{dt}\phi(t,f) = P(t,\phi(t,f))$$

has a unique solution with initial condition $\phi(0,f) = f$ on $[-\varepsilon,\varepsilon]$. Moreover if P is $\mathcal{C}^p$, $t \to \phi(t,x)$ is $\mathcal{C}^p$ for any $f \in \tilde{\mathcal{U}}$

What are the Tools from Functional Analysis?

Cauchy Theorem in the Fréchet case

Let $P \ : \mathcal{U} \subset \mathcal{F} \to \mathcal{F}$ be a **smooth Banach map**. Then $\forall f_0 \in \mathcal{U}$, $\exists \tilde{\mathcal{U}} \ni f_0$ and $\varepsilon > 0$ s.t. $\forall f \in \tilde{\mathcal{U}}$

$$\frac{d}{dt} f = P(f)$$

has a unique solution with initial condition $f(0) = f$ on $0 \leq t \leq \varepsilon$ depending smoothly on t and f.

We say that P is a smooth Banach map between two Fréchet spaces $\mathcal{F}$ and $\mathcal{G}$ if we can factor $P = Q \circ R$ where $R : U \subset \mathcal{F} \to W \subset B$ and $Q : W \subset B \to V \subset \mathcal{G}$ are smooth maps and B is a Banach space.

Reference: R.S. Hamilton, *The inverse function Theorem of Nash and Moser*

What are the Tools from Functional Analysis?

Inverse function Theorem

Theorem

Let $f : \mathcal{U} \subset B_1 \to B_2$ be a $\mathcal{C}^1$-map between ***Banach*** *spaces. If $Df(a)$ is invertible at $a \in \mathcal{U}$, then there exists an open neighborhood $\mathcal{V}_a$ of $a \in \mathcal{U}$ and an open neighborhood $\mathcal{V}_{f(a)} \subset B_2$ suth that $f : \mathcal{V}_a \to \mathcal{V}_{f(a)}$ is a $\mathcal{C}^1$-diffeomorphism.*

Counterexample : $\exp : \mathrm{Lie}(\mathrm{Diff}(\mathbb{S}^1)) \to \mathrm{Diff}(\mathbb{S}^1)$ not locally onto.

Theorem (Nash-Moser)

Let $f : \mathcal{U} \subset F_1 \to F_2$ be a smooth tame map between ***Fréchet*** *spaces. Suppose that the equation for the derivative $Df(x)(h) = k$ has a unique solution $h = L(x)k$ for all $x \in \mathcal{U}$ and $\forall k \in F_2$ and that the family of inverses $L : \mathcal{U} \times F_2 \to F_1$ is a smooth tame map. Then f is locally invertible and each local inverse is a smooth tame map.*

What are the Tools from Functional Analysis?

Theorems :	Hilbert	Banach	Fréchet	Locally Convex
Banach-Picard	$\surd$	$\surd$	$\surd$	X
Open Mapping	$\surd$	$\surd$	$\surd$	F webbed G limit of Baire
Hahn-Banach	$\surd$	$\surd$	$\surd$	$\surd$
Cauchy Theorem	$\surd$	$\surd$	Hamilton	X
Inverse function	$\surd$	$\surd$	Nash-Moser	X

Which **Geometric structures** can we consider?

Riemannian $\subset$ Symplectic $\subset$ Poisson Geometry

Riemannian metric = smoothly varying inner product on a manifold M

$$\begin{array}{rccl} g_x : & T_xM \times T_xM & \to & \mathbb{R} \\ & (U, V) & \mapsto & g_x(U, V) \end{array}$$

strong Riemannian metric = for every $x \in M$, $g_x : T_xM \to (T_xM)^*$ is an isomorphism

weak Riemannian metric = for every $x \in M$, $g_x : T_xM \to (T_xM)^*$ is just injective

Levi-Cevita connection may not exist for a weak Riemannian metric.

