

Homogenous Boundaries of Geometric Structures

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Homogeneous Boundaries of Symplectic Manifolds

(M, ω) a symplectic manifold

$S \subseteq M$ a hypersurface (generalized boundary)

$Z \in \mathfrak{X}(M)$ a vector field s.t. $Z \lrcorner S$

①

$$(\mathcal{L}_Z \omega - \omega)|_S = 0$$

CONTACT BOUNDARY
THM

$\Rightarrow S$ possesses a contact structure $\left(\eta := i_S^*(\mathcal{L}_Z \omega) \text{ contact form} \right)$

②

$$\mathcal{L}_Z \omega|_S = 0$$

COSYMPLECTIC BOUNDARY
THM

$\Rightarrow S$ possesses a cosymplectic structure $\left((\eta := i_S^*(\mathcal{L}_Z \omega), \Omega := i_S^*(\omega)) \text{ cosymplectic pair} \right)$

Homogeneous Boundaries of Poisson Manifolds

(M, π) a Poisson manifold

$S \subseteq M$ a hypersurface (generalized boundary)

$Z \in \mathfrak{X}(M)$ a vector field s.t. $Z \lrcorner S$

$$\textcircled{1} \quad (\mathcal{L}_Z \pi + \pi) |_S = 0$$

$\Rightarrow S$ possesses a Jacobi structure (actually a Jacobi pair)

$$\textcircled{2} \quad \mathcal{L}_Z \pi |_S = 0$$

$\Rightarrow S$ possesses a Hamiltonian structure $\left(\begin{array}{l} \text{Poisson bivector} \\ + \\ \text{Poisson vector field} \end{array} \right)$

Homogeneous Boundaries of Complex Manifolds

(M, I) a complex manifold

$S \subseteq M$ a real hypersurface

$Z \in \mathfrak{X}(M)$ a vector field s.t. $Z \nmid S$

ODD COMPLEX BOUNDARY
THM

$\mathcal{L}_Z I|_S = 0 \Rightarrow S$ possesses a normal almost contact structure
(odd dimensional analogue of complex)

[TASHIRO 1963], [KENTARO, SHIGERU 1965]

How about more general / different geometries ?

- Dirac structures
- Generalized Complex structures
- G -structures
- ...

PLAN : Discussing a general Theorem that puts all these cases under the same umbrella.

Contact / Cosymplectic Geometry Revisited

STANDARD	ANALOGUES	LINE BUNDLE	PRINCIPAL $\mathbb{R}^*$ -BUNDLE
symplectic structure	contact structure	symplectic $\mathbb{L}$ -valued Atiyah Form	$\deg = 1$ homogeneous symplectic structure
symplectic structure	cosymplectic structure	symplectic $\mathbb{R}$ -valued Atiyah Form	$\deg = 0$ homogeneous symplectic structure
Poisson	Jacobi	✓	✓
complex	odd complex	✓	✓
...	...	...	...

Derivations

$E \rightarrow N$ a vector bundle

Derivation of E :

$$D : \Gamma(E) \rightarrow \Gamma(E)$$

- D is $\mathbb{R}$ -linear

- Leibniz rule: $D(f e) = X(f) e + f D(e)$, $X =: \sigma(D)$

symbol

Remarks:

- ▷ Derivations can be seen as infinitesimal automorphisms of E
- ▷ Derivations are sections of the gauge algebroid: $DE \Rightarrow N$;
- ▷ DE acts on E and on $\mathbb{R}_N \rightsquigarrow \Omega^*(DE, E)$, $\Omega^*(DE, \mathbb{R}_N)$
- ▷ $D\mathbb{R}_N \cong TN \oplus \mathbb{R}_N$

Atiyah forms

Contact Structures

N a manifold

Contact structure on N :

$$H \subseteq TN$$

maximally non-integrable hyperplane distribution

$L := TN/H$ line bundle

$\theta: TN \rightarrow L$ the projection

$$\theta \in \Omega^1(N, L)$$

$$\left\{ \begin{array}{l} \text{contact structures} \\ \text{with } TN/H = L \end{array} \right\} \iff \left\{ \begin{array}{l} \text{closed non-degenerate} \\ \omega \in \Omega^2(DL, L) \\ \text{symplectic Atiyah forms} \end{array} \right\}$$

$$\begin{array}{c} H \\ \sigma(\ker \theta|_H) \end{array}$$

$$\longrightarrow$$

$$\longleftarrow$$

$$\begin{array}{c} d_{DL}(\theta \circ \sigma) \\ \omega \end{array}$$

Cosymplectic Structures

Cosymplectic Structure on N :

