# Non-lorentzian spacetimes

## Homogeneous spacetimes with low coisotropy

José Miguel Figueroa-O'Farrill, IFWGP XXXIV, Universitat Rovira i Virgili, Tarragona 2/9/2026


Exactly 7 years ago
in IFWGP XXVIII!
ICMAT

# Homogeneous kinematical spacetimes

José Miguel Figueroa-O'Farrill

**XXVIII International Fall Workshop on Geometry and Physics**
**ICMAT, Madrid, 2 September 2019**

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# Based on the following papers

reviews KLAs spatially isotropic Lifshitz/conformal/coisotropy-one

arXiv:1711.05676, arXiv:1711.06111, arXiv:1711.07363, arXiv:1809.03603, arXiv:2204.13609

arXiv:1802.04048 (with **Tomasz Andrzejewski**)

arXiv:1809.01224 (with **Stefan Prohazka**)

arXiv:1905.00034, arXiv:2206.11806 (with **Ross Grassie** and **Stefan Prohazka**)

arXiv:2206.12177 (with **Eric Bergshoeff** and **Joaquim Gomis**)

work in progress (with **Jarah Fluxman**, **Ross Grassie** and **Stefan Prohazka**)

work in progress (with **Juanma Lorenzo-Naveiro**)

# Plan

- The original spacetime
- Kinematical Lie algebras
- Spatially isotropic homogeneous spacetimes
- Breaking isotropy:
  - Lifshitz Lie algebras
  - Classification of $(3,2)$-KLAs
- Outlook

# The original spacetime

# The birth of "spacetime"

"Von Stund' an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und noch eine Art Union der beiden soll Selbständigkeit bewahren."

Hermann Minkowski

# The birth of “spacetime”

> “Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.”
>
> Hermann Minkowski

# Minkowski spacetime

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- $c$ the ***speed of light***

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$$
\begin{aligned}
[L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho} + \eta_{\mu\sigma} L_{\nu\rho} \\
[L_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\
[P_\mu, P_\nu] &= 0
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- Lie algebra $\mathfrak{g} = \langle L_{ab}, B_a := L_{0a}, P_a, H := P_0 \rangle$ with $a, b = 1,2,3$

$$[L_{ab}, H] = 0$$
$$[L_{ab}, B_c] = \delta_{bc} B_a - \delta_{ac} B_b$$
$$[L_{ab}, P_c] = \delta_{bc} P_a - \delta_{ac} P_b$$
$$[L_{ab}, L_{cd}] = \delta_{bc} L_{ad} - \delta_{ac} L_{bd} - \delta_{bd} L_{ac} + \delta_{ad} L_{bc}$$

$$[B_a, B_b] = c^2 L_{ab}$$
$$[B_a, P_b] = \delta_{ab} H$$
$$[B_a, H] = c^2 P_a$$
$$[P_a, P_b] = 0$$
$$[H, P_a] = 0$$

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- A **kinematical Lie algebra** $\mathfrak{g}$ is a real Lie algebra with generators $L_{ab}, B_a, P_a, H$ such that $L_{ab} = -L_{ba}$ generates a subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$, relative to which $\mathfrak{g} = \mathbf{ad} \oplus 2V \oplus \mathbb{R}$, where $\mathbf{ad} \cong \wedge^2 V$ and $V \cong \mathbb{R}^d$

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- **Morand** (2023) introduced a general class of Lie algebras — $(s, v)$**-KLAs** — with a similar definition except that $\mathfrak{g} = \mathbf{ad} \oplus vV \oplus s\mathbb{R}$. So kinematical Lie algebras are $(1,2)$-KLAs

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- **JMF** (2018): $d > 3$
- **Andrzejewski + JMF** (2018): $d = 2$