Which **Geometric structures** can we consider?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Symplectic form = smoothly varying skew-symmetric bilinear form

$$\begin{array}{cccc} \omega_x & : & T_xM \times T_xM & \rightarrow & \mathbb{R} \\ & & (U, V) & \mapsto & \omega_x(U, V) \end{array}$$

with $dw = 0$ and $(T_xM)^{\perp_w} = \{0\}$

strong symplectic form = for every $x \in M$, $\omega_x \ : T_xM \to (T_xM)^*$ is an isomorphism

weak symplectic form = for every $x \in M$, $\omega_x \ : T_xM \to (T_xM)^*$ is just injective

Darboux Theorem does not hold for a weak symplectic form

Which **Geometric structures** can we consider?

Riemannian $\subset$ **Symplectic** $\subset$ Poisson Geometry

Hamiltonian Mechanics

(M, g) **strong Riemannian manifold**
$\Rightarrow (T^*M, \omega)$ **strong symplectic manifold**

Kinetic energy $=$ **Hamiltonian**

$$\begin{array}{cccc} H : & T^*M & \rightarrow & \mathbb{R} \\ & \eta_x & \mapsto & g_x(\eta_x^\sharp, \eta_x^\sharp) \end{array}$$

geodesic flow $=$ **flow of Hamiltonian vector field** $X_H : dH = \omega(X_H, \cdot)$

Which **Geometric structures** can we consider?

$$\left.\begin{array}{l}\text{Riemannian}\\ \text{Symplectic}\\ \text{Complex}\end{array}\right\} \subset \text{ Kähler} \subset \text{hyperkähler Geometry}$$

Complex structure $=$ smoothly varying endomorphism J of the tangent space s.t. $J^2 = -1$.

Integrable complex structure : s. t. there exists an holomorphic atlas
Formally integrable complex structure : with Nijenhuis tensor $= 0$

Newlander-Nirenberg Theorem is not true in general:
formal integrability does not imply integrability.

Which **Geometric structures** can we consider?

Riemannian $\subset$ Symplectic $\subset$ **Poisson Geometry**

Poisson bracket = family of bilinear maps
$\{\cdot,\cdot\}_U \; : \mathcal{C}^\infty(U) \times \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$, U open in the manifold M with

- skew-symmetry $\{f,g\}_U = -\{g,f\}_U$
- Jacobi identity $\{f,\{g,h\}_U\}_U + \{g,\{h,f\}_U\}_U + \{h,\{f,g\}_U\}_U = 0$
- Leibniz rule $\{f,gh\}_U = \{f,g\}_U h + g\{f,h\}_U$

A strong symplectic form defines a Poisson bracket by
$\{f,g\} = \omega(X_f, X_g)$ where $df = \omega(X_f,\cdot)$ and $dg = \omega(X_g,\cdot)$

A Poisson bracket may not be given by a bivector field

Reference: D. Beltita, T. Golinski, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics.

Which **Geometric structures** can we consider?

- **Contact structures in infinite-dimensions** (Poster of Fraser Sanders)
- **Multisymplectic** (Lecture of Narciso Román-Roy, talk of Marco Díaz Maceda)
- **Cosymplectic, Dirac** (Talk of Luca Vitagliano)
- k-**contact** (Talk of Javier de Lucas)
- **Jacobi** (Poster of Ángel Martinez-Muñoz)
- ...*the most important people are always forgotten*...

What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is :
"Never believe anything you have not proved yourself!"

- Differentials at $x \in M$ of globally defined scalar functions may not span $T_x^* M$
- The distance function associated to a Riemannian metric may by the zero function [Michor and Mumford].
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general : geodesic completeness $\neq$ metric completeness
- Darboux Theorem does not apply to weak symplectic forms in general
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations (even on a Hilbert space)
- a Poisson bracket may not be given by a bivector field (even on a Hilbert space)
- there are Lie algebras that can not be enlarged to Lie groups [Milnor or Neeb]
- partitions of unity may not exist
- at least 14 differents ways to define tensor products

Darboux's Theorem in infinite-dimension

Strong vs. Weak Symplectic Forms

- **Strong Form:** Induces an isomorphism $\omega^\flat : E \to E^*$.
- **Weak Form:** Induces only an injective mapping.