$$\Omega \in \Omega^2(N), \quad \eta \in \Omega^1(N)$$

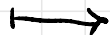
$\eta \neq 0$ & $\Omega|_{\ker \eta}$ non degenerate

$$(\Omega, \eta)$$

$$+ \quad d\Omega = d\eta = 0$$

$$\{\text{cosymplectic structures on } N\} \iff \left\{ \begin{array}{l} \text{closed, non-degenerate} \\ \omega \in \Omega^2(DR_N, \mathbb{R}_N^{\text{trivial}}) \end{array} \right\}$$

$$(\Omega, \eta)$$



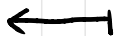
$$\begin{matrix} TN & \mathbb{R}_N \\ \mathbb{R}_N & \end{matrix} \begin{pmatrix} \Omega & \eta \\ -\eta & 0 \end{pmatrix}$$

ω

remember

$$DR_N \cong TN \oplus \mathbb{R}_N$$

$$(\omega|_{TN}, \pm \omega|_{TN})$$



"Conceptual" Cosymplectic structures = $\left\{ \begin{array}{l} \text{closed, non-degenerate} \\ \omega \in \Omega^2(DL, \mathbb{R}_N) \end{array} \right\}$
 on a line bundle $L \rightarrow N$ (ω -symplectic impls w/ non-orient. singular locus)

Homogeneity Structures

$$\begin{array}{ccc}
 \{\text{line bundles}\} & \rightleftharpoons & \{\text{principal } \mathbb{R}^* \text{-bundles}\} \\
 L & \xrightarrow{\quad} & L^* \setminus 0 =: \tilde{L} \quad \cup \mathbb{R}^* : h \\
 (P \times \mathbb{R}) / \mathbb{R}^* & \xleftarrow{\quad} & P
 \end{array}$$

Dictionary

$$r \in \mathbb{R}^*$$

$$\begin{array}{lll}
 \Gamma(L) & \rightleftharpoons & \text{functions of degree 1 on } \tilde{L} : h_r^* f = r \cdot f \\
 \Gamma(DL) & \rightleftharpoons & \text{vector fields of degree 0 on } \tilde{L} : h_r^* X = X \\
 \Omega^1(DL, L) & \rightleftharpoons & \text{differential forms of degree 1 : } h_r^* \omega = r \omega \\
 \Omega^1(DL, \mathbb{R}_N) & \rightleftharpoons & \text{differential forms of degree 0 : } h_r^* \omega = \omega
 \end{array}$$

$\xi \in \mathfrak{X}(\tilde{L})$ the infinitesimal generator of the principal action h

Homogeneous Symplectic Manifolds

$$\left\{ \begin{array}{l} \text{contact structures} \\ \text{with } \mathbb{T}N/H = L \end{array} \right\} \iff \left\{ \begin{array}{l} \text{homogeneous symplectic structures} \\ \text{on } \tilde{L} \text{ of } \deg = 1 \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{cosymplectic structures} \\ \text{on } L \rightarrow N \end{array} \right\} \iff \left\{ \begin{array}{l} \text{homogeneous symplectic structures} \\ \text{on } \tilde{L} \text{ of } \deg = 0 \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{odd complex structures} \\ \text{on } L \rightarrow N \end{array} \right\} \iff \left\{ \begin{array}{l} \text{homogeneous complex structures} \\ \text{on } \tilde{L} \text{ of } \deg = 0 \end{array} \right\}$$

complex structure in $DL \Rightarrow N$

homogeneity = equivariance

$$\begin{array}{ccc} \begin{array}{c} \tilde{L} \\ \uparrow \omega \\ \wedge^2 T^* \tilde{L} \end{array} & \deg \omega = 0 \iff \omega \text{ equivariant wrt } h^{\text{ctan}} : \mathbb{R}^* \curvearrowright \wedge^2 T^* \tilde{L} & \\ & h_r^{\text{ctan}}(\omega) = h_{r^{-1}}^*(\omega) & \\ \begin{array}{c} \tilde{L} \\ \downarrow \omega \\ \wedge^2 T^* \tilde{L} \end{array} & \deg \omega = 1 \iff \omega \text{ equivariant wrt } h^{\text{ph}} : \mathbb{R}^* \curvearrowright \wedge^2 T^* \tilde{L} & \\ & h_r^{\text{ph}}(\omega) = r h_{r^{-1}}^*(\omega) & \end{array}$$

Contact Boundaries of Symplectic Manifolds

(M, ω) a symplectic manifold

$S \subseteq M$ a hypersurface (generalized boundary)

