Geodesics, electrostatics and polyhedra

Gonçalo Oliveira
(joint work with Chris Fillmore and Herbert Edelsbrunner)

Instituto Superior Técnico, Universidade de Lisboa

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  ... but the metrics are not known explicitly ...

Act 1: A problem in electrostatics.

Playing with electric charges

- Let $P_1, \ldots, P_n \in \mathbb{R}^3$, and ϕ the *electric potential* generated by charge strengths $k_1, \ldots, k_n \in \mathbb{R}$ located at these points

$$\phi(x) = \varepsilon^{-1} + \sum_{i=1}^{n} \frac{k_i}{|x - P_i|},$$

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$$E = -\nabla\phi$$

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- How many electrostatic points are there?

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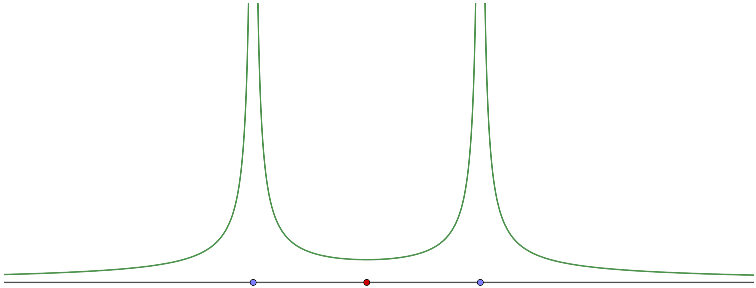

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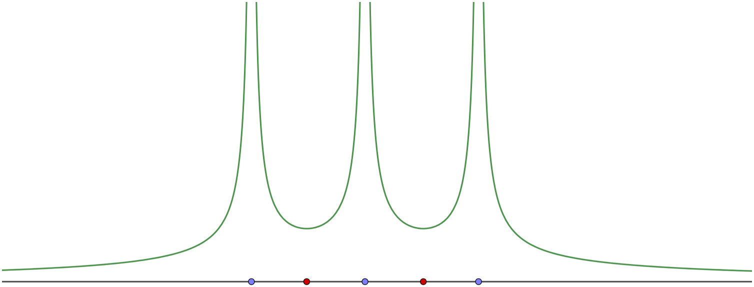

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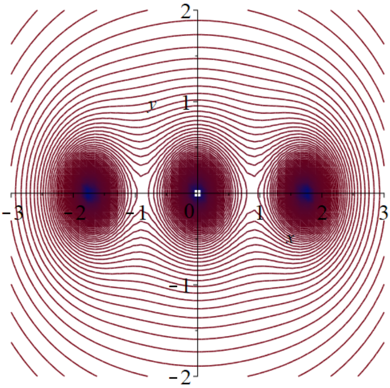

Figure: There are two electrostatic points (saddles of index 2).

$$c_1 = 0, \quad c_2 = 2.$$

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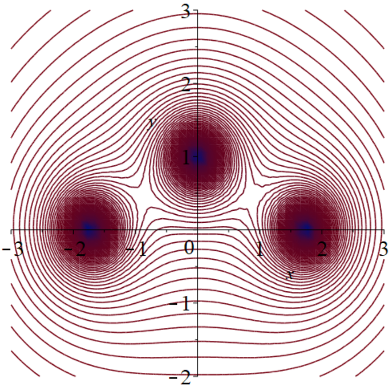

Figure: Move one a little, there are still two electrostatic points.

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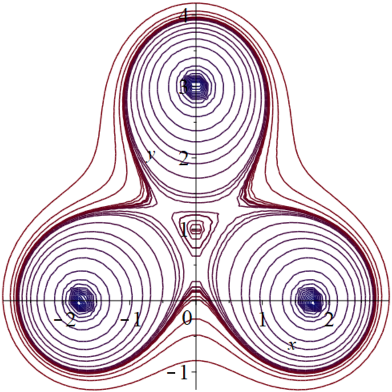

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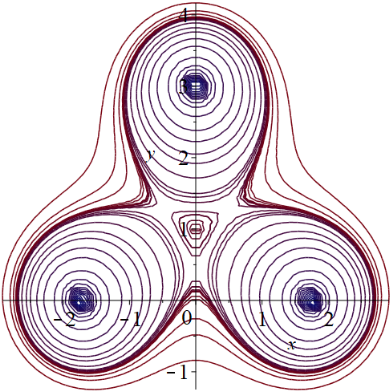

Figure: When the charges form an equilateral triangle $\Longrightarrow$ 4 electrostatics points.