# Kinematical Lie algebras (generic $d$)

| Name | Additional nonzero Lie brackets | | | | | Comments |
|---|---|---|---|---|---|---|
| $\mathfrak{s}$ | | | | | | |
| $\mathfrak{g}$ | $[\mathsf{H}, \mathbf{B}] = -\mathbf{P}$ | | | | | |
| $\mathfrak{n}^0$ | $[\mathsf{H}, \mathbf{B}] = \mathbf{B} + \mathbf{P}$ | $[\mathsf{H}, \mathbf{P}] = \mathbf{P}$ | | | | |
| $\mathfrak{n}^+_\gamma$ | $[\mathsf{H}, \mathbf{B}] = \gamma\mathbf{B}$ | $[\mathsf{H}, \mathbf{P}] = \mathbf{P}$ | | | | $\gamma \in [-1, 1]$ |
| $\mathfrak{n}^-_\chi$ | $[\mathsf{H}, \mathbf{B}] = \chi\mathbf{B} + \mathbf{P}$ | $[\mathsf{H}, \mathbf{P}] = \chi\mathbf{P} - \mathbf{B}$ | | | | $\chi \geqslant 0$ |
| $\mathfrak{c}$ | | | | $[\mathbf{B}, \mathbf{P}] = \mathsf{H}$ | | |
| $\mathfrak{iso}(d,1)$<br>$\mathfrak{iso}(d+1)$ | $[\mathsf{H}, \mathbf{B}] = -\varepsilon\mathbf{P}$ | | $[\mathbf{B}, \mathbf{B}] = \varepsilon\mathbf{L}$ | $[\mathbf{B}, \mathbf{P}] = \mathsf{H}$ | | $\varepsilon = \pm 1$ |
| $\mathfrak{so}(d+1, 1)$ | $[\mathsf{H}, \mathbf{B}] = \mathbf{B}$ | $[\mathsf{H}, \mathbf{P}] = -\mathbf{P}$ | | $[\mathbf{B}, \mathbf{P}] = \mathsf{H} + \mathbf{L}$ | | |
| $\mathfrak{so}(d,2)$<br>$\mathfrak{so}(d+2)$ | $[\mathsf{H}, \mathbf{B}] = -\varepsilon\mathbf{P}$ | $[\mathsf{H}, \mathbf{P}] = \varepsilon\mathbf{B}$ | $[\mathbf{B}, \mathbf{B}] = \varepsilon\mathbf{L}$ | $[\mathbf{B}, \mathbf{P}] = \mathsf{H}$ | $[\mathbf{P}, \mathbf{P}] = \varepsilon\mathbf{L}$ | $\varepsilon = \pm 1$ |

(Not shown are the Lie brackets involving the rotations, which are common to all KLAs)

# Spatially isotropic homogeneous spacetimes

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- A **Klein pair** $(\mathfrak{g}, \mathfrak{h})$ is **effective** if $\mathfrak{h}$ does not contain any (nonzero) ideal of $\mathfrak{g}$
- It is **geometrically realisable** if there exists a Lie group $G$ with Lie algebra $\mathfrak{g}$ such that the connected Lie subgroup corresponding to $\mathfrak{h}$ is closed
- Two Klein pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$ are **isomorphic** if there is a Lie algebra isomorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$ with $\varphi(\mathfrak{h}) = \mathfrak{h}'$

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- They fall into a small number of geometries: **riemannian**, **lorentzian**, **galilean**, **carrollian**, **aristotelian**,...
- Aristotelian Lie algebras have no boosts: they are the $(1,1)$-KLAs.
- An aristotelian Klein pair $(\mathfrak{a}, \mathfrak{r})$ is automatically effective and geometrically realisable.