$Z \in \mathfrak{X}(M)$ a vector field s.t. $Z \lrcorner S$

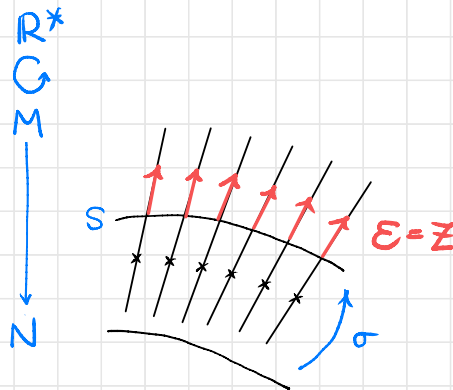
$$(\mathcal{L}_Z \omega - \omega)|_S = 0$$

Using a tubular neighborhood we can assume

▷ $M = \tilde{L}$ for some (trivial) line bundle $L \rightarrow N$

▷ $S = \text{im } \sigma$ for some section $M \rightrightarrows N$

▷ $Z = \mathcal{E}$

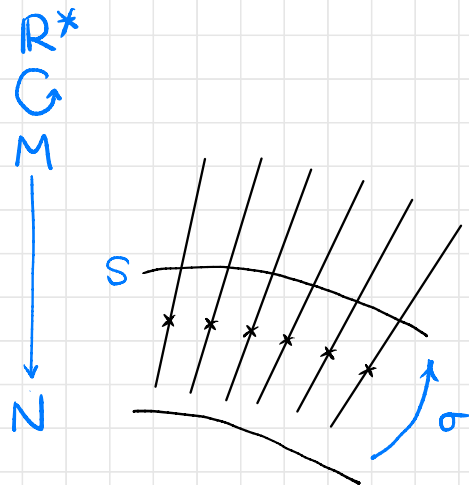


Consider the action

$$\mathbb{R}^{Ph}: \mathbb{R}^* \hookrightarrow \wedge^2 T^*M$$

denote $\mathcal{E}: \Omega^2(M) \rightarrow \Omega^2(M)$

the infinitesimal generator: $\mathcal{E}(\omega) = \mathcal{L}_Z \omega - \omega$



Remark. $(\mathcal{L}_Z \omega - \omega)|_S = 0 \iff \dot{\omega} := \mathcal{E}\omega|_S = 0$

Extend $\omega|_S$ equivariantly to a homogeneous $\tilde{\omega} \in \Omega^2(M)$ of $\deg = 1$

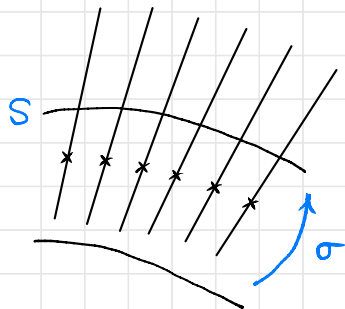
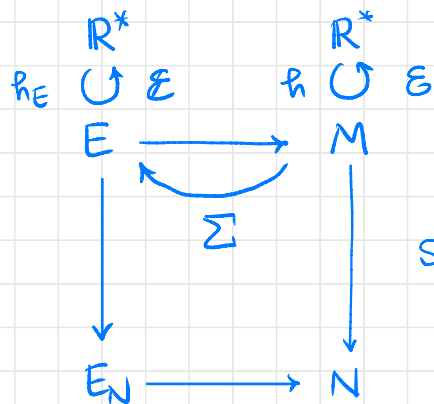
$\Rightarrow \tilde{\omega}$ is a homogeneous almost symplectic structure.

It is homogeneous symplectic if $d\tilde{\omega} = 0$ but

$$d\tilde{\omega} = \widetilde{d\omega} + \mathcal{O}(\dot{\omega}) = 0 \Rightarrow \mathcal{S} \text{ is contact}$$



An Abstract Picture



- ▷ M a principal $\mathbb{R}^*$ -bundle
- ▷ S the image of a section σ
- ▷ E an $\mathbb{R}^*$ -equivariant VB (e.g. $\Lambda^2 T^*M$)
- ▷ Σ a section of $E \rightarrow M$ (e.g. ω)

$$\mathcal{D}(\Sigma) = 0$$

for some (non-linear)

$\mathbb{R}^*$ -equivariant

k -th order DO $\mathcal{D}$ (e.g. d)

Extend $\Sigma|_S$ equivariantly to a $\tilde{\Sigma} \in \Gamma(E)$.

Question: when is $\mathcal{D}(\tilde{\Sigma}) = 0$?