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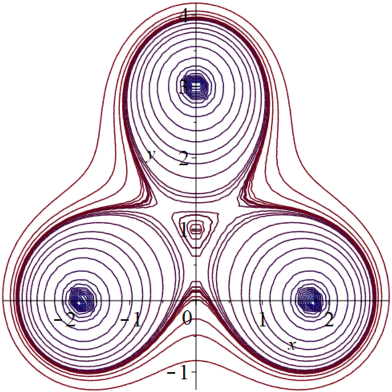


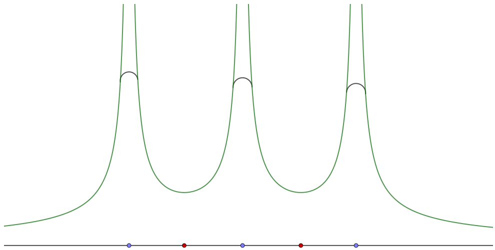
Figure: When the charges form an equilateral triangle $\Longrightarrow$ 4 electrostatics points.

- The one at the center has index 1, and the remaining three have index 2.

$$c_1 = 1, \quad c_2 = 3.$$

The Maxwell problem

- If all the charges are positive, modify ϕ to $\tilde{\phi} : S^3 \to \mathbb{R}$:



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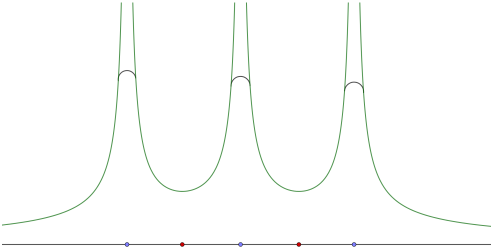

such that

$$\tilde{\phi}(x) = \begin{cases} \phi(x) \text{ if } |x - P_i| > \varepsilon, \\ \cap \qquad \text{if } |x - P_i| \leq \varepsilon, \\ 0 \qquad \text{if } x = \infty, \end{cases}$$

introduces no extra crit pts other than max at the P_i's and a min at ∞.

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- (Maxwell 1873) What is the maximum number of electrostatics points?

$$\text{Maxwell conjectures} \leq (n-1)^2.$$

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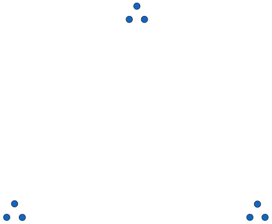

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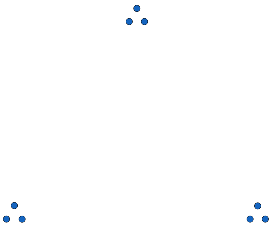

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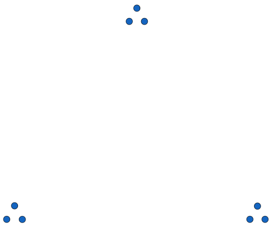
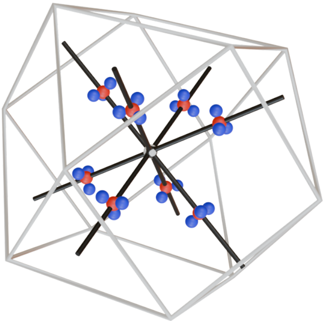


- Has ratio $\frac{3\times 4+4}{3\times 3} = 1.77(7)$.
- Iterating, the ratio converges to 2.