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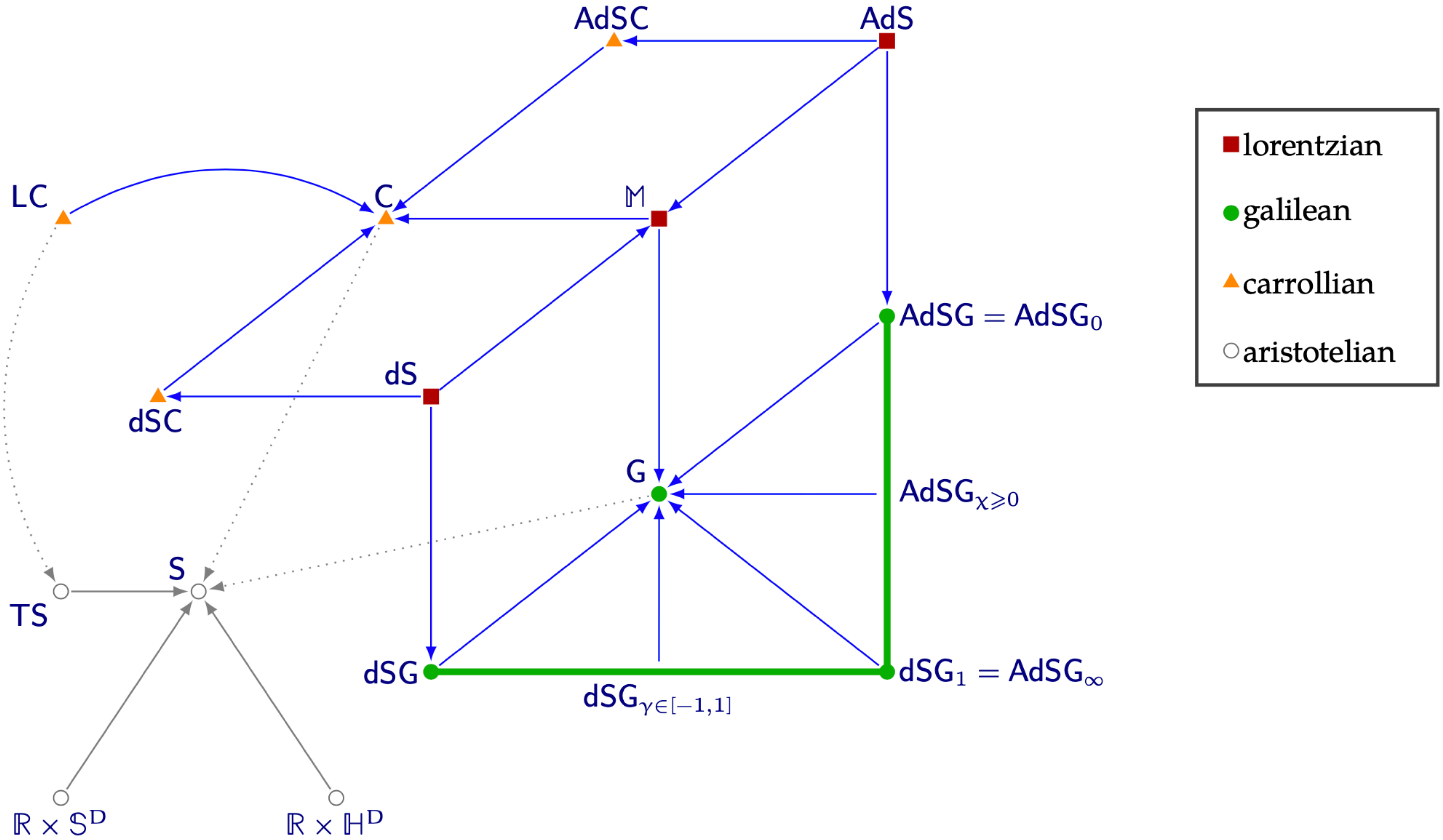
- It is the Cartan geometry modelled on Galilei spacetime

# Spatially isotropic homogeneous spacetimes

$(d+1)$-dimensional and generic $d$ and in a basis where $\mathfrak{h} = \langle L_{ab}, B_a \rangle$