Darboux's Theorem

- **Strong case:** Using Moser's trick, Weinstein showed that one can locally transform a strong symplectic form into a constant form via a chart (A. Weinstein).
- **Weak case:** This theorem **fails** for weak symplectic forms (J.E. Marsden, 1972)

References:

Alan Weinstein, Symplectic structures on Banach manifolds, Bulletin of the American Mathematical Society, 75 (1969), 1040-1041.
J. Marsden, Darboux's Theorem fails for weak symplectic forms, Proceedings of the American Mathematical Society, Volume 32, Number 2, April 1972

The Mechanics of Moser's Trick (Weinstein's Adaptation)

Interpolate between constant form ω_0 and variable form ω_1 via $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. We seek a flow ϕ_t generated by a vector field X_t such that $\phi_t^* \omega_t = \omega_0$.

1. **Differentiate the Flow Condition:**

$$\frac{d}{dt}(\phi_t^* \omega_t) = \phi_t^* \left(\mathcal{L}_{X_t} \omega_t + \frac{d}{dt} \omega_t \right) = 0 \implies \mathcal{L}_{X_t} \omega_t + (\omega_1 - \omega_0) = 0$$

2. **Apply Cartan's Magic Formula:**
   Since ω_t is closed ($d\omega_t = 0$), the Lie derivative simplifies:

$$\mathcal{L}_{X_t} \omega_t = d(i_{X_t} \omega_t) + i_{X_t}(d\omega_t) = d(i_{X_t} \omega_t)$$

3. **Invoke the Poincaré Lemma:**
   Since $d(\omega_1 - \omega_0) = 0$, locally $\omega_1 - \omega_0 = d\alpha$. Substituting back yields:

$$d(i_{X_t} \omega_t + \alpha) = 0 \qquad \Longleftarrow \qquad i_{X_t} \omega_t = -\alpha$$

4. **The Strong Symplectic Inversion:**
   Identify $i_{X_t} \omega_t$ with the musical morphism $\omega_t^\flat(X_t)$. If ω is **strong**:

$$X_t = -(\omega_t^\flat)^{-1}(\alpha)$$

The Counterexample Setup

Moser's trick fails for weak symplectic forms

If ω is a **weak** form, $\omega_t^\flat$ is not surjective and can not be inverted. Consequently the vector field X_t cannot be defined.

Let H be a real Hilbert space.

- **The Operator:** Choose a compact, self-adjoint, positive operator $S : H \to H$.
- **Dense Range:** The range of S is dense but proper (e.g., $S = (1 - \Delta)^{-1}$).
- **The Weak Metric:** Define a variable metric $g(x)$ via the operator:

$$A_x = S + \|x\|^2 I$$

- **Split Behavior:**
  - By Fredholm alternative, A_x is **onto** for $x \neq 0$ (Strong).
  - At the origin ($x = 0$), $A_0 = S$ is **not onto** (Weak).

Symplectic Forms Induced by Metrics

Let M be a Banach manifold and $g(x) = \langle \cdot, \cdot \rangle_x$ a smooth, weak Riemannian metric on M. This metric induces a weak symplectic form Ω on the tangent bundle TM.

The Canonical Construction

- **Injection:** The metric g defines an injective smooth bundle map $g^\flat : TM \to T^*M$.
- **Canonical symplectic form:** Let ω be the canonical symplectic form on T^*M.
- **The Pullback Form:** $\Omega = \left(g^\flat\right)^* \omega$ is a weak symplectic form on TM

The Proof by Contradiction

- $M = H$ Hilbert space
- ω the canonical symplectic form on $T^*M = H \times H^*$
- Ω be the weak symplectic form induced by g on $H \times H$.

The Contradiction

- **Assumption:** A local chart exists making Ω constant.
- **Requirement:** The range of $\Omega^\flat_x$ must remain topologically constant across the chart.
- **At** $x \neq 0$**:** $\Omega^\flat_{(x,0)}$ is fully surjective (onto the dual of $H \times H$).
- **At** $x = 0$**:** $\Omega^\flat_{(0,0)}$ is **not surjective** because g_0 is weak.