The cuboctahedron (C. Fillmore, H. Edelsbrunner, –, 2025)

- $n = 12$ unit charges at the vertices of a (perturbed) cuboctahedron yield:

$$n = 12, \quad c_1 = 8 + 5 = 13, \quad c_2 = 24.$$



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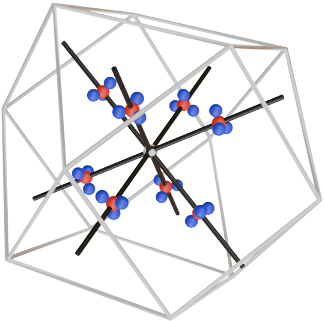

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- Ratio of $\frac{37}{12}$. When iterated, the ratio converges to $\frac{37}{11} = 3.(36)$.

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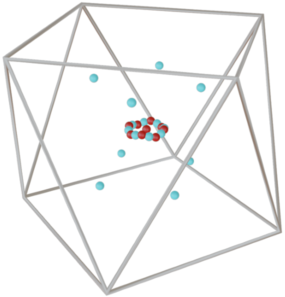
- $n = 8$ unit charges at the vertices of a quadrangular antiprism yields:

$$n = 8, \quad c_1 = 9, \quad c_2 = 16.$$



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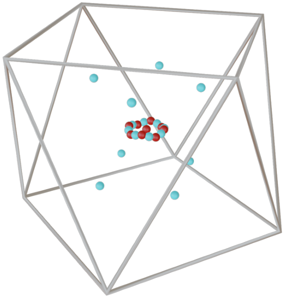

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- Ratio of $\frac{25}{8} = 3.125$. When iterated, the ratio converges to $\frac{25}{7} \approx 3.57$

Act 2: Relation to geodesics on Calabi–Yau manifolds.

Gibbons–Hawking A_{n-1}-manifolds

- Let ϕ be generated by unit charges at $P_1, \ldots, P_n$. $\exists$ circle bundle
$$\mu : \hat{X} \to \mathbb{R}^3 \backslash \{P_1, \ldots, P_n\},$$
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- Recall that
$$\Delta\phi = \sum_{i=1}^{n} \delta_{P_i},$$
and so
$$d *_{\mathbb{R}^3} d\phi = \sum_{i=1}^{n} \delta_{P_i} \mathrm{vol}_g = 0, \text{ away from } \{P_1, \ldots, P_n\}.$$

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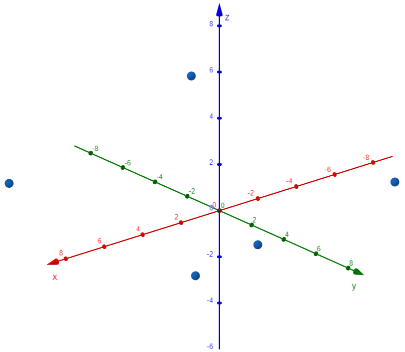

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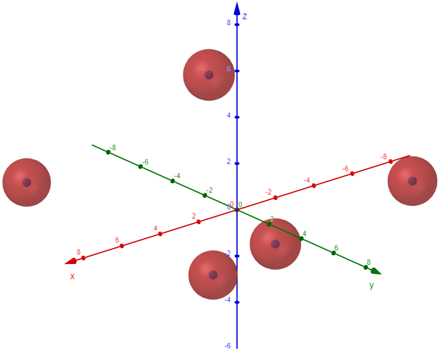

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with a connection η whose curvature is $*_{\mathbb{R}^3} d\phi$.
- Consider the metric
$$g = \phi^{-1}\eta^2 + \phi\mu^* g_{\mathbb{R}^3},$$
which is Calabi–Yau and complete if we add points to $\hat{X}$ above the P_i's.

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- Consider the metric

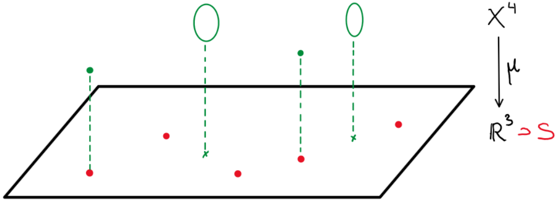
$$g = \phi^{-1}\eta^2 + \phi\mu^* g_{\mathbb{R}^3},$$

  which is Calabi–Yau and complete if we add points to $\hat{X}$ above the P_i's.