Thm. $\mathcal{D}(\tilde{\Sigma}) = \tilde{\mathcal{D}}(\Sigma) + \mathcal{O}(\Sigma^{(1)}, \dots, \Sigma^{(k)})$ where $\Sigma^{(q)} = \mathcal{E}^q \Sigma|_S$

Generalized Complex Manifolds

M a manifold

$\pi M = TM \oplus T^*M$ generalized tangent bundle

$\triangleright$ $pr : \pi M \rightarrow TM$ (a Courant algebroid)

$\triangleright$ $\langle\langle -, - \rangle\rangle : TM \otimes TM \rightarrow \mathbb{R}_M$

$$\langle\langle \begin{pmatrix} x_1 \\ \eta_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \eta_2 \end{pmatrix} \rangle\rangle = \langle \eta_1, x_2 \rangle + \langle \eta_2, x_1 \rangle$$

$\triangleright$ $[-, -] : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$

$$[\begin{pmatrix} x_1 \\ \eta_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ \eta_2 \end{pmatrix}] = \begin{pmatrix} [x_1, x_2] \\ \mathcal{L}_{x_1} \eta_2 - \mathcal{L}_{x_2} \eta_1 \end{pmatrix}$$

Generalized Complex Structure on M

$\mathbb{I} : \pi M \rightarrow \pi M$, $\mathbb{I}$ orthogonal wrt $\langle\langle -, - \rangle\rangle$ & $\mathbb{I}^2 = -1$ & $\tau_{\mathbb{I}} = 0$

$$\mathbb{I} = \begin{matrix} TM & T^*M \\ \begin{pmatrix} \phi \\ \omega \end{pmatrix} \\ T^*M \end{matrix} \begin{matrix} TM \\ \pi \\ T^*M \end{matrix} \begin{matrix} \phi^t \\ -\phi^t \end{matrix} \rightarrow \text{Poisson structure on } M$$

non-linear
integrability
condition

Generalized Contact Bundles

[IGLESIAS, WADE 2005]
[V., WADE 2016]

$$L \rightarrow N$$

line bundle

$$\mathbb{D}L = \mathbb{D}L \oplus J^1 L \quad \text{omni-Lie algebroid}$$

$$J^1 L \cong \text{Hom}(\mathbb{D}L, L)$$

$$\triangleright \text{pr} : \mathbb{D}L \rightarrow \mathbb{D}L \quad (\text{an } L\text{-Courant algebroid})$$

$$\triangleright \langle\langle -, - \rangle\rangle : \mathbb{D}L \otimes \mathbb{D}L \rightarrow L$$

$$\langle\langle (\psi_1), (\psi_2) \rangle\rangle = \langle \psi_1, D_2 \rangle + \langle \psi_2, D_1 \rangle$$

$$\triangleright [-, -] : \Gamma(\mathbb{D}L) \times \Gamma(\mathbb{D}L) \rightarrow \Gamma(\mathbb{D}L)$$

$$[(\psi_1), (\psi_2)] = \left(\begin{matrix} [D_1, D_2] \\ \iota_{D_1} \psi_2 - \iota_{D_2} \psi_1 \end{matrix} \right)$$

Generalized Contact Structure on L

$$K : \mathbb{D}L \rightarrow \mathbb{D}L, \quad K \text{ orthogonal wrt } \langle\langle -, - \rangle\rangle \text{ \& } K^2 = -1 \text{ \& } \tau_K = 0$$

$$K = \begin{matrix} \mathbb{D}L \\ \mathbb{D}L \end{matrix} \begin{pmatrix} \begin{matrix} \mathbb{D}L \\ \chi \end{matrix} \\ \begin{matrix} J^1 L \\ \omega \end{matrix} \end{pmatrix} \begin{matrix} \begin{matrix} J^1 L \\ J \end{matrix} \\ -\chi^t \end{matrix} \rightarrow \text{Jacobi structure on } L$$

non-linear
integrability
condition

Generalized Contact Boundaries

$$\left\{ \begin{array}{l} \text{generalized contact} \\ \text{structures on } \tilde{L} \end{array} \right\} \iff \left\{ \begin{array}{l} \text{homogeneous generalized} \\ \text{complex structures on } \tilde{L} \\ \text{wrt. } h^{\tan} \oplus h^{ph} : \pi \tilde{L} \rightarrow \pi \tilde{L} \end{array} \right\}$$