Conclusion

No such coordinate chart can exist.

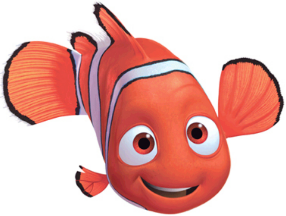
Bambusi-Darboux Theorem

Under some extra conditions (reflexive model space, the range of $\Omega_x^\flat$ is locally topologically constant) there exists a chart in which the weak symplectic form is constant.

Reference: Bambusi, *On the Darboux Theorem for weak symplectic manifolds*, Proceedings of the American Mathematical Society, Volume 127, Number 11, Pages 3383–3391.

Poisson bracket not given by a Poisson tensor

Queer Poisson Bracket = Poisson bracket not given by a Poisson tensor



Reference :

D. Beltiță, T. Goliński, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics, 2018.

Poisson bracket not given by a Poisson tensor

H separable Hilbert space

Kinetic tangent vector $X \in T_x H$ equivalence classes of curves $c(t)$, $c(0) = x$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Operational tangent vector $x \in H$ is a linear map $D : C_x^\infty(H) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(fg)(x) = Df\ g(x) + f(x)\ Dg$$

Poisson bracket not given by a Poisson tensor

Ingredients :

- Riesz Theorem
- Hahn-Banach Theorem
- compact operators $\mathcal{K}(H) \subsetneq \mathcal{B}(H)$ bounded operators
  $\Rightarrow \exists \ell \in \mathcal{B}(H)^*$ such that $\ell(\mathrm{id}) = 1$ and $\ell_{| \ \mathcal{K}(H)} = 0$.

Queer tangent vector [Kriegl-Michor]

Define $D_x : C_x^\infty(H) \to \mathbb{R}$, $D_x(f) = \ell(d^2(f)(x))$, where the bilinear map $d^2(f)(x)$ is identified with an operator $A \in \mathcal{B}(H)$ by Riesz Theorem

$$d^2(f)(x)(X, Y) = \langle X, AY \rangle$$

Then D_x is an operational tangent vector at $x \in H$ of order 2

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

$$d(fg)(x) = df(x).g(x) + f(x).dg(x)$$

$$\begin{aligned} d^2(fg)(x) &= d^2f(x).g(x) + df(x) \otimes dg(x) \\ &\quad + dg(x) \otimes df(x)) + f(x)d^2g(x) \end{aligned}$$

$$\begin{aligned} D_x(fg) &= \ell(d^2(fg)(x)) \\ &= \ell(d^2f(x)).g(x) + f(x)\ell(d^2g(x)) \\ &\quad + \ell(df(x) \otimes dg(x)) + \ell(dg(x) \otimes df(x)) \\ &= D_xf\ g(x) + f(x)\ D_xg \end{aligned}$$

Poisson bracket not given by a Poisson tensor

Theorem (D. Beltita, T. Golinski, A.B.Tumpach)

Consider $\mathcal{M} = H \times \mathbb{R}$. Denote points of $\mathcal{M}$ as (x, λ). The following Poisson bracket on $H \times \mathbb{R}$ can not be represented by a bivector field $\Pi : T^*\mathcal{M} \times T^*\mathcal{M} \to \mathbb{R}$:

$$\{f, g\}(x, \lambda) := D_x\left(f(\cdot, \lambda)\right) \frac{\partial g}{\partial \lambda}(x, \lambda) - \frac{\partial f}{\partial \lambda}(x, \lambda) D_x\left(g(\cdot, \lambda)\right)$$

The Hamiltonian vector field associated to $h(x, \lambda) = -\lambda$ is the queer operational vector field

$$X_h = \{h, \cdot\} = D_x$$

acting on $f \in C^\infty_x(H)$ by $D_x(f) = \ell(d^2(f)(x))$.