- Obtain

$$\mu : X \to \mathbb{R}^3,$$

  with $\mu^{-1}(p)$ a point iff $p \in \{P_1, \ldots, P_n\}$.



Geodesics on A_{n-1}-manifolds (Jason Lotay, –, 2020)

- Recall that $g = \phi^{-1}\eta^2 + \phi\mu^* g_{\mathbb{R}^3}$.

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- If $|P_i - P_j| \gg 1$, we find that it has index

$$3 - \mathrm{ind}_p(\phi).$$

Act 3: Moving to the K3 surface.

General architecture of Foscolo's K3 surfaces (2019)

As a Riemannian manifold

$$X_\varepsilon = K_\varepsilon \cup \bigcup_{i=1}^{n} N_\varepsilon^i \cup \bigcup_{j=1}^{8} M_\varepsilon^j,$$

with hyperkähler metric $\approx$ gluing of Gibbons–Hawking metrics (as $\varepsilon \searrow 0$).

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$$\sum_{i=1}^{n} 2k_i + \sum_{j=1}^{8} (2m_j - 4) = 0. \quad \text{(balancing condition)}$$

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- The Bubble Regions: $\varepsilon^{-2} N_\varepsilon^i$ and $\varepsilon^{-2} M_\varepsilon^j$ converge to fixed ALF gravitational instantons N^i of type $A_{k_i - 1}$ and M^j of type D_{m_j}.

Main Theorem: Geodesics on Collapsing K3 Surfaces

Let $n \geq 1$, $k_1, \ldots, k_n$, $m_1, \ldots, m_8$ satisfy the balancing condition and a (generic) collection of bubbles. Then, $\exists\, \varepsilon_0 \ll 1$ s.t. $\forall \varepsilon < \varepsilon_0$:

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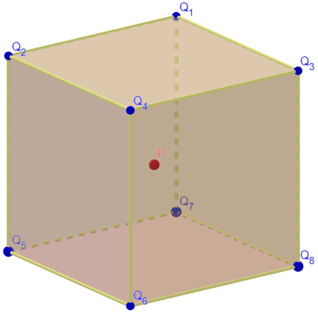
- In M^j_ε: $\exists$ at least 3 geodesics if $m_j = 0$ (don't know their index).

Example

$$X_\varepsilon = K_\varepsilon \cup N_\varepsilon \cup \bigcup_{j=1}^{8} M_\varepsilon^j \ , \quad \text{with}$$

- K_ε a circle bundle over a $\mathbb{Z}_2$-quotient of the punctured torus



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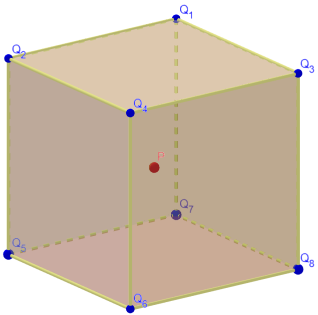

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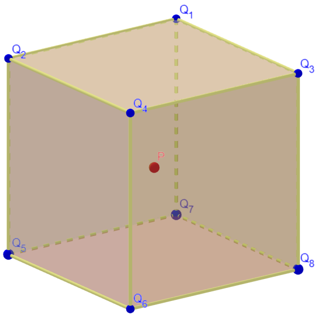

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- Main thm $\implies$ 5 geodesics of index one & 2 of index two in K_ε.

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- Results in

$$\#\{\text{index one geodesics in } N_\varepsilon\} \geq 16 \times 2 + 1 = 33$$
$$\#\{\text{index two geodesics in } N_\varepsilon\} \geq 8 \times 2 = 16.$$

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- In total, we found

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\#\{\text{index one geodesics in } X_\varepsilon\} &\geq \#\{\text{index one geodesics in } N_\varepsilon\} \\
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\#\{\text{index two geodesics in } X_\varepsilon\} &\geq \#\{\text{index two geodesics in } N_\varepsilon\} \\
&\quad + \#\{\text{index two geodesics in } K_\varepsilon\} \\
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&= 21.
\end{aligned}$$

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Thank you!