Thm

(M, Π)

a generalized complex manifold

$S \subseteq M$

a hypersurface

$Z \in \mathfrak{X}(M)$

a vector field s.t. $Z \lrcorner S$

GENERALIZED CONTACT BOUNDARY
THM

$$\mathcal{L}(\Pi)|_S = 0$$

$$\left(\mathcal{L} \begin{pmatrix} \phi & \pi \\ \omega & -\phi^* \end{pmatrix} = \begin{pmatrix} \mathcal{L}_Z \phi & \mathcal{L}_Z \pi + \pi \\ \mathcal{L}_Z \omega - \omega & -\mathcal{L}_Z \phi^* \end{pmatrix} \right)$$

$\Rightarrow$ S is generalized contact

Proof

$$E = \text{End } \pi^* TM, \quad h_E = \text{End}(h^{\tan} \oplus h^{ph})$$

Generalized Cosymplectic Bundles

$$L \rightarrow N$$

line bundle

$$\mathbb{D}^{\text{cs}} L = DL \oplus (DL)^*$$

double of the gauge algebroid

$$\triangleright \rho : \mathbb{D}^{\text{cs}} L \rightarrow TM$$

$$\triangleright \langle\langle -, - \rangle\rangle : \mathbb{D}^{\text{cs}} L \otimes \mathbb{D}^{\text{cs}} L \rightarrow \mathbb{R}_M$$

$$\langle\langle \begin{pmatrix} D_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} D_2 \\ \psi_2 \end{pmatrix} \rangle\rangle = \langle \psi_1, D_2 \rangle + \langle \psi_2, D_1 \rangle$$

$$\triangleright [[-, -]] : \Gamma(\mathbb{D}^{\text{cs}} L) \times \Gamma(\mathbb{D}^{\text{cs}} L) \rightarrow \Gamma(\mathbb{D}^{\text{cs}} L)$$

$$[[\begin{pmatrix} D_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} D_2 \\ \psi_2 \end{pmatrix}]] = \begin{pmatrix} [D_1, D_2] \\ i_{D_1} \psi_2 - i_{D_2} \psi_1 \end{pmatrix}$$

Generalized Cosymplectic Structure on L

$$J : \mathbb{D}^{\text{cs}} L \rightarrow \mathbb{D}^{\text{cs}} L, \quad J \text{ orthogonal wrt } \langle\langle -, - \rangle\rangle \text{ \& } J^2 = -1 \text{ \& } \tau_J = 0$$

$$J = \begin{pmatrix} DL & (DL)^* \\ \chi & \pi \\ (DL)^* & -\chi^+ \end{pmatrix} \rightarrow \text{Hamiltonian structure}$$

non-linear
integrability
condition

Generalized Cosymplectic Boundaries

$$\left\{ \text{generalized } \text{cosymplectic} \text{ contact structures on } L \right\} \iff \left\{ \begin{array}{l} \text{homogeneous generalized} \\ \text{complex structures on } \tilde{L} \\ \text{wrt. } h^{\text{tan}} \oplus h^{\text{ctan}} : \pi\tilde{L} \rightarrow \pi\tilde{L} \end{array} \right\}$$

Thm

(M, Π)

a generalized complex manifold

$S \subseteq M$

a hypersurface

$Z \in \mathfrak{X}(M)$

a vector field s.t. $Z \pitchfork S$

GENERALIZED COSYMPLECTIC BOUNDARY
THM

$$\mathcal{E}(\Pi)|_S = 0$$

$$\left(\mathcal{E} \begin{pmatrix} \phi & \pi \\ \omega & -\phi^* \end{pmatrix} = \begin{pmatrix} \mathcal{L}_Z \phi & \mathcal{L}_Z \pi + \pi \\ \mathcal{L}_Z \omega & -\mathcal{L}_Z \phi^* \end{pmatrix} \right)$$

$\Rightarrow$ S is generalized ~~contact~~ cosymplectic

Proof

$$E = \text{End } \pi^* TM, \quad h_E = \text{End} (h^{\text{tan}} \oplus h^{\text{ctan}})$$

Homogeneous Boundary of G -structures

$h: \mathbb{R}^* \curvearrowright \tilde{L} \rightarrow N$ a principal $\mathbb{R}^*$ -bundle

There is a notion of **HOMOGENEOUS G -STRUCTURE** on $\tilde{L}$
contact, cosymplectic, odd complex, ... [TORTORELLA, YUDILEVICH, V. 2020]

Conjecture

(M, Σ) a manifold with a brussly integrable G -structure

$S \subseteq M$ a hypersurface

$Z \in \mathfrak{X}(M)$ a vector field s.t. $Z \lrcorner S$

APPROPRIATE INFINITESIMAL EQUIVARIANCE CONDITION ALONG S

$\Rightarrow S$ possesses a brussly integrable homogeneous G -structure

Thank You !!!