| Name | Klein pair | Nonzero Lie brackets in addition to $[\mathbf{L},\mathbf{L}]=\mathbf{L}$, $[\mathbf{L},\mathbf{B}]=\mathbf{B}$, $[\mathbf{L},\mathbf{P}]=\mathbf{P}$ | | | | |
|---|---|---|---|---|---|---|
| Minkowski | $(\mathfrak{iso}(d,1),\mathfrak{so}(d,1))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | | $[\mathbf{B},\mathbf{B}]=\mathbf{L}$ | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | |
| de Sitter | $(\mathfrak{so}(d+1,1),\mathfrak{so}(d,1))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=-\mathbf{B}$ | $[\mathbf{B},\mathbf{B}]=\mathbf{L}$ | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | $[\mathbf{P},\mathbf{P}]=-\mathbf{L}$ |
| anti de Sitter | $(\mathfrak{so}(d,2),\mathfrak{so}(d,1))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=\mathbf{B}$ | $[\mathbf{B},\mathbf{B}]=\mathbf{L}$ | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | $[\mathbf{P},\mathbf{P}]=\mathbf{L}$ |
| euclidean | $(\mathfrak{iso}(d+1),\mathfrak{so}(d+1))$ | $[\mathrm{H},\mathbf{B}]=\mathbf{P}$ | | $[\mathbf{B},\mathbf{B}]=-\mathbf{L}$ | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | |
| sphere | $(\mathfrak{so}(d+2),\mathfrak{so}(d+1))$ | $[\mathrm{H},\mathbf{B}]=\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=-\mathbf{B}$ | $[\mathbf{B},\mathbf{B}]=-\mathbf{L}$ | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | $[\mathbf{P},\mathbf{P}]=-\mathbf{L}$ |
| hyperbolic | $(\mathfrak{so}(d+1,1),\mathfrak{so}(d+1))$ | $[\mathrm{H},\mathbf{B}]=\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=\mathbf{B}$ | $[\mathbf{B},\mathbf{B}]=-\mathbf{L}$ | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | $[\mathbf{P},\mathbf{P}]=\mathbf{L}$ |
| Galilei | $(\mathfrak{g},\mathfrak{iso}(d))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | | | | |
| de Sitter–Galilei | $(\mathfrak{n}^+_{\gamma=-1},\mathfrak{iso}(d))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=-\mathbf{B}$ | | | |
| torsional de Sitter–Galilei | $(\mathfrak{n}^+_{\gamma\in(-1,1)},\mathfrak{iso}(d))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=\gamma\mathbf{B}+(1+\gamma)\mathbf{P}$ | | | |
| torsional de Sitter–Galilei | $(\mathfrak{n}^0,\mathfrak{iso}(d))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=\mathbf{B}+2\mathbf{P}$ | | | |
| anti de Sitter–Galilei | $(\mathfrak{n}^-_{\chi=0},\mathfrak{iso}(d))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=\mathbf{B}$ | | | |
| torsional anti de Sitter–Galilei | $(\mathfrak{n}^-_{\chi>0},\mathfrak{iso}(d))$ | $[\mathrm{H},\mathbf{B}]=-\mathbf{P}$ | $[\mathrm{H},\mathbf{P}]=(1+\chi^2)\mathbf{B}+2\chi\mathbf{P}$ | | | |
| Carroll | $(\mathfrak{c},\mathfrak{iso}(d))$ | | | | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | |
| de Sitter–Carroll | $(\mathfrak{iso}(d+1),\mathfrak{iso}(d))$ | | $[\mathrm{H},\mathbf{P}]=-\mathbf{B}$ | | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | $[\mathbf{P},\mathbf{P}]=-\mathbf{L}$ |
| anti de Sitter–Carroll | $(\mathfrak{iso}(d,1),\mathfrak{iso}(d))$ | | $[\mathrm{H},\mathbf{P}]=\mathbf{B}$ | | $[\mathbf{B},\mathbf{P}]=\mathrm{H}$ | $[\mathbf{P},\mathbf{P}]=\mathbf{L}$ |
| lightcone | $(\mathfrak{so}(d+1,1),\mathfrak{iso}(d))$ | $[\mathrm{H},\mathbf{B}]=\mathbf{B}$ | $[\mathrm{H},\mathbf{P}]=-\mathbf{P}$ | | $[\mathbf{B},\mathbf{P}]=\mathrm{H}+\mathbf{L}$ | |



AdSC
AdS
LC
C
$\mathbb{M}$
dS
dSC
G
AdSG = AdSG$_0$
AdSG$_{\chi \geqslant 0}$
S
TS
dSG
dSG$_1$ = AdSG$_\infty$
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# Breaking isotropy

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- Let $\mathfrak{k} = \mathfrak{so}(V) \oplus 2V \oplus \mathbb{R}$ be a kinematical Lie algebra with spatial isotropy: $\mathrm{SO}(V)$ acts transitively on $\mathbb{P}V$ and consider an effective geometrically realisable Klein pair $(\mathfrak{k}, \mathfrak{h})$

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- Fix a line $\ell = \mathbb{R}v_0 \in \mathbb{P}V$
- Let $\mathfrak{g} \subset \mathfrak{k}$ be the centraliser of $\ell$
- There are precisely three spatially isotropic Klein pairs $(\mathfrak{k}, \mathfrak{h})$ for which $(\mathfrak{g}, \mathfrak{h} \cap \mathfrak{g})$ is the Klein pair of a homogeneous spacetime: those of Minkowski, Galilei and Carroll spacetimes

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# Two kinds of Lie algebras

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- So we get either a $(2,2)-$KLA or a $(3,2)-$KLA
- This suggests classifying $(2,2)-$ and $(3,2)-$KLAs

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(Anisotropy under scaling)

JMF+Grassie+Prohazka (2022)

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JMF+Grassie+Prohazka (2022)

# Boost-extended Lifshitz Lie algebras

(Adding boosts of Lifshitz Lie algebras)

JMF+Fluxman+Grassie+Prohazka (w.i.p.)

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JMF+Fluxman+Grassie+Prohazka (w.i.p.)

# Classification of $(3,2)-$KLAs

JMF+Lorenzo-Naveiro (w.i.p.)