Definition

Consider a unital subalgebra $\mathcal{A} \subset \mathcal{C}^\infty(M)$ of smooth functions on a Banach manifold M, i.e. $\mathcal{A}$ is a vector subspace of $\mathcal{C}^\infty(M)$ containing the constants and stable under pointwise multiplication. An $\mathbb{R}$-bilinear operation $\{\cdot,\cdot\} \ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a **Poisson bracket** on the set of **admissible functions** $\mathcal{A}$ if it satisfies :

(i) anti-symmetry : $\{f, g\} = -\{g, f\}$;

(ii) Jacobi identity : $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$;

(iii) Leibniz formula : $\{f, gh\} = \{f, g\}h + g\{f, h\}$;

Let us recall the definition of a duality pairing.

Definition

Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ be two normed vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$$

be a continuous bilinear map. One says that the map $\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-}$ is a **duality pairing** between $\mathfrak{g}_+$ and $\mathfrak{g}_-$ if and only if it is **non-degenerate**, i.e. if the following two conditions hold :

$$\begin{array}{l} \left(\langle x, y \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} = 0, \quad \forall x \in \mathfrak{g}_+\right) \Rightarrow y = 0 \quad \text{and} \\ \left(\langle x, y \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} = 0, \quad \forall y \in \mathfrak{g}_-\right) \Rightarrow x = 0. \end{array}$$

Definition

Let M be a Banach manifold and $\mathcal{A}$ be a unital subalgebra of $\mathcal{C}^\infty(M)$. Denote by $\mathrm{J}^1(\mathcal{A})$ the subbundle of the cotangent bundle T^*M whose fiber over $p \in M$ is the space of differentials of functions in $\mathcal{A}$,

$$\mathrm{J}^1(\mathcal{A})_p = \{df_p \ : f \in \mathcal{A}\}.$$

We will say that the unital subalgebra $\mathcal{A}$ is **large** if the duality pairing between TM and T^*M restricts to a duality pairing between TM and $\mathrm{J}^1(\mathcal{A})$.

Remark

A unital subalgebra $\mathcal{A}$ of $\mathcal{C}^\infty(M)$ is large if and only if, for every $p \in M$, the pairing

$$\begin{array}{rcccc} \langle \cdot, \cdot \rangle_{J^1(\mathcal{A}), TM} & : & J^1(\mathcal{A})_p \times T_pM & \to & \mathbb{R} \\ & & (df_p, X) & \mapsto & \langle df_p, X \rangle_{TM^*, TM} \end{array}$$

is non-degenerate. This is the case if and only if $\mathrm{J}^1(\mathcal{A})_p$ separates points in the tangent space T_pM for any $p \in M$, i.e. if for every $p \in M$, a tangent vector $X \in T_pM$ such that

$$\langle df_p, X \rangle_{T^*M, TM} = 0, \quad \forall f \in \mathcal{A}$$

necessarly vanishes.

Definition

Let M be a Banach manifold endowed with a Poisson bracket on the space of admissible functions $\mathcal{A}$. A function $h \in \mathcal{A}$ is called **Hamiltonian** if the derivation $\{\cdot, h\}$ is represented by a smooth vector field, i.e. if there exists a smooth vector field X_h on M such that

$$X_h(f) := \langle df, X_h \rangle_{TM^*, TM} = \{f, h\},$$

for any $f \in \mathcal{A}$, in particular $\{\cdot, h\}$ is a first order operator. One calls X_h a Hamiltonian vector field associated to f.

Proposition:

Let M be a Banach manifold endowed with a Poisson bracket on the space of admissible functions $\mathcal{A}$. If $\mathcal{A}$ is large, then any Hamiltonian function $h \in \mathcal{A}$ is associated to a **unique** Hamiltonian vector field X_h.

Proof.