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- For $n = 1$, the adjoint is trivial and the scalar and vector are isomorphic: any 5-dimensional real Lie algebra is a $(3,2)-$KLA

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- We will concentrate in $n \geq 3$ which we will break up into the *generic* $n > 3$ case and the *special* $n = 3$ case
- Recall that these are transitive Lie algebras for coisotropy-one homogeneous spacetimes of dimension $n + 2$

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$$
\begin{aligned}
[X, v \otimes w] &= Xv \otimes w \\
[B, v \otimes w] &= v \otimes \rho(B)w
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Let $(\mathfrak{b}, \rho, \alpha, \varphi)$ and $(\mathfrak{b}', \rho', \alpha', \varphi')$ be the data of two $(3,2)-$KLAs $\mathfrak{g}$ and $\mathfrak{g}'$ with $n = \dim V > 3$.

Fix an $\mathfrak{so}(V)$ subalgebra of $\mathfrak{g}$ and $\mathfrak{g}'$ and consider w.l.o.g. isomorphisms $f\colon \mathfrak{g} \to \mathfrak{g}'$ which are the identity on $\mathfrak{so}(V)$: **relative isomorphisms**. Every relative isomorphism $f$ is given by

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$$\rho' \circ \psi = \mathrm{Ad}(h) \circ \rho, \qquad \alpha = h^*\alpha', \qquad \psi \circ \varphi = h^*\varphi'$$

# Remark about equivalent representations

The condition

$$\rho' \circ \psi = \mathrm{Ad}(h) \circ \rho$$

is ***strictly weaker*** than equivalence of the representations $\rho$ and $\rho'$.

So we need only classify real 2-dimensional representations of Bianchi Lie algebras up to the above notion of equivalence.

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There are tables in the paper with all isomorphism classes of such Lie algebras.

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where now $\alpha\colon W \times W \to \mathbb{R}$ is uniquely characterised by

$$\rho\left(\varphi(w_1, w_2)\right) w_3 + \sigma(\sigma(w_2, w_3), w_1) - \sigma(\sigma(w_1, w_3), w_2) = \alpha(w_2, w_3) w_1 - \alpha(w_1, w_3) w_2$$

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2. $0 \neq [V \otimes W, V \otimes W] \subset \mathfrak{b}$, so that $\varphi \neq 0$ but $\alpha = 0$ and $\sigma = 0$
3. $0 \neq [V \otimes W, V \otimes W] \subset \mathfrak{so}(V) \oplus \mathfrak{b}$, so that $\alpha \neq 0$ but $\sigma = 0$
4. $V \otimes W$ is a nonabelian ideal, so that $\alpha = 0$ and $\varphi = 0$, but $\sigma \neq 0$

# Classification for $n = 3$

- Let $\mathfrak{g}$ be a $(3,2)-$KLA with $n = 3$ with data $(\mathfrak{b}, \rho, \alpha, \varphi, \sigma)$. Then up to a relative isomorphism, $\mathfrak{g}$ belongs to exactly one of the following four classes of Lie algebras:

1. $V \otimes W$ is abelian, so that $\varphi = 0$ and $\sigma = 0$ (and hence $\alpha = 0$)

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4. $V \otimes W$ is a nonabelian ideal, so that $\alpha = 0$ and $\varphi = 0$, but $\sigma \neq 0$

Tables are given in the paper of the isomorphism classes of such $(3,2)-$KLAs.

# Outlook

# Next steps in the programme

**(Work in progress with Juanma Lorenzo-Naveiro)**

# Next steps in the programme

- Classification of $(3,2)-$KLAs with $n = \dim V = 2$

(Work in progress with Juanma Lorenzo-Naveiro)

# Next steps in the programme

- Classification of $(3,2)-$KLAs with $n = \dim V = 2$
- Classification of effective, geometrically realisable Klein pairs $(\mathfrak{g}, \mathfrak{h})$ where

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# Next steps in the programme

- Classification of $(3,2)-$KLAs with $n = \dim V = 2$
- Classification of effective, geometrically realisable Klein pairs $(\mathfrak{g}, \mathfrak{h})$ where
  - $\mathfrak{g}$ is a $(2,2)-$KLA and $\mathfrak{h} = \mathfrak{so}(V) \oplus V$ (as a vector space)

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  - $\mathfrak{g}$ is a $(3,2)-$KLA and $\mathfrak{h} = \mathfrak{so}(V) \oplus V \oplus \mathbb{R}$ (as a vector space)

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# Next steps in the programme

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  - $\mathfrak{g}$ is a $(3,2)-$KLA and $\mathfrak{h} = \mathfrak{so}(V) \oplus V \oplus \mathbb{R}$ (as a vector space)
- For each such Klein pair, study the linear isotropy representation and determine the invariant geometric structures on the coisotropy-one homogeneous spacetimes

**(Work in progress with Juanma Lorenzo-Naveiro)**

Watch this space!