The uniqueness of the Hamiltonian vector field is a consequence of the hypothesis that $\mathcal{A}$ is large. Indeed, let X_1 and X_2 be two Hamiltonian vector fields associated to $h \in \mathcal{A}$. Then

$$\begin{aligned}(X_1 - X_2)(f) &= \langle df, X_1 - X_2\rangle_{TM^*,TM}\\ &= \langle df, X_1\rangle_{TM^*,TM} - \langle df, X_2\rangle_{TM^*,TM}\\ &= \{f,h\} - \{f,h\} = 0,\end{aligned}$$

for any $f \in \mathcal{A}$. Since $\mathcal{A}$ is large, the differentials of functions in $\mathcal{A}$ separates points in TM. It follows that $X_1 - X_2 = 0$. □

Definitions of infinite-dimensional Poisson manifolds

Reference	[7]	[8]	[26]	[3]	[36]	[24]
Definition number	11	14	18	20	23	27
1. Jacobi for $\mathcal{A}$	√	√	√	√	√	√
2. Jacobi for π on $\mathbb{F}^{\mathcal{A}}$					√	√
3. Localizability					√	√
4. Leibniz for $\mathcal{A}$	√	√	√	√	√	√
5. Separability of $\Gamma(TM)$			√	√	√	√
6. Separability of TM			√		√	√
7. $\mathbb{F}^{\mathcal{A}}$ vector bundle	√	√			√	√
8. Poisson anchor	√	√	√	√		
9. Poisson tensor	√	√			√	√
10. Hamiltonian v.f.	√	√	√	√		

Table 1. Summary of the properties of the different definitions of Poisson structures presented in Section 4. A √ sign means that the property is implied by the definition. Otherwise, the property is not necessarily satisfied.

Tomasz Goliński, Praful Rahangdale and A.B.T., *Poisson structures in the Banach setting: comparison of different approaches*

Appropriate definition of Banach Poisson manifold for our purpose

Definition of a Poisson tensor :
M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha, \beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) = 0.$$

Definition of a Poisson Manifold : A Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Banach symplectic manifold

Any Banach symplectic manifold (M, ω) is naturally a Banach Poisson manifold $(M, \mathbb{F}, \pi)$ with

1. $\mathbb{F} = \omega^\flat(TM)$;
2. $\pi \ : \omega^\flat(TM) \times \omega^\flat(TM) \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \omega(X_\alpha, X_\beta)$ where X_α and X_β are uniquely defined by $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$.

Definition

We will say that a Banach space $\mathfrak{b}$ is a **Banach Lie-Poisson space with respect to $\mathfrak{u}$** if $\mathfrak{u}$ is in duality with $\mathfrak{b}$ and is a Banach Lie algebra $(\mathfrak{u}, [\cdot, \cdot]_\mathfrak{u})$ which acts continuously on $\mathfrak{b}$ by coadjoint action.

Theorem [A. A. Odzijewicz, T. Ratiu] [A.B.T]

Consider a duality pairing $\langle \cdot, \cdot \rangle_{\mathfrak{u},\mathfrak{b}} : \mathfrak{u} \times \mathfrak{b} \to \mathbb{K}$ between two Banach spaces. Suppose that $\mathfrak{b}$ is a Banach Lie–Poisson space with respect to $\mathfrak{u}$. Denote by $\mathcal{A}$ the unital subalgebra of $\mathcal{C}^\infty(\mathfrak{b})$ generated by $\mathfrak{u}$. Define the Poisson bracket of two functions f, h in $\mathcal{A}$ by

$$\{f, h\}(x) = \pi_x(df_x, dh_x) = \langle x, [df_x, dh_x]_{\mathfrak{u}} \rangle_{\mathfrak{b},\mathfrak{u}},$$

where $x \in \mathfrak{b}$, and df and dh denote the Fréchet derivatives of f and h respectively. Then $(\mathfrak{b}, \mathrm{J}^1(\mathcal{A}), \pi)$ is a Banach Poisson manifold. If h is a smooth function on $\mathfrak{b}$ belonging to $\mathcal{A}$, the associated Hamiltonian vector field is given by

$$X_h(x) = -\mathrm{ad}^*_{dh_x} x \in \mathfrak{b}.$$

Finite-dimensional Poisson–Lie groups

References for finite-dimensional Poisson-Lie groups

V.G. Drinfel'd, '83
Y. Kosmann-Schwarzbach, F. Magri, '88
J.-H. Lu, '91

From Quantum groups to Poisson-Lie groups

Kirillov, 99, Merits and Demerits of the orbit method:

The first approximation to a quantum group is a so-called **Poisson-Lie group** which is an ordinary Lie group G with an additional structure: the Poisson brackets in the space of functions on G. It arises when we consider the (non-commutative) multiplication in the function algebra on a quantum group as a deformation of the ordinary multiplication in the function algebra on G.

Namely, this deformation has the form

$$f \circ g = fg + h\{f,\ g\} + o(h),$$

where h is the deformation parameter (considered either as a real number or as a formal variable) and $\{\ ,\ \}$ is the Poisson bracket on G compatible with group multiplication.

In terms of the corresponding bivector c compatibility means

$$c(xy) = L_x^* c(y) + R_y^* c(x),$$

where L_x, R_y are the operators of left and right shifts on G. It follows that c vanishes at the unit element $e \in G$ and is completely determined by its first order jet at e. The latter gives a Lie algebra structure $[\ ,\ ]_*$ on the space $T_e^*(G) = \mathfrak{g}^*$.

Poisson–Lie groups in the finite-dimensional case

Manin triples

$$\Updownarrow$$

Lie-bialgebras

$$\Updownarrow$$

connected simply connected Poisson–Lie groups

Poisson–Lie groups

Let us start with an example of a Manin triple...

$\mathfrak{u}(n)$ = Lie-algebra of the unitary group $\mathrm{U}(n)$
= space of skew-symmetric matrices

$\mathfrak{b}(n)$ = Lie-algebra of the Borel group $\mathrm{B}(n, \mathbb{C})$
= space of upper triangular matrices with real coef. on diagonal

Then the space $M(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$ of all complex matrices decomposes :

$$M(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$$

and for the non-degenerate symmetric bilinear continuous map $\langle \cdot, \cdot \rangle$ given by

$$\langle A, B \rangle = \operatorname{Im} \operatorname{Tr}(AB) = \text{ imaginary part of trace}(AB)$$

the blocks $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$ are both isotropic.

Poisson–Lie groups

Definition of a Manin triple

A **Banach Manin** triple consists of a triple of Banach Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ over a field $\mathbb{K}$ and a **non-degenerate symmetric bilinear** continuous map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that

1. the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is invariant with respect to the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$, i.e.

$$\langle [x, y]_{\mathfrak{g}}, z \rangle_{\mathfrak{g}} + \langle y, [x, z]_{\mathfrak{g}} \rangle_{\mathfrak{g}} = 0, \quad \forall x, y, z \in \mathfrak{g}; \tag{1.1}$$

2. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Banach spaces;
3. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Banach Lie subalgebras of $\mathfrak{g}$;
4. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Let G be a finite-dimensional Lie group.

Poisson-Lie groups

A **Poisson-Lie group** G is a Lie group equipped with a Poisson structure compatible with the group multiplication.

Example

Any Lie group G with $\{\cdot, \cdot\} = 0$ is a Poisson Lie group
Any compact Lie group is a Poisson-Lie group in a non-trivial way.

Definition

For any Poisson-Lie group (G, π), with Lie algebra $\mathfrak{g}$, one defines $\Pi_r^G := R_g^* \pi \ : G \to \Lambda^2 \mathfrak{g}$ as

$$\Pi_r^G(g)(\alpha, \beta) := \pi(R_g^* \alpha, R_g^* \beta), \quad \alpha, \beta \in \mathfrak{g}^*$$

Let (G, π) be a finite-dimensional Poisson-Lie group.

Facts

1. the fact that the Poisson tensor π is compatible with the group multiplication implies the following cocycle condition on $\Pi_r^G := R_g^* \pi$

$$\Pi_r^G(gh) = \Pi_r^G(g) + \mathrm{Ad}_g \Pi_r^G(h)$$

2. the derivative of $\Pi_r^G \ : G \to \Lambda^2 \mathfrak{g}$ at the unit of the group is a cocycle $\theta \ : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2 \mathfrak{g}$

$$\begin{aligned} \theta\left([x, y]\right)(\alpha, \beta) = \; & \theta(y)(\mathrm{ad}_x^* \alpha, \beta) + \theta(y)(\alpha, \mathrm{ad}_x^* \beta) \\ & - \theta(x)(\mathrm{ad}_y^* \alpha, \beta) - \theta(x)(\alpha, \mathrm{ad}_y^* \beta) \end{aligned}$$

   where $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.

3. the Jacobi identity verified by the Poisson structure implies that $\theta^* := [\cdot, \cdot]_{\mathfrak{g}}^* \ : \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$

Finite-dimensional Lie bialgebras

Lie bialgebra

Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. One says that $(\mathfrak{g}, \mathfrak{g}^*)$ form a **Lie bialgebra** if there is a Lie bracket $\Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ whose dual map $\mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$

$$\begin{aligned}\theta\left([x,y]\right)(\alpha,\beta) = \; & \theta(y)(\mathrm{ad}^*_x\alpha,\beta) + \theta(y)(\alpha,\mathrm{ad}^*_x\beta) \\ & -\theta(x)(\mathrm{ad}^*_y\alpha,\beta) - \theta(x)(\alpha,\mathrm{ad}^*_y\beta)\end{aligned}$$

Poisson–Lie groups in the finite-dimensional case

Manin triples

$$\Updownarrow$$

Lie-bialgebras

$$\Updownarrow$$

connected simply connected Poisson–Lie groups

Example

- $M(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$ with $\langle A, B\rangle = \mathrm{Im}\ \mathrm{Tr} AB$ is a Manin triple.
- $\mathrm{U}(n)$ and $\mathrm{B}(n,\mathbb{C})$ are dual Poisson-Lie groups with

$$\Pi_r^G(g)(x_1, x_2) = \langle p_{\mathfrak{u}}(g^{-1}x_1 g), p_{\mathfrak{b}}(g^{-1}x_2 g)\rangle.$$

- Moreover $GL(n,\mathbb{C}) = U(n) \times B(n)$ because of Iwasawa dec.
- This gives a dressing action

$$\varphi : B(n) \times U(n) \to U(n)$$

by $\varphi(b)(k) = k'$ where k' is the unique element of $U(n)$ such that $bk = k'b'$ with $b' \in B(n)$.

Reference :

J.-H. Lu, A. Weinstein, *Poisson Lie groups, Dressing Transformations, and Bruhat Decompositions*, Journal of Differential Geometry, 1990.

Bruhat-Poisson structure of finite-dimensional Grassmannians

Proposition :

- $\mathrm{U}(n)$ and $\mathrm{B}(n, \mathbb{C})$ are dual Poisson-Lie groups with

$$\Pi_r^G(g)(x_1, x_2) = \Im \mathrm{Tr}\, p_{\mathfrak{u}}(g^{-1}x_1 g)p_{\mathfrak{b}}(g^{-1}x_2 g).$$

- the Grassmannians $\mathrm{Gr}(p, n) = \mathrm{U}(n)/(U(p) \times U(n-p))$ are Poisson homogeneous spaces
- the right action of $\mathrm{B}(n, \mathbb{C})$ on $\mathrm{Gr}(p, n)$ is a Poisson map
- the symplectic leaves of $\mathrm{Gr}(p, n)$ are the orbits under the action of $\mathrm{B}(n, \mathbb{C})$

Reference :

J.-H. Lu, A. Weinstein, *Poisson Lie groups, Dressing Transformations, and Bruhat Decompositions*, Journal of Differential Geometry, 1990.

Summary

- Momentum maps can take values in Poisson-Lie groups
- $\mathfrak{u}(n)$ is part of a Manin triple $M(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$
- the induced structure on the group $\mathrm{U}(n)$ is that of a Poisson-Lie gp
- this Poisson structure goes down to the Grassmannians
- the Grassmannians inherites a Poisson action of the dual gp $\mathrm{B}(n, \mathbb{C})$